<table>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2010), B22: 61-70</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2010-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/177041">http://hdl.handle.net/2433/177041</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Wavelet characterization of weighted spaces

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1 Wavelets

In this article we investigate a generalization of the Sobolev-Lieb-Thirring inequality or Lieb's inequality for Bessel potentials. First we recall the definition of Meyer's wavelet basis. Let \( \theta \) be a function which satisfies the following conditions.

(i) \( \theta \) is a real valued and even function in \( C_0^\infty(\mathbb{R}) \).

(ii) \( 0 \leq \theta(\xi) \leq 1 \) and \( \text{supp} \theta \subset [-4\pi/3, 4\pi/3] \).

(iii) \( \theta(\xi) = 1 \) for all \( \xi \in [-2\pi/3, 2\pi/3] \).

(iv) \( \theta(\xi)^2 + \theta(2\pi - \xi)^2 = 1 \) for all \( \xi \in [0, 2\pi] \).

We define a function \( \psi \in S(\mathbb{R}) \) by

\[
\hat{\psi}(\xi) = \int_{\mathbb{R}} \psi(x)e^{-7\xi x}dx = \{ \theta(\xi/2)^2 - \theta(\xi)^2 \}^{1/2}e^{-7\xi/2}.
\]

For integers \( j, k \) we set \( \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \). Then it turns out that \( \{ \psi_{j,k} \}_{j,k \in \mathbb{Z}} \) is an orthonormal basis of \( L^2(\mathbb{R}) \) which we call Meyer's wavelet basis([8]).

We define \( n \)-dimensional Meyer's wavelet basis as follows. Let \( \varphi \) be a function in \( S(\mathbb{R}) \) such that \( \hat{\varphi}(\xi) = \theta(\xi) \). Set \( E = \{0, 1\}^n \setminus \{0\} \), \( \psi^0(x) = \varphi(x) \), and \( \psi^1(x) = \psi(x) \). For \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) we define

\[
\psi^\varepsilon(x) = \psi^{\varepsilon_1}(x_1) \cdots \psi^{\varepsilon_n}(x_n).
\]

Let \( \Lambda = \{ (\varepsilon, j, k) : \varepsilon \in E, j \in \mathbb{Z}, k \in \mathbb{Z}^n \} \). For \( \lambda = (\varepsilon, j, k) \in \Lambda, x \in \mathbb{R}^n \), set

\[
\psi_\lambda(x) = 2^{nj/2}\psi^\varepsilon(2^j x - k).
\]

Then \( \{ \psi_\lambda \}_{\lambda \in \Lambda} \) is an orthonormal basis of \( L^2(\mathbb{R}^n) \) which we call \( n \)-dimensional Meyer's wavelet basis([8]).

We can construct another orthonormal basis by \( \varphi \) and \( \psi \). Let

\[
\Lambda_0 = \{ (\varepsilon, j, k) : \varepsilon \in E, j \in \mathbb{Z}, j \geq 0, k \in \mathbb{Z}^n \},
\]

\[
\Phi(x) = \varphi(x_1) \cdots \varphi(x_n), \quad \text{and} \quad \Phi_k(x) = \Phi(x - k) \quad (k \in \mathbb{Z}^n).
\]
Then we can prove that
\[
\{ \Phi_k, \psi_{\lambda} : \lambda \in \Lambda_0, k \in \mathbb{Z}^n \}
\]
is an orthonormal basis of \(L^2(\mathbb{R}^n)([8]) \). The function \(\Phi\) is called a scaling function.

2 Weighted spaces

We recall the definition of \(A_p\)-weights. By a cube in \(\mathbb{R}^n\) we mean a cube which sides are parallel to coordinate axes. A locally integrable function \(w > 0\) a.e. on \(\mathbb{R}^n\) is an \(A_p\)-weight for some \(p \in (1, \infty)\) if there exists a positive constant \(C\) such that
\[
\frac{1}{|Q|} \int_Q w(x) \ dx \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \ dx \right)^{p-1} \leq C
\]
for all cubes \(Q \subset \mathbb{R}^n\), where \(|Q|\) is the volume of \(Q\).

