FINAL STATE PROBLEM FOR SOME KDV TYPE EQUATION

福岡教育大学・教育学部　瀬片　純市 (Jun-ichi Segata)
Faculty of Education,
Fukuoka University of Education

1. Introduction

In this note we consider the large time behavior of solutions to the third order Korteweg-de Vries type equation (Hirota equation) of the form:

$$i\partial_t u + \partial_x^2 u + i\mu\partial_x^3 u = -\frac{1}{2}|u|^2 u - \frac{3}{2}i\mu|u|^2\partial_x u, \qquad t, x \in \mathbb{R}, \tag{1.1}$$

where $i = \sqrt{-1}$, $\partial_t = \dfrac{\partial}{\partial t}$, $\partial_x^j = \dfrac{\partial^j}{\partial x^j}$ ($j \in \mathbb{N}$), $u = u(t,x)$ is an unknown function, and μ is a non-zero constant.

(1.1) arises in various situations in Mathematics and Physics. For example, the equation (1.1) describes the three-dimensional motion of an isolated vortex filament embedded in an inviscid incompressible fluid filling an infinite region. Fukumoto-Miyazaki [4] proposed this equation as some detailed model taking account of the effect from the higher order corrections of the Da Rios model (nonlinear Schrödinger equation):

$$i\partial_t u + \partial_x^2 u = -\frac{1}{2}|u|^2 u, \qquad t, x \in \mathbb{R}. \tag{1.2}$$

which is the classical model for the motion of the vortex filament. We note that (1.2) is obtained by letting $\mu = 0$ in (1.1). On the other hand, neglecting the second term in the left hand side and the first term in the right hand side of (1.1) we obtain the complex-valued modified Korteweg-de Vries (modified KdV) equation:

$$\partial_t u + \mu\partial_x^3 u = -\frac{3}{2}\mu|u|^2\partial_x u, \qquad t, x \in \mathbb{R}, \tag{1.3}$$

Therefore we expect that the equation (1.1) will be similar to the nonlinear Schrödinger equation and the modified KdV equation.

In this note we show the unique existence of the solution to (1.1) which tends to the given asymptotic profiles. Concerning the local and global existence of solutions to (1.1) see e,g., Laurey [11] and Tani-Nishiyama [13].

The large time behavior of solutions to the nonlinear dispersive equations are determined by the balance between those dispersions and nonlinearities. We call the nonlinear dispersive equation is the short range type if those solutions behave like a free solution, and otherwise we call the long range type. To seek the criterion for the short range type we consider the following nonlinear dispersive equation:

$$i\partial_t u + \mathcal{P}(-i\partial_x)u = \mathcal{N}(u, \partial_x u), \qquad t, x \in \mathbb{R}. \tag{1.4}$$

Let $\{e^{it\mathcal{P}}\}_{t\in\mathbb{R}}$ be a unitary group generated by $i\mathcal{P}(-i\partial_x)$. We assume that

$$\begin{aligned} \|e^{it\mathcal{P}}\phi\|_{L^\infty_x} &\leq Ct^{-m}, \\ \mathcal{N}(u,v) &= \mathcal{O}(|u|^p+|v|^p), \ near\ (u,v)=(0,0). \end{aligned}$$

Since (1.4) is rewritten as the integral equation

$$u(t) = e^{i(t-s)\mathcal{P}}u(s) - i\int_s^t e^{i(t-\tau)\mathcal{P}}\mathcal{N}(u,\partial_x u)(\tau)d\tau,$$

we have from the assumption

$$\begin{aligned} \|e^{-it\mathcal{P}}u(t) - e^{-is\mathcal{P}}u(s)\|_{L^2_x} &\leq \int_s^t \|\mathcal{N}(u,\partial_x u)(\tau)\|_{L^2_x}dt' \\ &\sim \int_s^t \tau^{-m(p-1)}d\tau \sim (t-s)^{-m(p-1)+1}, \end{aligned}$$

for $t > s$. Therefore we expect that

$$\|e^{-it\mathcal{P}}u(t) - e^{-is\mathcal{P}}u(s)\|_{L^2_x} \sim \begin{cases} 0 & (\text{if } p > 1+\frac{1}{m}) \\ \infty & (\text{if } p < 1+\frac{1}{m}), \end{cases}$$

as $t,s\to\infty$. Hence we expect that when $p > 1+1/m$, (1.4) is the short range type and when $p < 1+1/m$, (1.4) is the long range one. Concerning this observation, we consider the Schrödinger equation with power type interaction:

