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Continuous wavelet transforms and non-commutative Fourier analysis

By

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Abstract

We discuss continuous wavelet transforms for the semidirect product group of a unimodular (not necessarily commutative) normal subgroup $N$ with a closed subgroup $H$ of Aut($N$), which is a generalization of the wavelet theory for an affine transformation group on a vector space. The operator-valued Fourier transform for $N$ plays a substantial role in the arguments.

§1. Introduction

Let $G$ be a locally compact group, and $(\pi, \mathcal{H})$ an irreducible unitary representation of $G$. The representation $\pi$ is said to be square-integrable if there exists a vector $\phi \in \mathcal{H}$ for which $\int_G |(\pi(g)\phi|\phi)_{\mathcal{H}}|^2 dg < +\infty$, where $dg$ is a left Haar measure on $G$. Such $\phi$ is called an admissible vector of $\pi$. If $(\pi, \mathcal{H})$ is square-integrable, there exists a unique positive self-adjoint operator (the Duflo-Moore operator) $C_\pi$ with the following two properties ([7], [13]):
(C1) $\phi$ is admissible $\iff \phi \in \text{dom}(C_\pi)$,
(C2) For $f_1, f_2 \in \mathcal{H}$ and $\phi_1, \phi_2 \in \text{dom}(C_\pi)$, one has

$$\int_G (f_1|\pi(g)\phi_1)_{\mathcal{H}} (\pi(g)\phi_2|f_2)_{\mathcal{H}} \, dg = (f_1|f_2)_{\mathcal{H}} (C_\pi\phi_1|C_\pi\phi_2)_{\mathcal{H}}.$$

When $G$ is unimodular, $C_\pi$ is a scalar operator. Furthermore, if $G$ is a compact group, then $C_\pi = (\dim \mathcal{H})^{-1/2}\text{Id}$. Thus, $C_\pi^{-2}$ is called the formal degree of $\pi$ in general.
Regarding the equality in (C2) as an identity for \( f_2 \in \mathcal{H} \), we obtain

\[
(1.1) \quad f_1 = \frac{1}{(C_{\pi}\phi_1 | C_{\pi}\phi_2)_{\mathcal{H}}} \int_G (f_1 | \pi(g)\phi_1)_{\mathcal{H}} \pi(g)\phi_2 \, dg,
\]

where the integral is taken in the weak sense. For an admissible vector \( \phi \in \text{dom}(C_{\pi}) \), the continuous wavelet transform \( W_\phi \) is defined as a linear map from the Hilbert space \( \mathcal{H} \) into the space \( C(G) \) of continuous functions on the group \( G \) defined by

\[
W_\phi f(g) := (f | \pi(g)\phi)_{\mathcal{H}} \quad (f \in \mathcal{H}, \ g \in G).
\]

Then (1.1) tells us that the inverse formula of \( W_\phi \) is given by

\[
(1.2) \quad f = \frac{1}{c_\phi} \int_G W_\phi f(g) \pi(g)\phi \, dg,
\]

where \( c_\phi = \|C_{\pi}\phi\|^2_{\mathcal{H}} \).

For instance, let us consider the affine transformation group \( G_{\text{aff}} \) on the real line \( \mathbb{R} \) consisting of the maps \( g_{b,a} : \mathbb{R} \ni x \mapsto ax + b \in \mathbb{R} \) with \( a \in \mathbb{R} \setminus \{0\} \) and \( b \in \mathbb{R} \). The unitary representation \( L \) of \( G_{\text{aff}} \) is defined on the Hilbert space \( L^2(\mathbb{R}) \) by \( L(g_{b,a})f(x) := |a|^{-1/2} f(g_{b,a}^{-1}x) \) \( (x \in \mathbb{R}) \). This representation is square-integrable, and the Duflo-Moore operator is given by

\[
(C_L\phi)^\wedge(\xi) = \sqrt{\frac{2\pi}{|\xi|}} \hat{\phi}(\xi) \quad (\xi \in \mathbb{R}),
\]

where \( \wedge \) stands for the Fourier transform given by \( \hat{\phi}(\xi) := \int_{\mathbb{R}} e^{ix\xi} \phi(x) \, dx \) \( (\xi \in \mathbb{R}) \). Then \( c_\phi = \|C_L\phi\|^2 \) equals \( \int_{\mathbb{R}\setminus\{0\}} |\hat{\phi}(\xi)|^2 |\xi|^{-1} \, d\xi \), and if \( c_\phi < +\infty \), the equality (1.2) yields the Calderón formula

\[
(1.3) \quad f = \frac{1}{c_\phi} \int_{\mathbb{R}} \int_{\mathbb{R}^2} W_\phi f(b,a) L(g_{b,a})\phi \frac{db \, da}{|a|^2} \quad (f \in L^2(\mathbb{R})),
\]

where \( W_\phi f(b,a) = W_\phi f(g_{b,a}) = |a|^{-1/2} \int_{\mathbb{R}} f(x) \overline{\phi(\frac{x-b}{a})} \, dx \).

The results about the wavelet transform for \( G_{\text{aff}} \) have been generalized to a multidimensional affine group \( G = H \rtimes \mathbb{R}^n \), where \( H \) is a closed subgroup of \( GL(\mathbb{R}^n) \) (see [4], [8] and [10] for example). In this article, we consider a further generalization to the case that \( G \) is a semidirect product group \( N \rtimes H \), where \( N \) is a unimodular (not necessarily commutative) group, and \( H \) is a closed subgroup of \( \text{Aut}(N) \) satisfying certain conditions. Although the situation becomes complicated if \( N \) is not commutative, the argument goes in parallel with the commutative case, where the operator-valued Fourier transform for \( N \) plays a substantial role instead of the ordinary Fourier transform.