We say that \(w\) is an \(A_1\)-weight if there exists a positive constant \(C\) such that
\[
\frac{1}{|Q|} \int_Q w(y) \ dy \leq C w(x) \quad \text{a.e. } x \in Q
\]
for all cubes \(Q \subset \mathbb{R}^n\).

We write \(A_p\) for the class of \(A_p\)-weights. An example of \(A_p\)-weight for \(1 < p < \infty\) is given by \(w(x) = |x|^\alpha \in A_p\) where \(x \in \mathbb{R}^n\) and \(-n < \alpha < p(n-1)\). The inclusion \(A_p \subset A_q\) holds for \(p < q\).

For \(w \in A_p\) we set
\[
L^p(w) = \{ f : \text{measurable}, \|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \ dx \right)^{1/p} < \infty \}.
\]
For \(\lambda = (\epsilon, j, k) \in \Lambda\) set
\[
Q(\lambda) = \{(x_1, \ldots, x_n) : k_i \leq 2^j x_i < k_i + 1, i = 1, \ldots, n\}
\]
and
\[
\tilde{\chi}_{\lambda}(x) = |Q(\lambda)|^{-1/2} \chi_{Q(\lambda)}(x),
\]
where \(\chi_{Q(\lambda)}(x)\) is the characteristic function of \(Q(\lambda)\). The cube as above is called a dyadic cube.

Now we give the definition of an unconditional basis in a Banach space \(B\) over \(\mathbb{C}\). Let \(\{e_i\}_{i=1}^{\infty}\) be a family of elements in \(B\). We say \(\{e_i\}_{i=1}^{\infty}\) is a Schauder basis of \(B\) if every \(f \in B\) can be written
\[
f = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_k e_k + \cdots
\]
where the $\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$ are uniquely determined coefficients in $\mathbb{C}$ and the convergence in (1) is defined by

$$\lim_{N \to \infty} \|f - \alpha_1 e_1 - \alpha_2 e_2 - \cdots - \alpha_N e_N\|_B = 0.$$ 

A Schauder basis $\{e_i\}_{i=1}^\infty$ in $B$ is an unconditional basis if the following property is satisfied: for all $f \in B$ $f = \sum_{i=1}^\infty \alpha_{\sigma(i)} e_{\sigma(i)}$ in $B$ for any permutation $\sigma$ of $\mathbb{N}$, where $\alpha_i$ are coefficients given by (1).

The following theorem is a simple modification of results by Lemarié and Meyer([4],[5],[8]), where we use the notation

$$(f, g) = \int_{\mathbb{R}^n} f(x) \overline{g(x)} \, dx.$$ 

**Theorem 2.1.** Let

$$1 < p < \infty \quad \text{and} \quad w \in A_p.$$ 

Then $\{\psi_\lambda\}_{\lambda \in \Lambda}$ is an unconditional basis of $L^p(w)$. Furthermore for $f \in L^p(w)$ we have

$$f = \sum_{\lambda \in \Lambda} (f, \psi_\lambda) \psi_\lambda \quad \text{in} \quad L^p(w)$$ 

and

$$\|f\|_{L^p(w)} \approx \left\| \left( \sum_{\lambda \in \Lambda} |(f, \psi_\lambda)|^2 \tilde{\chi}_\lambda^2 \right)^{1/2} \right\|_{L^p(w)}.$$ 

