On the existence of ground states for the Pauli-Fierz model with a variable mass

By

Takeru Hidaka *

Abstract

The purpose of this paper is to review [9]. The existence of ground states of the Pauli-Fierz model with a variable mass is considered. This paper presents the outline of the proof of it under the infrared regularity condition.

§ 1. Introduction

The Pauli-Fierz model describes a minimal interaction between a low energy electron and a quantized radiation field, where the electron is governed by a Schrödinger operator. The Pauli-Fierz Hamiltonian is the physical quantity corresponding to the energy of the system and is realized as a self-adjoint operator on a certain Hilbert space and its bottom of the spectrum is called the ground state energy. An eigenvector associated with the ground state energy is called a ground state, if it exists.

The existence of ground states of the Pauli-Fierz Hamiltonian is investigated in [1, 2, 4, 8, 10, 12]. In [2, 8], the infrared regularity condition is not assumed. In [4, 8], the existence of ground states is shown for arbitrary values of coupling constants. The uniqueness of the ground state of the Pauli-Fierz Hamiltonian is proven in [11].

The Pauli-Fierz Hamiltonian with a variable mass is considered in this paper. It is derived from the analogy of the Nelson model on a pseudo Riemannian manifold [5, 6, 7]. Under the infrared regularity condition, this Hamiltonian has ground states for all values of a coupling constant when a variable mass decays sufficiently fast.
§ 2. Definition of the Pauli-Fierz model

§ 2.1. Hilbert space of states

We consider the Hilbert space of states of total system as

$$\mathcal{H} := \mathcal{H}_P \otimes \mathcal{F},$$

where

$$\mathcal{H}_P := L^2(\mathbb{R}^3)$$

describes state space of one electron and $\mathcal{F}$ is the boson Fock space over $L^2(\mathbb{R}^3; \mathbb{C}^2)$ defined by

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \left( \bigotimes_{s}^{n} L^2(\mathbb{R}^3; \mathbb{C}^2) \right).$$

Here $\otimes_s^n L^2(\mathbb{R}^3; \mathbb{C}^2)$ denotes the n-fold symmetric tensor product of $L^2(\mathbb{R}^3; \mathbb{C}^2)$ with $\otimes_s^0 L^2(\mathbb{R}^3; \mathbb{C}^2) = \mathbb{C}$. The inner product on $\mathcal{F}$ is given by

$$(\Psi, \Phi)_{\mathcal{F}} = \overline{\Psi(0)} \Phi^{(0)} + \sum_{n=1}^{\infty} \int_{\mathbb{R}^{3n}} \overline{\Psi^{(n)}(k_1, \cdots, k_n)} \Phi^{(n)}(k_1, \cdots, k_n) dk_1 \cdots dk_n.$$ 

The Hilbert space $\mathcal{H}$ can be identified with

$$\mathbb{H} \cong \bigsqcup_{\mathbb{R}^3} \mathcal{F} dx \cong L^2(\mathbb{R}^3) \oplus \left( \bigoplus_{n=1}^{\infty} L_{\text{sym}}^2(\mathbb{R}^{3+3n}; \mathbb{C}^2) \right).$$

Here $L_{\text{sym}}^2(\mathbb{R}^{3+3n}; \mathbb{C}^2)$ is the set of $L^2(\mathbb{R}^{3+3n}; \mathbb{C}^2)$-functions such that

$$f(x, k_1, \cdots, k_n) = f(x, k_{\sigma(1)}, \cdots, k_{\sigma(n)})$$

for an arbitrary permutation $\sigma$.

Let $T$ be a densely defined closable operator on $L^2(\mathbb{R}^3; \mathbb{C}^2)$. Then $\Gamma(T)$ and $d\Gamma(T)$ are defined by

$$\Gamma(T) := \bigoplus_{n=0}^{\infty} \otimes^n T, \quad d\Gamma(T) := \bigoplus_{n=0}^{\infty} \otimes^n T^{(n)},$$

where $\otimes^0 T = 1$, $T^{(n)} := \sum_{k=1}^{n} 1 \otimes \cdots 1 \otimes \frac{kth}{T} \otimes 1 \cdots \otimes 1$ and $T^{(0)} = 0$. The number operator is defined by

$$N := d\Gamma(1).$$
The annihilation operator $a(f)$ and the creation operator $a^\dagger(f)$ smeared by $f \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ on $\mathcal{F}$ are defined by

\begin{align}
D(a^\dagger(f)) &= \left\{ \Psi \in \mathcal{F} \mid \sum_{n=1}^\infty n \| S_n(f \otimes \Psi^{(n-1)}) \|^2 < \infty \right\}, \\
(a^\dagger(f)\Psi)^{(n)} &= \sqrt{n}S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1, \quad (a^\dagger(f)\Psi)^{(0)} = 0, \\
a(f) &= (a^\dagger(\overline{f}))^*,
\end{align}

where $S_n$ denotes the symmetrization operator of degree $n$ and $D(T)$ the domain of $T$. $\Omega := (1,0,0,\cdots) \in \mathcal{F}$ is called the Fock vacuum. Let

\begin{equation}
(a(k)\Psi)^{(n)}(k_1, \cdots, k_n) := \sqrt{n+1}\Psi^{(n+1)}(k, k_1, \cdots, k_n)
\end{equation}

for $\Psi \in D(N^{1/2})$. Then for almost every $k$, $a(k)\Psi \in \mathcal{F}$.