$$i\partial_t u + \partial_x^2 u = \lambda|u|^{p-1}u, \qquad t,x\in\mathbb{R}, \tag{1.5}$$

where λ is a real constant. From the above observation and the decay property of the free Schrödinger group:

$$\|e^{it\partial_x^2}\phi\|_{L^\infty_x} \leq Ct^{-\frac{1}{2}}\|\phi\|_{L^1_x},$$

we expect that the power of the borderline between the short range case and the long range one will be 3. In this regard, Y. Tsutsumi-Yajima [14] proved that (1.5) is the short range type if $p > 3$ and Barab [1] proved that (1.5) with $p \leq 3$ is the long range type. Therefore the above observation is true for the nonlinear Schrödinger equation of the form (1.5). Concerning the critical case ((1.5) with $p = 3$), Ozawa [12] proved that for given "final data" ϕ_+ there exists a solution u to (1.5) such that

$$u(t,x) \sim \sqrt{\frac{1}{2it}}\hat{\phi}_+\left(\frac{x}{2t}\right)\exp\left(\frac{ix^2}{4t} + \frac{i}{4}\left|\hat{\phi}_+\left(\frac{x}{2t}\right)\right|^2\log t\right), \tag{1.6}$$

as $t\to+\infty$ in $L^2(\mathbb{R})$, where $\hat{\phi}_+$ is the Fourier transform of ϕ_+. Furthermore Hayashi-Naumkin [6] showed that for given "initial" data u_0 there exists a function ϕ_+ such that a solution to (1.5) with the initial condition $u = u_0$ at $t = 0$ behaves like (1.6) as $t\to\infty$.

Next we consider the modified KdV equation (1.3). Since the solution to the corresponding linear equation (Airy equation) decays like

$$\|e^{-t\partial_x^3}\phi\|_{L^\infty_x} \leq Ct^{-\frac{1}{3}}\||\partial_x|^{-1/2}\phi\|_{L^1_x},$$

(see Hayashi-Naumkin [9]), the modified KdV equation (1.3) belongs to the borderline between the short and long range cases. Hayashi-Naumkin [7] (see also [8]) proved that the solution to (1.3) behaves like

$$\begin{aligned} u(t,x) \sim \operatorname{Re}\Bigg[& \sqrt{\frac{2}{3it}}\left(-\frac{x}{3\mu t}\right)^{-\frac{1}{4}} \hat{\phi}_+\left(\sqrt{-\frac{x}{3\mu t}}\right) \\ & \times \exp\left(\frac{2}{3}ix\sqrt{-\frac{x}{3\mu t}} - \frac{i}{4}\left|\hat{\phi}_+\left(\sqrt{-\frac{x}{3\mu t}}\right)\right|^2 \log t\right)\Bigg], \end{aligned} \tag{1.7}$$

as $t \to +\infty$ in $L^2(x<0)$. For the detail see [7],[8] for the initial value problem and [9] for the final value problem.

In the two asymptotic formulae (1.6) and (1.7), the logarithmic terms are the contributions from those nonlinear terms because those free solution behave like

$$e^{it\partial_x^2}\phi_+ \sim \sqrt{\frac{1}{2it}}\hat{\phi}_+\left(\frac{x}{2t}\right)\exp\left(\frac{ix^2}{2t}\right),$$

and

$$e^{-t\partial_x^3}\phi_+ \sim \operatorname{Re}\left[\sqrt{\frac{2}{3it}}\left(-\frac{x}{3\mu t}\right)^{-\frac{1}{4}}\hat{\phi}_+\left(\sqrt{-\frac{x}{3\mu t}}\right)\exp\left(\frac{2}{3}ix\sqrt{-\frac{x}{3\mu t}}\right)\right],$$

as $t \to \infty$ respectively. Therefore we expect that (1.1) will be the long range type. In Section 2 we answer this question.