Moreover

$$\{ \Phi_k, \psi_\lambda : \lambda \in \Lambda_0, k \in \mathbb{Z}^n \}$$

is an unconditional basis of $L^p(w)$. For $f \in L^p(w)$ we have

$$f = \sum_{k \in \mathbb{Z}^n} (f, \Phi_k) \Phi_k + \sum_{\lambda \in \Lambda_0} (f, \psi_\lambda) \psi_\lambda$$

in $L^p(w)$ and

$$\|f\|_{L^p(w)} \approx \left( \sum_k |(f, \Phi_k)|^p w(Q_k) \right)^{1/p} + \left\| \left( \sum_{\lambda \in \Lambda_0} |(f, \psi_\lambda)| \tilde{\chi}_\lambda(x) \right)^2 \right\|_{L^p(w)}^{1/2},$$

where

$$Q_k = \{(x_1, \ldots, x_n) : k_i \leq x_i < k_i + 1, i = 1, \ldots, n\},$$

and

$$w(Q_k) = \int_{Q_k} w \, dx.$$ 

We will use this result in the proofs of Theorem 3.2 in Section 3 and Theorem 6.2 in Section 6.
3 The Sobolev-Lieb-Thirring inequality

In 1976 Lieb and Thirring proved the following inequality([7]).

**Theorem 3.1 (The Sobolev-Lieb-Thirring inequality).** Suppose that \( n \in \mathbb{N}, \ f_{i} \in H^{1}(\mathbb{R}^{n}) \ (i = 1, \ldots, N), \) and that \( \{f_{i}\}_{i=1}^{N} \) is an orthonormal family in \( L^{2}(\mathbb{R}^{n}) \). Then we have

\[
\int_{\mathbb{R}^{n}} \rho^{1+2/n} dx \leq c_{\eta} \sum_{i=1}^{N} \int_{\mathbb{R}^{n}} |\nabla f_{i}|^{2} dx,
\]

where

\[
\rho(x) = \sum_{i=1}^{N} |f_{i}(x)|^{2}.
\]

In the statement of the Sobolev-Lieb-Thirring inequality \( H^{1}(\mathbb{R}^{n}) \) denotes the Sobolev space of order one. The Sobolev-Lieb-Thirring inequality has important applications such as the stability of matter or the estimates of the dimension of attractors of nonlinear equations([7]).

In this section we give a weighted version of the Sobolev-Lieb-Thirring inequality. Let \( w \in A_{2} \) and \( \mathcal{H}^{1}(w) \) be the completion of \( C_{0}^{\infty}(\mathbb{R}^{n}) \) with respect to the norm

\[
\|f\|_{\mathcal{H}^{1}(w)} = \left\{ \int_{\mathbb{R}^{n}} |\nabla f(x)|^{2} w(x) dx + \|f\|^{2} \right\}^{1/2},
\]

where \( \|\cdot\| \) denotes the norm in \( L^{2}(\mathbb{R}^{n}) \). We have the following generalization of the Sobolev-Lieb-Thirring inequality for \( n \geq 3 \)(c.f.[9]).

**Theorem 3.2.** Let \( n \in \mathbb{N}, \ n \geq 3, \ w \in A_{2} \) and \( w^{-n/2} \in A_{n/2} \). Suppose that \( f_{i} \in \mathcal{H}^{1}(w) \ (i = 1, \ldots, N), \) and \( \{f_{i}\}_{i=1}^{N} \) is orthonormal in \( L^{2}(\mathbb{R}^{n}) \). Then we have

\[
\int_{\mathbb{R}^{n}} \rho(x)^{1+2/n} w(x) dx \leq c \sum_{i=1}^{N} \int_{\mathbb{R}^{n}} |\nabla f_{i}(x)|^{2} w(x) dx,
\]

where

\[
\rho(x) = \sum_{i=1}^{N} |f_{i}(x)|^{2}
\]

and \( c \) is a positive constant depending only on \( n \) and \( w \).

An example of \( w \) which satisfies the conditions in Theorem 3.2 is given by \( w(x) = |x|^{\alpha} \) for \( -n + 2 < \alpha < 2 \).

We explain about the outline of a proof of Theorem 3.2 in the next section. We use the estimates of some weighted integrals by means of wavelets. These estimates enable us to prove a weighted version of the Sobolev-Lieb-Thirring inequality.
4 Proof of Theorem 3.2

For $f \in L_{loc}^{1}(\mathbb{R}^{n})$, we define the Hardy-Littlewood maximal operator as

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_{Q} |f(y)| \, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^{n}$ such that $x \in Q$.

The proof of the following proposition is in [3].

