EXTENSIONS IN SPACES WITH VARIABLE EXPONENTS — THE HALF SPACE

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Abstract. In this article we study the Hardy–Littlewood maximal operator in variable exponent spaces. For a given variable exponent p on a domain Ω for which the maximal operator is continuous on $L^{p(\cdot)}(\Omega)$ we construct an extension of p to $\mathbb{R}^n$ such that the maximal operator is continuous on $\mathbb{R}^n$. The variable exponent will be constructed by means of a Whitney type extension. In this paper we restrict ourselves to the case where Ω is the half space. But the technique applies to less regular domains. The application to ε-δ-domains (Jones domains) is the subject of a forthcoming article.

1. Introduction

Spaces of variable integrability have been the subject of quite a lot of interest recently, as surveyed in [DHN04, Sam05]. The spaces can be traced back to W. Orlicz [Orl31], but the modern development started with the paper [KR91] of Kováčik and Rákosník. Apart from interesting theoretical considerations, the motivation to study such function spaces comes from applications to fluid dynamics [Růž00], image processing [CLS03], PDE, and the calculus of variation [Zhi86, AM01].

A crucial step in the development of the theory was establishing the boundedness of the Hardy–Littlewood maximal operator on $L^{p(\cdot)}$. Many important properties, like density of smooth functions, continuity of singular integrals, Sobolev embeddings, can be deduced solely from the boundedness of the maximal operator, see [Die04b, DR03, CUFMP06]. Of particular interest in this context is the article of Cruz-Uribe, Fiorenza, Martell, and Pérez [CUFMP06]. In that paper the authors showed that it is possible to transfer many results known for weighted Lebesgue spaces by the extrapolation technique of Rubio de Francia to the spaces of variable exponent. The only requirements

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needed are some assumptions on the boundedness of the maximal operator. This includes the boundedness of the maximal operator on the dual space together with some left-openness result. In particular it is needed that the maximal operator is bounded on $L^{(p(\cdot)/s)'}(\mathbb{R}^n)$ for some $s > 1$.

The boundedness of the maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$ was originally proved by Diening [Die04a] assuming that the variable exponent is locally log-Hölder continuous, constant outside a compact set, and bounded away from 1 and ∞. Pick and Růžička [PR01] complemented this result by showing that the local log-Hölder continuity is the optimal continuity modulus for this assertion. The assumption that p be constant outside a compact set was replaced by Cruz-Uribe, Fiorenza, and Neugebauer [CUFN03] by a decay condition at infinity. The local continuity condition and the decay condition are summarized in the term *globally* log-*Hölder continuity* of p. These authors also showed that their decay condition is the optimal decay condition for the boundedness of the maximal operator. Nekvinda [Nek04] independently proved the same result under a slightly weaker decay assumption at infinity, replacing the continuity modulus by an integral condition. While the condition $p^- > 1$ is necessary for the continuity of the maximal operator [CUFN04, Die07] the condition $p^+ < \infty$ was only needed for technical reason, which is reflected in the fact that $M : L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is obviously true. The case including $p^+ = \infty$ was proved by Diening, Harjulehto, Hästö, Mizuta, and Shimomura [DHHMS09], see Proposition 2.2 below.

Although the global log-Hölder continuity of the variable exponent p with $p^- > 1$ is sufficient for the boundedness of the maximal operator, it is not necessary. Lerner [Ler05] constructed examples of variable exponents which are not continuous at zero and infinity but for which the maximal operator is nevertheless bounded. At the same time Diening [Die05] gave a full charaterization of the boundedness of the maximal operator for variable eponents $p \in \mathcal{P}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. This characterization is closely connected to the Muckenhoupt condition for weighted Lebesgue spaces and is stated in terms of averaging operators over families of disjoint cubes. Based on this characterization Diening also showed that the maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if it is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$ and $L^{(p(\cdot)/s)'}(\mathbb{R}^n)$ for some $s > 1$. This ensures that the application of the extrapolation results of Cruz-Uribe, Fiorenza, and Neugebauer [CUFN03] requires only the boundedness of the maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$. Note that the additional boundedness of the maximal operator on $L^{p'(\cdot)}(\mathbb{R}^n)$ and

$L^{(p(\cdot)/s)'}(\mathbb{R}^n)$ for some $s > 1$ is immediate for globally log-Hölder continuous variable exponents with $1 < p^- \leqslant p^+ < \infty$ but for general variable exponents this is obvious by no means.

Many results for bounded domains Ω can be deduced from the result for the whole space $\mathbb{R}^n$ in combination with an extension result. Assume for example that Ω and $q \in (1, n)$ are such that there exists a continuous extension operator $\mathcal{E} : W^{1,q}(\Omega) \to W^{1,q}(\mathbb{R}^n)$. Then the Sobolev embedding $W^{1,q}(\Omega) \to L^{q^*}(\Omega)$ with $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$ follows immediately from the estimate $\|f\|_{L^{q^*}(\mathbb{R}^n)} \leqslant \|\nabla f\|_{L^q(\mathbb{R}^n)}$. This raises the question under which conditions on Ω and the variable exponent p there exists an extension operator $\mathcal{E} : W^{1,p(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\mathbb{R}^n)$.

We have to distinguish two cases. In the first case the variable exponent p is given a priori for the whole space $\mathbb{R}^n$ and the question is only to find a proper Sobolev extension operator $\mathcal{E}$. If we have proper control of the maximal operator, then this can be achieved by extrapolation and the corresponding results for weighted Lebesgue spaces [Chu06]. The reduction to the boundedness of the maximal operator was used in [DH07] for the Sobolev extensions from the half space to $\mathbb{R}^n$ and in [CUFMP06] for Sobolev extentions for domains satisfying the uniform, interior cone condition.

In the second case the variable exponent p is only given on Ω and has to be extended as well. This should be done in such a way that the maximal operator will be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. If p is globally log-Hölder continuous on Ω, then it is not difficult to extend p such that p is globally log-Hölder continuous on $\mathbb{R}^n$. This can for example be achieved with the help of the extension theorem of McShane [McS34] to $\mathbb{R}^n$, see [Die04a], [CUFMP06], and [DHHMS09] for corresponding extension results. In [CUFMP06] this was used to get a suitable extension of the variable exponent to $\mathbb{R}^n$ before the Sobolev extension result was considered. The situation is much more difficult for measurable p. This problem of a suitable extension of the variable exponent p was the only restriction in [CUFMP06] that circumvented a Sobolev extension theorem for measurable p. Due to this reason Cruz-Uribe, Fiorenza, Martell, and Pérez stated the following question:

> **(EP) Extension-Problem** [CUFMP06, Remark 4.4]**:**
>
> It would be interesting to determine if every exponent $p \in \mathcal{B}(\Omega)$ can be extended to an exponent function in $\mathcal{B}(\mathbb{R}^n)$.

Hereby, $\mathcal{B}(\Omega)$ denotes the set of variable exponents $p \in \mathcal{P}(\Omega)$ such that M is bounded on $L^{p(\cdot)}(\Omega)$.

It is the purpose of this paper to provide a positive (partial) answer to this question. Our final goal is to solve (EP) for ε-δ-domain. These domains were introduced by Jones [Jon81] and are therefore also called *Jones domains*. They are the natural domains for the extension of Sobolev functions and it is therefore our aim to solve (EP) for the same type of domains. Once this problem is solved, the extension result for variable exponent Sobolev spaces will follow immediately from extrapolation and the result of Chua [Chu06].

