Brézis-Wainger type inequalities in the Hölder spaces with double logarithmic terms and their sharp constants

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This is a joint work with Professor Tokushi Sato and Professor Hidemitsu Wadade.

Abstract. We consider the sharp constants in a Brézis-Gallouet-Wainger type inequality with a double logarithmic term in the Hölder space in a bounded domain in $\mathbb{R}^n$. Ibrahim, Majdoub and Masmoudi gave the sharp constant in the 2-dimensional case. We make precise estimates to give the sharp constants in the higher dimensions $n \geq 2$. Solving a minimizing problem of the L^n-norm of the gradients in a ball with a unilateral constraint plays an essential role for the proof of our results. When the domain is a ball, we also show the existence of an extremal function of that inequality with some suitable constants.

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1. Introduction and main results

This paper is based on the joint work with T. Sato and H. Wadade [9].

In this paper, we are mainly concerned with Brézis-Gallouet-Wainger type inequalities with sharp constants to the embeddings of the critical Sobolev space $W_0^{1,n}(\Omega)$ with the aid of the homogeneous Hölder space $\dot{C}^\alpha(\Omega)$ for any bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$. Here, $\dot{C}^\alpha(\Omega)$ denotes the subspace of the homogeneous Hölder space of order α endowed with the seminorm

$$\|u\|_{\dot{C}^\alpha(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

with $0 < \alpha \leq 1$.

First we recall the Sobolev embedding theorem. Namely, for $s \geq 0$ and $1 < p < \infty$, the embedding $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ holds if

(i) $0 \leq s < n/p$ and $p \leq q \leq 1/(1/p - s/n)$,
(ii) $s = n/p$ and $p \leq q < \infty$,
(iii) $s > n/p$ and $p \leq q \leq \infty$.

In addition, if $n/p < s < n/p + 1$ in (iii), then $W^{s,p}(\mathbb{R}^n) \hookrightarrow \dot{C}^\alpha(\mathbb{R}^n)$ holds with $\alpha = s - n/p < 1$. We also remark that $W^{n/p,p}(\mathbb{R}^n)$ cannot be embedded into $L^\infty(\mathbb{R}^n)$ in the critical case (ii). However, with the partial aid of the $W^{s,r}$-norm with $s > n/r$

and $1 \leq r \leq \infty$, we can estimate the L^∞-norm by the $W^{n/p,p}$-norm as follows:

$$\|u\|_{L^\infty(\mathbb{R}^n)}^{p/(p-1)} \leq C(1 + \log(1 + \|u\|_{W^{s,r}(\mathbb{R}^n)})) \tag{1.1}$$

holds for all $u \in W^{n/p,p}(\mathbb{R}^n) \cap W^{s,r}(\mathbb{R}^n)$ with $\|u\|_{W^{n/p,p}(\mathbb{R}^n)} = 1$, which is known as the Brézis-Gallouet-Wainger inequality. Originally, Brézis-Gallouet [2] proved (1.1) for the case $n = p = r = s = 2$. Later on, Brézis-Wainger [3] obtained (1.1) for the general case, and remarked that the power $p/(p-1)$ in (1.1) is optimal in the sense that one cannot replace it by any larger power. However, little is known about the sharp constants in Brézis-Gallouet-Wainger type inequalities.

In the special case $p = n$, if Ω is a domain in $\mathbb{R}^n$ satisfying the strong local Lipschitz condition, then the inequality (1.1) holds for all $u \in W_0^{1,n}(\Omega) \cap W^{s,r}(\Omega)$ with $\|u\|_{W^{1,n}(\Omega)} = 1$, where $s > n/r$, $1 \leq r \leq \infty$. If $s > 0$ and $n/s < r < n/(s-1)_+$, then the embedding $W^{s,r}(\Omega) \hookrightarrow \dot{C}^\alpha(\Omega)$ holds with $\alpha = s - n/r$, and we can consider a slightly better inequality

$$\|u\|_{L^\infty(\Omega)}^{n/(n-1)} \leq C(1 + \log(1 + \|u\|_{\dot{C}^\alpha(\Omega)})) \tag{1.2}$$

for $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|u\|_{W^{1,n}(\Omega)} = 1$, with $0 < \alpha < 1$. In the case $n = 2$, Ibrahim-Majdoub-Masmoudi [6] investigated the sharp constant in the inequality (1.2) with $\Omega = B_1$. Moreover, they also studied the crucial case more precisely as follows. We remark that they also proved similar estimates on an arbitrary bounded domain Ω in $\mathbb{R}^2$ instead of B_1. Here, B_1 denotes the unit open ball centered at the origin in $\mathbb{R}^n$ with $n \geq 2$.

Theorem A (Ibrahim-Majdoub-Masmoudi [6, Theorems 1.3 and 1.4]). *Let $n = 2$ and $0 < \alpha < 1$.*

(i) *If $\lambda_1 > 1/(2\pi\alpha)$, then there exists a constant $C > 0$ such that*

$$\|u\|_{L^\infty(B_1)}^2 \leq \lambda_1 \log(\|u\|_{\dot{C}^\alpha(B_1)} + C) \tag{1.3}$$

holds for all $u \in W_0^{1,2}(B_1) \cap \dot{C}^\alpha(B_1)$ with $\|\nabla u\|_{L^2(B_1)} = 1$. Furthermore, if $\lambda_1 \leq 1/(2\pi\alpha)$, then the inequality (1.3) *does not hold for some $u \in W_0^{1,2}(B_1) \cap \dot{C}^\alpha(B_1)$ with $\|\nabla u\|_{L^2(B_1)} = 1$.*

(ii) *If $\lambda_1 = 1/(2\pi\alpha)$, then there exists a constant $C > 0$ such that*

$$\|u\|_{L^\infty(B_1)}^2 \leq \lambda_1 \log(e^3 + C\|u\|_{\dot{C}^\alpha(B_1)}(\log(2e + \|u\|_{\dot{C}^\alpha(B_1)}))^{1/2}) \tag{1.4}$$

holds for all $u \in W_0^{1,2}(B_1) \cap \dot{C}^\alpha(B_1)$ with $\|\nabla u\|_{L^2(B_1)} = 1$. Furthermore, if $\lambda_1 < 1/(2\pi\alpha)$, then the inequality (1.4) *does not hold for some $u \in W_0^{1,2}(B_1) \cap \dot{C}^\alpha(B_1)$ with $\|\nabla u\|_{L^2(B_1)} = 1$.*

In this paper, in general dimensions $n \geq 2$, we consider a similar inequality on an arbitrary bounded domain Ω in $\mathbb{R}^n$. Instead of the inequalities (1.3) and (1.4), we introduce a new formulation of the inequality:

$$\|u\|_{L^\infty(\Omega)}^{n/(n-1)} \leq \lambda_1 \log(1 + \|u\|_{\dot{C}^\alpha(\Omega)}) + \lambda_2 \log(1 + \log(1 + \|u\|_{\dot{C}^\alpha(\Omega)})) + C \tag{1.5}$$

for $u \in W_0^{1,n}(\Omega)\cap\dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$. We are here concerned with the sharpness of both constants λ_1 and λ_2, where C is a constant which may depend on Ω, α, λ_1 and λ_2. We remark that the power $n/(n-1)$ in (1.5) is also optimal in the sense that one cannot replace it by any larger power (see also Remark 3.4 below).

Our main purpose is to show that $\lambda_1 = \Lambda_1/\alpha$ and $\lambda_2 = \Lambda_2/\alpha$ are the sharp constants in (1.5). Here, we define

$$\Lambda_1 = \frac{1}{\omega_{n-1}^{1/(n-1)}}, \ \Lambda_2 = \frac{\Lambda_1}{n} = \frac{1}{n\omega_{n-1}^{1/(n-1)}}$$

and $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n;\ |x| = 1\}$. More precisely, we have the following theorems.