**Proposition 4.1.** (i) Let $1 < p < \infty$ and $w \in A_{p}$. Then $M$ is bounded on $L^{p}(w)$.

(ii) Let $0 < \tau < 1$, $f \in L_{loc}^{1}(\mathbb{R}^{n})$, and $M(f)(x) < \infty$ a.e.. Then $(M(f)(x))^{\tau} \in A_{1}$.

(iii) Let $1 < p < \infty$ and $w_{1}, w_{2} \in A_{1}$. Then $w_{1}w_{2}^{1-p} \in A_{p}$.

We may assume $f_{i} \in C_{0}^{\infty}(\mathbb{R}^{n})$ for $i = 1, \ldots, N$. Let $V(x) = \delta \rho(x)^{2/n}w(x)$ where $\delta$ is a positive constant. Then we get

$$\int_{\mathbb{R}^{n}} V^{1+n/2}w^{-n/2} \, dx < \infty$$

and $w^{-n/2} \in A_{(1+n/2)/\kappa} = A_{\frac{n}{2}}$ for $\kappa = 1+2/\iota$.

The following lemma is essentially proved by Frazier and Jawerth (c.f.[9]).

**Lemma 4.1.** Let $w \in A_{2}$. Then there exists a $\alpha > 0$ such that

$$\alpha \sum_{\lambda \in \Lambda} |Q(\lambda)|^{-2/n} |(f, \psi_{\lambda})|^{2} \int_{Q(\lambda)} w \, dx \leq \int_{\mathbb{R}^{n}} |\nabla f|^{2} w \, dx$$

for all $f \in C_{0}^{\infty}(\mathbb{R}^{n})$.

The next lemma is a corollary of Theorem 2.1.

**Lemma 4.2.** Let $v \in A_{2}$. Then there exists a $\beta > 0$ such that

$$\int_{\mathbb{R}^{n}} |f|^{2} v \, dx \leq \beta \sum_{\lambda \in \Lambda} |(f, \psi_{\lambda})|^{2} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx$$

for all $f \in C_{0}^{\infty}(\mathbb{R}^{n})$.

By Lemmas 4.1 and 4.2 we have for $f \in C_{0}^{\infty}(\mathbb{R}^{n})$

$$\int_{\mathbb{R}^{n}} |\nabla f|^{2} w \, dx - \int_{\mathbb{R}^{n}} V |f|^{2} \, dx$$

$$\geq \alpha \sum_{\lambda \in \Lambda} |Q(\lambda)|^{-2/n} |(f, \psi_{\lambda})|^{2} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w \, dx - \beta \sum_{\lambda \in \Lambda} |(f, \psi_{\lambda})|^{2} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx.$$
Let
\[ I = \{ \lambda \in \Lambda : \beta \int_{Q(\lambda)} v \, dx > \alpha |Q(\lambda)|^{-2/n} \int_{Q(\lambda)} w \, dx \} \]
and \( \{ \mu_k \}_{1 \leq k} \) be the non-decreasing rearrangement of
\[ \left\{ \alpha |Q(\lambda)|^{-2/n-1} \int_{Q(\lambda)} w \, dx - \beta |Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx \right\}_{\lambda \in I} \]
When
\[ \mu_k = \alpha |Q(\lambda)|^{-2/n-1} \int_{Q(\lambda)} w \, dx - \beta |Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx, \]
we define \( \Psi_k = \psi_\lambda \). Then we get
\[
\sum_{i=1}^{N} \int_{\mathbb{R}^n} |\nabla f_i|^2 w \, dx - \sum_{i=1}^{N} \int_{\mathbb{R}^n} V|f_i|^2 \, dx
\geq \sum_{i=1}^{N} \sum_{\lambda \in I} |(f_i, \psi_\lambda)|^2 \left\{ \alpha |Q(\lambda)|^{-2/n-1} \int_{Q(\lambda)} w \, dx - \beta |Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx \right\}
\geq \sum_{i=1}^{N} \sum_{k} \mu_k |(f_i, \Psi_k)|^2 = \sum_{k} \mu_k \sum_{i=1}^{N} |(f_i, \Psi_k)|^2
\geq -c \sum_{k} |\mu_k|.
\]

Now we use the following lemma in [9].