The extensions of Sobolev functions for ε-δ-domains and constant exponent are constructed by the use of the Whitney extension, see [Jon81, Chu06]. Therefore, it is natural to use a Whitney type extension to extend the variable exponent p. (Note that other methods like the method of reflection require higher regularity of the boundary of Ω.) However, instead of working with ε-δ-domains we will restrict ourselves in this article to the case where Ω is the half space. We are aware of the fact that in the case of the half space (EP) can be solved far easier by reflection of the exponent. However, the use of a Whitney type extension shows the strong potential of the method in the sense that it can also be applied in the context of ε-δ-domains. The solution of (EP) for ε-δ-domains will be the subject of a forthcoming paper. Therefore, we will keep the level of details low and concentrate on presenting the essential ideas of the method. Detailed calculations for the case of the half space can be found in [Frö08] and in the forthcoming article [DF08].

2. Notation and Basic Properties

By c we denote a generic constant, i.e. its value may change from line to line. We write $f \sim g$ if there exist constants $c_1, c_2 > 0$ so that $c_1 f \leqslant g \leqslant c_2 g$. For a measurable subset $\Omega \subset \mathbb{R}^n$ with positive (Lebesgue) measure we denote by $L^0(\Omega)$ the space of real-valued, measurable functions on Ω and by $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the space of real-valued, locally integrable functions on $\mathbb{R}^n$. We use χ_Ω for the characteristic function of Ω. By $L^p(\Omega)$ with $p \in [1, \infty]$ we denote the usual Lebesgue spaces. We use $|\Omega|$ for the n-dimensional Lebesgue measure of Ω. For $f \in L^1(\Omega)$ we write

$$f_\Omega = \fint_\Omega f(y)\,dy := |\Omega|^{-1} \int_\Omega f(y)\,dy.$$

By M we denote the (uncentered) Hardy–Littlewood maximal operator, i.e for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ let

$$Mf(x) := \sup_{Q \ni x} \fint_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes which contain x. Throughout the paper all cubes will have sides parallel to the axes. If $f \in L^1(\Omega)$ with measurable $\Omega \subset \mathbb{R}^n$, then we often implicitly extend f outside of Ω by zero, so that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. This is in particular used in the definition of Mf for $f \in L^1(\Omega)$. For a cube $W \subset \mathbb{R}^n$ and $\lambda > 0$ we denote by λW the cube with the same center and λ-times the diameter. Analogously, for a family $\mathcal{W}$ of cubes and $\lambda > 0$ we denote by $\lambda\mathcal{W}$ the familiy $\{\lambda W \,:\, W \in \mathcal{W}\}$.

A measurable function $p\colon \Omega \to [1,\infty]$ will be called a variable exponent on Ω. We write $\mathcal{P}(\Omega)$ for the set of all variable exponents on Ω. For $p \in \mathcal{P}(\Omega)$ we define $p^+_\Omega = \operatorname{esssup}_{x\in\Omega} p(x)$ and $p^-_\Omega = \operatorname{essinf}_{x\in\Omega} p(x)$, and abbreviate $p^+ = p^+_{\mathbb{R}^n}$ and $p^- = p^-_{\mathbb{R}^n}$. For $p \in \mathcal{P}(\Omega)$ we define the *dual exponent* $p' \in \mathcal{P}(\Omega)$ pointwise by $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. Hereby and in the following we use the convention $\frac{1}{\infty} = 0$. For $q \in [1,\infty]$ and $t \geqslant 0$ we define $\varphi_q \,:\, [0,\infty) \to [0,\infty]$ by

$$\varphi_q(t) := \begin{cases} t^q & \text{for } 1 \leqslant q < \infty, \\ 0 & \text{for } q = \infty, t \in (0,1], \\ \infty & \text{for } q = \infty, t \in (1,\infty). \end{cases}$$

The reason to define $\rho_\infty(1) = 0$ is to get a left-continuous function, as in the general theory of Orlicz–Musielak spaces. Note that $\rho_\infty(t) \leqslant \lim_{q\to\infty} \rho_q(t) \leqslant \rho_\infty(2t)$ for all $t \geqslant 0$. Let $q,r,s \in [1,\infty]$ with $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$. Then Young's inequality reads

$$\varphi_q(ab) \leqslant \varphi_r(a) + \varphi_s(b)$$

for all $a, b \geqslant 0$.

The variable exponent *modular* is defined for measurable functions by

$$\varrho_{p(\cdot)}(f) = \int_\Omega \varphi_{p(x)}(|f(x)|)\, dx.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of measurable functions $f\colon \Omega \to \mathbb{R}$ with $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg norm on this space by the formula

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\big\{\lambda > 0\colon \varrho_{p(\cdot)}(f/\lambda) \leqslant 1\big\}.$$

The abbreviation $\|f\|_{p(\cdot)}$ is used for the norm $\|f\|_{L^{p(\cdot)}(\Omega)}$ over all of Ω. We use $\varphi_{p(\cdot)}$ for the function $(x,t) \mapsto \varphi_{p(x)}(t)$.

The following results are standard in the context of Orlicz spaces, see e.g. [RR91], [Mus83], [Die07, Section 1.1]. A convex, left-continuous function $\rho : [0,\infty) \to [0,\infty]$ with $\rho(0) = 0$, $\lim_{t\to\infty} \rho(t) = \infty$, and $\lim_{t\searrow 0} \rho(t) = 0$ is called a *φ-function*. The *complementary function* ρ^* is defined by $\rho^*(u) := \sup_{t\geqslant u}(tu - \rho(t))$. Then $(\rho^*)^* = \rho$. Note that $\varphi_q^*(t) \sim \varphi_{q'}(t)$ for $1 \leqslant q \leqslant \infty$ and all $t \geqslant 0$. We define $\varphi_{p(\cdot)}^*$ pointwise with respect to x.

The following has emerged as a central condition in the theory of variable exponent spaces.

Definition 2.1. *Let $\alpha \in C(\Omega)$. We say that α is* locally log-Hölder continuous *if there exists $c_{\log} > 0$ so that*

$$|\alpha(x) - \alpha(y)| \leqslant \frac{c_{\log}}{\ln(e + 1/|x-y|)} \tag{2.1}$$

for all $x, y \in \Omega$.

*We say that α is (*globally*)* log-Hölder continuous *if it is locally* log-*Hölder continuous and there exists $\alpha_\infty \in \mathbb{R}$ so that the decay condition*

$$|\alpha(x) - \alpha_\infty| \leqslant \frac{c_{\log}}{\ln(e + |x|)} \tag{2.2}$$

holds for all $x \in \Omega$. The smallest constant $c_{\log}$ that satisfies (2.1) *and* (2.2) *is called the* log-*Hölder constant of α.*

The notation $\mathcal{P}^{\ln}(\Omega)$ is used for the set of variable exponents $p \in \mathcal{P}(\Omega)$ for which $1/p$ is globally log-Hölder continuous. Then $p \in \mathcal{P}^{\ln}(\Omega)$ if and only if $p' \in \mathcal{P}^{\ln}(\Omega)$. If $\Omega \subset \mathbb{R}^n$ is an unbounded, open set and $p \in \mathcal{P}^{\ln}(\Omega)$, then we define p_∞ to be the limit $\lim_{|x|\to\infty} p(x)$, which is well defined, since $\frac{1}{p}$ is globally log-Hölder continuous. We have $(p_\infty)' = (p')_\infty$.

The importance of $\mathcal{P}^{\ln}(\mathbb{R}^n)$ becomes clear by the following result.

