RIMS Kôkyûroku Bessatsu
B21 (2010), 035–044

Time reversal symmetries and properties of ground states in nonrelativistic QED

By

Michael Loss*, Tadahiro Miyao**and Herbert Spohn***

Abstract

Time reversal symmetries of the Pauli-Fierz Hamiltonian are investigated in both Fock and Schrödinger representations. As an application, we investigate some properties of ground states expectations.

§ 1. Introduction

Let us consider one electron coupled with the quantized radiation field. This system is described by the so-called Pauli-Fierz Hamiltonian. By now some spectral properties of this model were succesfully investigated by several authors [1, 2, 3, 4, 5, 6, 8, 10, 15, 16]. In particular the existence of a ground state is rather well understood. If we ignore the spin, the ground state is unique [12, 21]. A typical way to see the uniqueness is to prove that the heat kernel of the Hamiltonian improves the positivity in the Schrödinger representation [12]. Then the uniqueness is a direct consequence of the Perron-Frobenius theorem. On the other hand if the spin is included, it was shown that the ground state is always degenerate by the Kramers degeneracy theorem comming from the time reversal symmetry [14, 17, 18].

Usually we analyze the Pauli-Fierz Hamiltonian in the standard Fock representation where fields operators are described by the annihilation- and creation operators. However as mentioned above there is another important representation space for the

Received December 20, 2009. Revised September 11, 2010.
2000 Mathematics Subject Classification(s): 81T10, 47A05
Key Words: Time reversal symmetries, Nonrelativistic QED, Ground states:
*Georgia Institute of Technology, School of Mathematics, Atlanta, Georgia 30332-0160.
e-mail: `loss@math.gatech.edu`
**Setsunan University, Ikeda-Naka-machi 17-8, Neyagawa 572-8508, Japan.
e-mail: `miyao@mpg.setsunan.ac.jp`
***Zentrum Mathematik, Technische Universität München, D-85747 Garching, Germany.
e-mail: `spohn@ma.tum.de`

Pauli-Fierz Hamiltonian, often called the Schrödinger representation where the fields operators are expressed as real valued multiplication operators in some L^2-space. Each representation has its own advantage. For instance, since the quantized vector potential is a real multiplication operator in the Schrödinger representation, we can construct a path integral formula by modifying arguments about the standard Pauli operator. The point is that some properties could be easily proven in the Schrödinger representation even though these are hard to see in the Fock representation, and vice versa. Therefore choice of a suitable representation depends on each problem.

In previous works [17, 18], the time reversal symmetry was discussed in the Fock representation. Then it is rather natural to investigate the time reversal symmetry in the Schrödinge representation. Comparing results obtained in both representations, we could discover new aspects of the Pauli-Fierz Hamiltonian. This is a motivation of this little note. Indeed we will find a time reversal symmetry which is defferent from the one in the Fock representation. Then applying differences between two positively, we will investigate some properties of the ground states expectations.

§ 2. Pauli-Fierz Hamiltonian with spin 1/2

The Pauli-Fierz Hamiltonian is given by

$$H = \frac{1}{2}\big(-\mathrm{i}\nabla_x + eA(x)\big)^2 + \frac{e}{2}\sigma\cdot B(x) + V(x) + H_{\mathrm{f}} \tag{2.1}$$

acting in $L^2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathfrak{F}$, where $\mathfrak{F}$ is the photon Fock space

$$\mathfrak{F} = \sum_{n\geq 0}^{\oplus} L^2(\mathbb{R}^3\times\{1,2\})^{\otimes_{\mathrm{s}} n}, \tag{2.2}$$

where $\mathfrak{h}^{\otimes_{\mathrm{s}} n}$ means the n-fold symmetric tensor product of $\mathfrak{h}$ with the convention $\mathfrak{h}^{\otimes_{\mathrm{s}} 0} = \mathbb{C}$. The quantized vector potential $A(x) = (A_1(x), A_2(x), A_3(x))$ is given by

$$A(x) = \sum_{\lambda=1,2}(2\pi)^{-3/2}\int_{|k|\leq\Lambda}\mathrm{d}k\,\frac{\varepsilon(k,\lambda)}{\sqrt{2|k|}}\{\mathrm{e}^{\mathrm{i}k\cdot x}a(k,\lambda) + \mathrm{e}^{-\mathrm{i}k\cdot x}a(k,\lambda)^*\}, \tag{2.3}$$

where $\varepsilon(k,\lambda)$ is a polarization vector which is real valued and measurable, Λ is the ultraviolet cutoff. Here $a(k,\lambda), a(k,\lambda)^*$ are the annihilation and creation operators which satisfy the standard commutation relations

$$[a(k,\lambda), a(q,\mu)^*] = \delta_{\lambda\mu}\delta(k-q),\ \ [a(k,\lambda), a(q,\mu)] = 0 = [a(k,\lambda)^*, a(q,\mu)^*]. \tag{2.4}$$