**Lemma 4.3.** There exists a positive constant \( c \) such that
\[
\sum_{k} |\mu_k| \leq c \int_{\mathbb{R}^n} v^{1+n/2} w^{-n/2} \, dx,
\]
where \( c \) depends only on \( n \) and \( w \).

Hence by Lemma 4.3 we have
\[
\sum_{i=1}^{N} \int_{\mathbb{R}^n} |\nabla f_i|^2 w \, dx - \sum_{i=1}^{N} \int_{\mathbb{R}^n} V|f_i|^2 \, dx
\geq -c \int_{\mathbb{R}^n} V^{1+n/2} w^{-n/2} \, dx = -c \int_{\mathbb{R}^n} V^{1+2/n} w \, dx.
\]

Therefore
\[
\sum_{i=1}^{N} \int_{\mathbb{R}^n} |\nabla f_i|^2 w \, dx \geq \delta \int_{\mathbb{R}^n} V^{1+2/n} w \, dx - c \int_{\mathbb{R}^n} V^{1+2/n} w \, dx 
= \{ \delta - c \} \int_{\mathbb{R}^n} V^{1+2/n} w \, dx.
\]

If we take \( \delta \) small enough, then we get the inequality in Theorem 3.2.
5 \textit{L}^p \textit{ Sobolev-Lieb-Thirring inequality}

By Theorem 3.2 we are able to prove the following \textit{L}^p version of the Sobolev-Lieb-Thirring inequality.

\textbf{Theorem 5.1 ([10]).} Let \( n \in \mathbb{N}, \ n \geq 3 \) and \( 2n/(n + 2) < p < n \). Then there exists a positive constant \( c \) such that for every family \( \{f_i\}_{i=1}^{N} \) in \( L^2(\mathbb{R}^n) \) which is orthonormal and \( |\nabla f_i(x)| \in L^p(\mathbb{R}^n), \) \( (i=1,\ldots,N) \), we have
\[
\int_{\mathbb{R}^n} \rho(x)^{(1+2/n)p/2} dx \leq c \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\nabla f_i(x)|^2 \right)^{p/2} dx,
\]
where
\[
\rho(x) = \sum_{i=1}^{N} |f_i(x)|^2
\]
and \( c \) depends only on \( n \) and \( p \).