The content of Sections 2–4 is essentially a summary of [16], while Section 5 is devoted to discuss a concrete example that \( N \) is the Heisenberg group. Such Heisenberg
case is already studied by He-Liu [14], whereas we shall present a new example of admissible vector $\phi_{\pm,\alpha} \in L^2(N)$. In this article, we write $\mathbb{T}$ for the set of complex numbers with absolute value 1. For a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$, (resp. $\mathcal{B}_{\mathrm{HS}}(\mathcal{H})$, $\mathcal{B}_{\mathrm{T}r}(\mathcal{H}), \mathcal{U}(\mathcal{H})$) the space of bounded (resp. Hilbert-Schmidt, trace class, unitary) operators on $\mathcal{H}$.

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§ 2. Preliminaries

Let $N$ be a separable locally compact unimodular group of type I. We denote by $\hat{N}$ the unitary dual of $N$, that is, the set of equivalence classes of irreducible unitary representations of $N$. For each $\lambda \in \hat{N}$, we take a unitary representation $(\pi_{\lambda}, \mathcal{H}_{\lambda})$ for which $\lambda$ equals the equivalence class $[\pi_{\lambda}]$ of $\pi_{\lambda}$. Let us fix a Haar measure $\nu$ on $N$. For $f \in L^1(N)$, we define the bounded operator $\pi_{\lambda}(f) \in \mathcal{B}(\mathcal{H}_{\lambda})$ by $\pi_{\lambda}(f) := \int_{N} f(n) \pi_{\lambda}(n) d\nu(n)$. It is known that the Plancherel measure $\mu$ on $\hat{N}$ is uniquely determined by the abstract Plancherel formula [6]:

$$\int_{N} |f(n)|^2 d\nu(n) = \int_{\hat{N}} \|\pi_{\lambda}(f)\|_{\mathrm{HS}}^2 d\mu(\lambda) \quad (f \in L^1(N) \cap L^2(N)), \tag{2.1}$$

where $\| \cdot \|_{\mathrm{HS}}$ stands for the Hilbert-Schmidt norm. We define the operator-valued Fourier transform $\mathbf{F} : L^2(N) \rightarrow \bigoplus_{\lambda \in \hat{N}} \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_{\lambda}) d\mu(\lambda)$ as the unitary isomorphism which is the extension of the map $L^1(N) \cap L^2(N) \ni f \mapsto (\pi_{\lambda}(f))_{\lambda \in \hat{N}} \in \bigoplus_{\lambda \in \hat{N}} \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_{\lambda}) d\mu(\lambda)$. The inverse formula of $\mathbf{F}$ is given as follows:

**Proposition 2.1** ([11, Theorem 4.15]). Let $(A(\lambda))_{\lambda \in \hat{N}}$ be an element of the direct integral $\int_{\hat{N}}^\oplus \mathcal{B}_{\mathrm{T}r}(\mathcal{H}_{\lambda}) d\mu(\lambda)$ of the Banach space $\mathcal{B}_{\mathrm{T}r}(\mathcal{H}_{\lambda})$. Define a function $f$ on $N$ by

$$f(n) := \int_{\hat{N}} \operatorname{tr} A(\lambda) \pi_{\lambda}(n)^* d\mu(\lambda) \quad (n \in N).$$

Then $f$ belongs to $L^2(N)$ if and only if $(A(\lambda))_{\lambda \in \hat{N}} \in \int_{\hat{N}}^\oplus \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_{\lambda}) d\mu(\lambda)$. In that case, one has $\mathbf{F} f(\lambda) = A(\lambda)$ (a.a. $\lambda \in \hat{N}$).

Let $H$ be a closed subgroup of the automorphism group $\operatorname{Aut}(N)$ on $N$, and $G$ the semidirect product group $N \rtimes H$. We write the action of $h$ to $n \in N$ as $h \cdot n$. For $h \in H$, we have a positive number $\delta(h)$ for which $d\nu(h \cdot n) = \delta(h) d\nu(n) \ (n \in N)$. Clearly, $\delta : H \rightarrow \mathbb{R}_+$ is a representation of $H$. Define the unitary representation $L$ of $G$ on $L^2(N)$ by

$$L(h)f(n_0) := \delta(h)^{-1/2} f(h^{-1} \cdot n_0),$$

$$L(n)f(n_0) := f(n^{-1} n_0) \quad (f \in L^2(N), h \in H, n, n_0 \in N). \tag{2.2}$$
It is easy to see that the representation $L$ is equivalent to the induced representation $\text{Ind}_H^G 1$, where $1$ is the trivial representation of $H$.

We define the action of $H$ on the unitary dual $\hat{N}$ by $h \cdot \lambda := [\pi_{\lambda} \circ h^{-1}]$ ($h \in H$, $\lambda \in \hat{N}$). Then we have a unitary operator $C(h, \lambda) : \mathcal{H}_\lambda \to \mathcal{H}_{h \cdot \lambda}$ with the property

$$C(h, \lambda)\pi_{\lambda}(h^{-1} \cdot n) = \pi_{h \cdot \lambda}(n)C(h, \lambda) \quad (n \in N).$$