$B(x)$ is the quantized magnetic field defined by

$$\begin{aligned} B(x) &= \mathrm{rot}A(x) \\ &= \mathrm{i}\sum_{\lambda=1,2}(2\pi)^{-3/2}\int_{|k|\leq\Lambda}\mathrm{d}k\frac{k\times\varepsilon(k,\lambda)}{\sqrt{2|k|}}\{\mathrm{e}^{\mathrm{i}k\cdot x}a(k,\lambda) - \mathrm{e}^{-\mathrm{i}k\cdot x}a(k,\lambda)^*\}. \end{aligned} \tag{2.5}$$

H_{f} is the field energy given by

$$H_{\mathrm{f}} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \mathrm{d}k\, |k| a(k,\lambda)^* a(k,\lambda). \tag{2.6}$$

In this note, we assume the following:

(V) V is infestimally small with respec to $-\Delta_x$. Moreover $V(-x) = V(x)$ for a.e. $x \in \mathbb{R}^3$.

Then, by [11, 13], H is self-adjoint on $\mathrm{dom}(-\Delta_x) \cap \mathrm{dom}(H_{\mathrm{f}})$, bounded from below.

§ 3. Time reversal symmetry in the Fock representation

On $L^2(\mathbb{R}^3;\mathbb{C}^2) \otimes \mathfrak{F} = \oplus^2 L^2(\mathbb{R}^3;\mathfrak{F})$, we take the following involution:

$$J_F \varphi = j\varphi_\uparrow \oplus j\varphi_\downarrow, \tag{3.1}$$

$$j\varphi_\sigma = \sum\nolimits_{n\ge 0}^{\oplus} \overline{\varphi^{(n)}}_\sigma(-x; k_1, \lambda_1, \dots, k_n, \lambda_n), \quad x \in \mathbb{R}^3, \quad (k_i, \lambda_i) \in \mathbb{R}^3 \times \{1,2\} \tag{3.2}$$

for $\varphi = \varphi_\uparrow \oplus \varphi_\downarrow \in \oplus^2 L^2(\mathbb{R}^3;\mathfrak{F})$ with $\varphi_\sigma = \sum\nolimits_{n\ge 0}^{\oplus} \varphi_\sigma^{(n)}(x; k_1, \lambda_1, \dots, k_n, \lambda_n) \in L^2(\mathbb{R}^3;\mathfrak{F})$, $\sigma = \uparrow, \downarrow$. In this note, we often denote the linear operator $X \oplus X$ acting in $\mathfrak{h} \oplus \mathfrak{h}$ simply by X.

Since the annihilation operator $a(k,\lambda)$ acts by

$$a(k,\lambda)\varphi_\sigma = \sum\nolimits_{n\ge 0}^{\oplus} \sqrt{n+1}\,\varphi_\sigma^{(n+1)}(x; k, \lambda, k_1, \lambda_1, \dots, k_n, \lambda_n) \tag{3.3}$$

for $\varphi_\sigma = \sum_{n\ge 0}^{\oplus} \varphi_\sigma^{(n)} \in L^2(\mathbb{R}^3;\mathfrak{F})$ with $\sigma = \uparrow, \downarrow$, one has

$$J_F a(k,\lambda) = a(k,\lambda) J_F, \quad J_F a(k,\lambda)^* = a(k,\lambda)^* J_F. \tag{3.4}$$

Namely the annihilation and creation operators are reality preserving w.r.t. J_F. As a consequence, we obtain

$$J_F(-\mathrm{i}\nabla_x) = (-\mathrm{i}\nabla_x) J_F, \tag{3.5}$$

$$J_F A(x) = A(x) J_F, \tag{3.6}$$

$$J_F B(x) = -B(x) J_F, \tag{3.7}$$

$$J_F H_{\mathrm{f}} = H_{\mathrm{f}} J_F, \tag{3.8}$$

$$J_F V(x) = V(-x) J_F. \tag{3.9}$$

Now let introduce a time reversal operator in the Fock representation by

$$\vartheta_F = \sigma_2 J_F. \tag{3.10}$$

Clearly $\vartheta_F^2 = -\mathbb{1}$.