Proposition 2.2 (Theorem 3.6, [DHHMS09])**.** *Let $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ \leqslant \infty$. Then M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, i.e. to*

$$\|Mf\|_{p(\cdot)} \leqslant K\, \|f\|_{p(\cdot)}.$$

Here $K > 0$ depends only on the dimension n, the constant of log-*Hölder continuity of $\frac{1}{p}$, and p^-.*

3. The extension of the variable exponents

In this section we will construct the extension of the variable exponent. By $\mathbb{H}$ we denote the (positive) half space of $\mathbb{R}^n$, i.e.

$$\mathbb{H} := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0\}.$$

By $-\mathbb{H}$ we denote the (negative) half space, i.e. $-\mathbb{H} = \{-x : x \in \mathbb{H}\}$. We make in the following assumption on our variable exponent p for the rest of the paper.

Assumption 3.1. *Let $p \in \mathcal{P}(\mathbb{H})$ with $1 < p^- \leqslant p^+ < \infty$ be such that the maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{H})$, i.e. for some $K > 0$ holds*

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{H})} \leqslant K \|f\|_{L^{p(\cdot)}(\mathbb{H})}. \tag{3.1}$$

for all $f \in L^{p(\cdot)}(\mathbb{H})$, where f is extended by zero outside of $\mathbb{H}$.

Note that (3.1) is equivalent to

$$\|\chi_{\mathbb{H}} M(f\chi_{\mathbb{H}})\|_{p(\cdot)} \leqslant K \|f\chi_{\mathbb{H}}\|_{p(\cdot)} \tag{3.2}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

3.1. **The extension.** We want to extend p outside of $\mathbb{H}$ by means of a Whitney type extension. Herefore, we take a suitable Whitney decompositions $\mathcal{W}_1$ and $\mathcal{W}_2$ of $\mathbb{H}$ and $-\mathbb{H}$, respectively. In particular, $\mathcal{W}_1$ and $\mathcal{W}_2$ are families of open cubes from $\mathbb{R}^n$ that satisfy:

- (W1) $\bigcup_{W \in \mathcal{W}_1} 125W \subset \mathbb{H} \subset \bigcup_{W \in \mathcal{W}_1} \frac{16}{17}W$.
- (W2) $\bigcup_{W \in \mathcal{W}_2} 125W \subset -\mathbb{H} \subset \bigcup_{W \in \mathcal{W}_2} \frac{16}{17}W$.
- (W3) $\frac{1}{2}W \cap \frac{1}{2}Z = \emptyset$ for all $W, Z \in \mathcal{W}_j$ $(j = 1, 2)$ with $W \neq Z$.
- (W4) The family $125\mathcal{W}_j$ $(j = 1, 2)$ can be written as the finite union of pairwise disjoint families of cubes. The number of families only depends on the dimension n.
- (W5) $\sum_{W \in \mathcal{W}_j} \chi_{125W} \leqslant c$ for $j = 1, 2$.
- (W6) For all $W, Z \in \mathcal{W}_j$ $(j = 1, 2)$ with $W \cap Z \neq \emptyset$ holds $Z \subset 5W$.
- (W7) There exist $c_1, c_2 > 0$ such that

  $$c_1 \operatorname{diam}(W) \leqslant \operatorname{dist}(x, \partial\mathbb{H}) \leqslant c_2 \operatorname{diam}(W)$$

  for all $W \in \mathcal{W}_1$ $(j = 1, 2)$ and all $x \in W$.

A construction of the families $\mathcal{W}_1$ and $\mathcal{W}_2$ can be found in the appendix. For $W \in \mathcal{W}_2$ we denote by W^* the reflected cube $W^* := \{(x_1, \dots, x_{n-1}, -x_n) : x \in W\}$. We assume that the families $\mathcal{W}_1$ and $\mathcal{W}_2$ are chosen such that $W^* \in \mathcal{W}_1$ for every $W \in \mathcal{W}_2$ (for example use $\mathcal{W}_1 := \{W^* : W \in \mathcal{W}_2\}$).

To the family $\mathcal{W}_2$ we find a subordinate partition of unity $\{\eta_W\}_{W\in\mathcal{W}_2}$ with $\eta_W \in C_0^\infty(\frac{33}{34}W)$, $0 \leqslant \eta_W \leqslant 1$, $\|\nabla\eta_W\|_\infty \leqslant c\,\operatorname{diam}(W)^{-\frac{1}{n}}$ for all $W \in \mathcal{W}_2$ and $\sum_{W\in\mathcal{W}_2}\eta_W = \chi_{-\mathbb{H}}$. Property (W3) implies that $\eta_W(x) = 1$ for all $x \in \frac{1}{2}W$ with $W \in \mathcal{W}_2$.

For a subset $U \subset \mathbb{R}^n$ with positive measure and $p \in \mathcal{P}(U)$ we define $p_U \in [1,\infty]$ by

$$\frac{1}{p_U} := \fint_U \frac{1}{p(y)}\,dy. \tag{3.3}$$

We are now able to define the extension of p to the complement of $\mathbb{H}$. For $x \in -\mathbb{H}$ we extend p by

$$\frac{1}{p(x)} := \sum_{H\in\mathcal{W}_2} \eta_H(x)\frac{1}{p_{5W^*}}. \tag{3.4}$$

Recall that $5W^*$ is the cube with the same center as W^* but five times the diameter. Since $\mathbb{R}^n \setminus (\mathbb{H} \cup (-\mathbb{H}))$ is a null set, this defines an extension $p \in \mathcal{P}(\mathbb{R}^n)$. We use the same symbol p for $p \in \mathcal{P}(\mathbb{H})$ and its extension $p \in \mathcal{P}(\mathbb{R}^n)$.

Remark 3.2. *The extension of p can be interpreted as a Whitney extension of $\frac{1}{p}$. One can ask the question: "Why do we extend $\frac{1}{p}$ rather than p?" The reason is that $\frac{1}{p}$ behaves much better than p, especially with respect to duality. For example if we extend the dual exponent p' by means of* (3.4)*, then we will get the same result as if we take the dual exponent of the extended variable exponent. So our extension operator for p commutes with duality.*

We can now state our main result:

Theorem 3.3. *Let $p \in \mathcal{P}(\mathbb{H})$ with $1 < p^- \leqslant p^+ < \infty$ be as in Assumption 3.1, i.e. the maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{H})$. Then* (3.4) *defines an extension of p to $\mathbb{R}^n$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

This theorem provides a positive answer to (EP) for $\Omega = \mathbb{H}$. The proof of our main result is based on a fundamental characterization of those variable exponents $p \in \mathcal{P}(\mathbb{R}^n)$ for which the maximal operator is bounded. To explain this we need a few more notations.

Definition 3.4. *Let*

$$\mathcal{Y}^n := \{\mathcal{W} : \mathcal{W} \textit{ is a family of pairwise disjoint cubes from } \mathbb{R}^n\}.$$

Then for $\mathcal{Q} \in \mathcal{Y}^n$ *we define* $\mathcal{T}_{\mathcal{Q}} : L^1_{\mathrm{loc}}(\mathbb{R}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *by*

$$T_{\mathcal{Q}} f := \sum_{Q \in \mathcal{Q}} \chi_Q M_Q f,$$

where $M_Q f := \fint_Q |f(x)|\, dx$.

Let $U \subset \mathbb{R}^n$ *be an open set. We say that* $\varphi_{p(\cdot)} \in \mathcal{A}(U)$ *if there exists* $K > 0$ *such that* $\|\chi_U \mathcal{T}_{\mathcal{Q}}(f\chi_U)\|_{L^{p(\cdot)}(U)} \leqslant K\, \|f\chi_U\|_{L^{p(\cdot)}(U)}$ *for all* $\mathcal{Q} \in \mathcal{Y}^n$ *and all* $f \in L^{p(\cdot)}(\mathbb{R}^n)$. *The smallest constant* K *is called the* $\mathcal{A}(U)$*-constant of* $\varphi_{p(\cdot)}$.