2. Main Result

In this section we state the main Theorem on the large time behavior of solution to (1.1). In this note we only treat the final state problem of (1.1). Before we state our main result precisely, we introduce several notations. Let $\{W(t)\}_{t\in\mathbb{R}}$ be the unitary group generated by the linear operator $i\partial_x^2 - \mu\partial_x^3$:

$$W(t)\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi+it(-\xi^2+\mu\xi^3)}\hat{\phi}(\xi)d\xi.$$

For $\alpha > 0$ we define the function space $\mathcal{A}^\alpha$ by

$$\begin{aligned} \mathcal{A}^\alpha &= \{\phi \in \mathcal{S}'(\mathbb{R});\ \|\phi\|_{\mathcal{A}^\alpha} < \infty\}, \\ \|\phi\|_{\mathcal{A}^\alpha} &= \left\|\frac{\langle 3\mu\xi - 1\rangle^{\alpha+3}}{|3\mu\xi - 1|^\alpha}\hat{\phi}(\xi)\right\|_{L^\infty_\xi} + \left\|\frac{\langle 3\mu\xi - 1\rangle^{\alpha+2}}{|3\mu\xi - 1|^{\alpha-1}}\hat{\phi}'(\xi)\right\|_{L^\infty_\xi}, \end{aligned}$$

where $\langle\xi\rangle^\alpha = (1+|\xi|^2)^{\alpha/2}$.

Let be $\varphi = \varphi(t,x)$ the characteristic function on the domain $\{(t,x); 1-3\mu\frac{x}{t} \geq 0\}$:

$$\varphi(t,x) = \begin{cases} 1 & if\ \ 1-3\mu\frac{x}{t} \geq 0, \\ 0 & if\ \ 1-3\mu\frac{x}{t} < 0. \end{cases}$$

We note that the oscillatory integral associated to $W(t)\phi$ has no stationary points on $\{(t,x); 1-3\mu\frac{x}{t} < 0\}$ and has two stationary points on $\{(t,x); 1-3\mu\frac{x}{t} > 0\}$. By a direct calculation, we easily see that the stationary points in the later domain are given by $\chi_{\pm} = \frac{1}{3\mu}\left(1 \pm \sqrt{1 - 3\mu\frac{x}{t}}\right)$.

Our main Theorem in this note is the following.

Theorem 2.1. *Let $\frac{1}{2} < \alpha \le 1$. If $\phi_+ \in \mathcal{A}^\alpha$ and $\|\phi_+\|_{\mathcal{A}^\alpha} < \epsilon$ sufficiently small, then there exists a unique solution $u \in L^\infty([0,\infty); H_x^2(\mathbb{R})) \cap C([0,\infty); H_x^1(\mathbb{R}))$ to (1.1) such that*

$$\sum_{j=0}^{2} \|\partial_x^j u(t) - u_+^j(t)\|_{L_x^2} \le Ct^{-\beta},$$

for $t \ge 2$ and $\frac{1}{3} < \beta < \frac{\alpha}{3} + \frac{1}{6}$, where

$$\begin{aligned} u_+^j(t,x) &\equiv \sum_{\pm} t^{-\frac{1}{2}} \sqrt{\frac{i}{2(3\mu\chi_\pm - 1)}}\varphi(x)(i\chi_\pm)^j \hat{\phi}_+(\chi_\pm) \\ &\qquad \times \exp\left(-2i\mu t\chi_\pm^3 + it\chi_\pm^2 \mp \frac{i}{4}\left|\hat{\phi}_+(\chi_\pm)\right|^2 \log t\right), \end{aligned}$$

and $\sum_{\pm}$ means that $\sum_{\pm} A_\pm = A_+ + A_-$.

Remark. Since (χ_+, χ_-) approaches to $\left(+\infty, \frac{x}{2t}\right)$ as $\mu \to 0$, $u_+^0(t,x)$ convergences to $\sqrt{\frac{1}{2it}}\hat{\phi}_+\left(\frac{x}{2t}\right)\exp\left(\frac{ix^2}{4t} + \frac{i}{4}\left|\hat{\phi}_+\left(\frac{x}{2t}\right)\right|^2 \log t\right)$, as $\mu \to 0$ which is the leading term of (1.2). In this sense our result coincides the result due to Ozawa [12].

Roughly speaking, in Theorem 2.1 we assumed that $\hat{\phi}_+(\xi)$ vanishes at $\xi = \frac{1}{3\mu}$. We explain this reason by the following heuristic argument. By the method of stationary phase, the leading term of $W(t)\phi_+$ is given by

$$\begin{aligned} W(t)\phi_+(x) &\sim t^{-\frac{1}{2}}\sqrt{\frac{i}{2(3\mu\chi_+ - 1)}}\varphi(x)\hat{\phi}_+(\chi_+)\exp\left(-2i\mu t\chi_+^3 + it\chi_+^2\right) \\ &+ t^{-\frac{1}{2}}\sqrt{\frac{i}{2(3\mu\chi_- - 1)}}\varphi(x)\hat{\phi}_+(\chi_-)\exp\left(-2i\mu t\chi_-^3 + it\chi_-^2\right). \end{aligned}$$

Therefore if $|3\mu\xi - 1|^{-\alpha}|\hat{\phi}_+(\xi)|$ is bounded for some $\alpha > 0$, then $W(t)\phi_+$ decays like $t^{-1/2}$. Therefore cubic nonlinearity belongs to the borderline between the short and long range cases. That is why we assumed that $\hat{\phi}_+(\xi)$ vanishes at $\xi = \frac{1}{3\mu}$ in Theorem 2.1.