By the above relations, one arrives at the following:

Proposition 3.1. *[16] H has a time reversal symmetry in the Fock representation:*

$$\vartheta_F H = H \vartheta_F. \tag{3.11}$$

§ 4. Time reversal symmetry in the Schrödinger representation

There is a natural way to regard the fields operators $A(x)$ and $B(x)$ as real valued multiplication operators on L^2-space. Such a representation is called the *Schrödinger representation.* To explain this, let us introduce

$$A(\mathbf{f}) = (A_1(f_1), A_2(f_2), A_3(f_3)), \tag{4.1}$$

$$A_j(f_j) = \sum_{\lambda=1,2} \int_{|k|\le\Lambda} \mathrm{d}k\, \frac{\varepsilon_j(k,\lambda)}{\sqrt{2|k|}} \{\hat{f}_j(-k)a(k,\lambda) + \hat{f}_j(k)a(k,\lambda)^*\} \tag{4.2}$$

for $\mathbf{f} = (f_1, f_2, f_3)$ with $|k|^{-1/2}\hat{f}_j \in L^2(\mathbb{R}^3)$ and $\hat{f}_j$ is real valued, where $\hat{f}_j$ means the Fourier transformation of f_j. Then one sees

$$\langle \Omega, \mathrm{e}^{\mathrm{i}\sum_{j=1}^3 A_j(f_j)}\Omega\rangle = \mathrm{e}^{-\frac{1}{2}q(\mathbf{f},\mathbf{f})}, \tag{4.3}$$

$$q(\mathbf{f},\mathbf{g}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{|k|}\, \overline{\hat{\mathbf{f}}}(k)\big(\mathbb{1} - |\hat{k}\rangle\langle\hat{k}|\big)\hat{\mathbf{g}}(k), \tag{4.4}$$

where $\hat{k} = k/|k|$ and $\Omega = 1\oplus 0 \oplus 0 \oplus \cdots$ is the Fock vacuum. The left hand side of (4.3) is the characteristic functional of a Gaussian measure, $\mathrm{d}\mu(A)$ with mean 0 and covariance q. Hence, by Minlos' theorem, Fock space $\mathfrak{F}$ can be identified with $L^2(Q, \mathrm{d}\mu)$, where Q is the dual space of $\oplus^3\mathcal{S}_{\mathrm{real}}(\mathbb{R}^3)$, the set of real valued Schwarz test functions. By the construction, $A(x)$ and $B(x)$ are multiplication operators on $L^2(Q, \mathrm{d}\mu)$. More precise explanation can be found in [7], see also [12, 21].

Let J_S be a natural involution on $L^2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathfrak{F} = \oplus^2 L^2(\mathbb{R}^3\times Q)$ defined by $J_S\varphi = \overline{\varphi}_\uparrow \oplus \overline{\varphi}_\downarrow$ for each $\varphi = \varphi_\uparrow\oplus\varphi_\downarrow \in \oplus^2 L^2(\mathbb{R}^3\times Q)$. Then since $A(x), B(x)$ and $V(x)$ are real valued multiplication operators, one sees

$$J_S A(x) = A(x) J_S, \tag{4.5}$$

$$J_S B(x) = B(x) J_S, \tag{4.6}$$

$$J_S V(x) = V(x) J_S, \tag{4.7}$$

$$J_S(-\mathrm{i}\nabla_x) = -(-\mathrm{i}\nabla_x) J_S. \tag{4.8}$$

Moreover since $\mathrm{e}^{-\beta H_{\mathrm{f}}}$ preserves the positivity [20], one has

$$J_S H_{\mathrm{f}} = H_{\mathrm{f}} J_S. \tag{4.9}$$

Let $\xi = \mathrm{e}^{\mathrm{i}\pi N_\mathrm{f}/2}$ with $N_\mathrm{f} = \sum_{\lambda=1,2}\int_{\mathbb{R}^3} \mathrm{d}k\, a(k,\lambda)^* a(k,\lambda)$, the number operator. Remark the following important relation

$$J_S\xi = \mathrm{e}^{\mathrm{i}\pi N_\mathrm{f}}\xi J_S. \tag{4.10}$$

For a linear operator Z, let us denote $\tilde{Z} = \xi Z \xi^{-1}$. Then, by (4.10), one obtains

$$J_S\tilde{A}(x) = -\tilde{A}(x)J_S, \tag{4.11}$$

$$J_S\tilde{B}(x) = -\tilde{B}(x)J_S. \tag{4.12}$$

On the other hand, since $\tilde{V}(x) = V(x), \widetilde{(-\mathrm{i}\nabla_x)} = -\mathrm{i}\nabla_x$ and $\tilde{H}_\mathrm{f} = H_\mathrm{f}$, we have

$$J_S\tilde{V}(x) = \tilde{V}(x)J_S, \quad J_S\widetilde{(-\mathrm{i}\nabla_x)} = -\widetilde{(-\mathrm{i}\nabla_x)}J_S, \quad J_S\tilde{H}_\mathrm{f} = \tilde{H}_\mathrm{f}J_S. \tag{4.13}$$