The operator $C(h, \lambda)$ is unique up to multiple by elements of $T$ by Schur’s lemma. Moreover, for $h, h' \in H$ and $\lambda \in \hat{N}$, we have

$$C(hh', \lambda) = s_{h, h', \lambda}C(h, h' \cdot \lambda)C(h', \lambda)$$

with $s_{h, h', \lambda} \in T$. Thus we have a uniquely determined operator $D(h, \lambda) : \mathcal{B}(\mathcal{H}_\lambda) \ni T \mapsto C(h, \lambda)TC(h, \lambda)^* \in \mathcal{B}(\mathcal{H}_{h \cdot \lambda})$, which satisfies the chain rule

$$D(hh', \lambda) = D(h, h' \cdot \lambda)D(h', \lambda).$$

Using the operator-valued Fourier transform $\mathbf{F}$, we describe the representation $(L, L^2(N))$ of $G$ as follows:

**Proposition 2.2.** For $f \in L^2(N)$, $h \in H$ and $n \in N$, one has

$$\mathbf{F}[L(h)f](\lambda) = \delta(h)^{1/2}D(h, h^{-1} \cdot \lambda)\mathbf{F}f(h^{-1} \cdot \lambda),$$

$$\mathbf{F}[L(n)f](\lambda) = \pi_{\lambda}(n)\mathbf{F}f(\lambda)$$

for almost all $\lambda \in \hat{N}$ with respect to the Plancherel measure $\mu$.

§ 3. Decomposition of $L^2(N)$

For $\lambda \in \hat{N}$, let $O^*_\lambda \subset \hat{N}$ be the $H$-orbit through $\lambda$. Now let us assume that there exist elements $\lambda_1, \ldots, \lambda_K \in \hat{N}$ satisfying the following conditions:

(H1) $\mu(O^*_\lambda) > 0$ ($k = 1, \ldots, K$),

(H2) The stabilizer $H_k := \{h \in H ; h \cdot \lambda_k = \lambda_k\}$ is compact for all $k = 1, \ldots, K$.

(H3) The map $H/H_k \ni hH_k \mapsto h \cdot \lambda_k \in O^*_\lambda$ is a homeomorphism for $k = 1, \ldots, K$, where the topology on $O^*_\lambda$ is induced from the Fell topology on $\hat{N}$.

(H4) $O^*_\lambda \cap O^*_\lambda = \emptyset$ ($k \neq l$) and $\mu(\hat{N} \setminus \bigcup_{k=1}^K O^*_\lambda) = 0$.

Under this assumption, we shall see in Sections 3 and 4 that the unitary representation $(L, L^2(N))$ of $G$ is decomposed into the direct sum of countably many irreducible subrepresentations, and such subrepresentations are all square-integrable. Their Duflo-Moore operators are described by using the operator-valued Fourier transform $\mathbf{F}$.

Thanks to (2.3), we have a projective unitary representation $\tau_k : H_k \ni h \mapsto C(h, \lambda_k) \in U(\mathcal{H}_{\lambda_k})$ of the group $H_k$ for $k = 1, \ldots, K$. Since $H_k$ is compact, we have an
irreducible decomposition \( \mathcal{H}_{\lambda_k} = \bigoplus_{\alpha \in A_k} \mathcal{H}_{\lambda_k, \alpha} \), where \( A_k \) is an at most countable index set. The subspaces \( \mathcal{H}_{\lambda_k, \alpha} \) are finite dimensional. For \( \lambda = \tilde{h} \cdot \lambda_k \in \mathcal{O}_{\lambda_k}^* \) with \( \tilde{h} \in H \), we put \( \mathcal{H}_{\lambda, \alpha} := C(\tilde{h}, \lambda_k) \mathcal{H}_{\lambda_k, \alpha} \) (\( \alpha \in A_k \)), where the right-hand side is independent of the choice of \( \tilde{h} \). Moreover we have \( \mathcal{H}_\lambda = \bigoplus_{\alpha \in A_k} \mathcal{H}_{\lambda, \alpha} \), which gives an irreducible decomposition of the projective representation \( \tau_\lambda : H \ni h \mapsto C(h, \lambda) \in U(\mathcal{H}_\lambda) \) of the compact group \( H_\lambda := \{ h \in H ; h \cdot \lambda = \lambda \} \). On the other hand, the relation
\[
(3.1) \quad C(h, \lambda) \mathcal{H}_{\lambda, \alpha} = \mathcal{H}_{h \cdot \lambda, \alpha}
\]
holds for \( h \in H, \lambda \in \mathcal{O}_{\lambda_k}^* \) and \( \alpha \in A_k \) in general. Using the orthogonal projection \( P_{\lambda, \alpha} : \mathcal{H}_{\lambda} \to \mathcal{H}_{\lambda, \alpha} \), we define
\[
(3.2) \quad \mathcal{B}_{\lambda, \alpha} := \{ T \in \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_{\lambda}) ; TP_{\lambda, \alpha} = T \}.
\]
If we identify \( \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_{\lambda}) \) with the tensor product \( \mathcal{H}_{\lambda} \otimes \overline{\mathcal{H}}_{\lambda} \), the space \( \mathcal{B}_{\lambda, \alpha} \) is nothing but \( \mathcal{H}_{\lambda} \otimes \overline{\mathcal{H}}_{\lambda, \alpha} \). Thus we see that
\[
(3.3) \quad \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_{\lambda}) = \bigoplus_{\alpha \in A_k} \mathcal{B}_{\lambda, \alpha},
\]
while (3.1) yields
\[
(3.4) \quad D(h, \lambda) \mathcal{B}_{\lambda, \alpha} = \mathcal{B}_{h \cdot \lambda, \alpha}.
\]
Now we set
\[
L_{k, \alpha}(N) := \mathbf{F}^{-1} \left( \int_{\mathcal{O}_{\lambda_k}^*} \mathcal{B}_{\lambda, \alpha} \, d\mu(\lambda) \right)
\]
\[= \left\{ f \in L^2(N) ; \begin{aligned}
\mathbf{F} f(\lambda) &= \mathbf{F} f(\lambda) P_{\lambda, \alpha} \\
\mathbf{F} f(\lambda) &= 0
\end{aligned} \quad \text{(a.a. \( \lambda \in \mathcal{O}_{\lambda_k}^* \))}
\]
\[\text{otherwise)} \right\}
\]
for \( k = 1, \ldots, K \) and \( \alpha \in A_k \). By (3.3) and (H4), we have
\[
(3.6) \quad L^2(N) = \bigoplus_{1 \leq k \leq K} \bigoplus_{\alpha \in A_k} L_{k, \alpha}(N),
\]
and each \( L_{k, \alpha}(N) \) is \( G \)-invariant thanks to Proposition 2.2 and (3.4).