3. Choice of Asymptotic Profile

In Theorem 2.1 we stated that there exists a solution to (1.1) which behaves like u_+^0 as $t \to \infty$. In this section we explain why asymptotic profile of a solution to (1.1) is given by u_+^0 by employing the analogue method of Ozawa [12]. For the simplicity we put $\mathcal{L} = i\partial_t + \partial_x^2 + i\mu\partial_x^3$ and $\mathcal{N}(u, \partial_x u) = -\frac{1}{2}|u|^2 u - \frac{3}{2} i\mu |u|^2 \partial_x u$. We consider the final state problem

$$\begin{cases} \mathcal{L}u = \mathcal{N}(u, \partial_x u), \\ u(t) \to u_+^0(t), \qquad \text{as } t \to +\infty, \end{cases}$$

where u_+^0 is determined later. This problem is rewritten as the integral equation

$$u(t) - u_+^0(t) = i \int_t^{+\infty} W(t-\tau)\{\mathcal{N}(u, \partial_x u) - \mathcal{L}u_+^0\}(\tau) d\tau.$$

We guarantee Theorem 2.1 by applying the contraction mapping principle to the integral equation (3.1). More precisely, we prove that if $\|\phi_+\|_{\mathcal{A}^\alpha}$ is sufficiently small, then the map

$$(\Phi u)(t) = u_+^0(t) + i \int_t^{+\infty} W(t-\tau)\{\mathcal{N}(u, \partial_x u) - \mathcal{L}u_+^0\}(\tau) d\tau \tag{3.1}$$

is a contraction on the function space

$$\begin{aligned} X_T &\equiv \{u \in \mathcal{S}'(\mathbb{R}^2); \|u\|_{X_T} \le 2\epsilon\}, \\ \|u\|_{X_T} &\equiv \sup_{t \ge T} t^\beta (\|u(t) - u_+^0(t)\|_{H_x^2} + \|u(\tau) - u_+^0(\tau)\|_{L_\tau^6(t,\infty; H_x^{1,\infty})}). \end{aligned}$$

To prove this we split the right hand side of (3.1) into two pieces:

$$\begin{aligned} (\Phi u)(t) - u_+^0(t) &= i \int_t^{+\infty} W(t-\tau)\{\mathcal{N}(u, \partial_x u) - \mathcal{N}(u_+^0, \partial_x u_+^0)\}(\tau) d\tau \\ &\quad - i \int_t^{+\infty} W(t-\tau)\{\mathcal{L}u_+^0 - \mathcal{N}(u_+^0, \partial_x u_+^0)\}(\tau) d\tau. \end{aligned} \tag{3.2}$$

The X_T norm of the first term on the right hand side of (3.2) is evaluated by the combination of the Strichartz inequality

$$\left\| \int_\tau^{+\infty} W(\tau - \tau') F(\tau') d\tau' \right\|_{L_\tau^6(t,\infty; L_x^\infty)} \le C \|F\|_{L_\tau^{6/5}(t,\infty; L_x^1)}$$

(see e.g., [11], [10]) and the standard energy method. To estimate the second term in the right hand side of (3.1), we need the following inequality

$$\left\| \int_t^{+\infty} W(t-\tau)\{\mathcal{L}u_+^0 - \mathcal{N}(u_+^0, \partial_x u_+^0)\}(\tau) d\tau \right\|_{H_x^2} \le C t^{-\beta}. \tag{3.3}$$