Note that Assumption 3.1 implies that $\varphi_{p(\cdot)} \in \mathcal{A}(\mathbb{H})$. The condition $\varphi_{p(\cdot)} \in \mathcal{A}(\mathbb{R}^n)$ is closely related to the Muckenhoupt classes for weighted Lebesgue space, see the remarks after Definition 3.1 in [Die05]. The following characterization is due to Diening.

Theorem 3.5 (Theorem 8.1, [Die05])**.** *Let* $p \in \mathcal{P}(\mathbb{R}^n)$ *with* $1 < p^- \leqslant p^+ < \infty$. *The following are equivalent*

(a) $\varphi_{p(\cdot)} \in \mathcal{A}(\mathbb{R}^n)$.
(b) M *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$.
(c) M *is continuous on* $L^{\frac{p(\cdot)}{q}}(\mathbb{R}^n)$ *for some* $q > 1$ *("left–openness").*
(d) $\varphi_{p'(\cdot)} \in \mathcal{A}(\mathbb{R}^n)$.
(e) M *is bounded on* $L^{p'(\cdot)}(\mathbb{R}^)$.

So instead of proving the boundedness of M on $L^{p(\cdot)}(\mathbb{R}^n)$ for our extended variable exponent p, it suffices to prove the simpler condition $\varphi_{p(\cdot)} \in \mathcal{A}(\mathbb{R}^n)$. The assumptions in Theorem 3.5 are the reason that we exclude the case $p^+ = \infty$ in Theorem 3.3. On the other hand we get additionally the boundedness of M on $L^{p'(\cdot)}(\mathbb{R}^n)$ and for this $p^+ < \infty$ is necessary.

Remark 3.6. *If* $p \in \mathcal{P}(\mathbb{R}^n)$ *is constant outside a large ball, then it has been shown by Kopaliani* [Kop07] *that the condition* $\varphi_{p(\cdot)} \in \mathcal{A}(\mathbb{R}^n)$ *can be simplified. Instead of the boundedness of the* $T_{\mathcal{Q}}$ *for all* $\mathcal{Q} \in \mathcal{Y}^n$ *it is enough to verify the boundedness of the* $T_{\{Q\}}$*, where the* $\{Q\}$ *are just the families consisting of one single cube.*

3.2. **Boundedness of** $T_{\mathcal{W}_2}$**.** In order to apply Theorem 3.5 we have to show that $\varphi_{p(\cdot)} \in \mathcal{A}(\mathbb{R}^n)$. In particular, we have to show that the averaging operators $T_{\mathcal{Q}}$ are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ uniformly in $\mathcal{Q} \in \mathcal{Y}^d$. As a crucial step we begin with the boundedness of $\mathcal{T}_{\mathcal{W}_2}$. It will be an important tool for the general case.

The idea is the following. First, we take the average on the left side $-\mathbb{H}$ but evaluate them on the right side $\mathbb{H}$. Second, we show how to tranfer these averages back to the left side $-\mathbb{H}$. In particular, our proof for the boundedness of $T_{\mathcal{W}_1}$ will look as follows

$$\Big\|\sum_{W\in\mathcal{W}_1}\chi_W M_W f\Big\|_{p(\cdot)} \leqslant c\,\Big\|\sum_{W\in\mathcal{W}_1}\chi_{W^*} M_W f\Big\|_{p(\cdot)} \leqslant c\,\|f\|_{p(\cdot)}. \tag{3.5}$$

We explain both step in (3.5) separately. Let us begin with the second part of (3.5). We will need the following assertion from [DHHMS09, Lemma 6.1].

Lemma 3.7. *For a cube $Q\subset\mathbb{R}^n$ and $p\in\mathcal{P}(Q)$ holds*

$$\left(\frac{t}{2}\right)^{p_Q} \leqslant \fint_Q t^{p(y)}\,dy \tag{3.6}$$

for all $t\geqslant 0$.

For $p\in\mathcal{P}(\mathbb{R}^n)$ and a cube $Q\subset\mathbb{R}^n$ we define

$$M_Q\varphi_{p(\cdot)}(t) := \fint_Q \varphi_{p(x)}(t)\,dx = \fint_Q t^{p(x)}\,dx.$$

for $t\geqslant 0$. Analgously, we define $M_Q\varphi^*_{p(\cdot)}(t) = \fint_Q \varphi^*_{p(x)}(t)\,dx$. Then $t\mapsto M_Q\varphi^*_{p(\cdot)}(t)$ is a φ-function. We denote its complementary function by $(M_Q\varphi^*_{p(\cdot)})^*$. It was proved in [Kop07] and [Die07, Lemma 4.6] that

$$(M_Q\varphi^*_{p(\cdot)})^*(t) = \inf_{f\,:\,M_Q f = t}\fint_Q |f(x)|^{p(x)}\,dx \tag{3.7}$$

and

$$(M_Q\varphi^*_{p(\cdot)})^*\left(\frac{t}{2}\right) \leqslant t^{p_Q} \leqslant M_Q\varphi_{p(\cdot)}(2\,t) \tag{3.8}$$

for all $t\geqslant 0$.

Let us get back to the proof of the second part of (3.5). Let $f\in L^{p(\cdot)}(-\mathbb{H})$ with $\|f\|_{p(\cdot)}\leqslant 1$, so $\int|f(x)|^{p(x)}\,dx\leqslant 1$. First, we decompose f into functions $f_W\in L^{p(\cdot)}(W)$ such that $f=\sum_{W\in\mathcal{W}_2} f_W$ and

$$\sum_{W\in\mathcal{W}_2}\int_W |f_W(x)|^{p_{5W^*}}\,dx \leqslant c.$$

This is possible, since the extension of p ensures

$$\min_{W\in\mathcal{W}_2\,:\,x\in W}|f(x)|^{p_{5W^*}} \leqslant |f(x)|^{p(x)} \tag{3.9}$$

Then (3.8) and Jensen's inequality imply

$$\sum_{W\in\mathcal{W}_2} |5W|\big(M_{5W^*}\varphi^*_{p(\cdot)}\big)^*\big(\tfrac{1}{2}M_{5W}f_W\big)$$
$$\leqslant c \sum_{W\in\mathcal{W}_2} |5W|\big(M_{5W}f_W\big)^{p_{5W^*}} \leqslant c.$$

Now, $\varphi_{p(\cdot)} \in \mathcal{A}(\mathbb{H})$ (by Assumption 3.1) together with the characterization (3.7) imply that

$$\sum_{W\in\mathcal{W}_2} |5W|\big(M_{5W^*}\varphi_{p(\cdot)}\big)\big(\tfrac{1}{2}M_{5W}f_W\big) \leqslant c.$$

This step is in detail explained in [Die05, Theorem 4.2]. Therefore,

$$\int_{\mathbb{R}^n} \Bigg(\sum_{W\in\mathcal{W}_2} \chi_{5W^*}M_{5W}f_W\Bigg)^{p(x)} dx \leqslant c.$$

The construction of the f_W finally ensures that

$$\int_{\mathbb{R}^n} \Bigg(\sum_{W\in\mathcal{W}_2} \chi_{W^*}M_{W}f\Bigg)^{p(x)} dx \leqslant c.$$

This proves the second part of (3.5). Let us summarize that the main ingredient in the proof was (3.8). It is important for (3.8) that p_{5W^*} is defined via the reciprocal mean value of p, see (3.3).