§ 2.2. Definition of the Pauli-Fierz model

Let $v$ be a multiplication operator on $L^2(\mathbb{R}^3)$. We introduce assumptions on $v$.

**Assumption 1.**

1. $\sigma_P(-\Delta + v) \subset (0, \infty)$;
2. $v(x) \leq \text{const.}\langle x \rangle^{-\beta}$ with $\beta > 3$, where $\langle x \rangle = \sqrt{1+|x|^2}$.

Here $\sigma_P(T)$ denotes the set of eigenvalues of $T$.

Then there exists a unique function $\Psi(k, x)$ such that for $k \neq 0$,

\begin{equation}
(-\Delta_x + v(x))\Psi(k, x) = |k|^2\Psi(k, x)
\end{equation}

and $\Psi(k, x)$ satisfies the Lippman-Schwinger equation:

\begin{equation}
\Psi(k, x) = e^{ikx} - \frac{1}{4\pi} \int e^{ik||x-y||}v(y) \Psi(k, y)dy.
\end{equation}

We will use the regularity properties of $\Psi(k, x)$ below to show the existence of ground states.

**Lemma 2.1.** Suppose Assumption 1. Then

(a) 

\begin{equation}
|\Psi(k, x) - e^{ikx}| \leq \text{const.}\langle x \rangle^{-1}
\end{equation}
holds.

(b) $\Psi(k, x)$ is continuously differentiable in $x$ for each fixed $k$ but $k \neq 0$ and

$$
(2.11) \quad \frac{\partial}{\partial x_{\mu}} \Psi(k, x) - ik_{\mu}e^{ikx} = -\frac{1}{4\pi} \int_{\mathbb{R}^{3}} \left( \frac{e^{i|x-y|}(x_{\mu} - y_{\mu})}{|x-y|^{3}} - \frac{i|x|e^{i|x-y|}(x_{\mu} - y_{\mu})}{|x-y|^{2}} \right) v(y) \Psi(k, y) dy.
$$

In particular, for any compact set $D$ but $0 \notin D$, $\sup_{k \in D, x} \left| \frac{\partial \Psi}{\partial x_{\mu}}(k, x) \right| < \infty$.

(c) For $k \neq 0$ and $k + h \neq 0$,

$$
(2.12) \quad \frac{1}{|h|} \left| \Psi(k + h, x) - \Psi(k, x) \right| \leq \text{const.} (1 + |x|),
$$

$$
(2.13) \quad \frac{1}{|h|} \left| \frac{\partial}{\partial x_{\nu}} \Psi(k + h, x) - \frac{\partial}{\partial x_{\nu}} \Psi(k, x) \right| \leq \text{const.} (1 + |k| + |x| + |k||x|)
$$

hold, and $\Psi(k, x)$ and $\frac{\partial}{\partial x_{\nu}} \Psi(k, x)$ are differentiable in $k \in \mathbb{R}^{3} \backslash \{0\}$ for each fixed $x$.

Let us introduce the dispersion relation and the quantized radiation field with a variable mass $v$.

**Definition 2.2.** The dispersion relation with a variable mass is given by

$$
(2.14) \quad \hat{\omega} := \sqrt{-\Delta + v}
$$
on $L^{2}(\mathbb{R}^{3}; \mathbb{C}^{2})$, where $v$ is called a variable mass. The free Hamiltonian is defined by the second quantization of $\hat{\omega}$:

$$
(2.15) \quad H_{f} = d\Gamma(\hat{\omega}).
$$

Let $m \geq 0$ and $\hat{\omega}_{m} := \sqrt{-\Delta + v + m^{2}}$. We set

$$
H_{f}(m) = d\Gamma(\hat{\omega}_{m}).
$$

In order to define the quantized radiation field, we introduce a cutoff functions: $\hat{\varphi}_{j}^{\mu}, j = 1, 2, \mu = 1, 2, 3$.