We note that (3.3) fails for $u_+^0 = W(t)\phi$. We put

$$\begin{aligned} u_+^0(t,x) \equiv & \sum_{\pm} t^{-\frac{1}{2}} \sqrt{\frac{i}{2(3\mu\chi_\pm - 1)}} \varphi(x) \hat{\phi}_+(\chi_\pm) \\ & \times \exp\left(-2i\mu t \chi_\pm^3 + it\chi_\pm^2 + iD^\pm\right), \end{aligned}$$

and choose $D^{\pm}$ so that the inequality (3.3) holds. For the notational convenience we put

$$\begin{aligned} B_{\pm} &= t^{-\frac{1}{2}}\sqrt{\frac{i}{2(3\mu\chi_{\pm}-1)}}\varphi(x), \\ C_{\pm} &= -2\mu t\chi_{\pm}^3 + t\chi_{\pm}^2. \end{aligned}$$

Then u_+^0 is rewritten as

$$u_+^0 = \sum_{\pm} B_{\pm}\hat{\phi}_+(\chi_{\pm})e^{iC_{\pm}+iD_{\pm}}.$$

We calculate $\mathcal{L}u_+^0 - \mathcal{N}(u_+^0, \partial_x u_+^0)$. By a direct computation, the leading terms of $\mathcal{L}u_+^0$ and $\mathcal{N}(u_+^0, \partial_x u_+^0)$ are given by

$$\mathcal{L}u_+^0 \sim -\sum_{\pm}\partial_t D^{\pm} B_{\pm}\hat{\phi}_+(\chi_{\pm})e^{iC_{\pm}+iD_{\pm}}, \tag{3.4}$$

$$\begin{aligned} &\mathcal{N}(u_+^0, \partial_x u_+^0) \\ &\quad\sim \frac{1}{4t}\sum_{\pm} \pm B_{\pm}|\hat{\phi}_+(\chi_{\pm})|^2\hat{\phi}_+(\chi_{\pm})e^{iC_{\pm}+iD_{\pm}} \\ &\qquad + \frac{1}{2}\sum_{\pm}(3\mu\chi_{\pm}-1)B_{\pm}^2\overline{B_{\mp}}\hat{\phi}_+(\chi_{\pm})^2\overline{\hat{\phi}_+(\chi_{\mp})}e^{i(2C_{\pm}-C_{\mp})+i(2D_{\pm}-D_{\mp})}, \end{aligned} \tag{3.5}$$

respectively. The oscillation factors $e^{iC_{\pm}}$ of (3.4) are equal to those of the first summand of (3.5), but not equal to those of the second summand of (3.5). In this sense we call that (3.4) and the first terms of (3.5) are the resonance terms, while the second terms of (3.5) are the non-resonance terms.

Concerning the non-resonance terms, thanks to the difference between the oscillations of the resonance and non-resonance terms, those terms decay faster than t^{-1} by integrating by parts with respect to t variable. Therefore we regard the non-resonance parts as the remainder terms. However the resonance terms do not have such a fine property. Therefore we cannot expect that the resonance parts decay faster than t^{-1}. To overcome this, we eliminate the resonance terms by the factors $D_{\pm}$. To eliminate those terms it suffices to choose

$$\partial_t D_{\pm} = \mp\frac{1}{4t}|\hat{\phi}_+(\chi_{\pm})|^2.$$

By choosing

$$D_{\pm} = \mp\frac{1}{4}|\hat{\phi}_+(\chi_{\pm})|^2\log t$$

we are able to drive the estimate (3.3).

To show that the non-resonance terms (the second summand on (3.5) decay faster than t^{-1}, we need to control the oscillation factors $e^{i(2C_{\pm}-C_{\mp})}$. To do so we use the following two asymptotic formulae: For $k = 0, 1$, we define

$$W_k(t)\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\left\{x\xi + t(-\xi^2+\mu\xi^3) - \frac{8}{243\mu^2}kt(3\mu\xi-1)^3\right\}}\hat{\phi}(\xi)d\xi, \tag{3.6}$$

$$\chi_{k,\pm} = \frac{1}{3\mu}\left\{1 \pm (2k+1)\sqrt{1 - 3\mu\frac{x}{t}}\right\}, \tag{3.7}$$

$$F_{k,\pm} = \sqrt{\frac{i}{2\left(1-\frac{8}{9}k\right)(3\mu\chi_{k,\pm}-1)}}\varphi(x), \tag{3.8}$$

$$E_{k,\pm} = \exp\left[i\left\{-2\left(1-\frac{8}{9}k\right)\mu t\chi_{k,\pm}^3 + \left(1-\frac{8}{9}k\right)t\chi_{k,\pm}^2 + \frac{8}{243\mu^2}kt\right\}\right]. \tag{3.9}$$