The proof of the first part of (3.5) relies on the following important fact: The mapping $\frac{1}{q} \mapsto qt^q$ is convex for any $t > 0$. This and the definition of p immediately imply that

$$\frac{1}{p(x)}t^{p(x)} \leqslant \sum_{W\in\mathcal{W}_2} \eta_W(x)\frac{1}{p_{5W^*}}t^{p_{5W^*}}.$$

The left hand side of this estimate corresponds to $\sum_{W\in\mathcal{W}_2}\chi_W(x)t^{p(x)}$ and the right hand side to $\sum_{W\in\mathcal{W}_2}\chi_{\frac{1}{2}W^*}(x)t^{p(x)}$. The rest of the proof of (3.5) is straightforward.

3.3. **Proof of Class $\mathcal{A}$.** Let us now explain how to prove the boundedness of $T_{\mathcal{Q}}$ for general $\mathcal{Q} \in \mathcal{Y}^n$. First of all, it is important to realize that we have to distinguish two cases of cubes. Such cubes with are small compared to the cubes of our Whitney decompositions $\mathcal{W}_1$ and $\mathcal{W}_2$ and the other cubes which are called big cubes.

Definition 3.8. *Let $\mathcal{Q} \in \mathcal{Y}^n$.*

(a) *A cube $Q \in \mathcal{Q}$ is called* small*, if there exists $W \in \mathcal{W}_1 \cup \mathcal{W}_2$ such that $Q \subset W$. We define*

$$\begin{aligned} \mathcal{Q}_{1,small} &:= \{Q \in \mathcal{Q} \,:\, \exists W \in \mathcal{W}_1 \text{ with } Q \subset W\}, \\ \mathcal{Q}_{2,small} &:= \{Q \in \mathcal{Q} \,:\, \exists W \in \mathcal{W}_2 \text{ with } Q \subset W\}, \\ \mathcal{Q}_{small} &:= \mathcal{Q}_{1,small} \cup \mathcal{Q}_{2,small}. \end{aligned}$$

(b) *A cube $Q \in \mathcal{Q}$ is called* big*, if it is not small.*

$$\mathcal{Q}_{big} := \mathcal{Q} \setminus \mathcal{Q}_{small}.$$

We will see later that big cubes are automatically close to $\partial\mathbb{H}$ with respect to their size. But let us concentrate in the first step on small cubes. The family $\mathcal{Q}_{1,\text{small}}$ is no problem, since these cubes are completely inside $\mathbb{H}$ and therefore the continuity of $T_{\mathcal{Q}_{1,\text{small}}}$ is guaranteed by Assumption 3.1.

3.4. **Small cubes.** So let $Q \in \mathcal{Q}_{2,\text{small}}$. Then Q is contained in some $W_Q \in \mathcal{W}_2$. If $Q \subset \frac{1}{2}W_Q$, then p is constant on Q (namely $p = p_{5W_Q^*}$) and we are in the case of a constant exponent which is quite simple. But if Q lies more at the boundary of W_Q, then p is variable. For example if $Z \in \mathcal{W}_2$ is another cube with $Q \subset Z$, then p varies on Q between $p_{5W_Q^*}$ and p_{5Z^*}. However, the regularity of p depends mostly on the local regularity of the partition of unity η_W. So the bigger W_Q is, the more regular p is on W_Q and Q. Certainly, the regularity of p on Q also depends on $|\frac{1}{p_{5W_Q^*}} - \frac{1}{p_{5Z^*}}|$, but we will see that this difference is rather nice. Indeed, the following Proposition 3.9 shows that oscillation of $\frac{1}{p}$ has a similar behaviour than log-Hölder continuous functions if averaging operators over single cubes are bounded.

Proposition 3.9. *Let $s \in \mathcal{P}(\mathbb{R}^n)$, $K > 0$, and let $Q \subset \mathbb{R}^n$ be an arbitrary cube such that $\|\chi_Q M_Q f\|_{s(\cdot)} \leqslant K \|f\|_{s(\cdot)}$ for all $f \in L^{s(\cdot)}(\mathbb{R}^n)$. Then*

$$\fint_Q \fint_Q \left|\frac{1}{s(y)} - \frac{1}{s(z)}\right| dy\, dz \leqslant \frac{\ln(40K^2)}{\ln(e + |Q| + \frac{1}{|Q|})}. \tag{3.10}$$

Proof. Note that $T_{\{Q\}} f := \chi_Q M_Q f$. So by assumption $\|T_{\{Q\}} f\|_{p(\cdot)} \leqslant K \|f\|_{p(\cdot)}$ for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Therefore, we get

$$\begin{aligned}
\|T_{\{Q\}} g\|_{p'(\cdot)} &\leqslant 2 \sup_{\|h\|_{p(\cdot)} \leqslant 1} \int_Q T_{\{Q\}} g |h| \, dx \\
&= 2 \sup_{\|h\|_{p(\cdot)} \leqslant 1} \int_Q |g| T_{\{Q\}} h \, dx \\
&\leqslant 2\, K \|g\|_{p'(\cdot)},
\end{aligned}$$

for all $g \in L^{p'(\cdot)}(Q)$, where we have used in the first step the characterization of the dual space of $L^{p(\cdot)}$, see Lemma 2.9 [Die07].

Define $f := \chi_Q |Q|^{-1/p}$ and $u := \chi_Q |Q|^{-1/p'}$. Then $\varrho_{p(\cdot)}(f) \leqslant 1$ and $\varrho_{p'(\cdot)}(u) \leqslant 1$, which implies $\|f\|_{p(\cdot)} \leqslant 1$ and $\|u\|_{p'(\cdot)} \leqslant 1$. So with Hölder's inequality and the continuity of $T_{\mathcal{Q}}$, we get

$$\begin{aligned}
|Q| M_Q f M_Q g &= \int_{\mathbb{R}^n} \chi_Q M_Q f M_Q g \, dx \\
&\leqslant 2\, \|T_{\{Q\}} f\|_{p(\cdot)} \|T_{\{Q\}} u\|_{p'(\cdot)} \\
&\leqslant 4\, K^2 \|f\|_{p(\cdot)} \|u\|_{p'(\cdot)} \\
&\leqslant 4\, K^2.
\end{aligned}$$

By definition of f and u this implies

$$\fint_Q \fint_Q |Q|^{-\frac{1}{p(y)} + \frac{1}{p(z)}} \, dy \, dz \leqslant 4\, K^2.$$

By symmetry in y and z we get

$$\fint_Q \fint_Q \max \left\{ |Q|^{\left| \frac{1}{p(y)} - \frac{1}{p(z)} \right|}, |Q|^{-\left| \frac{1}{p(y)} - \frac{1}{p(z)} \right|} \right\} dy \, dz \leqslant 8K^2.$$

Since $\max\{t, 1/t\} \geqslant c(e + t + 1/t)$ for all $t > 0$, we get

$$\fint_Q \fint_Q \left(e + |Q| + \frac{1}{|Q|} \right)^{\left| \frac{1}{p(y)} - \frac{1}{p(z)} \right|} dy \, dz \leqslant 40K^2.$$

The mapping $s \mapsto (e + |Q| + 1/|Q|)^s$ is convex, so by Jensen's inequality we get

$$\left(e + |Q| + \frac{1}{|Q|} \right)^{\fint_Q \fint_Q \left| \frac{1}{p(y)} - \frac{1}{p(z)} \right| dy \, dz} \leqslant 40\, K^2.$$

Taking the logarithm proves the claim. $\square$

Remark 3.10. *As we have mentioned above (see* [Ler05] *and* [Die05]*),* log-*Hölder continuity of* $\frac{1}{p}$ *is not necessary for the boundedness of the maximal operator* M*. However, Propositon 3.9 shows that the boundedness of* M *implies that the oscillations of* $\frac{1}{p}$ *satisfy the same estimates as a* log-*Hölder continuous function. Since* $\int_0^1 \frac{1}{t\,\ln(e+t)}\,dt = \infty$*, it follows from* [Spa65] *that the estimates for the oscillations do not imply the corresponding Hölder estimates. Note that the construction by Lerner* [Ler05] *of a discontinuous, variable exponent p such that the maximal operator* M *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$ *is based on oscillation estimates similar to* (3.10)*.*

Propositon 3.9 and the properties of the decomposition of unity can be used to deduce the following interesting result.