Theorem 1.1. *Let $n \geq 2$, $0 < \alpha \leq 1$ and Ω be a bounded domain in $\mathbb{R}^n$. Assume that either*

$$\text{(I) } \lambda_1 > \frac{\Lambda_1}{\alpha} \ (\text{and } \lambda_2 \in \mathbb{R}) \quad \text{or} \quad \text{(II) } \lambda_1 = \frac{\Lambda_1}{\alpha} \text{ and } \lambda_2 \geq \frac{\Lambda_2}{\alpha}$$

holds. Then there exists a constant C such that the inequality (1.5) *holds for all $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$.*

Theorem 1.2. *Let $n \geq 2$, $0 < \alpha \leq 1$ and Ω be a bounded domain in $\mathbb{R}^n$. Assume that either*

$$\text{(III) } \lambda_1 < \frac{\Lambda_1}{\alpha} \ (\text{and } \lambda_2 \in \mathbb{R}) \quad \text{or} \quad \text{(IV) } \lambda_1 = \frac{\Lambda_1}{\alpha} \text{ and } \lambda_2 < \frac{\Lambda_2}{\alpha}$$

holds. Then for any constant C, the inequality (1.5) *does not hold for some $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$.*

We are also interested in the existence of an extremal function of the inequality (1.5). Here, for fixed λ_1 and λ_2 such that (1.5) holds, the supremum of

$$\|u\|_{L^\infty(\Omega)}^{n/(n-1)} - \lambda_1 \log(1 + \|u\|_{\dot{C}^\alpha(\Omega)}) - \lambda_2 \log(1 + \log(1 + \|u\|_{\dot{C}^\alpha(\Omega)}))$$

over $\{u \in W_0^{1,n}(\Omega)\cap\dot{C}^\alpha(\Omega);\ \|\nabla u\|_{L^n(\Omega)} = 1\}$ is called the best constant for (1.5), and u_0 is called an extremal function of (1.5) if u_0 attains its supremum. Since the inequality (1.5) corresponds to the critical embedding, we cannot expect any compactness property for treating that maximizing problem, and it is difficult to ensure the existence of an extremal function, in general. However, in the special case $\Omega = B_1$, we can find an extremal function in some cases.

Theorem 1.3. *Let $n \geq 2$, $0 < \alpha \leq 1$ and $\Omega = B_1$. Fix $\lambda_1, \lambda_2 \geq 0$ satisfying the assumption* (I) *or* (II) *in Theorem 1.1. If the best constant C for the inequality* (1.5) *(with $\Omega = B_1$) is positive, then there exists an extremal function $u_0 \in W_0^{1,n}(B_1)\cap\dot{C}^\alpha(B_1)$ with $\|\nabla u_0\|_{L^n(B_1)} = 1$ of* (1.5)*.*

Now we give some remarks on our results. The following two remarks are concerned with Theorems 1.1 and 1.2.

Remark 1.4. (i) In our formulation of the problem, the behavior of the right hand side as $\|u\|_{\dot{C}^\alpha(\Omega)} \to \infty$ with the normalization $\|\nabla u\|_{L^n(\Omega)} = 1$ is essential. In the inequality (1.4) with $\lambda_1 = 1/(2\pi\alpha)$ (and $n = 2$), the right hand side behaves like

$$\frac{1}{2\pi\alpha} \log\|u\|_{\dot{C}^\alpha(B_1)} + \frac{1}{4\pi\alpha} \log(\log\|u\|_{\dot{C}^\alpha(B_1)}) + O(1)$$

as $\|u\|_{\dot{C}^\alpha(B_1)} \to \infty$ with the same normalization. Hence Theorem A (ii) essentially claims that Theorem 1.1 (II) holds in the case $n = 2$ and $\Omega = B_1$. Indeed, we can derive Theorem A (ii) from the special case of Theorem 1.1 (II). Similarly, Theorem A (i) essentially claims that Theorem 1.1 (I) and Theorem 1.2 (III) hold in the same case.

(ii) In Theorem A, it is not mentioned whether the power $1/2$ of the inner logarithmic factor in the right hand side of (1.4) is optimal or not. On the other hand, we can assert that the power $1/2$ in (1.4) must be optimal by virtue of Theorem 1.2 (IV).

Remark 1.5. When we consider the inequality (1.5) without the double logarithmic term, i.e., $\lambda_2 = 0$, Theorem 1.1 (I) and Theorem 1.2 (III) claim that Λ_1/α is the sharp constant for λ_1, and (1.5) with $\lambda_1 = \Lambda_1/\alpha$ (and $\lambda_2 = 0$) fails to hold by virtue of Theorem 1.2 (IV). Hence, only in this case, it is essentially meaningful to consider the inequality with the double logarithmic term. Then Theorem 1.1 (II) and Theorem 1.2 (IV) claim that Λ_2/α is the sharp constant for λ_2 in the case $\lambda_1 = \Lambda_1/\alpha$, and (1.5) holds with these sharp constants. Therefore, even in the crucial case $\lambda_1 = \Lambda_1/\alpha$ and $\lambda_2 = \Lambda_2/\alpha$, it is essentially meaningless to consider an inequality with any weaker term such as the triple logarithmic term; see also Remark 3.5 below.

The following remark is concerned with Theorem 1.3.

Remark 1.6. (i) The assumption of the positivity of the best constant C for the inequality (1.5) (with $\Omega = B_1$) in Theorem 1.3 seems to be technical.

(ii) In the case that n is not so large and α is sufficiently close to 1, the best constant C for the inequality (1.5) with $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 = \Lambda_2/\alpha$ (and $\Omega = B_1$) is positive, and hence there exists an extremal function of (1.5); see Remark 4.4 below.

We here mention that Ozawa [11] gave another proof of the Brézis-Gallouet-Wainger inequality (1.1). First he established refinement of a Gagliardo-Nirenberg inequality, which states that

$$\|u\|_{L^q(\mathbb{R}^n)} \le C q^{1-1/p} \|u\|_{L^p(\mathbb{R}^n)}^{p/q} \|(-\Delta)^{n/(2p)} u\|_{L^p(\mathbb{R}^n)}^{1-p/q} \tag{1.6}$$

holds for all $u \in W^{n/p,p}(\mathbb{R}^n)$ with $p \le q < \infty$, where $1 < p < \infty$ and the constant C is independent of q. We note that the growth order $q^{1-1/p}$ of the coefficient in the right hand side as $q \to \infty$ is optimal. Then, by applying (1.6), he proved the Brézis-Gallouet-Wainger inequality (1.1).

Furthermore, Kozono-Ogawa-Taniuchi [8] and Ogawa [10] recently studied similar estimates to (1.1) in the Besov or the Triebel-Lizorkin spaces, or BMO. They also gave applications to the Navier-Stokes equations and the Euler equations.

Let us describe the outline of the proof of our results. First we note that the inequality (1.5) holds for all $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$ if and only if there exists a constant C such that

$$\begin{aligned}(1.7)\qquad &\left(\frac{\|u\|_{L^\infty(\Omega)}}{\|\nabla u\|_{L^n(\Omega)}}\right)^{n/(n-1)} - \lambda_1 \log\left(1 + \frac{\|u\|_{\dot{C}^\alpha(\Omega)}}{\|\nabla u\|_{L^n(\Omega)}}\right) \\ &\qquad -\lambda_2 \log\left(1 + \log\left(1 + \frac{\|u\|_{\dot{C}^\alpha(\Omega)}}{\|\nabla u\|_{L^n(\Omega)}}\right)\right) \le C\end{aligned}$$

holds for all $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega) \setminus \{0\}$. The key point of the proof of Theorems 1.1 and 1.2 is that we can explicitly determine the minimizer of the minimizing problem with a unilateral constraint

$$(1.8)\qquad \inf\{\|\nabla u\|_{L^n(B_1)}^n;\ u \in W_0^{1,n}(B_1), u \ge h_T \text{ a.e. on } B_1\}$$

for $0 < T \le 1$. Here the obstacle function h_T is given by

$$(1.9)\qquad h_T(x) = \tilde{h}_T(|x|) = 1 - \left(\frac{|x|}{T}\right)^\alpha \quad \text{for } x \in \mathbb{R}^n.$$

This approach is based on the argument by Ibrahim-Majdoub-Masmoudi [6] in the case $n = 2$. Since $W_0^{1,n}(B_1)$ is not a Hilbert space for $n \ge 3$, we are not able to use several tools for treating such a variational problem. Compared to the case in $W_0^{1,2}(B_1)$, little seems to be known on its regularity of a minimizer in the space $W_0^{1,n}(B_1)$ for $n \ge 3$, and we are not able to assume any regularity property of a minimizer. However, because of the uniqueness of a minimizer, it is radially symmetric and continuous on $\bar{B}_1 \setminus \{0\}$. Furthermore, we can show that the minimizer $u_T^\sharp$ is n-harmonic on the region $\{u_T^\sharp > h_T\}$. Then we can explicitly determine the shape of the minimizer with the aid of elementary one-dimensional calculi. Although we cannot assume any regularity of the minimizer, the explicit representation of the minimizer implies the C^1-regularity on $\bar{B}_1 \setminus \{0\}$ as a conclusion. Our method consists of calculating the norms of the minimizer and a simple scale argument. On the other hand, Ibrahim-Majdoub-Masmoudi [6] made use of the C^1-regularity of the minimizer and the theory of the rearrangement of functions to obtain Theorem A.