\section*{4. Main results}

Fixing a left Haar measure \( d_H \) on \( H \), we define a left Haar measure \( d_G \) on \( G \) by \( d_G(nh) := \delta(h)^{-1} \, dv(n) \, d_H(h) \) (\( n \in N, h \in H \)). Let us consider the square-integrability
of the matrix coefficients \((f|L(g)\phi)\) \((f, \phi \in L_{k, \alpha}(N), g \in G)\) of the representation \((L, L_{k, \alpha}(N))\) of \(G\). By (2.1) and (2.5), we have

\[
(f|L(n)L(h)\phi) = \int_{O_{\lambda_{k}}^{*}} \text{tr} \ Ff(\lambda)F[L(n)L(h)\phi](\lambda)^{*} d\mu(\lambda)
\]

\[
= \int_{O_{\lambda_{k}}^{*}} \text{tr} \ (Ff(\lambda)F[L(h)\phi](\lambda)^{*})\pi_{\lambda}(n)^{*} d\mu(\lambda).
\]

We note that the operator \(Ff(\lambda)F[L(h)\phi](\lambda)^{*}\) is of trace class because both \(Ff(\lambda)\) and \(F[L(h)\phi](\lambda)\) are Hilbert-Schmidt operators. Furthermore, \((Ff(\lambda)F[L(h)\phi](\lambda)^{*})_{\lambda \in O_{\lambda_{k}}^{*}}\) belongs to \(\int_{O_{\lambda_{k}}^{*}}^{\oplus} B_{\text{Tr}}(\mathcal{H}_{\lambda}) d\mu(\lambda)\) because

\[
\int_{O_{\lambda_{k}}^{*}} \| Ff(\lambda)F[L(h)\phi](\lambda)^{*} \|_{\text{Tr}} d\mu(\lambda)
\]

\[
\leq \int_{O_{\lambda_{k}}^{*}} \| Ff(\lambda) \|_{\text{HS}} \| F[L(h)\phi](\lambda) \|_{\text{HS}} d\mu(\lambda)
\]

\[
\leq \left\{ \int_{O_{\lambda_{k}}^{*}} \| Ff(\lambda) \|_{\text{HS}}^{2} d\mu(\lambda) \right\}^{1/2} \left\{ \int_{O_{\lambda_{k}}^{*}} \| F[L(h)\phi](\lambda) \|_{\text{HS}}^{2} d\mu(\lambda) \right\}^{1/2}
\]

\[
= \| f \| \| L(h)\phi \| = \| f \| \| \phi \| < +\infty,
\]

where \(\| \cdot \|_{\text{Tr}}\) denotes the Trace norm. Now we assume that

\[
\int_{G} |(f|L(g)\phi)|^{2} dG(g) = \int_{H} \int_{N} |(f|L(n)L(h)\phi)|^{2} \delta(h)^{-1} d\nu(n) d_{H}(h) < +\infty.
\]

Then \(\int_{N} |(f|L(n)L(h)\phi)|^{2} d\nu(n)\) is finite for almost all \(h \in H\). Thus, applying Proposition 2.1, we see from (4.1) and (2.1) that

\[
\int_{N} |(f|L(n)L(h)\phi)|^{2} d\nu(n) = \int_{O_{\lambda_{k}}^{*}} \| Ff(\lambda)F[L(h)\phi](\lambda)^{*} \|_{\text{HS}}^{2} d\mu(\lambda) \quad \text{(a.a. } h \in H)\).
\]

Therefore the integral \(\int_{G} |(f|L(g)\phi)|^{2} dG(g)\) is equal to

\[
\int_{H} \int_{O_{\lambda_{k}}^{*}} \| Ff(\lambda)F[L(h)\phi](\lambda)^{*} \|_{\text{HS}}^{2} \delta(h)^{-1} d\mu(\lambda) d_{H}(h).
\]

Now we observe that

\[
\int_{H} \| Ff(\lambda)F[L(h)\phi](\lambda)^{*} \|_{\text{HS}}^{2} \delta(h)^{-1} d_{H}(h)
\]

\[
= \int_{H} \int_{H} \left( \text{tr} \ Ff(\lambda)F[L(h_{1}h)\phi](\lambda)^{*} F[L(h_{1}h)\phi](\lambda) Ff(\lambda)^{*} \right) \delta(h_{1}h)^{-1} dh_{1}d_{H}(h),
\]
where $dh_1$ is the normalized Haar measure on the compact group $H_{\lambda}$. Since $L(h_1 h)\phi$ belongs to $L_{k, \alpha}(N)$, we have by (3.5)

$$P_{\lambda, \alpha}F[L(h_1 h)\phi](\lambda)^*F[L(h_1 h)\phi](\lambda)P_{\lambda, \alpha} = F[L(h_1 h)\phi](\lambda)^*F[L(h_1 h)\phi](\lambda),$$