**Assumption 2.**

(1) The support of $\hat{\varphi}_{j}^{\mu}$ is compact;

(2) $\hat{\varphi}_{j}^{\mu}$ is differentiable and the derivative function is bounded;

(3) (infrared regularity condition)

It holds that

$$
(2.16) \quad \int_{\mathbb{R}^{3}} \frac{|\hat{\varphi}_{j}^{\mu}(k)|^{2p}}{|k|^{5p}} dk < \infty \quad \text{for all} \quad 0 < p < 1.
$$
Let the test function $\rho_x^\mu = (\rho_x^{\mu,1}, \rho_x^{\mu,2}) \in L^2(\mathbb{R}^3; \mathbb{C}^2) \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ be such that

$$\rho_x^{\mu,j}(y) := (2\pi)^{-3/2} \int \overline{\Psi(k, x)} \Psi(k, y) \hat{\varphi}_j^\mu(k) dk.$$ 

The quantized radiation field with a variable mass is given by

$$(2.17) \quad A_\mu(x) := \frac{1}{\sqrt{2}} \left( a^\dagger (\hat{\omega}^{-1/2}\rho_x^\mu) + a (\overline{\hat{\omega}^{-1/2}\rho_x^\mu}) \right), \quad \mu = 1, 2, 3,$$

for each $x \in \mathbb{R}^3$.

**Definition 2.3.** Let $V$ be a multiplication operator, and $V_+$ and $V_-$ the positive part and the negative part of $V$, respectively. Then the quadratic form $q_m^V$ is defined by

$$(2.18) \quad q_m^V(\Psi, \Phi) = \frac{1}{2} \sum_{\mu=1}^3 \left( (p_\mu + \sqrt{\alpha} A_\mu) \Psi, (p_\mu + \sqrt{\alpha} A_\mu) \Phi \right) + \left( \hat{H}_f^{1/2}(m) \Psi, \hat{H}_f^{1/2}(m) \Phi \right) + \left( V_+^{1/2} \Psi, V_+^{1/2} \Phi \right) - \left( V_-^{1/2} \Psi, V_-^{1/2} \Phi \right)$$

with the form domain

$$(2.19) \quad Q(q_m^V) = D(|p|) \cap D(\hat{H}_f^{1/2}(m)) \cap D(|V|^{1/2}).$$

Here $\alpha$ is a coupling constant. When $m = 0$, we denote $q^V$ for $q_0^V$.

**§ 2.3. Generalized Fourier transformation**

By [14], under Assumption 1, the generalized Fourier transformation is defined by

$$(2.20) \quad f \mapsto \mathcal{F}f(\cdot) := (2\pi)^{-3/2} \mathrm{i}. \mathrm{m.} \int f(x) \overline{\Psi(\cdot, x)} dx,$$

which is a unitary transformation on $L^2(\mathbb{R}^3)$. By $1 \otimes \Gamma(\mathcal{F}) : \mathcal{H} \to \mathcal{H}$, the quadratic form $q_m^V$ is transformed as

$$(2.21) \quad \hat{q}_m^V(\Psi, \Phi) = q_m^V(1 \otimes \Gamma(\mathcal{F}) \Psi, 1 \otimes \Gamma(\mathcal{F}) \Phi)$$

$$= \frac{1}{2} \sum_{\mu=1}^3 \left( (p_\mu + \sqrt{\alpha} \hat{A}_\mu) \Psi, (p_\mu + \sqrt{\alpha} \hat{A}_\mu) \Phi \right) + \left( \hat{H}_f^{1/2}(m) \Psi, \hat{H}_f^{1/2}(m) \Phi \right)$$

$$+ \left( V_+^{1/2} \Psi, V_+^{1/2} \Phi \right) - \left( V_-^{1/2} \Psi, V_-^{1/2} \Phi \right)$$

with the form domain

$$(2.22) \quad Q(\hat{q}_m^V) = D(|p|) \cap D(\hat{H}_f^{1/2}(m)) \cap D(|V|^{1/2}).$$
Here

\begin{equation}
\hat{A}_{\mu}(x) := \frac{1}{\sqrt{2}} \sum_{j=1,2} \left( a^{\dagger} \left( \frac{\hat{\varphi}_{j}^{\mu}(\cdot, x)}{\sqrt{\omega}} \right) + a \left( \frac{\hat{\varphi}_{j}^{\mu}(\cdot, x)}{\sqrt{\omega}} \right) \right), \quad \omega(k) = |k|,
\end{equation}

and

\begin{equation}
\hat{H}_{\mathrm{f}}(m) := d \Gamma(\omega_{m}), \quad \omega_{m}(k) := \sqrt{k^{2}+m^{2}}.
\end{equation}

We introduce following assumptions on $V$:

**Assumption 3.**

1. $V$ is a measurable function and for almost every $x \in \mathbb{R}^{3}$, $-\infty < V(x) < \infty$;
2. For all $\epsilon > 0$, there exists a positive constant $C_{\epsilon}$ such that for $\Psi \in D(|p|)$,

\begin{equation}
\| V^{-1/2} \Psi \|^{2} \leq \epsilon \| p \Psi \|^{2} + C_{\epsilon} \| \Psi \|^{2};
\end{equation}

3. $Q(\hat{q}_{m}^{V})$ is dense.