Lemma 3.11. *Let* $W \in \mathcal{W}_2$*. Then there exists* $q \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *with* $p = q$ *on* W*,* $q = p_{5W^*}$ *on* $\mathbb{R}^n \setminus (5W)$ *such that the* log-*Hölder constant of* q *only depends on the* $\mathcal{A}(\mathbb{H})$*-constant of* p*.*

So for our small cube $Q \in \mathcal{Q}_{1,\mathrm{small}}$ we can use the known results for log-Hölder continuous exponents. In particular, we can use Proposition 2.2 and related results. As a minor modification of Lemma 3.3 of [DHHMS09] we get the following result.

Lemma 3.12. *Let* $q \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *with* $1 < q^- \leqslant q^+ < \infty$*. Then there exist constants* $\beta, K > 0$ *(only depending on* q^-*,* q^+*, and the* log-*Hölder constant of* $\frac{1}{q}$*) such that for all* $f \in L^{p(\cdot)}(\mathbb{R}^n)$ *with* $\|f\|_{p(\cdot)} \leqslant 1$*, all cubes* $Z \subset \mathbb{R}^n$ *with* $M_Z f \geqslant 1$ *holds*

$$\int\limits_Z (M_Z f)^{q(x)}\,dx \leqslant K \int\limits_Z |f(x)|^{q(x)}\,dx. \tag{3.11}$$

Note that Lemma 3.3 of [DHHMS09] has an additional $+1$ on the right hand side of (3.11) but does not require the assumption $M_Q f > 1$. The idea is that if $M_Q f > 1$, then $\int_Z |f(x)|^{q(x)}\,dx \geqslant c$, so we can omit the $+1$ on the right hand side of (3.11).

Based on (3.7) it has been shown in [Die05, Theorem 4.2] and [Die07, Theorem 4.17] that $\varphi_{p(\cdot)} \in \mathcal{A}(\mathbb{R}^n)$ if and only if for all families $\mathcal{Z} \in \mathcal{Y}^n$ of disjoint cubes

$$\sum_{Z \in \mathcal{Z}} |Z| (M_Z \varphi^*_{p(\cdot)})^*(t_Z) \leqslant 1$$

implies

$$\sum_{Z \in \mathcal{Z}} |Z| (M_Z \varphi_{p(\cdot)})(c\, t_Z) \leqslant 1$$

for all families $t_Z \geqslant 0$. Due to (W4) the cubes $Z \in \mathcal{Z}$ in the formulas above can be replaced by $25Z$. Based on these estimate it follows exactly as in [Die05, (8.24)] and [Die07, Theorem 4.54] that there exists a constant $K > 0$ and a family $b_W \geqslant 0$ for $W \in \mathcal{W}_1$ with

$$\sum_{W \in \mathcal{W}_1} |W| b_W \leqslant c \tag{3.12}$$

such that for all $t \geqslant 0$ with $|W|\,(M_{125W}\varphi^*_{p(\cdot)})^*(t) \leqslant 1$ or(!) $t \leqslant 1$ holds

$$(M_{125W}\varphi_{p(\cdot)})(t) \leqslant K\,(M_{125W}\varphi^*_p)^*(t) + b_W. \tag{3.13}$$

If $|W|\,(M_{25W}\varphi^*_{p(\cdot)})^*(t) \leqslant 1$ and $t \geqslant 1$, then additionally

$$(M_{125W}\varphi_{p(\cdot)})(t) \leqslant K\,(M_{125W}\varphi^*_p)^*(t). \tag{3.14}$$

Note that b and K also depends only p^- and p^+ and the $\mathcal{A}(\mathbb{H})$-constant of $\varphi_{p(\cdot)}$ (which depends on (3.1)) . Based on (3.13) and (3.14) we are able to show the boundedness of $\mathcal{T}_{\mathcal{Q}_{2,\text{small}}}$.

Lemma 3.13. *There holds*

$$\left\| T_{\mathcal{Q}_{2,small}} f \right\|_{p(\cdot)} \leqslant c\, \|f\|_{p(\cdot)}.$$

The idea hereby is the following. First, we use Lemma 3.12 for $Q \in \mathcal{Q}_{2,\text{small}}$ with $M_Q f \geqslant 1$. So the additional regularity of p on $-\mathbb{H}$ is quite crucial for our proof. Second, we use (3.13) for $Q \in \mathcal{Q}_{2,\text{small}}$ with $M_Q f < 1$ and

$$\left(M_{125Z^*}\varphi^*_{p(\cdot)}\right)^*(t) \leqslant c \inf_{x \in W} t^{p(x)}$$

for all $t \geqslant 0$ and all $Z, W \in \mathcal{Z}$ with $Z \cap W \neq \emptyset$.

Together with the boundedness of $T_{\mathcal{Q}_{1,\text{small}}}$ we get the boundedness of $T_{\mathcal{Q}_{\text{small}}}$ on $L^{p(\cdot)}(\mathbb{R}^n)$.

Corollary 3.14. *There holds*

$$\left\| T_{\mathcal{Q}_{small}} f \right\|_{p(\cdot)} \leqslant c\, \|f\|_{p(\cdot)}.$$

3.5. **Big cubes.** Let us now turn to the boundedness of $T_{\mathcal{Q}_{\text{big}}}$. Since the $Q \in \mathcal{Q}_{\text{big}}$ are covered by the familiy $\{\frac{16}{17}W\}_{W \in \mathcal{W}_1 \cup \mathcal{W}_2}$ but are contained in none of the $W \in \mathcal{W}_1 \cup \mathcal{W}_2$, it follows easily that

$$\operatorname{dist}(x, \partial\mathbb{H}) \leqslant c \operatorname{diam}(Q), \tag{3.15}$$

for all $Q \in \mathcal{Q}_{\text{big}}$ and $x \in Q$. Moreover, if $Q \in \mathcal{Q}_{\text{big}}$ and $W \in \mathcal{W}_1 \cup \mathcal{W}_2$ with $\frac{16}{17}W \cap Q \neq \emptyset$, then $W \subset \gamma Q$ for some $\gamma > 0$ (independent of W and Q).

For $Q \in \mathcal{Q}_{\text{big}}$ we define

$$Q^\sharp := \bigcup_{W \in \mathcal{W}_2 \,:\, \frac{16}{17}W \cap Q \neq \emptyset} W^*.$$

Note that $Q^\sharp \subset \mathbb{H}$, since $W^* \subset \mathbb{H}$ for $W \in \mathcal{W}_2$. Moreover, if $Q \in \mathcal{Q}_{\text{big}}$, then $Q^\sharp \subset \gamma_2 Q$ for some $\gamma_2 > 0$ (independent of Q). So the sets Q and $Q^\sharp$ have a similar size and are both close to the boundary with respect to the size of Q. Hence, $Q^\sharp$ can be interpreted as a reflection of Q, although it is not a cube itself.