The organization of this paper is as follows. In Section 2, we investigate the minimizing problem (1.8). Then we can give the proof of Theorems 1.1 and 1.2, which will be described in Section 3. In Section 4, for λ_1 and λ_2 such that (1.5) holds, we consider the existence of an extremal function of (1.5) with the best constant C in the special case $\Omega = B_1$.

2. Minimizing problem

Throughout this paper, let the dimension $n \ge 2$ and $0 < \alpha \le 1$. First of all, we introduce some function spaces. Let Ω be a bounded domain in $\mathbb{R}^n$. In what follows, we regard a function u on Ω as the function on $\mathbb{R}^n$ extended by $u = 0$ on $\mathbb{R}^n \setminus \Omega$, and we denote

$$\|u\|_p = \|u\|_{L^p(\mathbb{R}^n)},\ \|\nabla u\|_p = \|\,|\nabla u|\,\|_p$$

for $1 \le p \le \infty$,

$$\|u\|_{(\alpha)} = \|u\|_{\dot{C}^\alpha(\mathbb{R}^n)} = \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha},$$

for simplicity. Note that we have

$$\|\nabla u\|_p = \|\nabla u\|_{L^p(\Omega)},\ \|u\|_{(\alpha)} = \|u\|_{\dot{C}^\alpha(\Omega)}$$

for all $u \in W_0^{1,p}(\Omega)$, and $u \in \dot{C}^\alpha(\Omega)$ with $\operatorname{supp} u \subset \bar{\Omega}$, respectively. We also note that the norm of $W_0^{1,p}(\Omega)$ is equivalent to $\|\nabla u\|_p$ if Ω is bounded and $1 \le p < \infty$, because of the Poincaré inequality. We denote by B_R the open ball in $\mathbb{R}^n$ centered at the origin with the radius $R > 0$, i.e., $B_R = \{x \in \mathbb{R}^n;\ |x| < R\}$.

In order to prove our results, we examine a problem of minimizing $\|\nabla u\|_n^n$ with a unilateral constraint. More generally, for $1 < p < \infty$, we formulate the following minimizing problem:

$$(\mathrm{M}^p;\ \Omega, h) \qquad m[\Omega, h] = \inf\{\|\nabla u\|_p^p;\ u \in K[\Omega, h]\},$$

where the obstacle h is a measurable function on Ω and

$$K[\Omega, h] = \{u \in W_0^{1,p}(\Omega);\ u \ge h \ \text{ a.e. on } \Omega\}.$$

In this section, we prove three propositions. The first one ensures the existence of a unique minimizer whenever the set $K[\Omega, h]$ is nonempty. Since the functional $K[\Omega, h] \ni u \mapsto \|\nabla u\|_p^p \in [0, \infty)$ is continuous, strictly convex, coercive, and $K[\Omega, h]$ is convex, (weakly) closed, we can obtain the following proposition with the aid of [4, Chapter II, Proposition 1.2].

Proposition 2.1. *Let $1 < p < \infty$, Ω be a bounded domain in $\mathbb{R}^n$, h be a measurable function defined on Ω, and assume that $K[\Omega, h]$ is nonempty. Then there exists a minimizer $u^\sharp = u^\sharp[\Omega, h] \in K[\Omega, h]$ of $(\mathrm{M}^p;\ \Omega, h)$ uniquely, that is, $\|\nabla u^\sharp\|_p^p = m[\Omega, h]$.*

The second one shows that the minimizer is p-harmonic on the (open) set $\{u^\sharp > h\}$ in the weak sense. We can prove the proposition below by a similar argument to [5] and we omit the proof in this paper; see [9] for details. This property is well-known for the case $p = 2$; see e.g. [5] and [7].

Proposition 2.2. *Let $1 < p < \infty$, Ω be a bounded domain in $\mathbb{R}^n$ and $h \in C(\bar{\Omega})$. Assume that $K[\Omega, h]$ is nonempty and the minimizer $u^\sharp = u^\sharp[\Omega, h]$ of $(\mathrm{M}^p;\ \Omega, h)$ is continuous on $\hat{\Omega}$ for some open subset $\hat{\Omega}$ of Ω. Then it holds*

$$\int_{O[\Omega,\hat{\Omega},h]} |\nabla u^\sharp(x)|^{p-2} \nabla u^\sharp(x) \cdot \nabla \phi(x)\, dx = 0 \ \text{ for all } \phi \in C_c^1(O[\Omega, \hat{\Omega}, h]), \tag{2.1}$$

where

$$O[\Omega, \hat{\Omega}, h] = \{x \in \hat{\Omega};\ u^\sharp(x) > h(x)\}.$$

The goal of this section is to prove the following proposition, which explicitly gives the minimizer $u_T^\sharp$ of the specific minimizing problem $(\mathrm{M}^n;\, B_1, h_T)$ with a parameter $0 < T \le 1$, where h_T is defined by (1.9). We also denote $K_T = K[B_1, h_T]$. Since the function $(0,1] \ni s \mapsto s(\alpha \log(1/s) + 1)^{1/\alpha} \in (0,1]$ is increasing, we can determine $0 < \tau \le T$ uniquely by

$$T = \tau \left(\alpha \log \frac{1}{\tau} + 1 \right)^{1/\alpha}.$$

Proposition 2.3. *For any $0 < T \le 1$, the (unique) minimizer $u_T^\sharp$ of $(\mathrm{M}^n;\, B_1, h_T)$ is given by*

$$u_T^\sharp(x) = \tilde{u}_T^\sharp(|x|) = \begin{cases} h_T(x) & \text{for } x \in \bar{B}_\tau, \\ \alpha \left(\dfrac{\tau}{T} \right)^\alpha \log \dfrac{1}{|x|} & \text{for } x \in B_1 \setminus B_\tau. \end{cases} \tag{2.2}$$

In what follows, we prove Proposition 2.3. We need several lemmas; see [9] for the proof of Lemma 2.5.

Lemma 2.4. *Let $h \in C(\bar{B}_1)$ be a radially symmetric function and assume that $K[B_1, h]$ is nonempty.*

(i) *The minimizer $u^\sharp = u^\sharp[B_1, h]$ of $(\mathrm{M}^n;\, B_1, h)$ is radially symmetric and continuous on $\bar{B}_1 \setminus \{0\}$.*

(ii) *The set $O = O[B_1, B_1 \setminus \{0\}, h]$ can be decomposed into a disjoint (at most countable) union $\{O_j\}_j$ of annuli, that is,*

$$O = \bigcup_j O_j,\ O_j = \{r\omega;\, a_j \le r \le b_j,\, \omega \in S^{n-1}\} = (a_j, b_j) \times S^{n-1},$$

where $0 \le a_j < b_j \le 1$, and $\{(a_j, b_j)\}_j$ is disjoint.