**Proposition 2.4.** Suppose Assumptions 1, 2 and 3. Then there exists the unique self-adjoint operator $\hat{H}_{m}^{V}$ such that $Q(\hat{q}_{m}^{V}) = D(|\hat{H}_{m}^{V}|^{1/2})$ and for all $\Psi$ and $\Phi \in Q(\hat{q}_{m}^{V})$,

\begin{equation}
\hat{q}_{m}^{V}(\Psi, \Phi) - E^{V}(m)(\Psi, \Phi) = \left( (\hat{H}_{m}^{V} - E^{V}(m))^{1/2} \Psi, (\hat{H}_{m}^{V} - E^{V}(m))^{1/2} \Phi \right).
\end{equation}

Here we denote the ground state energy of $\hat{q}_{m}^{V}$ by

\begin{equation}
E^{V}(m) := \inf_{\Psi \in Q(\hat{q}_{m}^{V}), \| \Psi \| = 1} \hat{q}_{m}^{V}(\Psi, \Psi).
\end{equation}

Formally, the Pauli-Fierz Hamiltonian $H_{m}^{V}$ is given by

\begin{equation}
H_{m}^{V} := \frac{1}{2} \sum_{\mu, \nu} \left( p_{\mu} + \sqrt{\alpha}A_{\mu} \right) a_{\mu\nu} \left( p_{\nu} + \sqrt{\alpha}A_{\nu} \right) + H_{\mathrm{f}}(m) + V.
\end{equation}

Here $\{a_{\mu, \nu}\}_{\mu, \nu=1,2,3} = \{a_{\mu, \nu}(x)\}_{\mu, \nu=1,2,3}$ is positive definite. We consider only the case of $a_{\mu, \nu}(x) = \delta_{\mu, \nu}$ for simplicity.

### §3. Binding condition

We introduce functions $\phi_{R}$ and $\tilde{\phi}_{R}$ below. Let $\phi \in C^{\infty}(\mathbb{R}^{3})$ be such that for all $x \in \mathbb{R}^{3}$, $0 \leq \phi(x) \leq 1$ and

\begin{equation}
\phi(x) = \begin{cases} 
1 & \text{if } |x| < 1, \\
0 & \text{if } |x| > 2.
\end{cases}
\end{equation}
Let $\tilde{\phi} \in C^\infty(\mathbb{R}^3)$ be such that for all $x \in \mathbb{R}^3$, $0 \leq \tilde{\phi}(x) \leq 1$ and
\[ \phi(x)^2 + \tilde{\phi}(x)^2 = 1. \]

We set for $R > 0$,
\[ (3.1) \quad \phi_R(x) := \phi(x/R), \quad \tilde{\phi}_R(x) := \phi(x/R). \]

Let
\[ (3.2) \quad E^V(R, m) = \inf_{\|\tilde{\phi}_R\Psi\|=1, \Psi \in D(\hat{H}_m^V)} (\tilde{\phi}_R\Psi, \hat{H}_m^V\tilde{\phi}_R\Psi). \]

\[ \lim_{R \to \infty} E^V(R, m) - E^V(m) \] formally describes ionization energy by definition, it is expected that positive ionization energy yields ground state.

**Assumption 4 (Binding condition).**
\[ (3.3) \quad E^V(m) < \lim_{R \to \infty} E^V(R, m). \]

§ 4. Massive case

The existence of ground states in the case of $m > 0$ is considered in this section.

**Theorem 4.1.** Let $m > 0$. Suppose Assumptions 1-4. Then ground states of $\hat{H}_m^V$ exist for all values of a coupling constant.

**Outline of Proof.** Let $\{\Psi^j\}_j \subset Q(\hat{q}_m^V)$ be a sequence such that weakly converges to 0. It suffices to show that
\[ (4.1) \quad \lim_{j \to \infty} \inf_{j} \hat{q}_m^V(\Psi^j, \Psi^j) > E^V(m). \]

