ON WEIGHTED STRICHARTZ ESTIMATES AND NLS FOR RADIAL DATA IN SOBOLEV SPACES OF NEGATIVE INDICES

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1. Introduction

In this paper we discuss the Strichartz-type estimates for radial solutions to the homogeneous and the inhomogeneous Schrödinger equations. We also consider the Cauchy problem for the nonlinear Schrödinger equation with low-regularity radial data.

In his seminal paper [27] Strichartz showed

$$\|e^{it\Delta}\varphi\|_{L^{q_s}(\mathbb{R}^{1+n})} \le C\|\varphi\|_{L^2(\mathbb{R}^n)} \quad (\varphi \in L^2(\mathbb{R}^n)) \tag{1.1}$$

for $n \ge 1$ and $q_s = 2 + (4/n)$ by proving the equivalent estimate for the restriction of the Fourier transform to the paraboloid $\{ (\xi', \xi) \in \mathbb{R} \times \mathbb{R}^n \,:\, \xi' = |\xi|^2 \}$

$$\|(\mathcal{F}\Phi)(|\xi|^2, \xi)\|_{L^2(\mathbb{R}^n, d\xi)} \le C\|\Phi\|_{L^{q_s'}(\mathbb{R}^{1+n})} \tag{1.2}$$

for $\Phi \in \mathcal{S}(\mathbb{R}^{1+n})$. (Here and in what follows we denote by q' the Hölder conjugate exponent of q.) Later, Ginibre and Velo [7] and Yajima [33] generalized the estimate (1.1), and they independently proved that, for any pair of exponents q and r satisfying

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r}\right) \quad \text{and} \quad \max\left\{0, \frac{n-2}{2n}\right\} < \frac{1}{r} \le \frac{1}{2}, \tag{1.3}$$

there exists a constant $C > 0$ and the estimate

$$\|e^{it\Delta}\varphi\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \le C\|\varphi\|_{L^2(\mathbb{R}^n)} \tag{1.4}$$

holds for all $\varphi \in L^2(\mathbb{R}^n)$. Both of the proofs of [7] and [33] used the point-wise (in time) estimate

$$\|e^{it\Delta}\varphi\|_{L^q(\mathbb{R}^n)} \le (4\pi|t|)^{-n(1/2-1/q)}\|\varphi\|_{L^{q'}(\mathbb{R}^n)}, \quad 2 \le q \le \infty, \tag{1.5}$$

the duality argument, and the 1-dimensional fractional integral inequality.

The end-point case $(1/q, 1/r) = (1/2, (n-2)/(2n))$ for $n \geq 3$ was investigated by Keel and Tao [15], and it was shown that the estimate (1.4) remains true for all $\varphi \in L^2(\mathbb{R}^n)$. The end-point case $(1/q, 1/r) = (1/2, 0)$ for $n = 2$ is subtle. Montgomery-Smith [18] showed that the end-point estimate fails in general, whereas Tao [29] proved that the end-point estimate is still true for radial $\varphi \in L^2(\mathbb{R}^2)$.

As these studies of the end-point estimates show, the Strichartz-type estimates stand a good chance of improving in the presence of radial symmetry. In this connection we refer to the works [17], [24], [26], [5], [10], and [3] where various Strichartz-type estimates specific to radial solutions have been obtained in the context of the wave equation or the semi-relativistic Hartree equation. In the first half of this paper we discuss some proofs of weighted and unweighted Strichartz-type estimates for radial solutions to the Schrödinger equation. In the presence of radial symmetry of data, one of the advantageous features lies in that there hold weighted and unweighted Strichartz-type estimates with the gain of derivatives for solutions to the free Schrödinger equation. Another feature is that, by virtue of the celebrated lemma of Christ and Kiselev, it is (partially at this point in time) possible to get weighted analogs of the inhomogeneous estimates of Kato [12], Foschi [6], and Vilela [32].

In the second half of this paper we turn our attention to the pure-power nonlinear Schrödinger (NLS) equation. Especially, we consider the Cauchy problem with low-regularity radial data. In the presence of radial symmetry of data we can take full advantages of our weighted linear estimates, and we see that the Cauchy problem admits a unique, global-in-time radial solution for small radial data in the scale-critical homogeneous Sobolev space, when the index of regularity is strictly bigger than $-1/2$.