We want to show the following result

Lemma 3.15. *There holds*

$$\big\|T_{\mathcal{Q}_{big}}(f\chi_{\mathbb{H}})\big\|_{p(\cdot)} \leqslant c\, \|f\chi_{\mathbb{H}}\|_{p(\cdot)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

The proof of this result is split into several parts.

$$\big\|\chi_{\mathbb{H}} T_{\mathcal{Q}_{\text{big}}}(f\chi_{\mathbb{H}})\big\|_{p(\cdot)} \leqslant c\, \|f\|_{p(\cdot)}, \tag{3.16}$$

$$\big\|\chi_{\mathbb{H}} T_{\mathcal{Q}_{\text{big}}}(f\chi_{-\mathbb{H}})\big\|_{p(\cdot)} \leqslant c\, \|f\|_{p(\cdot)}, \tag{3.17}$$

$$\big\|\chi_{-\mathbb{H}} T_{\mathcal{Q}_{\text{big}}}(f\chi_{\mathbb{H}})\big\|_{p(\cdot)} \leqslant c\, \|f\|_{p(\cdot)}, \tag{3.18}$$

$$\big\|\chi_{-\mathbb{H}} T_{\mathcal{Q}_{\text{big}}}(f\chi_{-\mathbb{H}})\big\|_{p(\cdot)} \leqslant c\, \|f\|_{p(\cdot)}. \tag{3.19}$$

The validity of (3.16) follows immediately from Assumption 3.1 but the other estimate require some work. Let us explain the proof for (3.19), since the others work similarily. The idea is to shift the averages $M_Q f$ from $-\mathbb{H}$ to $\mathbb{H}$ by means of the following estimate

$$\Bigg\|\chi_{-\mathbb{H}} \sum_{Q \in \mathcal{Q}_{\text{big}}} \chi_Q M_Q(f\chi_{-\mathbb{H}})\Bigg\|_{p(\cdot)} \leqslant c \Bigg\|\chi_{\mathbb{H}} \sum_{Q \in \mathcal{Q}_{\text{big}}} \chi_{Q^\sharp} M_Q(f\chi_{-\mathbb{H}})\Bigg\|_{p(\cdot)}.$$

Now, that we are on $\mathbb{H}$ we can use the pointwise estimate

$$\chi_{Q^\sharp} M_Q\big(f\chi_{-\mathbb{H}}\big) \leqslant c\, \chi_{\mathbb{H}} M\Bigg(\sum_{W \in \mathcal{W}_2} \chi_{W^*} M_W\big(f\, \chi_{-\mathbb{H}}\big)\Bigg)$$

and our Assumption 3.1 to get the estimate

$$\Big\|\chi_{-\mathbb{H}} \sum_{Q\in\mathcal{Q}_{\text{big}}} \chi_Q M_Q(f\chi_{-\mathbb{H}})\Big\|_{p(\cdot)} \leqslant c \Big\|\sum_{W\in\mathcal{W}_2} \chi_{W^*} M_W\big(f\,\chi_{-\mathbb{H}}\big)\Big\|_{p(\cdot)}.$$

The boundedness of $T_{\mathcal{W}_1}$ finally implies (3.19).

Overall, we have shown the following result.

Theorem 3.16. *Let $\mathcal{Q} \in \mathcal{Y}^n$. Then*

$$\big\|T_{\mathcal{Q}} f\big\|_{p(\cdot)} \leqslant c\,\|f\|_{p(\cdot)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, where c only depends on p^-, p^+, and the constant in (3.1)*. In particular, $\varphi_{p(\cdot)} \in \mathcal{A}(\mathbb{R}^n)$.*

Now, our main result follows easily.

Proof of Theorem 3.3. Due to Theorem 3.16 we know that our extended variable exponent satisfies $\varphi_{p(\cdot)} \in \mathcal{A}(\mathbb{R}^n)$. So the boundedness of M on $L^{p(\cdot)}(\mathbb{R}^n)$ follows by the characterization in Theorem 3.5. $\square$

Appendix

In this appendix we construct a Whitney decomposition as needed in Section 3.1 to ensure the properties see (W1)–(W7).

For a given cube Q in $\mathbb{R}^n$ we denote by $\ell(Q)$ its length. We say that Q is *dyadic*, if it is of the form $2^{-k}m + [0, 2^{-k}]^n$ for some $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. In particular, our dyadic cubes are closed.

The following proposition is a slight modification of the proposition in Appendix J of [Gra04]. Since our constants are slightly sharper, we include a proof.

Proposition 3.17. *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$. Then there exists a countable family $\mathcal{F}$ of closed, dyadic cubes such that*

(A1) $\bigcup_{Q\in\mathcal{F}} Q = \Omega$ *and the $Q \in \mathcal{F}$ have disjoint interiors.*
(A2) $\sqrt{n}\ell(Q) < \operatorname{dist}(Q, \Omega^{\complement}) \leqslant 4\sqrt{n}\ell(Q)$ *for all $Q \in \mathcal{F}$.*
(A3) If $Q, Q' \in \mathcal{F}$ intersect, then

$$\frac{1}{2} \leqslant \frac{\ell(Q)}{\ell(Q')} \leqslant 2.$$

(A4) For given $Q \in \mathcal{F}$, there exists at most $4^n - 2^n$ cubes $Q' \in \mathcal{F}$ touching Q (boundaries intersect but not the interiors).

Proof. Let $\mathcal{D}_m$ denote the collection of all dyadic cubes of length 2^{-m}. Each cube in $\mathcal{D}_m$ gives rise to 2^n cubes in $\mathcal{D}_{m+1}$ by bisecting each side. Decompose Ω into the sets

$$\Omega_m := \{x \in \Omega \,:\, 2\sqrt{n}2^{-m} < \operatorname{dist}(x, \Omega^\complement) \leqslant 4\sqrt{n}2^{-m}\}$$

for $m \in \mathbb{Z}$. Let $\mathcal{F}_m := \{Q \in \mathcal{D}_m \,:\, Q \cap \Omega_m \neq \emptyset\}$ for $m \in \mathbb{Z}$ and $\mathcal{F}' := \bigcup_{m\in\mathbb{Z}} \mathcal{F}_m$. Let $Q \in \mathcal{F}_m$ and $x \in Q \cap \Omega_m$. Then

$$\begin{aligned}\sqrt{n}2^{-m} &< \operatorname{dist}(x, \Omega^\complement) - \sqrt{n}2^{-m} \leqslant \operatorname{dist}(Q, \Omega^\complement)\\ &\leqslant \operatorname{dist}(x, \Omega^\complement) \leqslant 4\,\sqrt{n}2^{-m}.\end{aligned}$$

This proves (A2) for every $Q \in \mathcal{F}'$.

Next we observe that $\bigcup_{Q\in\mathcal{F}'} Q = \Omega$. Indeed, every $Q \in \mathcal{F}'$ is contained in Ω and every $x \in \Omega$ is contained in some Ω_m and in some dyadic cube $Q \in \mathcal{D}_m$.

Unfortunately, the cubes in the collection $\mathcal{F}'$ may not be disjoint and we need to eliminate the cubes that are contained in some other cubes of the collection. Observe that two dyadic cubes have either disjoint interiors or one contains the other. On the other hand if $Q \in \mathcal{F}'$, then by condition (A2) the cubes of $\mathcal{F}$ containing Q cannot be arbitrary large. Therefore, we can define for each $Q \in \mathcal{F}'$ a unique *maximal dyadic cube* $Q^{\max} \in \mathcal{F}'$ that is contained in no other cube of $\mathcal{F}$ and that contains Q. Now set $\mathcal{F} := \{Q^{\max} \,:\, Q \in \mathcal{F}'\}$. By maximality two different cubes of $\mathcal{F}$ have disjoint interiors. We still have $\bigcup_{Q\in\mathcal{F}} Q = \Omega$. This proves (A1) for the familiy $\mathcal{F}$.