We can suppose that $\sup_j \hat{q}_m^V(\Psi^j, \Psi^j) < \infty$. Let $\phi_R$ and $\tilde{\phi}_R$ be in (3.1).
\[ (4.2) \quad \hat{q}_m^V(\Psi^j, \Psi^j) = \hat{q}_m^V(\tilde{\Psi}^j_R, \tilde{\Psi}^j_R) + \hat{q}_m^V(\tilde{\Psi}^j_R, \tilde{\Psi}^j_R) \]
\[ -\frac{1}{2} \| (|\nabla \phi_R| \otimes 1) \Psi^j \|^2 - \frac{1}{2} \| (|\nabla \tilde{\phi}_R| \otimes 1) \Psi^j \|^2. \]

holds. Here $\tilde{\Psi}^j_R = \phi_R \Psi^j$ and $\tilde{\Psi}^j_R = \tilde{\phi}_R \Psi^j$. Let $j_1$ and $j_2$ be nonnegative, smooth functions on $\mathbb{R}^3$ such that
\[ (4.3) \quad j_1(k) = \begin{cases} 1 \text{ if } |k| < 1, \\ 0 \text{ if } |k| > 2 \end{cases} \text{ and } j_1(k)^2 + j_2(k)^2 = 1. \]
We set \( \hat{j}_{l,P} = j_l(-i\nabla_k/P) \), \( l = 1, 2 \), and
\[
(4.4) \quad \hat{j}_P \Psi = \hat{j}_{1,P} \Psi \oplus \hat{j}_{2,P} \Psi,
\]
for \( \Psi \in L^2(\mathbb{R}^3; \mathbb{C}^2) \). Let us define the isometric operator from \( \mathcal{F} \) to \( \mathcal{F} \otimes \mathcal{F} \) by
\[
(4.5) \quad d\tilde{\Gamma}(\hat{j}_P)a^{\dagger}(h_1)\cdots a^{\dagger}(h_n)\Omega
\]
\[= a^{\dagger}(\hat{j}_{1,P}h_1)\cdots a^{\dagger}(\hat{j}_{1,P}h_n)\Omega \oplus a^{\dagger}(\hat{j}_{2,P}h_1)\cdots a^{\dagger}(\hat{j}_{2,P}h_n)\Omega.
\]
By the localization argument (see [8]), it holds that
\[
(4.6) \quad \liminf_{j \to \infty} \hat{q}_m^V(\Psi_R^j, \Psi_R^j) \geq (E^V(m) + m) \liminf_{j \to \infty} \|\Psi_R^j\|^2 + o_P(0)
\]
and
\[
(4.7) \quad \hat{q}_m^V(\tilde{\Psi}_R^j, \tilde{\Psi}_R^j) \geq E_{R,m}^V \|\tilde{\Psi}_R^j\|^2 + o(R^0).
\]
Here \( o_P(0) \) goes to zero as \( P \to \infty \) for each fixed \( R > 0 \). By (4.2), (4.6) and (4.7), we can see that
\[
(4.8) \quad \liminf_{j \to \infty} \hat{q}_m^V(\Psi^j, \Psi^j) \geq E^V(m) + \min\{m, E^V(R,m) - E^V(m)\}.
\]
By the binding condition, we obtain (4.1). \(\square\)

§ 5. The case of \( m = 0 \)

Throughout in this section, we suppose Assumptions 1, 2, 3 and Assumption 4 with \( m = 0 \). \( \Phi_m \) denotes the normalized ground state of \( \hat{H}_m^V \). Similarly to the case of \( v = 0 \), the following lemma holds.

**Lemma 5.1.** Let \( \{m_j\}_{j=1}^{\infty} \) be a sequence converging to 0. Then
\[
\lim_{j \to \infty} E^V(m_j) = E^V(0)
\]
and for sufficiently small \( 0 < m \), the binding condition holds.

The pull through formula below leads to a photon number bound (Lemma 5.3 and Corollary 5.4) and a photon derivative bound (Lemma 5.6).

**Lemma 5.2 (Pull through formula).** Let \( f \in D(\omega_m) \). Then \( a(f)\Phi_m \in Q(\hat{q}_m^V) \) and for all \( \eta \in Q(\hat{q}_m^V) \),
\[
(5.1) \quad \hat{q}_m^V(\eta, a(f)\Phi_m) - E^V(m)(\eta, a(f)\Phi_m)
\]
\[= -\sqrt{\alpha}(\eta, (\overline{f}, \overline{G}) \cdot (p + \sqrt{\alpha}\overline{A})\Phi_m) + \frac{i\sqrt{\alpha}}{2}(\eta, (\overline{f}, \nabla_x \cdot \overline{G})\Phi_m) - (\eta, a(\omega_m f)\Phi_m).
\]
holds. Here
\[ G_{j}^{\mu}(k, x) := \frac{\hat{\varphi}_{j}^{\mu}(k) \Psi(k, x)}{\sqrt{2 \omega(k)}}. \]

**Lemma 5.3.** Let \( \theta = (\theta_1, \theta_2) \in L^\infty(\mathbb{R}^3; \mathbb{R}^2) \). Then
\[
\| d\Gamma(\theta^2)^{1/2} \Phi_m \|^2 \leq C\alpha \sum_{j,\mu} \int \frac{\hat{\varphi}_{j}^{\mu}(k)^2 \theta_j(k)^2}{\omega(k) \omega_m(k)^2} dk,
\]
where \( C \) is a constant independent of \( \alpha \) and sufficiently small \( m \).