Then we have

Theorem 3.1. *Let $W_0(t)$ and $W_1(t)$ be defined by (3.6). Then We have*

$$\begin{aligned} W_k(t)\phi(x) &= t^{-\frac{1}{2}}F_{k,+}(t,x)E_{k,+}(t,x)\hat{\phi}(\chi_{k,+}) \\ &\quad + t^{-\frac{1}{2}}F_{k,-}(t,x)E_{k,-}(t,x)\hat{\phi}(\chi_{k,-}) + R_k(t,x), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} &\|R_k(t)\|_{L^p_x} \\ &\le C_k t^{-\frac{\alpha}{3}-\frac{1}{3}\left(1-\frac{1}{p}\right)}\left\{\left\|\frac{\langle 3\mu\xi-1\rangle^{\alpha}}{|3\mu\xi-1|^{\alpha}}\hat{\phi}(\xi)\right\|_{L^\infty_\xi} + \left\|\frac{\langle 3\mu\xi-1\rangle^{\alpha-1}}{|3\mu\xi-1|^{\alpha-1}}\hat{\phi}'(\xi)\right\|_{L^\infty_\xi}\right\}, \end{aligned} \tag{3.11}$$

for $2 \le p \le \infty$, $0 < \alpha < 2\left(1-\frac{1}{p}\right)$ and $\chi_{k,\pm}$, $F_{k,\pm}$ and $E_{k,\pm}$ are given by (3.7), (3.8) and (3.9), respectively.

Remark. After I finished my talk in "Workshop on Harmonic Analysis and Nonlinear Partial Differential Equations" I have been kindly informed by Kenji Nakanishi that if $u(t,x)$ is the solution to the Hirota equation (1.1), then the function

$$v(t,x) = e^{-\frac{ix}{3\mu}-\frac{it}{27\mu^2}}u\left(t, x+\frac{t}{3\mu}\right)$$

satisfies the modified KdV equation (1.3). By using this transformation the proof of our main Theorem may be simpler. Hayashi-Naumkin [7] mentioned the large time behavior of the solution to the "real-valued" modified KdV equation. However the combination of above transform and their result does not imply our Theorem because the solution to (1.3) is a "complex-valued".

References

[1] J.E. Barab, *Nonexistence of asymptotically free solutions for a nonlinear Schrödinger equation*, J. Math. Phys. **25** (1984), 3270–3273.

[2] L.S. Da Rios, *On the motion of an unbounded fluid with a vortex filament of any shape.* [in Italian] Rend, Circ. Mat. Palermo. **22** (1906), 117-135.

[3] Y. Fukumoto, *Motion of a curved vortex filament: higher-order asymptotics.* In Proc. of IUTAM Symposium on Geometry and Statistics of Turbulence (eds. T. Kambe, T. Nakano and T. Miyauchi). (2001), 211-216, Kluwer

[4] Y. Fukumoto, and T. Miyazaki, *Three-dimensional distortions of a vortex filament with axial velocity.* J. Fluid Mech. **222** (1991), 369-416.

[5] H. Hasimoto, *A soliton on a vortex filament.* J. Fluid Mech. **51** (1972), 477-485.

[6] N. Hayashi and P.I. Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations*, Amer. J. Math. **120** (1998), 369–389.

[7] N. Hayashi and P.I. Naumkin, *Large time behavior of solutions for the modified Korteweg-de Vries equation*, Internat. Math. Res. Notices **1999** (1999), 395–418.

[8] N. Hayashi and P.I. Naumkin, *On the modified Korteweg-de Vries equation*, Math. Phys. Anal. Geom. **4** (2001), 197–227.

[9] N. Hayashi and P.I. Naumkin, *Final state problem for Korteweg-de Vries type equations*, J. Math. Phys. **47** (2006).

[10] C.E. Kenig, G. Ponce, and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. math J. **40** (1991), 33-69

[11] C. Laurey, *The Cauchy problem for a third order nonlinear Schrödinger equation.* Nonlinear Anal. T.M.A. **29** (1997), 121-158.

[12] T. Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Comm. Math. Phys. **139** (1991), 479–493.

[13] A. Tani, and T. Nishiyama, *Solvability of equations for motion of a vortex filament with or without axial flow.* Publ. RIMS, Kyoto Univ. **33** (1997), 509-526.

[14] Y. Tsutsumi and K. Yajima, *The asymptotic behavior of nonlinear Schrödinger equations*, Bull. Amer. Math. Soc. **11** (1984), 186–188.