2. Strichartz-type estimates

As was mentioned in Introduction, Strichartz himself obtained his original estimate (1.1) by showing the equivalent estimate for the restriction of the Fourier transform to the paraboloid. In this section we first set to work on a different proof of the Strichartz-type estimates. Our proof builds upon the Tomas-Stein lemma [30], [25] which concerns the restriction of the Fourier transform to the unit sphere $S^{n-1} = \{\, \omega \in \mathbb{R}^n : |\omega| = 1 \,\}$. In terms of the boundedness of the adjoint operator this lemma is stated as follows.

Lemma 2.1. *Suppose $n \geq 2$ and $2(n+1)/(n-1) \leq q < \infty$. There exists a constant $C > 0$ and the estimate*

$$\left\| \int_{S^{n-1}} e^{ix\cdot\omega} \psi(\omega) d\sigma(\omega) \right\|_{L^q(\mathbb{R}^n)} \leq C \|\psi\|_{L^2(S^{n-1})} \tag{2.1}$$

holds for $\psi \in L^2(S^{n-1})$.

Here and in what follows we let $d\sigma$ denote the standard measure on S^{n-1}. The estimate (2.1) obviously remains true also for $q = \infty$. As is well-known, this lemma improves in the presence of radial symmetry. We will see how the Strichartz-type estimates in turn improve in the presence of radial symmetry of data. Let us start with the proof of the following proposition. Here and in what follows we will employ the standard notation $|D_x|^s := \mathcal{F}^{-1}|\xi|^s\mathcal{F}$.

Proposition 2.2. *Suppose $n \geq 2$ and $2(n+1)/(n-1) \leq q < \infty$. There exists a constant $C > 0$ and the estimate*

$$\|e^{it\Delta}\varphi\|_{L^q(\mathbb{R}^{1+n})} \leq C \| |D_x|^s \varphi\|_{L^2(\mathbb{R}^n)} \tag{2.2}$$

holds for $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Here

$$s = \frac{n}{2} - \frac{n+2}{q}.$$

Proof. Set

$$U(t)\varphi := e^{it\Delta}\varphi = \mathcal{F}^{-1} e^{-it|\xi|^2} \mathcal{F}\varphi. \tag{2.3}$$

Fix $x \in \mathbb{R}^n$ for a moment. Using the Sobolev embedding theorem with respect to the t variable, we get

$$\|(U\varphi)(\cdot, x)\|_{L^q(\mathbb{R})} \leq C \left\| |D_t|^{(1/2)-(1/q)} \left(\int_{\mathbb{R}^n} e^{-it|\xi|^2 + ix\cdot\xi} (\mathcal{F}\varphi)(\xi) d\xi \right) \right\|_{L^2(\mathbb{R})}. \tag{2.4}$$

Making a change of variables $\xi = \rho\omega$ ($\rho > 0$, $\omega \in S^{n-1}$) first and then $\eta = \rho^2$, we continue the estimate by the Plancherel theorem as follows:

$$\cdots = C \left\| \eta^{\frac{1}{2} - \frac{1}{q} + \frac{n-1}{2} - \frac{1}{2}} \left(\int_{S^{n-1}} e^{ix\cdot\sqrt{\eta}\omega} (\mathcal{F}\varphi)(\sqrt{\eta}\omega) d\sigma \right) \right\|_{L^2((0,\infty),\, d\eta)}. \tag{2.5}$$

We therefore obtain by the Minkowski inequality

$$\begin{aligned} &\|U\varphi\|_{L^q(\mathbb{R}^{1+n})} \\ &\leq C \left\| \eta^{\frac{1}{2} - \frac{1}{q} + \frac{n-2}{2}} \left\| \int_{S^{n-1}} e^{ix\cdot\sqrt{\eta}\omega} (\mathcal{F}\varphi)(\sqrt{\eta}\omega) d\sigma \right\|_{L^q(\mathbb{R}^n,\, dx)} \right\|_{L^2((0,\infty),\, d\eta)} \\ &= C \left\| \eta^{\frac{1}{2} - \frac{1}{q} + \frac{n-2}{2} - \frac{n}{2q}} \left\| \int_{S^{n-1}} e^{iy\cdot\omega} (\mathcal{F}\varphi)(\sqrt{\eta}\omega) d\sigma \right\|_{L^q(\mathbb{R}^n,\, dy)} \right\|_{L^2((0,\infty),\, d\eta)}. \end{aligned} \tag{2.6}$$