which means that we can regard $F[L(h_1 h)\phi](\lambda)^*F[L(h_1 h)\phi](\lambda)$ as a linear operator on the finite dimensional vector space $H_{\lambda, \alpha}$. Furthermore, applying Schur’s lemma to the representation $(\tau_{\lambda}, H_{\lambda, \alpha})$ of $H_{\lambda}$, we get

$$\int_{H_{\lambda}} \delta(h_1 h)^{-1}F[L(h_1 h)\phi](\lambda)^*F[L(h_1 h)\phi](\lambda) dh_1 = (\dim H_{\lambda, \alpha})^{-1} \int_{H_{\lambda}} \delta(h)^{-1} \|F[L(h)\phi](\lambda)\|^2_{HS} P_{\lambda, \alpha} \in \text{End}(H_{\lambda, \alpha}),$$

see [16, pp. 43-44] for the detail. Note that $\dim H_{\lambda, \alpha} = \dim H_{\lambda, k, \alpha}$, which we denote by $n_{k, \alpha}$ in what follows. By (4.5) and (3.5), the integral (4.4) equals

$$\frac{1}{n_{k, \alpha}} \int_{\mathcal{O}_{\lambda_k}^*} \int_{H} (\text{tr} F[f(\lambda)P_{\lambda, \alpha}F[f(\lambda)]^*]) \|F[L(h)\phi](\lambda)\|^2_{HS}\delta(h)^{-1} d_H(h) d\mu(\lambda)$$

and the right-hand side is rewritten as

$$\frac{1}{n_{k, \alpha}} \int_{\mathcal{O}_{\lambda_k}^*} \|F[f(\lambda)]\|^2_{HS} \left(\int_{H} \|F[F(h^{-1}, \cdot \lambda)]\|^2_{HS} d(h)\right) d\mu(\lambda),$$

by (2.4). Thanks to [16, Proposition 3], the integral $\int_{H} \|F[F(h^{-1}, \cdot \lambda)]\|^2_{HS} d(h)$ does not depend on $\lambda \in \mathcal{O}_{\lambda_k}^*$, and it is equal to $\int_{\mathcal{O}_{\lambda_k}^*} \|F[F(\lambda)]\|^2_{HS} D_k(\lambda) d\mu(\lambda)$ with a certain $H$-relatively invariant function $D_k$ on $\mathcal{O}_{\lambda_k}^*$. Therefore we have

$$+\infty > \int_G |(f|L(g)\phi)|^2 d_G(g) = \int_{\mathcal{O}_{\lambda_k}^*} \|F[f(\lambda)]\|^2_{HS} d\mu(\lambda) \cdot \frac{1}{n_{k, \alpha}} \int_{H} \|F[F(h^{-1}, \cdot \lambda)]\|^2_{HS} d_H(h)$$

$$= \|f\|^2 \cdot \frac{1}{n_{k, \alpha}} \int_{\mathcal{O}_{\lambda_k}^*} \|F[F(\lambda)]\|^2_{HS} D_k(\lambda) d\mu(\lambda).$$

In particular, if $f \neq 0$, then

$$c(\phi) := \frac{1}{n_{k, \alpha}} \int_{\mathcal{O}_{\lambda_k}^*} \|F[F(\lambda)]\|^2_{HS} D_k(\lambda) d\mu(\lambda) < +\infty$$

and

$$\int_G |(f|L(g)\phi)|^2 d_G(g) = c(\phi) \|f\|^2.$$
Conversely, if $\phi$ satisfies the condition (4.6), the integral (4.4) converges for any $f \in L_{k,\alpha}(N)$, so that the right-hand side of (4.3) converges for almost all $h \in H$. Therefore Proposition 2.1 implies (4.3), so that we get (4.7) again and thus, $\int_{G} |(f|L(g)\phi)|^{2} \, d_{G}(g) < +\infty$.

**Theorem 4.1.** The unitary representation $(L, L_{k,\alpha}(N))$ of $G$ is irreducible.

**Proof.** Let $\mathcal{L}$ be a nonzero invariant subspace of $L_{k,\alpha}(N)$. The orthogonal complement $\mathcal{L}^\perp \subset L_{k,\alpha}(N)$ is also invariant. We take $\phi \in \mathcal{L} \setminus \{0\}$ and $f \in \mathcal{L}^\perp$. By (4.7) we have

$$0 = \int_{G} |(f|L(g)\phi)|^{2} \, d_{G}(g) = c(\phi) \|f\|^{2},$$

which implies $f = 0$. Actually, the argument is valid even if $c(\phi)$ is not finite. Therefore $\mathcal{L}^\perp = \{0\}$ and Theorem 4.1 is proved. \(\square\)

Furthermore, we deduce from (4.6) and (4.7) the following result.

**Theorem 4.2.** The representation $(L, L_{k,\alpha}(N))$ is square-integrable, whose Duflo-Moore operator $C_{k,\alpha}$ is given by

$$\mathbf{F}[C_{k,\alpha}\phi](\lambda) = \sqrt{\frac{D_{k}(\lambda)}{n_{k,\alpha}}} \mathbf{F}\phi(\lambda) \quad (\phi \in L_{k,\alpha}(N), \lambda \in O^*_k).$$

A classification of the representations $(L, L_{k,\alpha}(N))$ is also given in [16]. The result is as follows:

**Theorem 4.3** ([16, Theorem 4]). The unitary representations $(L, L_{k,\alpha}(N))$ and $(L, L_{k',\alpha'}(N))$ of $G$ are equivalent if and only if $k = k'$ and the projective representations $(\tau_{k}, \mathcal{H}_{k,\alpha})$ and $(\tau_{k}, \mathcal{H}_{k,\alpha'})$ of $H_{k}$ are equivalent, that is, there exists an isometry $A : \mathcal{H}_{k,\alpha} \rightarrow \mathcal{H}_{k,\alpha'}$ such that $\tau_{k}(h) \circ A = A \circ \tau_{k}(h)$ for all $h \in H_{k}$.