Let us now prove (A3). If $Q, Q' \in \mathcal{F}$ with $Q \cap Q' \neq \emptyset$, then using (A2) we estimate

$$\begin{aligned}\sqrt{n}\ell(Q) &< \operatorname{dist}(Q, \Omega^\complement) \leqslant \operatorname{dist}(Q, Q') + \operatorname{dist}(Q', \Omega^\complement)\\ &\leqslant 0 + 4\sqrt{n}\ell(Q').\end{aligned}$$

Thus, $\ell(Q) < 4\ell(Q')$. Since $\ell(Q)$ and $\ell(Q')$ are of the special form 2^{-m_1} and 2^{-m_2}, respectively, for some $m_1, m_2 \in \mathbb{Z}$, we get the stricter estimate $\ell(Q) \leqslant 2\,\ell(Q')$. This proves (A3).

Let $Q \in \mathcal{F}$. Then $Q \in \mathcal{D}_m$ for some $m \in \mathbb{Z}$. Due to (A3) every cube from $\mathcal{F}$ touching Q contains at least one cube from $\mathcal{D}_{m+1}$ touching Q. Since there are exactly $4^n - 2^n$ cubes from $\mathcal{D}_{m+1}$ touching Q, there are also at most $4^n - 2^n$ cubes from $\mathcal{F}$ touching Q. $\square$

Corollary 3.18. *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$. Then there exists a family $\mathcal{W}$ of open cubes such that*

(B1) $\bigcup_{W\in\mathcal{W}} \frac{16}{17}W = \bigcup_{W\in\mathcal{W}} 125W = \Omega$.
(B2) $\frac{1}{2}W \cap \frac{1}{2}Z = \emptyset$ *for all* $W, Z \in \mathcal{W}$ *with* $W \neq Z$.

(B3) *If $W, W' \in \mathcal{W}$ intersect, then $W \subset 5W'$ and $W' \subset 5W$.*
(B4) *The family $125\mathcal{W}$ can be written as the finite union of at most 256^n pairwise disjoint families of cubes.*
(B5) $\sum_{W\in\mathcal{W}} \chi_{125W} \leqslant 256^n$.
(B6) *There holds*

$$\Big(256 - \frac{1}{8}\Big)\,\mathrm{diam}(W) < \mathrm{dist}(W, \partial\Omega) \leqslant 1024\,\mathrm{diam}(W)$$

for every $W \in \mathcal{W}$.

Proof. Let $\mathcal{F}$ denote the family of closed dyadic cubes of Proposition 3.17. Let $\mathcal{F}^{\#}$ denote the family of closed dyadic cubes that we get if we slit every cube of $\mathcal{F}$ by repeated bisection into 256^n closed dyadic cubes of length $\frac{1}{256}\ell(Q_j)$. The new family still satisfies (A1), (A3), and (A4) of Proposition 3.17, but (A2) has to be replaced by

$$\text{(A2b)}\quad 256\sqrt{n}\ell(Q) < \mathrm{dist}(Q, \Omega^{\complement}) \leqslant 1024\sqrt{n}\ell(Q)$$

for all $Q \in \mathcal{F}^{\#}$. For $Q \in \mathcal{F}^{\#}$ define $W_Q := \frac{9}{8}\mathrm{int}(Q)$, where $\mathrm{int}(Q)$ denotes the interior of Q. Let $\mathcal{W} := \{W_Q : Q \in \mathcal{F}^{\#}\}$. Then it follows easily from (A1) and (A2b) that $\mathcal{W}$ satisfies (B1), (B2), and (B6).

We claim that

$$\text{(3.20)}\quad \Big(W_Q \cap W_P \neq \emptyset \quad \text{and} \quad P \neq Q\Big) \qquad \Rightarrow \qquad Q \text{ and } P \text{ touch}$$

for all $Q, P \in \mathcal{F}^{\#}$. Due to (A3) Q is completely surrounded by a belt of cubes of length at least $\frac{1}{2}\ell(Q)$. So $\frac{9}{8}Q$ can only penetrate the first quarter of this belt. This together with the same consideration for $\frac{9}{8}P$ implies that $\frac{9}{8}Q$ and $\frac{9}{8}P$ can only intersect if P is one of the cubes in the belt around Q. In particular, Q and P touch, which proves (3.20).

Let us prove (B3). Let $Q, P \in \mathcal{F}^{\#}$ with $W_Q \cap W_P \neq \emptyset$. Then by (3.20) Q and P touch. Now it follows with $\ell(P) \leqslant 2\ell(Q)$ that $P \subset 5Q$ and $W_P \subset 5W_Q$. This proves (B3).

It remains to prove (B4) and (B5). In the construction of $\mathcal{F}^{\#}$ have split every cube Q into 256^n closed dyadic cubes of length $\frac{1}{256}\ell(Q)$. We can sort these cubes lexicographically by the coordinates of its center. Now, let us place each subcube according to its position in the sorted list into the families $\mathcal{F}_1, \dots, \mathcal{F}_{256^n}$. In particular, we use up to translation and scaling for every splitting the same order of numbering, e.g. the "top-left-most" subcube has always the same index from the sorting. We claim that

$$\text{(3.21)}\quad \big(P_j, Q_j \in \mathcal{F}_j \quad \text{and} \quad P_j \neq Q_j\big) \qquad \Rightarrow \qquad W_{P_j} \cap W_{Q_j} = \emptyset$$

for $j \in \{1, \dots, 256^n\}$. Indeed, let $P, Q \in \mathcal{F}$ and let $P_j, Q_j \in \mathcal{F}_j$ be the subcubes of P, Q with the same number. Since $P_j \neq Q_j$, we have

$P \neq Q$ by construction ot the $\mathcal{F}_j$. It is easy to see that (3.21) holds if P and Q do not touch, so let us concentrate on the case, where P and Q touch. Without loss of generality $\ell(Q) \geqslant \ell(P)$ and therefore with (A3) $\ell(P_j) \leqslant \ell(Q_j) \leqslant 2\,\ell(P_j)$. By construction of the $\mathcal{F}_j$ we have $d_\infty(P_j, Q_j) \geqslant (256-1)\ell(P_j)$, where d_∞ is the l^∞-metric and therefore

$$\begin{aligned} d_\infty(125W_{P_j}, 125W_{Q_j}) &= d_\infty\big(\tfrac{125\cdot 9}{8}P_j, \tfrac{125\cdot 9}{8}Q_j\big) \\ &\geqslant d_\infty(P_j, Q_j) - \big(\tfrac{125\cdot 9}{8} - 1\big)\frac{\ell(P_j)}{2} - \big(\tfrac{125\cdot 9}{8} - 1\big)\frac{\ell(Q_j)}{2} \\ &\geqslant \Big(255 - \big(\tfrac{125\cdot 9}{8} - 1\big)\tfrac{1}{2} - \big(\tfrac{125\cdot 9}{8} - 1\big)\Big)\ell(P_j) = \tfrac{729}{16}\ell(P_j) > 0. \end{aligned}$$

This proves (3.21). Thus $125W_{P_j} \cap 125W_{Q_j} = \emptyset$. Therefore each family $125\mathcal{F}_j$ consists of pairwise disjoint cubes, which implies (B4). Now, (B5) is an immediate consequence of (B4) □

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