Here we have made a change of variables $y=\sqrt{\eta}x$. It is the last $L^q(\mathbb{R}^n,\,dy)$ norm that we use lemma 2.1 to handle. Assuming $q\ge 2(n+1)/(n-1)$, we get

(2.7) $$\begin{aligned}&\|U\varphi\|_{L^q(\mathbb{R}^{1+n})}\\ &\quad\le C\|\eta^{\frac12-\frac1q+\frac{n-2}{2}-\frac{n}{2q}}\|(\mathcal{F}\varphi)(\sqrt{\eta}\,\cdot)\|_{L^2(S^{n-1})}\|_{L^2((0,\infty),\,d\eta)}=C\||D_x|^{\frac n2-\frac{n+2}{q}}\varphi\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

The proof has been completed. □

Remark. Let $n\ge 2$ and set $q_w=2(n+1)/(n-1)$. The same argument as in the proof of Proposition 2.2 leads to the estimate

$$\|e^{it\sqrt{-\Delta}}\varphi\|_{L^{q_w}(\mathbb{R}^{1+n})}\le C\||D_x|^{1/2}\varphi\|_{L^2(\mathbb{R}^n)} \tag{2.8}$$

for $\varphi\in\mathcal{S}(\mathbb{R}^n)$. This estimate is just the same as Strichartz proved by establishing the equivalent inequality concerning the restriction of the Fourier transform to the cone $\{\,(\xi',\xi)\in\mathbb{R}\times\mathbb{R}^n\,:\,\xi'=|\xi|\,\}$

$$\|(\mathcal{F}\Phi)(|\xi|,\xi)\|_{L^2(\mathbb{R}^n,\,|\xi|^{-1}d\xi)}\le C\|\Phi\|_{L^{q_w'}(\mathbb{R}^{1+n})} \tag{2.9}$$

for $\Phi\in\mathcal{S}(\mathbb{R}^{1+n})$ [27].

We return to the estimate of $e^{it\Delta}\varphi$. The estimate (2.2) is not new in the least, for it also follows immediately from (1.4) and the Sobolev embedding. Our proof of (2.2) nevertheless seems worthy of being noted because it leads to refined Strichartz-type estimates for radial data in the following way by virtue of the improvement of Lemma 2.1 in the presence of radial symmetry. (Such a refinement for radial data would not follow from the standard proof using the $L^{q'}$-L^q estimate (1.5).) Going back to Lemma 2.1, we suppose that ψ is independent of ω. Since

$$\int_{S^{n-1}}e^{ix\cdot\omega}d\sigma=O(|x|^{-(n-1)/2})\quad(|x|\to\infty), \tag{2.10}$$

Lemma 2.1 turns out to be true for the larger range $q>2n/(n-1)$. Taking account of the radial symmetry of $\mathcal{F}\varphi$ for radial $\varphi\in\mathcal{S}(\mathbb{R}^n)$, we can handle the $L^q(\mathbb{R}^n,\,dy)$ norm appearing in (2.6) in exactly the same way. We thereby obtain

Theorem 2.3. *Suppose $n\ge 2$ and $2n/(n-1)<q<2(n+1)/(n-1)$. There exists a constant $C>0$ and the estimate*

$$\|e^{it\Delta}\varphi\|_{L^q(\mathbb{R}^{1+n})}\le C\||D_x|^s\varphi\|_{L^2(\mathbb{R}^n)} \tag{2.11}$$

holds for radially symmetric $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Here

$$s=\frac n2-\frac{n+2}{q}. \tag{2.12}$$

When the exponent q satisfies $2n/(n-1) < q < (2n+4)/n$ for $n \geq 3$, the index s in (2.12) is negative and the estimate (2.11) is hence the Strichartz-type estimate with the gain of derivatives.