Now we define a time reversal operator in the Schrödinger representation by

$$\vartheta_S = \sigma_2 J_S. \tag{4.14}$$

Note that $\vartheta_S^2 = -\mathbb{1}$. Summarizing the above discussion, we arrive at the following.

Proposition 4.1. *$\tilde{H}$ has a time reversal symmetry in the Schrödinger representation, namely,*

$$\vartheta_S\tilde{H} = \tilde{H}\vartheta_S. \tag{4.15}$$

§ 5. Ground states properties

Throughout this section we assume the following.

(G) The Schrödinger operator $h_\mathrm{at} = -\frac{1}{2}\Delta_x + V(x)$ has a unique strictly positive ground state ϕ_at with corresponding eigenvalue $E_\mathrm{at} < 0$.

Then H has a ground state [3, 4, 10, 15]. Let Ψ be a ground state of H. Then, by Proposition 3.1, $\vartheta_F\Psi$ is a ground state of H as well. Moreover we have

$$\langle\Psi, \vartheta_F\Psi\rangle = \langle\vartheta_F\Psi, \vartheta_F\vartheta_F\Psi\rangle^* = -\langle\vartheta_F\Psi, \Psi\rangle^* = -\langle\Psi, \vartheta_F\Psi\rangle$$

which means $\Psi \perp \vartheta_F\Psi$. Let P_g be an orthogonal projection onto a closed subspace spanned by the ground states of H. Then above argument tells us

$$\dim \mathrm{ran} P_\mathrm{g} \geq 2. \tag{5.1}$$

Namely H has degenerate ground states. Similar arguments are still valid when we consider $\tilde{H}$ instead of H in the Schrödinger representation and apply Proposition 4.1. (In general we can show each eigenvalue is degenerate by the parallel argument. The degeneracy comming from the time reversal symmetry is called the *Kramers' degeneracy* [17, 18].)

In this section, we consider one of the following.

(H $\uparrow$) $P_{\mathrm{g}} s_\uparrow \otimes \phi_{\mathrm{at}} \otimes \Omega \neq 0$,

(H $\downarrow$) $P_{\mathrm{g}} s_\downarrow \otimes \phi_{\mathrm{at}} \otimes \Omega \neq 0$,

where $s_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $s_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Remark that if e is sufficiently small, then we can actually check (H $\uparrow$) and (H $\downarrow$) by the pull-through arguments [4, 14].

Proposition 5.1. (H $\uparrow$) *implies* (H $\downarrow$). *Conversely* (H $\downarrow$) *implies* (H $\uparrow$).

Proof. By Proposition 3.1, one sees $\vartheta_F P_{\mathrm{g}} = P_{\mathrm{g}} \vartheta_F$. Assume (H $\uparrow$). Then

$$0 \neq \vartheta_F P_{\mathrm{g}} s_\uparrow \otimes \phi_{\mathrm{at}} \otimes \Omega = P_{\mathrm{g}} \vartheta_F s_\uparrow \otimes \phi_{\mathrm{at}} \otimes \Omega = \mathrm{i} P_{\mathrm{g}} s_\downarrow \otimes \phi_{\mathrm{at}} \otimes \Omega \tag{5.2}$$

which implies (H $\downarrow$). □

Henceforth we always assume (H $\uparrow$) or (H $\downarrow$). Define

$$\psi_\uparrow = \text{s-}\lim_{\beta\to\infty} \frac{\mathrm{e}^{-\beta H} s_\uparrow \otimes \phi_{\mathrm{at}} \otimes \Omega}{\|\mathrm{e}^{-\beta H} s_\uparrow \otimes \phi_{\mathrm{at}} \otimes \Omega\|}, \tag{5.3}$$

$$\psi_\downarrow = \text{s-}\lim_{\beta\to\infty} \frac{\mathrm{e}^{-\beta H} s_\downarrow \otimes \phi_{\mathrm{at}} \otimes \Omega}{\|\mathrm{e}^{-\beta H} s_\downarrow \otimes \phi_{\mathrm{at}} \otimes \Omega\|}. \tag{5.4}$$

Then both $\psi_\uparrow$ and $\psi_\downarrow$ are normalized ground states of H.