(iii) *For each j, there exist two constants $c_j, \bar{c}_j \in \mathbb{R}$ such that*

$$u^\sharp(x) = \tilde{u}^\sharp(|x|) = c_j \log \frac{1}{|x|} + \bar{c}_j \ \text{ for } x \in O_j.$$

Proof. (i) The minimizer $u^\sharp$ of $(\mathrm{M}^n;\, B_1, h)$ is radially symmetric because of the uniqueness. Then we can write $u^\sharp(x) = \tilde{u}^\sharp(|x|)$ for $x \in \bar{B}_1$ by using a one-variable function $\tilde{u}^\sharp$. Since $\tilde{u}^\sharp \in W^{1,n}_{\mathrm{loc}}((0,1])$, the Sobolev embedding theorem in one dimension implies that $\tilde{u}^\sharp$ is continuous on $(0,1]$, and hence $u^\sharp$ is continuous on $\bar{B}_1 \setminus \{0\}$.

(ii) By virtue of (i), there exists an open set $\tilde{O}$ in $(0,1)$ such that $O = \tilde{O} \times S^{n-1}$. Hence there exist disjoint (at most countable) open intervals $\{(a_j, b_j)\}_j$ such that $\tilde{O} = \bigcup_j (a_j, b_j)$. Then the assertion holds by putting $O_j = (a_j, b_j) \times S^{n-1}$.

(iii) Since the function $\mathbb{R}^n \ni x \mapsto \tilde{\phi}(|x|) \in \mathbb{R}$ belongs to $C^1_{\mathrm{c}}(O_j)$ for all $\tilde{\phi} \in C^1_{\mathrm{c}}((a_j, b_j))$, we have from (2.1) that

$$\omega_{n-1} \int_{a_j}^{b_j} |(\tilde{u}^\sharp)'(r) r|^{n-2} (\tilde{u}^\sharp)'(r) r \tilde{\phi}'(r) dr = 0 \ \text{ for all } \tilde{\phi} \in C^1_{\mathrm{c}}((a_j, b_j)).$$

By applying [1, Lemme VIII.1], there exists a constant $c_j \in \mathbb{R}$ such that

$$|(\tilde{u}^\sharp)'(r)r|^{n-2}(\tilde{u}^\sharp)'(r)r = -|c_j|^{n-2}c_j \text{ and } (\tilde{u}^\sharp)'(r)r = -c_j \text{ for a.e. } a_j < r < b_j,$$

because the function $\mathbb{R} \ni s \mapsto |s|^{n-2}s \in \mathbb{R}$ is bijective. Therefore, there exists a constant $\bar{c}_j \in \mathbb{R}$ such that $\tilde{u}^\sharp(r) = c_j \log(1/r) + \bar{c}_j$ for $a_j < r < b_j$, and then $u^\sharp(x) = c_j \log(1/|x|) + \bar{c}_j$ for $x \in O_j$. □

Lemma 2.5. *Let $0 < T \le 1$, $c, \bar{c} \in \mathbb{R}$ and $0 < a < b \le 1$. If $\tilde{u}(r) = c\log(1/r) + \bar{c}$ for $a \le r \le b$ and $\tilde{h}_T(a) = \tilde{u}(a)$, $\tilde{h}_T(b) = \tilde{u}(b)$, then $\tilde{h}_T > \tilde{u}$ on (a, b).*

Lemma 2.6. *For any $0 < T \le 1$ and $0 < a \le 1$, we define*

$$w_{T,a}(x) = \tilde{w}_{T,a}(|x|) = \begin{cases} h_T(x) & \text{for } x \in \bar{B}_a, \\ \dfrac{1-(a/T)^\alpha}{\log(1/a)} \log \dfrac{1}{|x|} & \text{for } x \in B_1 \setminus B_a. \end{cases}$$

(i) *There hold $w_{T,a} \in W_0^{1,n}(B_1)$ and*

$$\|\nabla w_{T,a}\|_n^n = \omega_{n-1}\alpha^{n-1}\left(\frac{(a/T)^{n\alpha}}{n} + \frac{|1-(a/T)^\alpha|^n}{(\alpha\log(1/a))^{n-1}}\right) \quad \text{for } \tau \le a \le 1.$$

(ii) *It holds $w_{T,a} \in K_T$ if and only if $\tau \le a \le 1$.*

Proof. (i) We can show the assertion by the direct calculation.

(ii) We define

$$\psi_T(a) = \frac{1-(a/T)^\alpha}{\log(1/a)} \quad \text{for } 0 < a \le T.$$

Then we can easily show that $\psi_T(a) \to 0$ as $a \searrow 0$, $\psi_T(T) = 0$ and ψ_T increases on $(0, \tau)$ and decreases on (τ, T). Hence for any $0 < a < \tau$, there exists $\tau < r_a < T$ uniquely such that $\psi_T(a) = \psi_T(r_a)$. This implies that $\tilde{w}_{T,a}(a) = \tilde{h}_T(a)$, $\tilde{w}_{T,a}(r_a) = \tilde{w}_{T,r_a}(r_a) = \tilde{h}_T(r_a)$ and

$$\tilde{w}_{T,a}(r) = \psi_T(a)\log\frac{1}{r} < \tilde{h}_T(r) \quad \text{for } a < r < r_a$$

by virtue of Lemma 2.5. This means $w_{T,a} \notin K_T$.

On the other hand, we can easily show that $\tilde{w}_{T,a} \ge \tilde{h}_T$ on $(0, 1)$ for $\tau \le a \le 1$, and $w_{T,a} \in K_T$ for $\tau \le a \le 1$. □

Lemma 2.7. *For any $0 < T \le 1$, there exists $\tau \le a_T \le 1$ such that $u_T^\sharp = w_{T,a_T}$ on B_1. In particular, $u_1^\sharp = h_1$ on B_1.*

Proof. We denote $O = O[B_1, B_1 \setminus \{0\}, h_T]$ as in Proposition 2.2 (or Lemma 2.4) and $O = \tilde{O} \times S^{n-1}$.

(Step 1) First we show that either $\tilde{O}$ is empty or $\tilde{O} = (a, 1)$ with some $0 < a < 1$. To prove this, we have only to show that $0 < a_j < b_j = 1$ for each j. If $0 < a_j < b_j < 1$, then $\tilde{u}_T^\sharp(a_j) = \tilde{h}_T(a_j)$ and $\tilde{u}_T^\sharp(b_j) = \tilde{h}_T(b_j)$, and it follows from Lemma 2.4 (iii) and Lemma 2.5 that

$$\tilde{u}_T^\sharp(r) = c_j \log\frac{1}{r} + \bar{c}_j < \tilde{h}_T(r) \quad \text{for } a_j < r < b_j,$$

which contradicts the definition of $\tilde{O}$. If $0 = a_j < b_j \le 1$, then Lemma 2.4 (iii) implies $\|\nabla u_T^\sharp\|_{L^n(O_j)} = \infty$, which is a contradiction. Consequently, the claim is proved.

(Step 2) The case $0 < T < 1$. Since $\tilde{u}_T^\sharp(1) = 0 > \tilde{h}_T(1)$, we see that $\tilde{O}$ is nonempty and $\tilde{O} = (a_T, 1)$ with some $0 < a_T < 1$. From the continuity of $\tilde{u}_T^\sharp$ on $(0, 1]$, Lemma 2.4 (iii) and Lemma 2.6 (ii), we have $\tau \le a_T < 1$ and $u_T^\sharp = w_{T,a_T}$ on B_1.

(Step 3) The case $T = 1$. Suppose that $\tilde{O}$ is nonempty, i.e. $\tilde{O} = (a_1, 1)$ with some $0 < a_1 < 1$. As we argued in Step 2, we have $\tau \le a_1 < 1$ and $u_1^\sharp = w_{1,a_1}$ on B_1. Since $\tau = 1$, this is a contradiction. Therefore, $\tilde{O}$ is empty, and hence $u_1^\sharp = h_1 = w_{1,1}$. □

We can determine a_T in Lemma 2.7 by using the following lemma. We shall omit the proof in this paper; see [9].

Lemma 2.8. *For $\rho > 0$, we define*

$$H(\sigma; \rho) = \frac{\sigma^n}{n} + \frac{(1-\sigma)^n}{(\rho - \log(\sigma(\rho+1)))^{n-1}} \quad \textit{for } \frac{1}{\rho+1} \le \sigma \le 1.$$

Then for any $\rho > 0$, $H(\sigma; \rho)$ attains its minimum only at $\sigma = 1/(\rho+1)$.