One might expect to obtain some $L^q(\mathbb{R}; L^r(\mathbb{R}^n))$ Strichartz-type estimates specific to radial solutions. Following Vilela [31] and Hidano and Kurokawa [10], we can indeed get

Theorem 2.4. *Suppose $n \geq 3$ and $(n-2)/(2n) \leq 1/r < (n-2)/(2(n-1))$. There exists a constant $C > 0$ and the estimate*

$$\|e^{it\Delta}\varphi\|_{L^2(\mathbb{R};L^r(\mathbb{R}^n))} \leq C\||D_x|^s\varphi\|_{L^2(\mathbb{R}^n)} \tag{2.13}$$

holds for radially symmetric $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Here

$$s = n\left(\frac{1}{2} - \frac{1}{r}\right) - 1.$$

Using the estimate (2.13) and the conservation law $\|e^{it\Delta}\varphi\|_{L^2(\mathbb{R}^n)} = \|\varphi\|_{L^2(\mathbb{R}^n)}$, we obtain by interpolation

Corollary 2.5. *Suppose $n \geq 3$,*

$$\frac{n}{2}\left(\frac{1}{2} - \frac{1}{r}\right) \leq \frac{1}{q} < (n-1)\left(\frac{1}{2} - \frac{1}{r}\right) \quad \textit{and} \quad 0 < \frac{1}{q} \leq \frac{1}{2}. \tag{2.14}$$

There exists a constant $C > 0$ and the estimate

$$\|e^{it\Delta}\varphi\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \leq C\||D_x|^s\varphi\|_{L^2(\mathbb{R}^n)} \tag{2.15}$$

holds for radially symmetric $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Here

$$s = n\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{2}{q}.$$

Proof of Theorem 2.4. We need a couple of lemmas.

Lemma 2.6. *Let $n \geq 2$ and $1/2 < \gamma < n/2$. There exists a constant $C > 0$ and the estimate*

$$\||x|^{-\gamma}|D_x|^{-\gamma+1}e^{it\Delta}\varphi\|_{L^2(\mathbb{R}^{1+n})} \leq C\|\varphi\|_{L^2(\mathbb{R}^n)} \tag{2.16}$$

holds for $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

For the proof of the estimate (2.16) we refer to Kato-Yajima [14], Ben-Artzi–Klainerman [1], Sugimoto [28], and Vilela [31]. We also need the following lemma.

Lemma 2.7. *Suppose $n \geq 2$. Let $1 < r_1 < r_2 < \infty$, $\alpha_1 < 1/r_1'$, $\alpha_2 < 1/r_2$, $\alpha_1 + \alpha_2 \geq 0$. There exists a constant $C > 0$ and the inequality*

$$\||x|^{-\alpha_2}|D_x|^{-\mu}v\|_{L^{r_2}(\mathbb{R}^n)} \leq C\||x|^{\alpha_1-(n-1)(1/r_1-1/r_2)}v\|_{L^{r_1}(\mathbb{R}^n)} \tag{2.17}$$

holds for radially symmetric function v. Here

$$\mu = \alpha_1 + \alpha_2 + \frac{1}{r_1} - \frac{1}{r_2}.$$

For the proof of Lemma 2.7 see [10]. We are ready to prove Theorem 2.4. Picking out $r_1 = 2$, $r_2 = r$ and $\alpha_1 = \alpha_2 = 0$, we get by Lemma 2.7