Proposition 5.2. *One has the following.*

(i) $\vartheta_F \psi_\uparrow = \mathrm{i}\psi_\downarrow, \quad \vartheta_F \psi_\downarrow = -\mathrm{i}\psi_\uparrow$.

(ii) Let $\tilde{\psi}_\uparrow = \xi\psi_\uparrow$ *and* $\tilde{\psi}_\downarrow = \xi\psi_\downarrow$. *Then* $\vartheta_S \tilde{\psi}_\uparrow = \mathrm{i}\tilde{\psi}_\downarrow, \quad \vartheta_S \tilde{\psi}_\downarrow = -\mathrm{i}\tilde{\psi}_\uparrow$.

(iii) $\langle \psi_\uparrow, \psi_\downarrow \rangle = 0$.

Proof. (i) and (ii) immediately follow from (5.3), (5.4) and Propositions 3.1, 4.1. To see (iii) we observe, by (i), that

$$\langle \psi_\uparrow, \psi_\downarrow \rangle = \langle \vartheta_F \psi_\downarrow, \vartheta_F \psi_\uparrow \rangle = -\langle \psi_\uparrow, \psi_\downarrow \rangle \tag{5.5}$$

which implies (iii). □

Remark that if we choose e sufficiently small, then $\dim \mathrm{ran} P_{\mathrm{g}} = 2$ by [14]. Hence, by Proposition 5.2 (iii), we have

$$P_{\mathrm{g}} = |\psi_\uparrow\rangle\langle\psi_\uparrow| + |\psi_\downarrow\rangle\langle\psi_\downarrow|$$

provided e sufficiently small, where $|\psi_\sigma\rangle\langle\psi_\sigma|$ stands for the orthogonal projection onto the one dimensional subspace spanned by ψ_σ.

Theorem 5.3. *Let $F \in L^\infty(\mathbb{R}^3)$. Assume $F^*(-x) = F(x)$. Then one has the following.*

(i) For all $i_1, \ldots, i_n \in \{1, 2, 3\}$ and $n \in \mathbb{N}$,

$$\begin{aligned} &\langle \psi_\uparrow, F(x)A_{i_1}(x)A_{i_2}(x)\cdots A_{i_n}(x)\psi_\uparrow\rangle \\ =&\langle \psi_\downarrow, F(x)A_{i_1}(x)A_{i_2}(x)\cdots A_{i_n}(x)\psi_\downarrow\rangle. \end{aligned} \tag{5.6}$$

(ii) For all $i_1, \ldots, i_n \in \{1, 2, 3\}$ and $n \in \mathbb{N}$,

$$\langle \psi_\uparrow, F(x)A_{i_1}(x)A_{i_2}(x)\cdots A_{i_n}(x)\psi_\downarrow\rangle = 0. \tag{5.7}$$

(iii) For all $i_1, \ldots, i_n \in \{1, 2, 3\}$ and $n \in \mathbb{N}$,

$$\begin{aligned} &\langle \psi_\uparrow, F(x)A_{i_1}(x)A_{i_2}(x)\cdots A_{i_{2n+1}}(x)\psi_\uparrow\rangle \\ =&\langle \psi_\downarrow, F(x)A_{i_1}(x)A_{i_2}(x)\cdots A_{i_{2n+1}}(x)\psi_\downarrow\rangle \\ =&0. \end{aligned} \tag{5.8}$$

Remark. In [19], similar properties played some important roles, where the spinless Pauli-Fierz Hamiltonian was studied.

Proof. It sufficies to show the assertions for each real $F \in L^\infty(\mathbb{R}^3)$ with $F(-x) = F(x)$.

(i) One has

$$\begin{aligned} \langle \psi_\uparrow, FA_{i_1}\cdots A_{i_n}\psi_\uparrow\rangle &= \langle \vartheta_F\psi_\uparrow, \vartheta_F FA_{i_1}\cdots A_{i_n}\psi_\uparrow\rangle^* \\ &= \langle \vartheta_F\psi_\uparrow, FA_{i_1}\cdots A_{i_n}\vartheta_F\psi_\uparrow\rangle^* \\ &= \langle \psi_\downarrow, FA_{i_1}\cdots A_{i_n}\psi_\downarrow\rangle \end{aligned}$$

by Proposition 5.2.