\textbf{Proof}

Our proof is very similar to that of the extrapolation theorem in harmonic analysis (c.f. [2, Theorem 7.8]). Let \( 2 < p < n \) and \( 2/p + 1/q = 1 \). Let \( u \in L^q \), \( u \geq 0 \) and \( \|u\|_{L^q} = 1 \). We take a \( \gamma \) such that \( n/(n-2) < \gamma < q \). Then we have \( u \leq M(u^\gamma)^{1/\gamma} \) a.e and \( M(u^\gamma)^{1/\gamma} \in A_1 \). Furthermore let \( \alpha = \frac{n}{(n-2)\gamma} \). Then \( 0 < \alpha < 1 \) and
\[
M(u^\gamma)^{-n/(2\gamma)} = \{M(u^\gamma)^{\alpha}\}^{1-n/2} \in A_{n/2},
\]
where we used \( M(u^\gamma)^{\alpha} \in A_1 \) and (iii) of Proposition 4.1. Therefore we have
\[
\int \rho^{1+2/n} u dx \leq \int \rho^{1+2/n} M(u^\gamma)^{1/\gamma} dx \leq c \left( \int \sum_{i=1}^{N} |\nabla f_i|^2 \right)^{p/2} M(u^\gamma)^{1/\gamma} dx
\]
\[
\leq c \left( \int \left( \sum_{i=1}^{N} |\nabla f_i|^2 \right)^{p/2} dx \right)^{2/p} \left( \int M(u^\gamma)^{q/\gamma} dx \right)^{1/q},
\]
where we used Theorem 3.2 and the inequality
\[
\int M(u^\gamma)^{q/\gamma} dx \leq c \int u^q dx = c.
\]
If we take the supremum for all \( u \in L^q \), \( u \geq 0 \) and \( \|u\|_{L^q} = 1 \), then we get
\[
\left( \int \rho^{(1+2/n)p/2} dx \right)^2 \leq c \left( \int \sum_{i=1}^{N} |\nabla f_i|^2 \right)^{p/2} dx \right)^2.
\]
Next we consider the case $2n/(n+2) < p < 2$. Let
\[ f = \left( \sum_{i=1}^{N} |\nabla f_i|^2 \right)^{1/2}. \]
We can take $\gamma$ such that $(2 - p)n/2 < \gamma < p$. Then we have
\[ M(f^\gamma)^{(2-p)/\gamma} \in A_2 \]
because
\[ M(f^\gamma)^{(2-p)/\gamma} \in A_1 \]
by (ii) of Proposition 4.1. Furthermore we have
\[ \{M(f^\gamma)^{-(2-p)/\gamma}\}^{-n/2} = M(f^\gamma)^{(2-p)n/(2\gamma)} \in A_1 \subset A_{n/2}. \]
Therefore
\[
\int \rho^{(1+2/n)p/2} dx = \int \rho^{(1+2/n)p/2} \left( M(f^\gamma)^{(2-p)/\gamma} \right) dx \\
\leq \left( \int \rho^{1+2/n} M(f^\gamma)^{-(2-p)/\gamma} dx \right)^{p/2} \left( \int M(f^\gamma)^{p/\gamma} dx \right)^{1-p/2} \\
\leq c \left( \int f^2 M(f^\gamma)^{-(2-p)/\gamma} dx \right)^{p/2} \left( \int f^p dx \right)^{1-p/2} \\
\leq c \left( \int M(f^\gamma)^{2/\gamma} M(f^\gamma)^{-(2-p)/\gamma} dx \right)^{p/2} \left( \int f^p dx \right)^{1-p/2} \leq c \int f^p dx,
\]
where we used Theorem 3.2 in the second inequality.

6 Lieb's inequality for Bessel potentials

Lieb proved the following inequality in [6].

**Theorem 6.1.** Let $n \in \mathbb{N}$, $s > 0$, $n > 2s$ and $m \geq 0$. Let $f_1, \ldots, f_N$ be orthonormal in $L^2(\mathbb{R}^n)$ and
\[ u_i = (-\Delta + m^2)^{-s/2} f_i. \]
Then
\[
\int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |u_i(x)|^2 \right)^{n/(n-2s)} dx \leq C_{n,s} N.
\]
Battle and Federbush ([1]) proved this inequality for \( n = 3 \) and \( s = 1 \) in 1982. They applied it to the quantum field theory. Lieb proved the case \( n \geq 4 \) and \( s > 0 \).

We can prove the following generalization of Lieb's inequality by means of Theorem 2.1.

**Theorem 6.2** (Tachizawa, 2007). Let \( n \in \mathbb{N} \), \( s > 0 \), \( n > 2s \) and \( m \geq 0 \). Let \( w \in A_{n/(n-2s)} \cap A_3 \) and \( w^{-n/(2s)} \in A_{n/(2s)} \). Let \( f_1, \ldots, f_N \) be orthonormal in \( L^2(\mathbb{R}^n) \), \( f_i \in L^2(w)_f \) and

\[
 u_i = (-\Delta + m^2)^{-s/2} f_i.
\]

Then

\[
 \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |u_i(x)|^2 \right)^{n/(n-2s)} w(x) dx \leq C \sum_{i=1}^{N} \int_{\mathbb{R}^n} |f_i(x)|^2 w(x) dx,
\]

where the constant \( C \) depends only on \( n, s, \) and \( w \).

The proof of Theorem 6.2 is given by a similar argument to that of Theorem 3.2. We use the characterization of weighted spaces by means of wavelets and scaling function. The detail will appear elsewhere.

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**-reference:**