(2.18)
$$\begin{aligned}
&\|e^{it\Delta}\varphi\|_{L^r(\mathbb{R}^n)} \\
&\leq C\||x|^{-(n-1)((1/2)-(1/r))}|D_x|^{(1/2)-(1/r)}(e^{it\Delta}\varphi)\|_{L^2(\mathbb{R}^n)} \\
&= C\||x|^{-(n-1)((1/2)-(1/r))}|D_x|^{-(n-1)((1/2)-(1/r))+1}e^{it\Delta}|D_x|^{n((1/2)-(1/r))-1}\varphi\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

for any fixed $t \in \mathbb{R}$. Since the inequality $1/2 < (n-1)((1/2)-(1/r)) < n/2$ holds for any r with $(n-2)/(2n) \leq 1/r < (n-2)/(2(n-1))$, we can use Lemma 2.6 with $\gamma = (n-1)((1/2)-(1/r))$, and obtain

$$\|e^{it\Delta}\varphi\|_{L^2(\mathbb{R};L^r(\mathbb{R}^n))} \leq C\||D_x|^{n((1/2)-(1/r))-1}\varphi\|_{L^2(\mathbb{R}^n)} \tag{2.19}$$

as desired. We have finished the proof of Theorem 2.4. □

One might wonder whether the condition $q > 2n/(n-1)$ of Theorem 2.3 or more generally the condition $1/q < (n-1)((1/2)-(1/r))$ of Corollary 2.5 is sharp or not for radial solutions. In regard to this point, it follows from Corollary 6.2 of the recent paper of Shao [22] and the Littlewood-Paley theory that Theorem 2.3 remains true for the larger range $2(2n+1)/(2n-1) < q < 2(n+1)/(n-1)$. It is therefore expected that the condition $1/q < (n-1)((1/2)-(1/r))$ of Corollary 2.5 can also be relaxed.

3. Weighted Strichartz-type estimates

Exploiting the decay property (2.11), we can actually obtain the following weighted estimates.

Theorem 3.1. *Suppose $n \geq 2$, $2 \leq q < \infty$ and $(n/q) - ((n-1)/2) < \alpha < n/q$. There exists a constant $C > 0$ and the estimate*

$$\||x|^{-\alpha}e^{it\Delta}\varphi\|_{L^q(\mathbb{R}^{1+n})} \leq C\||D_x|^s\varphi\|_{L^2(\mathbb{R}^n)} \tag{3.1}$$

holds for radially symmetric $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Here

$$s = \alpha + \frac{n}{2} - \frac{n+2}{q}. \tag{3.2}$$

We refer the readers to [9] for the details of the proof. It should be noticed that the estimate (3.1) remains true also for $q = \infty$ for radial φ. Indeed, recalling that for any α with $-(n-1)/2 < \alpha < 0$ there exists a constant $C > 0$ such that the Sobolev-type inequality

$$|x|^{-\alpha}|v(x)| \le C\||D_x|^{\alpha+(n/2)}v\|_{L^2(\mathbb{R}^n)} \tag{3.3}$$

holds for radially symmetric functions v (see, e.g., [8] for the proof), we get for any fixed $(t, x) \in \mathbb{R}^{1+n}$

$$\begin{aligned} |x|^{-\alpha}|e^{it\Delta}\varphi| &\le C\||D_x|^{\alpha+(n/2)}(e^{it\Delta}\varphi)\|_{L^2(\mathbb{R}^n)} \\ &= C\||D_x|^{\alpha+(n/2)}\varphi\|_{L^2(\mathbb{R}^n)}, \quad -\frac{n-1}{2} < \alpha < 0, \end{aligned} \tag{3.4}$$

as desired.

As has been brought to the attention of the author by the referee of the paper [9], the estimate (3.1) can be shown also by the interpolation between the one end-point inequality (3.4) and the other end-point inequality (see (2.16))

$$\||x|^{-\alpha}e^{it\Delta}\varphi\|_{L^2(\mathbb{R}^{1+n})} \le C\||D_x|^{\alpha-1}\varphi\|_{L^2(\mathbb{R}^n)}, \quad \frac{1}{2} < \alpha < \frac{n}{2}. \tag{3.5}$$