(ii) One has

$$\begin{aligned} \langle \psi_\uparrow, FA_{i_1}\cdots A_{i_n}\psi_\downarrow\rangle &= \langle \vartheta_F\psi_\uparrow, \vartheta_F FA_{i_1}\cdots A_{i_n}\psi_\downarrow\rangle^* \\ &= \langle \vartheta_F\psi_\uparrow, FA_{i_1}\cdots A_{i_n}\vartheta_F\psi_\downarrow\rangle^* \\ &= -\langle \psi_\uparrow, FA_{i_1}\cdots A_{i_n}\psi_\downarrow\rangle \end{aligned}$$

by Proposition 5.2.

(iii) To show (iii), we move to the Schrödinger representation $L^2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathfrak{F} = \oplus^2 L^2(\mathbb{R}^3\times Q)$. Recall $\vartheta_S\tilde{A}_i(x) = -\tilde{A}_i(x)\vartheta_S$ with $\tilde{A}_i(x) = \xi A_i(x)\xi^{-1}$. Then one sees

$$\begin{aligned}\langle\psi_\uparrow, FA_{i_1}\cdots A_{i_{2n+1}}\psi_\uparrow\rangle &= \langle\tilde{\psi}_\uparrow, F\tilde{A}_{i_1}\cdots\tilde{A}_{i_n}\tilde{\psi}_\uparrow\rangle\\ &= \langle\vartheta_S\tilde{\psi}_\uparrow, \vartheta_S F\tilde{A}_{i_1}\cdots\tilde{A}_{i_n}\tilde{\psi}_\uparrow\rangle^*\\ &= -\langle\vartheta_S\tilde{\psi}_\uparrow, F\tilde{A}_{i_1}\cdots\tilde{A}_{i_n}\vartheta_S\tilde{\psi}_\uparrow\rangle^*\\ &= -\langle\tilde{\psi}_\downarrow, F\tilde{A}_{i_1}\cdots\tilde{A}_{i_n}\tilde{\psi}_\downarrow\rangle \quad \text{(By Proposition 5.2)}\\ &= -\langle\psi_\downarrow, FA_{i_1}\cdots A_{i_n}\psi_\downarrow\rangle\\ &= -\langle\psi_\uparrow, FA_{i_1}\cdots A_{i_n}\psi_\uparrow\rangle. \quad \text{(By (i))}\end{aligned}$$

This proves (iii). ☐

Theorem 5.4. *One has the following.*

(i) For all $i_1,\ldots,i_n\in\{1,2,3\}$ and $n\in\mathbb{N}$,

$$\begin{aligned}&\langle\psi_\uparrow, A_{i_1}(x)\mathrm{e}^{-\beta_1 H}A_{i_2}(x)\mathrm{e}^{-\beta_2 H}\cdots A_{i_n}(x)\mathrm{e}^{-\beta_n H}\psi_\uparrow\rangle\\ =&\langle\psi_\downarrow, \mathrm{e}^{-\beta_n H}A_{i_n}(x)\mathrm{e}^{-\beta_{n-1}H}A_{i_{n-1}}(x)\cdots\mathrm{e}^{-\beta_1 H}A_{i_1}(x)\psi_\downarrow\rangle.\end{aligned}\tag{5.9}$$

(ii) For all $i_1,\ldots,i_n\in\{1,2,3\}$ and $n\in\mathbb{N}$,

$$\begin{aligned}&\langle\psi_\uparrow, A_{i_1}(x)\mathrm{e}^{-\beta_1 H}A_{i_2}(x)\mathrm{e}^{-\beta_2 H}\cdots A_{i_{2n+1}}(x)\mathrm{e}^{-\beta_{2n+1} H}\psi_\uparrow\rangle\\ =&\langle\psi_\downarrow, A_{i_1}(x)\mathrm{e}^{-\beta_1 H}A_{i_2}(x)\mathrm{e}^{-\beta_2 H}\cdots A_{i_{2n+1}}(x)\mathrm{e}^{-\beta_{2n+1} H}\psi_\downarrow\rangle\\ =&0.\end{aligned}\tag{5.10}$$