Keeping the decomposition (3.6) in mind, and applying (1.2) to each representation $(L, L_{k,\alpha}(N))$, we obtain

**Theorem 4.4.** For each $k = 1, \ldots, K$ and $\alpha \in A_{k}$, take admissible vectors $\phi_{k,\alpha} \in \text{dom}(C_{k,\alpha}) \subset L_{k,\alpha}(N)$. Then for all $f \in L^{2}(N)$ one has

$$f = \sum_{k=1}^{K} \sum_{\alpha \in A_{k}} \frac{1}{\|C_{k,\alpha}\phi_{k,\alpha}\|^{2}} \int_{G} W_{\phi_{k,\alpha}} f(g) L(g)\phi_{k,\alpha} \, d_{G}(g),$$

where $W_{\phi_{k,\alpha}} f(g) = (f|L(g)\phi_{k,\alpha})$. 
§ 5. Example

As an illustrative example, we shall consider the case that the unimodular group \( N \) is the Heisenberg group of \((2\ell + 1)\)-dimension and \( H \) is isomorphic to \( \mathbb{R}_+ \times U(\ell) \). The continuous wavelet transform for this case is first considered by He-Liu \([14]\).

Let \( N \) be the Lie group consisting of elements \( n(z, c) \) \((z \in \mathbb{C}^\ell, c \in \mathbb{R})\) with multiplication rule
\[
n(z, c)n(z', c') := (z + z', c + c' + \Im(z|z')) \quad (z, z' \in \mathbb{C}^\ell, c, c' \in \mathbb{R}),
\]
where \((z|z') := \sum_{k=1}^{\ell} z_k z_k'\). Define a Haar measure \( dv \) on \( N \) by \( dv(n(z, c)) := dm(z)dc \), where \( dm \) is the standard Euclidean measure on \( \mathbb{C}^\ell \).

For \( \zeta \in \mathbb{C}^\ell \), we define the unitary character \( \chi_\zeta \) by \( \chi_\zeta(n(z, c)) := e^{i\Re(z|\zeta)} \). For \( \lambda > 0 \), we define
\[
\begin{align*}
\mathcal{H}_{+\lambda} &:= \{ \varphi : \mathbb{C}^\ell \rightarrow \mathbb{C} \text{ (holomorphic)} ; \|\varphi\|^2 = \frac{\lambda^\ell}{\pi^\ell} \int_{\mathbb{C}^\ell} |\varphi(w)|^2 e^{-\lambda|w|^2} dm(w) < +\infty \}, \\
\mathcal{H}_{-\lambda} &= \{ \overline{\varphi} ; \varphi \in \mathcal{H}_{+\lambda} \}.
\end{align*}
\]
These \( \mathcal{H}_{\pm\lambda} \) are Hilbert spaces, on which we define the irreducible unitary representations \( \pi_{\pm\lambda} \) of \( N \) by
\[
\begin{align*}
\pi_{+\lambda}(n(z, c))\varphi(w) &:= e^{-i\lambda c + \lambda(z|w) - \lambda|z|^2/2}\varphi(w-z) \quad (\varphi \in \mathcal{H}_{+\lambda}), \\
\pi_{-\lambda}(n(z, c))\varphi(w) &:= e^{i\lambda c + \lambda(z|w) - \lambda|z|^2/2}\varphi(w-z) \quad (\varphi \in \mathcal{H}_{-\lambda}).
\end{align*}
\]
The Stone-von Neumann theorem states that every irreducible unitary representation \( \pi_\lambda \) of \( N \) is equivalent to one of \( \chi_\zeta \) \((\zeta \in \mathbb{C}^\ell)\) and \( \pi_\lambda \) \((\lambda \in \mathbb{R} \setminus \{0\})\), so that \( \hat{N} \) can be identified with \( \mathbb{C}^\ell \sqcup (\mathbb{R} \setminus \{0\}) \).

For \( f_1, f_2 \in L^1(N) \cap L^2(N) \), we have
\[
(f_1|f_2) = \frac{2^{\ell-1}}{\pi^{\ell+1}} \int_{\mathbb{R} \setminus \{0\}} \langle \pi_{+\lambda}(f_1)|\pi_{+\lambda}(f_2) \rangle_{\text{HS}} |\lambda|^\ell d\lambda
\]
by \([9, \text{Chapter I, section 5}]\), which implies that the Plancherel measure \( \mu \) on \( \hat{N} \) is given by \( \mu(\mathbb{C}^\ell) = 0 \) and \( d\mu(\lambda) = 2^{\ell-1} \pi^{\ell-1}|\lambda|^\ell d\lambda \) \((\lambda \in \mathbb{R} \setminus \{0\})\).