We are now in the position to prove Proposition 2.3.

Proof of Proposition 2.3. (Step 1) In view of Lemma 2.7, we may assume $0 < T < 1$. By the definition of $u_T^\sharp$, we can characterize a_T in Lemma 2.7 as

$$\|\nabla w_{T,a_T}\|_n^n = \min_{\tau \le a \le 1} \|\nabla w_{T,a}\|_n^n. \tag{2.3}$$

By virtue of Lemma 2.6 (i), we have that

$$\|\nabla w_{T,a}\|_n^n > \|\nabla w_{T,T}\|_n^n \quad \text{for } T < a \le 1,$$

and hence $\tau \le a_T \le T$.

(Step 2) By virtue of Lemma 2.8, we have that

$$H\left(\frac{(a/\tau)^\alpha}{\alpha \log(1/\tau) + 1}; \alpha \log \frac{1}{\tau}\right) \ge H\left(\frac{1}{\alpha \log(1/\tau) + 1}; \alpha \log \frac{1}{\tau}\right) \quad \text{for } \tau \le a \le T$$

and the equality holds only if $a = \tau$. Then we obtain

$$\begin{aligned} \|\nabla w_{T,a}\|_n^n &= \omega_{n-1}\alpha^{n-1}\left(\frac{(a/T)^{n\alpha}}{n} + \frac{(1-(a/T)^\alpha)^n}{(\alpha\log(1/a))^{n-1}}\right) \\ &= \omega_{n-1}\alpha^{n-1} H\left(\frac{(a/\tau)^\alpha}{\alpha \log(1/\tau) + 1}; \alpha \log \frac{1}{\tau}\right) \\ &\ge \omega_{n-1}\alpha^{n-1} H\left(\frac{1}{\alpha \log(1/\tau) + 1}; \alpha \log \frac{1}{\tau}\right) \\ &= \|\nabla w_{T,\tau}\|_n^n \quad \text{for } \tau \le a \le T, \end{aligned}$$

and $a_T = \tau$ follows. Therefore, we conclude that $u_T^\sharp = w_{T,\tau}$ on B_1. □

Remark 2.9. As is mentioned in the introduction, we cannot assume that the minimizer $u_T^\sharp$ is of class C^1 in $B_1 \setminus \{0\}$. However, in our argument, we determined a_T so that (2.3) holds, which yields necessary that $a_T = \tau$. As a conclusion, the minimizer has the C^1-regularity except for the origin. In fact, we see that $w_{T,a} \in C^1(B_1 \setminus \{0\})$ if and only if $a = \tau$.

3. Sharp constants for λ_1 and λ_2

In this section, we prove Theorems 1.1 and 1.2. We use the notation

$$\ell(s) = \log(1+s) \quad \text{for } s \geq 0,$$

for simplicity and then $\ell \circ \ell(s) = \log(1 + \log(1+s))$ for $s \geq 0$. In order to examine whether (1.7) holds or not, we may assume $\lambda_1 \geq 0$ and we define

$$F[u;\lambda_1,\lambda_2] = \left(\frac{\|u\|_\infty}{\|\nabla u\|_n}\right)^{n/(n-1)} - \lambda_1 \ell\left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right) - \lambda_2 \ell \circ \ell\left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right)$$
$$\text{for } u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega) \setminus \{0\}.$$

Note that

$$F[cu;\lambda_1,\lambda_2] = F[u;\lambda_1,\lambda_2] \quad \text{for all } c \in \mathbb{R} \setminus \{0\}.$$

Under the notation

$$F^*[\lambda_1,\lambda_2;\Omega] = \sup\{F[u;\lambda_1,\lambda_2];\ u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega) \setminus \{0\}\} \quad \text{for } \lambda_1 \geq 0,\ \lambda_2 \in \mathbb{R},$$

Theorems 1.1 and 1.2 are equivalent to the following:

Proposition 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then the following hold:*

(i) *For any $\lambda_1 > \Lambda_1/\alpha$ and $\lambda_2 \in \mathbb{R}$, it holds $F^*[\lambda_1,\lambda_2;\Omega] < \infty$;*
(ii) *For any $\lambda_2 \geq \Lambda_2/\alpha$, it holds $F^*[\Lambda_1/\alpha,\lambda_2;\Omega] < \infty$;*
(iii) *For any $0 \leq \lambda_1 < \Lambda_1/\alpha$ and $\lambda_2 \in \mathbb{R}$, it holds $F^*[\lambda_1,\lambda_2;\Omega] = \infty$;*
(iv) *For any $\lambda_2 < \Lambda_2/\alpha$, it holds $F^*[\Lambda_1/\alpha,\lambda_2;\Omega] = \infty$.*

In what follows, we shall concentrate to prove Proposition 3.1. Let us first reduce our problem on the general bounded domain Ω to that on the unit open ball B_1. We set

$$\hat{K} = \{u \in W_0^{1,n}(B_1) \cap \dot{C}^\alpha(B_1);\ \|u\|_\infty = u(0) = 1\}$$

and

$$\hat{F}^*[\lambda_1,\lambda_2] = \sup\{F[u;\lambda_1,\lambda_2];\ u \in \hat{K}\} \quad \text{for } \lambda_1 \geq 0,\ \lambda_2 \in \mathbb{R}.$$

Let s_+ denote the positive part of $s \in \mathbb{R}$, i.e., $s_+ = \max\{s, 0\}$.

Lemma 3.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $\lambda_1 \geq 0$, $\lambda_2 \in \mathbb{R}$. Then, $\hat{F}^*[\lambda_1,\lambda_2] < \infty$ holds if and only if $F^*[\lambda_1,\lambda_2;\Omega] < \infty$.*

Proof. (Step 1) For any $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega) \setminus \{0\}$, which is regarded as a function on $\mathbb{R}^n$, there exists $z_u \in \Omega$ such that $\|u\|_\infty = |u(z_u)| > 0$, and we define

$$v_u(x) = \frac{\operatorname{sgn} u(z_u)}{\|u\|_\infty} u(d_\Omega x + z_u) \ \text{ for } x \in \mathbb{R}^n,$$

where $d_\Omega = \operatorname{diam}\Omega = \sup\{|x-y|;\, x, y \in \Omega\}$. Then we have $v_u \in \hat{K}$ and

$$\|\nabla v_u\|_n = \frac{\|\nabla u\|_n}{\|u\|_\infty},\ \|v_u\|_{(\alpha)} = d_\Omega^\alpha \frac{\|u\|_{(\alpha)}}{\|u\|_\infty}.$$

Since $\max\{\ell(st), \ell(s+t)\} \le \ell(s) + \ell(t)$ for $s, t \ge 0$, we have

$$\begin{aligned}
F[u;\lambda_1,\lambda_2] &= \left(\frac{\|v_u\|_\infty}{\|\nabla v_u\|_n}\right)^{n/(n-1)} - \lambda_1 \ell\left(\frac{1}{d_\Omega^\alpha}\frac{\|v_u\|_{(\alpha)}}{\|\nabla v_u\|_n}\right) - \lambda_2 \ell\circ\ell\left(\frac{1}{d_\Omega^\alpha}\frac{\|v_u\|_{(\alpha)}}{\|\nabla v_u\|_n}\right)\\
&\le \left(\frac{\|v_u\|_\infty}{\|\nabla v_u\|_n}\right)^{n/(n-1)} - \lambda_1 \ell\left(\frac{\|v_u\|_{(\alpha)}}{\|\nabla v_u\|_n}\right) + \lambda_1 \ell(d_\Omega^\alpha)\\
&\quad - \lambda_2 \ell\circ\ell\left(\frac{\|v_u\|_{(\alpha)}}{\|\nabla v_u\|_n}\right) + |\lambda_2| \ell\circ\ell(d_\Omega^{\alpha \operatorname{sgn}\lambda_2})\\
&= F[v_u;\lambda_1,\lambda_2] + \lambda_1 \ell(d_\Omega^\alpha) + |\lambda_2| \ell\circ\ell(d_\Omega^{\alpha \operatorname{sgn}\lambda_2})\\
&\le \hat{F}^*[\lambda_1,\lambda_2] + \lambda_1 \ell(d_\Omega^\alpha) + |\lambda_2| \ell\circ\ell(d_\Omega^{\alpha \operatorname{sgn}\lambda_2}) \ \text{ for } u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega) \setminus \{0\}.
\end{aligned}$$

Therefore, if $\hat{F}^*[\lambda_1,\lambda_2] < \infty$, then $F^*[\lambda_1,\lambda_2;\Omega] < \infty$.