Proof. (i) Observe that

$$\begin{aligned}\langle\psi_\uparrow, A_{i_1}\mathrm{e}^{-\beta_1 H}\cdots A_{i_n}\mathrm{e}^{-\beta_n H}\psi_\uparrow\rangle &= \langle\vartheta_F\psi_\uparrow, \vartheta_F A_{i_1}\mathrm{e}^{-\beta_1 H}\cdots A_{i_n}\mathrm{e}^{-\beta_n H}\psi_\uparrow\rangle^*\\ &= \langle\vartheta_F\psi_\uparrow, A_{i_1}\mathrm{e}^{-\beta_1 H}\cdots A_{i_n}\mathrm{e}^{-\beta_n H}\vartheta_F\psi_\uparrow\rangle^*\\ &= \langle\psi_\downarrow, A_{i_1}\mathrm{e}^{-\beta_1 H}\cdots A_{i_n}\mathrm{e}^{-\beta_n H}\psi_\downarrow\rangle^*\\ &= \langle\psi_\downarrow, \mathrm{e}^{-\beta_n H}A_{i_n}\cdots\mathrm{e}^{-\beta_1 H}A_{i_1}\psi_\downarrow\rangle.\end{aligned}$$

(ii)

$$\begin{aligned}\langle\psi_\uparrow, A_{i_1}\mathrm{e}^{-\beta_1 H}\cdots A_{i_{2n+1}}\mathrm{e}^{-\beta_{2n+1} H}\psi_\uparrow\rangle &= \langle\tilde{\psi}_\uparrow, \tilde{A}_{i_1}\mathrm{e}^{-\beta_1\tilde{H}}\cdots\tilde{A}_{i_{2n+1}}\mathrm{e}^{-\beta_{2n+1}\tilde{H}}\tilde{\psi}_\uparrow\rangle\\ &= \langle\vartheta_S\tilde{\psi}_\uparrow, \vartheta_S\tilde{A}_{i_1}\mathrm{e}^{-\beta_1\tilde{H}}\cdots\tilde{A}_{i_{2n+1}}\mathrm{e}^{-\beta_{2n+1}\tilde{H}}\tilde{\psi}_\uparrow\rangle^*\\ &= -\langle\vartheta_S\tilde{\psi}_\uparrow, \tilde{A}_{i_1}\mathrm{e}^{-\beta_1\tilde{H}}\cdots\tilde{A}_{i_{2n+1}}\mathrm{e}^{-\beta_{2n+1}\tilde{H}}\vartheta_S\tilde{\psi}_\uparrow\rangle^*\\ &= -\langle\tilde{\psi}_\downarrow, \tilde{A}_{i_1}\mathrm{e}^{-\beta_1\tilde{H}}\cdots\tilde{A}_{i_{2n+1}}\mathrm{e}^{-\beta_{2n+1}\tilde{H}}\tilde{\psi}_\downarrow\rangle^*\\ &= -\langle\psi_\downarrow, \mathrm{e}^{-\beta_{2n+1}H}A_{i_{2n+1}}\cdots\mathrm{e}^{-\beta_1 H}A_{i_1}\psi_\downarrow\rangle\\ &= -\langle\psi_\uparrow, A_{i_1}\mathrm{e}^{-\beta_1 H}\cdots A_{i_{2n+1}}\mathrm{e}^{-\beta_{2n+1}H}\psi_\uparrow\rangle.\end{aligned}$$

In the last line, we used (i). ☐

Theorem 5.5. *Let F be a measurable function on $\mathbb{R}^3$ satisfying*

$$\sup_x |F(x)|\mathrm{e}^{-\varepsilon|x|} < \infty \tag{5.11}$$

for sufficiently small $\varepsilon > 0$. Then one has the following.

(i) $\langle\psi_\uparrow, F(x)\psi_\uparrow\rangle = \langle\psi_\downarrow, F(x)\psi_\downarrow\rangle$.

(ii) $\langle\psi_\uparrow, F(x)\psi_\downarrow\rangle = 0$.

(iii) If F is odd, i.e., $F(-x) = -F(x)$, then

$$\langle\psi_\uparrow, F(x)\psi_\uparrow\rangle = 0 = \langle\psi_\downarrow, F(x)\psi_\downarrow\rangle. \tag{5.12}$$

Proof. First we remark that, the ground states $\psi_\uparrow$ and $\psi_\downarrow$ have the exponential decay property $\|\mathrm{e}^{\varepsilon|x|}\psi_\sigma\| < \infty$ for sufficiently small $\varepsilon > 0$ [4, 9, 10]. Hence by the assumption (5.11), each ψ_σ belongs to the domain of the multiplcation operator F. We also remark that, it sufficies to show the assertions for real F.

(i)

$$\begin{aligned}\langle\psi_\uparrow, F\psi_\uparrow\rangle = \langle\tilde\psi_\uparrow, F\tilde\psi_\uparrow\rangle = \langle\vartheta_S\tilde\psi_\uparrow, \vartheta_S F\tilde\psi_\uparrow\rangle^* = \langle\vartheta_S\tilde\psi_\uparrow, F\vartheta_S\tilde\psi_\uparrow\rangle^* &= \langle\tilde\psi_\downarrow, F\tilde\psi_\downarrow\rangle^* \\ &= \langle\psi_\downarrow, F\psi_\downarrow\rangle.\end{aligned}$$

Similarly we can see (ii).