For \( a > 0 \) and \( u \in U(\ell) \), define \( h(a, u) \in \text{Aut}(N) \) by \( h(a, u) \cdot n(z, c) := n(auz, a^2c) \). Then we have
\[
h(a, u)h(a', u') = h(aa', uu') \quad (a, a' > 0, u, u' \in U(\ell)),
\]
so that \( H := \{ h(a, u) ; a > 0, u \in U(\ell) \} \) is a subgroup of \( \text{Aut}(N) \). For \( h = h(a, u) \in H \), we have \( \delta(h) = a^{2\ell+2} \), and we define the representation \((L, L^2(N))\) of \( G = N \rtimes H \) by (2.2). The action of \( H \) on \( \hat{N} \) is described as
\[
\begin{align*}
h(a, u) \cdot \zeta &= a^{-1}u\zeta \quad (\zeta \in \mathbb{C}^\ell), \\
h(a, u) \cdot \lambda &= a^{-2}\lambda \quad (\lambda \in \mathbb{R} \setminus \{0\}).
\end{align*}
\]
In particular, the intertwining operator \( C(h, \lambda) : \mathcal{H}_\lambda \to \mathcal{H}_{a^{-2}\lambda} \) is given by
\[
C(h, \lambda) \varphi(w) := \varphi(a^{-1}u^{-1}w) \quad (\varphi \in \mathcal{H}_\lambda).
\]
We denote by \( \mathcal{O}^{*}_\pm \) the \( H \)-orbit through \( \pm 1 \in \mathbb{R} \setminus \{0\} \subset \hat{N} \). Then \( \mathcal{O}^{*}_\pm = \{ \lambda ; \pm \lambda > 0 \} \), and the conditions (H1)–(H4) are satisfied for the two orbits. In particular, the stabilizer \( H_\lambda \) at any \( \pm 1 \in \mathbb{R} \setminus \{0\} \subset \hat{N} \) equals \( \{ h(1, u); u \in U(\ell) \} \simeq U(\ell) \).

For a non-negative integer \( \alpha \in \mathbb{Z}_{\geq 0} \), let \( \mathcal{P}_\alpha(\mathbb{C}^\ell) \) be the space of holomorphic polynomials of degree \( \alpha \) on \( \mathbb{C}^\ell \), and \( \overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell) \) the space \( \{ \overline{\varphi} ; \varphi \in \mathcal{P}_\alpha(\mathbb{C}^\ell) \} \). Then we have for \( \lambda > 0 \)
\[
\mathcal{H}_\lambda = \sum_{\alpha=0}^{\infty} \mathcal{P}_\alpha(\mathbb{C}^\ell) \infty, \quad \mathcal{H}_{-\lambda} = \sum_{\alpha=0}^{\infty} \overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell) \infty,
\]
which give the irreducible decomposition of the representation \( (\tau_{\pm\lambda}, \mathcal{H}_{\pm\lambda}) \) of \( H_{\pm\lambda} \) respectively. Let \( P_{\lambda,\alpha} : \mathcal{H}_\lambda \to \mathcal{P}_\alpha(\mathbb{C}^\ell) \) and \( P_{-\lambda,\alpha} : \mathcal{H}_{-\lambda} \to \overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell) \) be the orthogonal projections, and set
\[
L_{\pm,\alpha}(N) := \left\{ f \in L^2(N) ; \begin{array}{lcl}
\mathbf{F} f(\lambda) = \mathbf{F} f(\lambda) P_{\lambda,\alpha} & (\text{a.a. } \lambda \in \mathcal{O}^{*}_\pm) \\
\mathbf{F} f(\lambda) = 0 & (\text{otherwise}) \end{array} \right\}.
\]
Then an irreducible decomposition of the unitary representation \( (L, L^2(N)) \) of \( G \) is given by
\[
(5.1) \quad L^2(N) = \bigoplus_{\alpha=0}^{\infty} L_{+,\alpha}(N) \oplus \bigoplus_{\alpha=0}^{\infty} L_{-,\alpha}(N).
\]
The decomposition is multiplicity-free thanks to Theorem 4.3.

We define a Haar measure \( d_H \) on \( H \) by \( d_H(h(a, u)) := a^{-1} \, da \, du \), where \( du \) is the normalized Haar measure on \( U(\ell) \). For \( \lambda_0 \in \mathcal{O}^{*}_\pm \) and a measurable function \( p : \mathcal{O}^{*}_\pm \to \mathbb{R} \), we observe
\[
\int_H p(h^{-1} \cdot \lambda_0) d_H(h) = \int_0^{\infty} p(a^2 \lambda_0) \frac{da}{a} = \int_{\mathcal{O}^{*}_\pm} p(\lambda) \frac{d\lambda}{2|\lambda|} = \int_{\mathcal{O}^{*}_\pm} p(\lambda) \cdot 2 \left( \frac{\pi}{2|\lambda|} \right)^{\ell+1} \, d\mu(\lambda).
\]
Thus, setting \( D_{\pm}(\lambda) := 2 \left( \frac{\pi}{2|\lambda|} \right)^{\ell+1} \) \( (\lambda \in \mathcal{O}^{*}_\pm) \), the Duflo-Moore operator \( C_{\pm,\alpha} \) of the representation \( (L, L_{\pm,\alpha}(N)) \) is given by \( \mathbf{F}[C_{\pm,\alpha}\phi](\lambda) = \sqrt{\frac{D_{\pm}(\lambda)}{n_\alpha}} \mathbf{F}\phi(\lambda) \) \( (\phi \in L_{+,\alpha}(N), \lambda \in \mathcal{O}^{*}_\pm) \), where \( n_\alpha = \dim_{\mathbb{C}} \mathcal{P}_\alpha(\mathbb{C}^\ell) = \left( \begin{array}{c} \alpha + \ell - 1 \\ \alpha \end{array} \right) \).