(Step 2) Fix $z \in \Omega$ and $R > 0$ such that $B = \{x \in \mathbb{R}^n;\, |x - z| < 1/R\} \subset \Omega$. Assume that $\hat{F}^*[\lambda_1,\lambda_2] = \infty$. Then there exists a sequence $\{v_j\}_{j=1}^\infty \subset \hat{K}$ such that $F[v_j;\lambda_1,\lambda_2] \to \infty$ as $j \to \infty$. If we define $u_j(x) = v_j(R(x - z))$ for $x \in \mathbb{R}^n$, then $u_j \in W_0^{1,n}(B) \cap \dot{C}^\alpha(B) \subset W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ and we have

$$\|u_j\|_\infty = \|v_j\|_\infty,\ \|\nabla u_j\|_n = \|\nabla v_j\|_n,\ \|u_j\|_{(\alpha)} = R^\alpha \|v_j\|_{(\alpha)}.$$

A similar calculation to Step 1 yields

$$F[v_j;\lambda_1,\lambda_2] \le F[u_j;\lambda_1,\lambda_2] + \lambda_1 \ell(R^\alpha) + |\lambda_2| \ell\circ\ell(R^{\alpha \operatorname{sgn}\lambda_2}),$$

and it follows $F[u_j;\lambda_1,\lambda_2] \to \infty$ as $j \to \infty$. Therefore, $F^*[\lambda_1,\lambda_2;\Omega] = \infty$. □

For $\kappa > 0$ and $\mu_1, \mu_2 \ge 0$, we define

$$G_\kappa(s;\mu_1,\mu_2) = \left(\frac{(s+1)^n}{s+1/n}\right)^{1/(n-1)} - \mu_1 \ell\left(\frac{\kappa e^s}{(s+1/n)^{1/n}}\right) - \frac{\mu_2}{n} \ell\circ\ell\left(\frac{\kappa e^s}{(s+1/n)^{1/n}}\right) \quad \text{for } s \ge 0.$$

We also denote $G_\kappa(s) = G_\kappa(s;1,1)$ for simplicity. The following lemma tells us that the behavior of the function $G_\kappa(s;\mu_1,\mu_2)$ as $s \to \infty$ plays an essential role for proving Proposition 3.1. We shall omit the proof in this paper; see [9]. We shall use it also in Section 4.

Lemma 3.3. *Let $\kappa > 0$.*

(i) *If either* $\mu_1 > 1$, $\mu_2 \in \mathbb{R}$, *or* $\mu_1 = 1$, $\mu_2 > 1$, *then* $G_\kappa(s;\mu_1,\mu_2) \to -\infty$ *as* $s \to \infty$. *In particular, there exists* $s_\kappa[\mu_1,\mu_2] \geq 0$ *such that*

$$G_\kappa(s_\kappa[\mu_1,\mu_2];\mu_1,\mu_2) = \sup_{s\geq 0} G_\kappa(s;\mu_1,\mu_2). \tag{3.1}$$

(ii) *There exists* $\hat{s}_\kappa > 0$ *such that*

$$G'_\kappa(s) < 0 \ \ \textit{for } s > \hat{s}_\kappa.$$

Furthermore, there exist $\hat{G}_\kappa \in \mathbb{R}$ *and* $s_\kappa[1,1] \geq 0$ *such that* $G_\kappa(s) \to \hat{G}_\kappa$ *as* $s \to \infty$, *and* (3.1) *holds with* $\mu_1 = \mu_2 = 1$.

(iii) *If either* $\mu_1 < 1$, $\mu_2 \in \mathbb{R}$, *or* $\mu_1 = 1$, $\mu_2 < 1$, *then* $G_\kappa(s;\mu_1,\mu_2) \to \infty$ *as* $s \to \infty$.

We now show Proposition 3.1 by using Lemma 3.3.

Proof of Proposition 3.1. (Step 1) First we show that

$$\hat{K} = \bigcup_{0<T\leq 1} \hat{K}_T, \tag{3.2}$$

where

$$\hat{K}_T = \{u \in K_T \cap \dot{C}^\alpha(B_1);\ \|u\|_{(\alpha)} = 1/T^\alpha,\ \|u\|_\infty = u(0) = 1\}.$$

It is trivial that $\hat{K}_T \subset \hat{K}$ for all $0 < T \leq 1$. Conversely, for any $u \in \hat{K}$, we have

$$\|u\|_{(\alpha)} \geq \sup_{x\in\partial B_1} \frac{|u(x)-u(0)|}{|x|^\alpha} = 1,$$

and

$$u(x) = 1 - |u(x)-u(0)| \geq 1 - \|u\|_{(\alpha)}|x|^\alpha \ \text{ for } x \in \bar{B}_1.$$

Then, $u \in \hat{K}_T$ with $1/T^\alpha = \|u\|_{(\alpha)} \geq 1$, and hence we obtain (3.2).

(Step 2) Next we show that

$$F[u;\lambda_1,(\lambda_2)_+] \leq F[u_T^\sharp;\lambda_1,\lambda_2]_+ \ \text{ for } u \in \hat{K}_T. \tag{3.3}$$

Note that $\|\nabla u\|_n \geq \|\nabla u_T^\sharp\|_n$ for all $u \in K_T$. We also remark that $u_T^\sharp \in \hat{K}_T$ because $\|u_T^\sharp\|_{(\alpha)} = 1/T^\alpha$ and $\|u_T^\sharp\|_\infty = u_T^\sharp(0) = 1$. Since the functions $(0,\infty) \ni s \mapsto s^{n/(n-1)}\ell(1/s) \in (0,\infty)$ and $(0,\infty) \ni s \mapsto s^{n/(n-1)}\ell\circ\ell(1/s) \in (0,\infty)$ are both increasing, we have

$$\begin{aligned}
&\|\nabla u\|_n^{n/(n-1)} F[u;\lambda_1,(\lambda_2)_+] \\
&= 1 - \lambda_1\|\nabla u\|_n^{n/(n-1)}\ell\left(\frac{1}{T^\alpha}\frac{1}{\|\nabla u\|_n}\right) - (\lambda_2)_+\|\nabla u\|_n^{n/(n-1)}\ell\circ\ell\left(\frac{1}{T^\alpha}\frac{1}{\|\nabla u\|_n}\right) \\
&\leq 1 - \lambda_1\|\nabla u_T^\sharp\|_n^{n/(n-1)}\ell\left(\frac{1}{T^\alpha}\frac{1}{\|\nabla u_T^\sharp\|_n}\right) - \lambda_2\|\nabla u_T^\sharp\|_n^{n/(n-1)}\ell\circ\ell\left(\frac{1}{T^\alpha}\frac{1}{\|\nabla u_T^\sharp\|_n}\right) \\
&= \|\nabla u_T^\sharp\|_n^{n/(n-1)} F[u_T^\sharp;\lambda_1,\lambda_2] \\
&\leq \|\nabla u\|_n^{n/(n-1)} F[u_T^\sharp;\lambda_1,\lambda_2]_+ \ \text{ for } u \in \hat{K}_T,
\end{aligned}$$

which implies (3.3).