(iii) Note that if F is odd, then $\vartheta_F F = -F\vartheta_F$. Hence

$$\langle\psi_\uparrow, F\psi_\uparrow\rangle = \langle\vartheta_F\psi_\uparrow, \vartheta_F F\psi_\uparrow\rangle^* = -\langle\vartheta_F\psi_\uparrow, F\vartheta_F\psi_\uparrow\rangle^* = -\langle\psi_\downarrow, F\psi_\downarrow\rangle = -\langle\psi_\uparrow, F\psi_\uparrow\rangle.$$

This proves (iii). ☐

References

[1] W.K. Abou Salem, J. Faupin, J. Fröhlich, I.M. Sigal, On the theory of resonances in nonrelativistic QED and related models, Adv. App. Math. 43 (2009), 201-230.

[2] L. Amour, B. Grebert, J.C. Guillot, The dressed mobile atoms and ions, J. Math. Pures Appl. 86 (2006) 177-200.

[3] V. Bach, J. Fröhlich, A. Pizzo, Infrared-finite algorithms in QED: the groundstate of an atom interacting with the quantized radiation field. Comm. Math. Phys. 264 (2006), no. 1, 145–165.

[4] V. Bach, J. Fröhlich, I. M. Sigal, Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. Comm. Math. Phys. 207 (1999), no. 2, 249–290.

[5] V. Bach, T. Chen, J. Fröhlich, I. M. Sigal, The renormalized electron mass in non-relativistic quantum electrodynamics. J. Funct. Anal. 243 (2007), no. 2, 426–535.

[6] T. Chen, Infrared renormalization in non-relativistic QED and scaling criticality J. Funct. Anal. 254 (2008) 2555-2647

[7] G. M. Graf, C. Fefferman, J. Fröhlich, Stability of Ultraviolet-Cutoff Quantum Electrodynamics with Non-Relativistic Matter,Comm. Math. Phys. 190 (1997) 309-330

[8] M Konenberg, O. Matte, E. Stockmeyer Existence of ground states of hydrogen-like atoms in relativistic QED I: The semi-relativistic Pauli-Fierz operator, arXiv:0912.4223 (2009).

[9] M. Griesemer, Exponential decay and ionization thresholds in non-relativistic quantum electrodynamics, J. Funct. Anal. 210 (2004), 321-340.

[10] M. Griesemer, E. H. Lieb, M. Loss, Ground states in non-relativistic quantum electrodynamics, Invent. Math. 145 (2001) 557-595.

[11] D. Hasler, I. Herbst, On the self-adjointness and domain of Pauli-Fierz type Hamiltonians, Rev. Math. Phys. 20 (2008), 787-800.

[12] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics. II. J. Math. Phys. 41 (2000), 661–674.

[13] F. Hiroshima, Self-Adjointness of the Pauli-Fierz Hamiltonian for Arbitrary Values of Coupling Constants, Ann. Henri Poincare 3 (2003), 171-201.

[14] F. Hiroshima, H. Spohn, Ground state degeneracy of the Pauli-Fierz Hamiltonian with spin. Adv. Theor. Math. Phys. 5 (2001), 1091–1104.

[15] E. H. Lieb, M. Loss, Existence of atoms and molecules in non-relativistic quantum electrodynamics, Adv. Theor. Math. Phys. 7, (2003) 667-710.

[16] M. Loss, T. Miyao, H. Spohn, Lowest energy states in nonrelativistic QED: atoms and ions in motion, J. Funct. Anal. 243 (2007) 353-393.

[17] M. Loss, T. Miyao, H. Spohn, Kramers degeneracy theorem in nonrelativistic QED, Lett. Math. Phys. 89 (2009) 21-32.

[18] T. Miyao, H. Spohn, Spectral Analysis of the Semi-relativistic Pauli-Fierz Hamiltonian, J. Funct. Anal. 256 (2009) 2123-2156.

[19] T. Miyao, H. Spohn, The retarded van der Waals potential revisited, J. Math. Phys. 50 072103 (2009)

[20] B. Simon, The $P(\phi_2)$ Euclidean (Quantum) Field Theory, Princeton University Press, Princeton, 1974.

[21] H. Spohn, Dynamics of Charged Particles and their Radiation Field, Cambridge University Press 2004