We conclude this section by posing a problem. Our ultimate goal is to determine the optimal range of parameters q, r, and α for which the estimate

$$\begin{gathered} \||x|^{-\alpha}e^{it\Delta}\varphi\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \le C\||D_x|^s\varphi\|_{L^2(\mathbb{R}^n)}, \\ -\alpha + \frac{2}{q} + \frac{n}{r} = -s + \frac{n}{2} \end{gathered} \tag{3.6}$$

holds for radial $\varphi \in \mathcal{S}(\mathbb{R}^n)$. At this stage the full solution to this problem is out of the author's reach even for the unweighted estimates. In regard to the weighted estimates available so far we refer to the two papers [16], [10]. In [16] the weighted estimate

$$\||x|^{1/2}e^{it\Delta}\varphi\|_{L^4(\mathbb{R};L^\infty(\mathbb{R}^2))} \le C\|\varphi\|_{L^2(\mathbb{R}^2)} \tag{3.7}$$

has been obtained for radial φ. In [10] it has been shown that for any α with $-(1/2) + (1/n) < \alpha < (1/2) - (1/n)$ $(n \ge 3)$ there exists a constant $C > 0$ such that the weighted estimate

$$\||x|^{-\alpha}e^{it\Delta}\varphi\|_{L^2(\mathbb{R};L^{2n/(n-2)}(\mathbb{R}^n))} \le C\||D_x|^\alpha\varphi\|_{L^2(\mathbb{R}^n)} \tag{3.8}$$

holds for radial φ.

4. Weighted inhomogeneous estimates

Throughout this section the operator $e^{it\Delta}$ is denoted by $U(t)$. By virtue of the Christ-Kiselev lemma [4] (see also Smith and Sogge [23] and Tao [29]) we get the following weighted inhomogeneous estimates.

Theorem 4.1. *Suppose $n \geq 2$, $2 \leq q < \infty$, $2 \leq \tilde{q} < \infty$, $n/q-(n-1)/2 < \alpha < n/q$, $n/\tilde{q} - (n-1)/2 < \tilde{\alpha} < n/\tilde{q}$. Set*

$$s := -\alpha + \frac{n+2}{q} - \frac{n}{2}, \;\; \tilde{s} := -\tilde{\alpha} + \frac{n+2}{\tilde{q}} - \frac{n}{2}.$$

The estimate

$$\left\| |x|^{-\alpha} |D_x|^s \int_0^t U(t-\tau)F(\tau)d\tau \right\|_{L^q(\mathbb{R}\times\mathbb{R}^n)} \leq C \| |x|^{\tilde{\alpha}} |D_x|^{-\tilde{s}} F\|_{L^{\tilde{q}'}(\mathbb{R}\times\mathbb{R}^n)} \tag{4.1}$$

holds for radially symmetric (in x) F.

Theorem 4.2. *Suppose $n \geq 2$, $2 \leq \hat{q} < \infty$, $n/\hat{q} - (n-1)/2 < \hat{\alpha} < n/\hat{q}$. Set $\hat{s} := -\hat{\alpha} + (n+2)/\hat{q} - n/2$. The estimate*

$$\left\| |D_x|^{\hat{s}} \int_0^t U(t-\tau)F(\tau)d\tau \right\|_{L^\infty(\mathbb{R};L^2(\mathbb{R}^n))} \leq C \| |x|^{\hat{\alpha}} F\|_{L^{\hat{q}'}(\mathbb{R}\times\mathbb{R}^n)} \tag{4.2}$$

holds for radially symmetric (in x) F.

Corollary 4.3. *Suppose $n \geq 2$, $2 \leq q < \infty$, $2 \leq \tilde{q} < \infty$, $n/q-(n-1)/2 < \alpha < n/q$, $n/\tilde{q} - (n-1)/2 < \tilde{\alpha} < n/\tilde{q}$, and*

$$-\alpha + \frac{n+2}{q} - \frac{n}{2} - \tilde{\alpha} + \frac{n+2}{\tilde{q}} - \frac{n}{2} = 0.$$

The estimate

$$\begin{aligned} &\left\| |D_x|^{-\tilde{\alpha}+(n+2)/\tilde{q}-n/2} \int_0^t U(t-\tau)F(\tau)d\tau \right\|_{L^\infty(\mathbb{R};L^2(\mathbb{R}^n))} \\ &+ \left\| |x|^{-\alpha} \int_0^t U(t-\tau)F(\tau)d\tau \right\|_{L^q(\mathbb{R}\times\mathbb{R}^n)} \leq C \| |x|^{\tilde{\alpha}} F\|_{L^{\tilde{q}'}(\mathbb{R}\times\mathbb{R}^n)} \end{aligned} \tag{4.3}$$

holds for radially symmetric (in x) F.