(Step 3) We can calculate the norms of $u_T^\sharp$ as

$$\|u_T^\sharp\|_\infty = 1, \ \|u_T^\sharp\|_{(\alpha)} = \frac{1}{T^\alpha} = \frac{1}{\tau^\alpha(\alpha \log(1/\tau) + 1)},$$
$$\|\nabla u_T^\sharp\|_n^n = \left(\frac{\alpha}{\Lambda_1}\right)^{n-1} \frac{\alpha \log(1/\tau) + 1/n}{(\alpha \log(1/\tau) + 1)^n},$$

and hence

$$F[u_T^\sharp; \lambda_1, \lambda_2] = \frac{\Lambda_1}{\alpha} G_{(\Lambda_1/\alpha)^{1-1/n}} \left(\alpha \log \frac{1}{\tau}; \frac{\alpha}{\Lambda_1}\lambda_1, \frac{\alpha}{\Lambda_2}\lambda_2\right) \quad \text{for } 0 < T \leq 1 \tag{3.4}$$

(and then for $0 < \tau \leq 1$). Then we have

$$\hat{F}^*[\lambda_1, (\lambda_2)_+] \leq \frac{\Lambda_1}{\alpha} \sup_{s \geq 0} G_{(\Lambda_1/\alpha)^{1-1/n}} \left(s; \frac{\alpha}{\Lambda_1}\lambda_1, \frac{\alpha}{\Lambda_2}\lambda_2\right)_+ . \tag{3.5}$$

Indeed, combining (3.2)–(3.4) yields

$$\begin{aligned}
\sup_{u \in \hat{K}} F[u; \lambda_1, (\lambda_2)_+] &\leq \sup_{0<T\leq 1} \sup_{u \in \hat{K}_T} F[u; \lambda_1, (\lambda_2)_+] \\
&\leq \sup_{0<T\leq 1} F[u_T^\sharp; \lambda_1, \lambda_2]_+ \\
&= \frac{\Lambda_1}{\alpha} \sup_{0<\tau\leq 1} G_{(\Lambda_1/\alpha)^{1-1/n}} \left(\alpha \log \frac{1}{\tau}; \frac{\alpha}{\Lambda_1}\lambda_1, \frac{\alpha}{\Lambda_2}\lambda_2\right)_+ \\
&= \frac{\Lambda_1}{\alpha} \sup_{s \geq 0} G_{(\Lambda_1/\alpha)^{1-1/n}} \left(s; \frac{\alpha}{\Lambda_1}\lambda_1, \frac{\alpha}{\Lambda_2}\lambda_2\right)_+ .
\end{aligned}$$

By virtue of Lemma 3.2, the assertions (i) in the case $\lambda_2 \geq 0$ and (ii) follow from Lemma 3.3 (i) and (ii), respectively.

(Step 4) Consider the case $\lambda_1 > \Lambda_1/\alpha$ and $\lambda_2 < 0$. Since $\ell \circ \ell(s)/\ell(s) \to 0$ as $s \to \infty$, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\ell \circ \ell(s) \leq \varepsilon \ell(s) + C_\varepsilon \quad \text{for } s \geq 0.$$

By choosing $\delta > 0$ such that $\lambda_1 - \delta > \Lambda_1/\alpha$, we have from Step 3 that $\hat{F}^*[\lambda_1 - \delta, 0] < \infty$. Then

$$\begin{aligned}
\sup_{u \in \hat{K}} F[u; \lambda_1, \lambda_2] &= \sup_{u \in \hat{K}} \left(F[u; \lambda_1 - \delta, 0] - \lambda_2 \left(\frac{\delta}{\lambda_2} \ell\left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right) + \ell \circ \ell \left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right)\right)\right) \\
&\leq \hat{F}^*[\lambda_1 - \delta, 0] - \lambda_2 C_{-\delta/\lambda_2} \\
&< \infty,
\end{aligned}$$

and the assertion (i) (in the case $\lambda_2 < 0$) follows.

(Step 5) To prove (iii) and (iv), in view of Lemma 3.2, it suffices to show that $\limsup_{T \searrow 0} F[u_T^\sharp; \lambda_1, \lambda_2] = \infty$, because $u_T^\sharp \in \hat{K}$ for all $0 < T \leq 1$. This follows immediately from Lemma 3.3 (iii) and (3.4). □

Thus we have proved Theorems 1.1 and 1.2.

Remark 3.4. As is mentioned in the introduction, the power $n/(n-1)$ in the left hand side of (1.5) is optimal in the sense that $q = n/(n-1)$ is the largest power for which

$$\|u\|_\infty^q \le \lambda_1 \log(1 + \|u\|_{(\alpha)}) + C \tag{3.6}$$

can hold for all $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_n = 1$. Indeed, if $q > n/(n-1)$, then for any $\lambda_1 > 0$ and any constant C, (3.6) does not hold for some $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_n = 1$. On the contrary, if $1 \le q < n/(n-1)$, then for any $\lambda_1 > 0$, there exists a constant C such that (3.6) holds for all $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_n = 1$. To verify these facts, we have only to consider the behavior of the function

$$G_\kappa^q(s; \mu_1, \mu_2) = \left(\frac{(s+1)^n}{s+1/n} \right)^{q/n} - \mu_1 \ell \left(\frac{\kappa e^s}{(s+1/n)^{1/n}} \right) \quad \text{for } s \ge 0$$

as $s \to \infty$ instead of $G_\kappa(s; \mu_1, \mu_2)$.

Remark 3.5. As is mentioned in Remark 1.5, it is essentially meaningless to consider an inequality with any weaker term. More precisely, we can prove the following facts. We shall omit the proof because one can prove them by a slight modification of the proof of Lemma 3.3.

(i) We choose a continuous function $\gamma\colon [0,\infty) \to [0,\infty)$ such that

$$\gamma(s) \to \infty, \ \frac{\gamma(s)}{\ell \circ \ell(s)} \to 0 \ \text{ as } s \to \infty,$$

and consider the inequality

$$\|u\|_\infty^{n/(n-1)} \le \lambda_1 \ell(\|u\|_{(\alpha)}) + \lambda_2 \ell \circ \ell(\|u\|_{(\alpha)}) + \lambda \gamma(\|u\|_{(\alpha)}) + C$$

for $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_n = 1$. Then this inequality holds if and only if one of the following holds:

(I) $\lambda_1 > \Lambda_1/\alpha$ (and $\lambda_2, \lambda \in \mathbb{R}$);
(II-1) $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 > \Lambda_2/\alpha$ (and $\lambda \in \mathbb{R}$);
(II-2) $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 = \Lambda_2/\alpha$ and $\lambda \ge 0$.

(ii) Let $N \ge 3$ and consider the N-ple logarithmic inequality

$$\|u\|_\infty^{n/(n-1)} \le \sum_{j=1}^N \lambda_j \underbrace{\ell \circ \cdots \circ \ell}_{j}(\|u\|_{(\alpha)}) + C$$

for $u \in W_0^{1,n}(\Omega) \cap \dot{C}^\alpha(\Omega)$ with $\|\nabla u\|_n = 1$. Then this inequality holds if and only if one of the following holds:

(I) $\lambda_1 > \Lambda_1/\alpha$ (and $\lambda_2, \dots, \lambda_N \in \mathbb{R}$);
(II-1) $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 > \Lambda_2/\alpha$ (and $\lambda_3, \dots, \lambda_N \in \mathbb{R}$);
(II-2′) $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 = \Lambda_2/\alpha$, $\lambda_3 = \cdots = \lambda_{m-1} = 0$, $\lambda_m > 0$ for some $3 \le m \le N$ (and $\lambda_{m+1}, \dots, \lambda_N \in \mathbb{R}$);
(II-2″) $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 = \Lambda_2/\alpha$ and $\lambda_3 = \cdots = \lambda_N = 0$.

4. Existence of an extremal function

In this section, for fixed $\lambda_1, \lambda_2 \geq 0$ such that the inequality (1.5) holds, we consider the existence of an extremal function of (1.5) with the best constant C. Though it is difficult to ensure the existence of an extremal function for cases with general domains, we can find an extremal function in the special case $\Omega = B_1$ with constants λ_1 and λ_2 in a suitable region. Our method is due to the argument described in the previous section.