Finally, we give an example of admissible vector in \( L_{\pm,\alpha}(N) \). We set
\[
\phi_{\pm,\alpha}(n(z, c)) := \int_{\mathcal{O}^{*}_\pm} |\lambda| e^{-|\lambda|} \, \text{tr} \, P_{\lambda,\alpha} \pi_\lambda(n(z, c))^* \, d\mu(\lambda).
\]
Then \((|\lambda|e^{-|\lambda|}P_{\lambda, \alpha})_{\lambda \in \mathcal{O}_\pm}\) belongs to both of the direct integrals \(\int_{\mathcal{O}_\pm} \mathcal{B}_{\text{Tr}}(\mathcal{H}_\lambda) \, d\mu(\lambda)\) and \(\int_{\mathcal{O}_\pm} \mathcal{B}_{\text{HS}}(\mathcal{H}_\lambda) \, d\mu(\lambda)\). Applying Proposition 2.1, we see that \(\phi_{\pm, \alpha} \in L^2(N)\) and that

\[
F\phi_{\pm, \alpha}(\lambda) = \begin{cases} 
|\lambda|e^{-|\lambda|}P_{\lambda, \alpha} & (\text{a.a. } \lambda \in \mathcal{O}_\pm), \\
0 & \text{(otherwise)}. 
\end{cases}
\]

Thus \(\phi_{\pm, \alpha} \in L_{\pm, \alpha}(N)\). Furthermore we have

\[
\|C_{\pm, \alpha}\phi_{\pm, \alpha}\|^2 = n_{\alpha}^{-1} \int_{\mathcal{O}_\pm} \|F\phi_{\pm, \alpha}(\lambda)\|_{\text{HS}}^2 \frac{d\lambda}{2|\lambda|} = n_{\alpha}^{-1} \int_{\mathcal{O}_\pm} n_{\alpha}|\lambda|^2e^{-2|\lambda|} \frac{d\lambda}{2|\lambda|}
\]

so that \(\phi_{\pm, \alpha}\) is admissible.

The function \(\phi_{\pm, \alpha}\) can be calculated explicitly. By definition, we have \(\phi_{-, \alpha} = \overline{\phi_{+, \alpha}}\), so that we shall consider only \(\phi_{+, \alpha}\). For non-negative integers \(k\) and \(\alpha\), the Laguerre polynomial \(L^k_\alpha(s)\) is defined by

\[
L^k_\alpha(s) := \frac{e^s s^{-k}}{\alpha!} \frac{d^\alpha}{ds^\alpha}[e^{-s}s^{k+\alpha}] .
\]

By [3, Proposition 6.2], we have

\[
\text{tr} P_{\lambda, \alpha} \pi_\lambda(n(z, c))^* = e^{(ic-|z|^2/2)\lambda} L^{\ell-1}_\alpha(|\lambda|z^2) .
\]

Thus

\[
\phi_{+, \alpha}(n(z, c)) = \frac{2^{\ell-1}}{\pi^{\ell+1}} \int_0^{+\infty} e^{-(1-ic+|z|^2/2)\lambda} L^{\ell-1}_\alpha(|\lambda|z^2) \lambda^{\ell+1} d\lambda.
\]

On the other hand, for a parameter \(|r| < 1\), we have

\[
\sum_{\alpha=0}^\infty r^\alpha L^{\ell-1}_\alpha(s) = (1-r)^{-\ell} e^{-\frac{r^2 s}{1-r}}.
\]

Therefore

\[
\sum_{\alpha=0}^\infty r^\alpha \phi_{+, \alpha}(n(z, c)) = \frac{2^{\ell-1}}{\pi^{\ell+1}} \int_0^{+\infty} e^{-(1-ic+|z|^2/2)\lambda} (1-r)^{-\ell} e^{-\frac{r^2 |\lambda|^2}{1-r}} \lambda^{\ell+1} d\lambda
\]

\[
= \frac{2^{\ell-1}}{\pi^{\ell+1}} (1-r)^{-\ell} \cdot (\ell+1)! \left\{ \left(1 - \frac{(\alpha + \ell + 1)!}{(\ell+1)!} \right) \frac{r |\lambda|^2}{1-r} \right\}^{-(\ell+2)}
\]

\[
= \frac{2^{\ell-1}(\ell+1)!}{\pi^{\ell+1}} (1 - ic + |z|^2/2)^{-(\ell+2)} 
\times (1-r)^2 \left(1 + \frac{1 + ic + |z|^2/2}{1 - ic + |z|^2/2} \cdot r\right)^{-(\ell+2)} .
\]

Putting \(\theta := \frac{-1+ic+|z|^2/2}{1-ic+|z|^2/2}\), we have by the binomial theorem

\[
(1-r)^2(1 + \theta r)^{-(\ell+2)}
\]

\[
= \sum_{\alpha=0}^\infty \frac{(-1)^\alpha(\alpha + \ell - 1)!}{(\ell+1)!\alpha!} \left\{ \alpha(\alpha - 1)\theta^{\alpha-2} + 2\alpha(\alpha + \ell)\theta^{\alpha-1} + (\alpha + \ell)(\alpha + \ell + 1)\theta^\alpha \right\} r^\alpha .
\]
Hence
\[
\phi_{+, \alpha}(n(z, c)) = \frac{\pi^{\ell+1} (1 - ic + |z|^2/2)^{-(\ell+2)}}{2^{\ell-1}} \times \frac{(-1)^\alpha (\alpha + \ell - 1)!}{\alpha!} \{\alpha(\alpha - 1)\theta^{\alpha - 2} + 2\alpha(\alpha + \ell)\theta^{\alpha - 1} + (\alpha + \ell)(\alpha + \ell + 1)\theta^\alpha\}.
\]

References