**Outline of proof of Lemma 5.3.** Inserting \( \eta = a(f)\Phi_m \) into (5.2), we have
\[
\begin{align*}
(a(f) \Phi_m, a(\omega_m f) \Phi_m) & \leq -\sqrt{\alpha} \left( a(f) \Phi_m, (\bar{f}, \bar{G}) \cdot (p + \sqrt{\alpha}\hat{A}) \Phi_m \right) \\
& + \frac{i\sqrt{\alpha}}{2} (a(f) \Phi_m, (\bar{f}, \nabla_x \cdot \bar{G}) \Phi_m).
\end{align*}
\]
Let \( f := \omega_m \theta g_i \). Here \( \{g_i\}_{i=1}^\infty \) is a complete orthonormal system such that each \( g_i \in D(\omega_m^{1/2}) \). Note that
\[
\sum_{i=1}^\infty (a(\omega_m^{-1/2} \theta g_i) \Phi_m, a(\omega_m^{1/2} \theta g_i) \Phi_m)
= \sum_{j=1,2} \int_{\mathbb{R}^3} \theta_j(k)^2 \| a_j(k)\Phi_m \|^2 dk = \| d\Gamma(\theta^2)^{1/2} \Phi_m \|^2.
\]
Then by (5.3) and (5.4),
\[
\| d\Gamma(\theta^2)^{1/2} \Phi_m \|^2
\leq 2\alpha \int_{\mathbb{R}^3} \omega_m(k)^{-2} \| \theta(k)G(k) \cdot (p + \sqrt{\alpha}\hat{A}) \Phi_m \|^2 dk
+ \frac{\alpha}{2} \int_{\mathbb{R}^3} \omega_m(k)^{-2} \| \theta(k)\nabla_x \cdot G(k) \Phi_m \|^2 dk.
\]
can be estimated. Since \( \Psi(k, x) \) and \( \hat{\varphi}(k) \frac{\partial}{\partial x_{\mu}} \Psi(k, x) \) are bounded in \( k \) and \( x \), we can see that the lemma follows. \( \square \)

From Lemma 5.3, we can see that following facts hold.

**Corollary 5.4.** It holds that
\[
(1) \sup_{m < m_0} \| N^{1/2} \Phi_m \| < \infty,
(2) \text{supp } \Phi(m)(x, \cdot) \subset \Pi_{k=1}^n [\cup_{j,\mu}\text{supp } \hat{\varphi}_{j}^{\mu}].
\]

We can show the spatial exponentially decay of \( \Phi_m \) for many external potentials. See [13].
Assumption 5.

(1) For sufficiently large $|x|$, $V(x) > \text{const.}|x|^{2n}$.

(2) $\liminf_{|x| \to \infty} V(x) > \inf \sigma(H_p)$ and for all $t > 0$, $e^{-tH_P} : L^2 \to L^\infty$ with
\[
\|e^{-tH_P}f\|_{L^\infty(\mathbb{R}^3)} \leq \text{const.}\|f\|_{L^2(\mathbb{R}^3)},
\]
where $H_P = -\frac{1}{2}\Delta + V$.

**Theorem 5.5.** Suppose Assumption 5. Then for some $c$ and $m_0 > 0$,
\[
\sup_{0 < m < m_0} \|\exp(c|x|)\Phi_m\| < \infty.
\]
holds.

**Outline of Proof.** Since $\Phi_m = e^{tE}e^{-t\hat{H}_m^V}\Phi_m$, by the functional integral representation of $e^{-t\hat{H}_m^V}$, we can see that for all $t \geq 0$,
\[
\|\Phi_m(x)\| \leq Ce^{tE(m)}E^x\left[e^{-\int_0^t V(B_s)ds}\right]
\]
holds. Here $(B_t)_{t \geq 0}$ denotes Brownian motion starting from $x$. $C$ is a constant independent of $x$ and $m$.
\[
e^{t(x)E^V(m)}E^x\left[e^{-\int_0^{t(x)} V(B_s)ds}\right] \leq C_1 \exp(-C_2|x|^{n+1})
\]
and
\[
e^{t'(x)E^V(m)}E^x\left[e^{-\int_0^{t'(x)} V(B_s)ds}\right] \leq C_1' \exp(-C_2'|x|)
\]
hold. Here $t(x) = |x|^{1-n}$, $t'(x) = \beta|x|$. (5.8) and (5.9) are called Carmona’s estimate [3]. By (5.7), (5.8) and (5.9), the theorem can be proven.