Proposition 4.1. *Fix $\lambda_1, \lambda_2 \geq 0$ satisfying the assumption* (I) *or* (II) *in Theorem 1.1.*

(i) *If*

$$\sup_{s\geq 0} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s; \frac{\alpha}{\Lambda_1}\lambda_1, \frac{\alpha}{\Lambda_2}\lambda_2\right) \geq 0, \tag{4.1}$$

then there exists $0 < T_0 \leq 1$ such that

$$F^*[\lambda_1, \lambda_2; B_1] = F\left[\frac{u^\sharp_{T_0}}{\|\nabla u^\sharp_{T_0}\|_n}; \lambda_1, \lambda_2\right] = \frac{\Lambda_1}{\alpha}\max_{s\geq 0} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s; \frac{\alpha}{\Lambda_1}\lambda_1, \frac{\alpha}{\Lambda_2}\lambda_2\right). \tag{4.2}$$

In particular, $u^\sharp_{T_0}/\|\nabla u^\sharp_{T_0}\|_n$ is an extremal function of (1.5) *with $\Omega = B_1$.*

(ii) *The best constant C for the inequality* (1.5) *(with $\Omega = B_1$) is positive if and only if*

$$\sup_{s\geq 0} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s; \frac{\alpha}{\Lambda_1}\lambda_1, \frac{\alpha}{\Lambda_2}\lambda_2\right) > 0. \tag{4.3}$$

Because of Lemma 3.3 (i) and $\inf_{s\geq 0} \ell(\kappa e^s/(s+1/n)^{1/n}) > 0$, choosing a sufficiently large λ_1 forces (4.3) to fail for any fixed $\lambda_2 \geq 0$. In particular, we obtain the following corollary.

Corollary 4.2. *Let $n \geq 2$, $0 < \alpha \leq 1$ and $\Omega = B_1$. If $\lambda_1 \geq \Lambda_1/\alpha$ is sufficiently large, then the best constant C for the inequality* (1.5) *with $\lambda_2 = 0$ (and $\Omega = B_1$) is nonpositive. In particular,*

$$\|u\|_\infty^{n/(n-1)} \leq \lambda_1 \log(1 + \|u\|_{(\alpha)})$$

holds for all $u \in W_0^{1,n}(B_1) \cap \dot{C}^\alpha(B_1)$ with $\|\nabla u\|_{L^n(B_1)} = 1$.

We need the following lemma to prove Proposition 4.1.

Lemma 4.3. *If $\lambda_1, \lambda_2 \geq 0$, then $F^*[\lambda_1, \lambda_2; B_1] \leq \hat{F}^*[\lambda_1, \lambda_2]_+$.*

Proof. Since $u^*/\|u^*\|_\infty \in \hat{K}$ for all $u \in W_0^{1,n}(B_1) \cap \dot{C}^\alpha(B_1) \setminus \{0\}$, it suffices to show that

$$F[u; \lambda_1, \lambda_2] \leq F\left[\frac{u^*}{\|u^*\|_\infty}; \lambda_1, \lambda_2\right]_+ \quad \text{for } u \in W_0^{1,n}(B_1) \cap \dot{C}^\alpha(B_1) \setminus \{0\}. \tag{4.4}$$

Here, u^* is the symmetric decreasing rearrangement of u. It is known that

$$\|u^*\|_\infty = \|u\|_\infty,\ \|\nabla u^*\|_n \leq \|\nabla u\|_n,\ \|u^*\|_{(\alpha)} \leq \|u\|_{(\alpha)}.$$

Since the functions $(0,\infty) \ni s \mapsto s^{n/(n-1)}\ell(1/s) \in (0,\infty)$ and $(0,\infty) \ni s \mapsto s^{n/(n-1)}\ell \circ \ell(1/s) \in (0,\infty)$ are both increasing, we have

$$\begin{aligned}
&\|\nabla u\|_n^{n/(n-1)} F[u;\lambda_1,\lambda_2] \\
&= \|u\|_\infty^{n/(n-1)} - \lambda_1 \|\nabla u\|_n^{n/(n-1)} \ell\left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right) - \lambda_2 \|\nabla u\|_n^{n/(n-1)} \ell \circ \ell \left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right) \\
&\leq \|u^*\|_\infty^{n/(n-1)} - \lambda_1 \|\nabla u^*\|_n^{n/(n-1)} \ell\left(\frac{\|u^*\|_{(\alpha)}}{\|\nabla u^*\|_n}\right) - \lambda_2 \|\nabla u^*\|_n^{n/(n-1)} \ell \circ \ell \left(\frac{\|u^*\|_{(\alpha)}}{\|\nabla u^*\|_n}\right) \\
&= \|\nabla u^*\|_n^{n/(n-1)} F\left[\frac{u^*}{\|u^*\|_\infty};\lambda_1,\lambda_2\right] \\
&\leq \|\nabla u\|_n^{n/(n-1)} F\left[\frac{u^*}{\|u^*\|_\infty};\lambda_1,\lambda_2\right]_+ \quad \text{for } u \in W_0^{1,n}(B_1) \cap \dot{C}^\alpha(B_1) \setminus \{0\},
\end{aligned}$$

which implies (4.4). □

Proof of Proposition 4.1. (i) By virtue of Lemma 3.3 (i)–(ii), the function $s \mapsto G_{(\Lambda_1/\alpha)^{1-1/n}}(s;\alpha\lambda_1/\Lambda_1,\alpha\lambda_2/\Lambda_2)$ is bounded from above and there exists $s_0 \geq 0$ such that

$$G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s_0;\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right) = \sup_{s\geq 0} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s;\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right)$$

and we can define $0 < \tau_0 \leq T_0 \leq 1$ by

$$s_0 = \alpha \log \frac{1}{\tau_0},\ T_0 = \tau_0 \left(\alpha \log \frac{1}{\tau_0} + 1\right)^{1/\alpha}.$$

By applying (4.1), it holds

$$F\left[\frac{u^\sharp_{T_0}}{\|\nabla u^\sharp_{T_0}\|_n};\lambda_1,\lambda_2\right] = F\left[\frac{u^\sharp_{T_0}}{\|u^\sharp_{T_0}\|_\infty};\lambda_1,\lambda_2\right] = \hat{F}^*[\lambda_1,\lambda_2] \geq 0. \tag{4.5}$$

Indeed, in view of (3.5), we have

$$\begin{aligned}
\hat{F}^*[\lambda_1,\lambda_2] &\leq \frac{\Lambda_1}{\alpha} \sup_{s\geq 0} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s;\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right)_+ \\
&= \frac{\Lambda_1}{\alpha} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s_0;\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right) \\
&= F\left[\frac{u^\sharp_{T_0}}{\|u^\sharp_{T_0}\|_\infty};\lambda_1,\lambda_2\right],
\end{aligned}$$

which implies (4.5) because $u^\sharp_{T_0}/\|u^\sharp_{T_0}\|_\infty \in \hat{K}$. By virtue of Lemma 4.3, we obtain (4.2).

(ii) Note that the best constant C for the inequality (1.5) with $\Omega = B_1$ coincides with $F^*[\lambda_1,\lambda_2;B_1]$. If $F^*[\lambda_1,\lambda_2;B_1] > 0$, then we have from Lemma 4.3 and (3.5) that

$$0 < F^*[\lambda_1,\lambda_2;B_1] \leq \hat{F}^*[\lambda_1,\lambda_2] \leq \frac{\Lambda_1}{\alpha} \sup_{s\geq 0} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s;\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right)_+,$$

and (4.3) follows. Conversely, if (4.3) holds, then $F^*[\lambda_1, \lambda_2; B_1] > 0$ follows immediately from (i). $\square$

Remark 4.4. (i) If we define

$$\begin{aligned} A_0 &= \{0 < \alpha \leq 1;\ (4.1) \text{ holds with } \lambda_1 = \Lambda_1/\alpha \text{ and } \lambda_2 = \Lambda_2/\alpha\} \\ &= \{0 < \alpha \leq 1;\ \sup_{s \geq 0} G_{(\Lambda_1/\alpha)^{1-1/n}}(s) \geq 0\}, \end{aligned}$$

then it holds either $A_0 = \emptyset$ or $A_0 = [\alpha_0, 1]$ for some $0 < \alpha_0 \leq 1$. See [9] for details.

(ii) If $n \leq 131$, then $G_{\Lambda_1^{1-1/n}}(s_1) > 0$ for some $s_1 > 0$, which implies that $A_0 = [\alpha_0, 1]$ for some $0 < \alpha_0 < 1$. Indeed, we can observe it by choosing $s_1 = 6$.

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