**Lemma 5.6.** Suppose Assumption 5. Let $1 \leq p < 2$. Then
(a) $\Phi_m^{(n)} \in H^1(\mathbb{R}^{3+3n})$ for all $n \geq 0$;
(b) $\{\|\Phi_m^{(n)}\|_{W^{1,p}(\Omega)}\}_{0 < m \leq m_0}$ is bounded, where $m_0$ is sufficiently small number and $\Omega$ is any measurable and bounded set in $\mathbb{R}^{3+3n}$.
Here $W^{1,p}(\Omega)$ is the Sobolev space.

**Outline of proof.** Let $f = \omega_m^{-1/2}g_i$. By the pull through formula with $f(x)$ replaced by $f(x + h) - f(x)$, similarly to the proof of Lemma 5.3, we can see that for
almost every $k$ and sufficiently small $h$,

\begin{equation}
(5.10) \quad \| |h|^{-1}(a(k + h) - a(k)) \Phi_m \|^2 \\
\leq \frac{\text{const.}}{\omega_m(k)^2} \left( \sum_{\mu=1}^{3} (\| |h|^{-1}(\delta_h G_{\mu})(k) \Phi_m \|^2 + \| |h|^{-1}(\nabla_x \delta_h G_{\mu})(k) | \Phi_m \|^2) \right) + \| |h|^{-1}(\nabla_x \cdot \delta_h G)(k) \Phi_m \|^2 + \sum_{j, \mu} \frac{\hat{\varphi}_{j}^{\mu}(k + h)^{2}}{\omega_m(k + h)^2 \omega(k + h)} \frac{|\omega(k + h) - \omega(k)|^2}{|h|^2} \right).
\end{equation}

Here $(\delta_h G_{\mu})(k) = G_{\mu}(k + h, x) - G_{\mu}(k, x)$. By Lemma 2.1 (c) and Assumption 5,

\begin{equation}
(5.11) \quad \| |h|^{-1}(a(k + h) - a(k)) \Phi_m \|^2 \leq C \omega_m(k)^{-2} \sum_{\nu, j} \left( (1 + |k|^{-3}) \hat{\varphi}_{j}^{\nu}(k)^{2} + |k|^{-1} \sum_{\lambda} |\partial_{\lambda} \hat{\varphi}_{j}^{\nu}(k)|^{2} \right)
\end{equation}

holds for almost every $k$ and sufficiently small $|h|$. Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Thus by Alaoglu theorem, for almost every $k$, there exists the sequence $\{h_l(k)\}_{l=1}^{\infty}$ depending on $k$ so that

$$
\lim_{l \to \infty} h_l(k) = 0
$$

and $|h_l(k)|^{-1}(a(k - |h_l(k)|e_\mu) - a(k))\Phi_m$ weakly converges to some vector $v_\mu(k)$:

$$
v_\mu(k) := \text{w-} \lim_{l \to \infty} |h_l(k)|^{-1}(a(k - |h_l(k)|e_\mu) - a(k))\Phi_m.
$$

It can be proven that $v_\mu^{(n)}(k)(x, k_1, \cdots, k_n)$ is the weak derivative $\Phi_m^{(n+1)}(x, k, k_1, \cdots, k_n)$ with respect to $k_\mu$. Thus by (5.11), (a) and (b) are proven directly. 

**Theorem 5.7.** Let $m = 0$. Suppose Assumption 5. Then ground states of $\hat{H}^V$ exist for all values of a coupling constant.

By Lemmas 5.3, 5.6 and Theorem 5.5, Theorem 5.7 can be proven similarly to [8, Theorem 2.1].

### §6. Remarks on infrared cutoffs

We assumed the infrared regularity condition, but in the case of $v = 0$, we can show the existence of ground states of $\hat{H}$ without the infrared regularity condition. In the case of $v \neq 0$, 

\begin{equation}
(6.1) \quad \Psi(k, x) - e^{ikx} = \sum_{n=1}^{\infty} \left( \frac{1}{4\pi} \right)^n \int_{\mathbb{R}^3} \frac{e^{i|k|\sum_{j=1}^{n} |y_j - y_{j-1}|} \Pi_{j=1}^{n} v(y_j)}{\Pi_{j=1}^{n} |y_j - y_{j-1}|} dy_1 \cdots dy_n
\end{equation}
\[ \nabla_x \Psi(k, x) - ike^{ikx} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \left( \frac{e^{i|k||x-y|}(x-y)}{|x-y|^3} - \frac{i|k|e^{i|k||x-y|}(x-y)}{|x-y|} \right) v(y) \Psi(k, y) \, dy \]

hold. Here \( y_0 := x \). The right hand side of (6.1) is not \( O(|k|) \), \( (k \to 0) \). This is the reason that we assumed the infrared regularity condition. To see this, let us consider the case of \( v = 0 \). Set \( v = 0 \) and \( \hat{\phi}_j^\mu(k) = \chi_\Lambda(k) e_j^\mu(k) \), \( j = 1, 2, \mu = 1, 2, 3 \), where \( \chi_\Lambda \) is the characteristic function of the set \( \{k | |k| < \Lambda\} \) and \( e_1(k) \) and \( e_2(k) \), \( k \in \mathbb{R}^3 \setminus \{0\} \) are polarization vectors given by

\[ e_1(k) := \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}} \quad \text{and} \quad e_2(k) := \frac{k \times e_1(k)}{|k|}. \]

Note that the infrared regularity condition is not assumed in this case. Define the unitary operator \( U \) as

\[ U := \exp[i\sqrt{\alpha} x \cdot \hat{A}(0)]. \]

Put

\[ \tilde{q}_m^V(\Psi, \Phi) := q_m^V(U \Psi, U \Phi) \]

and

\[ \tilde{\hat{A}}(x) := \hat{\hat{A}}(x) - \hat{\hat{A}}(0). \]

Then

\[ \tilde{q}_m^V(\Psi, a(f)\tilde{\Phi}_m) - E^V(m)(\Psi, a(f)\tilde{\Phi}_m) = -\sqrt{\alpha}(\Psi, (\hat{f}, \hat{G})(p + \sqrt{\alpha} \hat{A})\tilde{\Phi}_m) - (\Psi, a(\omega_m f)\tilde{\Phi}_m) + i(\Psi, (\hat{f}, \omega_m w)\tilde{\Phi}_m) \]

follows. Here \( \tilde{\Phi}_m = U\Phi_m \), \( w_j := \frac{\chi_\Lambda(k)e_j(k) \cdot x}{\sqrt{\omega(k)}} \) and \( \tilde{G}_j^{\mu} := \frac{\chi_\Lambda(k)e_j^\mu(k)(e^{ikx} - 1)}{\sqrt{2\omega(k)}} \). Similarly to the proof of Theorem 5.3, we have

\[ \|a(k)\tilde{\Phi}_m\|^2 \leq \text{const.} \omega(k)^{-2} \left\{ \|\hat{G}\tilde{\Phi}_m\|^2 + \|\nabla_x \hat{G}\tilde{\Phi}_m\|^2 + \omega(k)^2 \|w\tilde{\Phi}_m\|^2 \right\} \chi_\Lambda(k). \]

Since \( |e^{ikx} - 1| \leq |k||x| \) and \( |\nabla_x e^{ikx}| = |k| \), by the exponential decay of \( \tilde{\Phi}_m \), it holds that

\[ \|d\Gamma(\theta^2)^{1/2}\tilde{\Phi}_m\|^2 \leq C\alpha \sum_j \int \frac{\chi_\Lambda(k) \theta_j(k)^2}{\omega(k)} \, dk. \]
Here $C$ is a constant independent of $\alpha$ and $m$ for sufficiently small $m$. Also by (6.6), for almost every $k$ and sufficiently small $h$,

\begin{equation}
\| (a(k+h) - a(k)) \tilde{\Phi}_m \|^2 \leq \frac{\text{const.}}{\omega(k)^2} \left\{ \| \delta h G \| \tilde{\Phi}_m \|^2 + \| \nabla_x \delta h \tilde{G} \| \tilde{\Phi}_m \|^2 + \omega(k)^2 \| \delta h (\omega w) \tilde{\Phi}_m \|^2 \\
+ \frac{1}{|k+h|} \| |x| \tilde{\Phi}_m \|^2 \omega(k+h) - \omega(k) |x| \chi_\Lambda (k+h) \right\}
\end{equation}

can be proven. Since $|e^{ikx} - 1| \leq |k||x|$ and $|\nabla_x e^{ikx}| = |k|$, by Assumption 5, we can see that

\begin{equation}
|h|^{-1} \| (a(k+h) - a(k)) \tilde{\Phi}_m \|^2 \leq \text{const.} \left\{ \frac{1}{|k|(k_1^2 + k_2^2)} + \frac{1}{|k-h|((k_1 - h_1)^2 + (k_2 - h_2)^2)} \right\}
\end{equation}

holds. This inequality implies that $\{ \| \tilde{\Phi}_m^{(n)} \|_{W^{1,p}(\Omega)} \}_{0 < m \leq m_0}$ with $1 \leq p < 2$ is bounded, where $\Omega$ is a bounded set and $m_0$ a sufficiently small number. Therefore in this case, the existence of ground states can be proven without the infrared regularity condition. Key inequalities are

\begin{equation}
|e^{ikx} - 1| \leq |k||x|
\end{equation}

and

\begin{equation}
|\nabla_x e^{ikx}| = |k|.
\end{equation}

Acknowledgment

I thank Prof. F. Hiroshima for his helpful advice.

References