Proof of Theorem 4.1. By Theorem 3.1 we know

$$\|U^* F\|_{L^2(\mathbb{R}^n)} \leq C \| |x|^{\tilde{\alpha}} |D_x|^{-\tilde{s}} F\|_{L^{\tilde{q}'}(\mathbb{R}\times\mathbb{R}^n)} \tag{4.4}$$

and hence

$$\||x|^{-\alpha}|D_x|^s UU^*F\|_{L^q(\mathbb{R}\times\mathbb{R}^n)} \le C\||x|^{\tilde\alpha}|D_x|^{-\tilde s}F\|_{L^{\tilde q'}(\mathbb{R}\times\mathbb{R}^n)}. \tag{4.5}$$

Here

$$(UU^*F)(t,x) = U(t)\int_{-\infty}^{\infty} U(-\tau)F(\tau)d\tau = \int_{-\infty}^{\infty} U(t-\tau)F(\tau)d\tau. \tag{4.6}$$

If $\tilde q' < q$, we get by virtue of the Christ-Kiselev lemma [4]

$$\left\| |x|^{-\alpha}|D_x|^s \int_{-\infty}^{t} U(t-\tau)F(\tau)d\tau \right\|_{L^q(\mathbb{R}\times\mathbb{R}^n)} \le C\||x|^{\tilde\alpha}|D_x|^{-\tilde s}F\|_{L^{\tilde q'}(\mathbb{R}\times\mathbb{R}^n)}. \tag{4.7}$$

This immediately yields for $\tilde q' < q$

$$\left\| |x|^{-\alpha}|D_x|^s \int_{0}^{t} U(t-\tau)F(\tau)d\tau \right\|_{L^q((0,\infty)\times\mathbb{R}^n)} \le C\||x|^{\tilde\alpha}|D_x|^{-\tilde s}F\|_{L^{\tilde q'}((0,\infty)\times\mathbb{R}^n)}. \tag{4.8}$$

Moreover we get by the change of variables $-\tau = \tilde\tau$, $-t = \tilde t$

$$\begin{aligned} &\left\| |x|^{-\alpha}|D_x|^s \int_{0}^{t} U(t-\tau)F(\tau)d\tau \right\|_{L^q((-\infty,0)\times\mathbb{R}^n)} \\ &= \left\| |x|^{-\alpha}|D_x|^s \int_{0}^{\tilde t} U(-\tilde t+\tilde\tau)F(-\tilde\tau)d\tilde\tau \right\|_{L^q((0,\infty)\times\mathbb{R}^n)} \\ &\le C\||x|^{\tilde\alpha}|D_x|^{-\tilde s}F(-\cdot)\|_{L^{\tilde q'}((0,\infty)\times\mathbb{R}^n)} = C\||x|^{\tilde\alpha}|D_x|^{-\tilde s}F\|_{L^{\tilde q'}((-\infty,0)\times\mathbb{R}^n)}. \end{aligned} \tag{4.9}$$

Here we have used the fact that the estimates (4.7)-(4.8) are obviously true with the operator $U(t-\tau)$ replaced by $U(-t+\tau)$. The proof of (4.1) has been finished in the case $\tilde q' < q$. It remains to consider the case $\tilde q' = q = 2$. Such a weighted L^2 estimate has been shown by Hoshiro [11], Sugimoto [28], and Vilela [31] even for non-radial F. The proof has been completed. □

The proof of Theorem 4.2 is similar, and therefore it is omitted here. Corollary 4.3 is a direct consequence of Theorems 4.1 and 4.2.

5. NLS equation with radial data in $\dot H^s_2$

The final section is concerned with the Cauchy problem for the nonlinear Schrödinger (NLS) equation

$$i\partial_t u + \Delta u = \lambda |u|^{p-1}u, \quad t\in\mathbb{R},\ x\in\mathbb{R}^n \tag{5.1}$$

($p > 1$, $\lambda \in \mathbb{C}$) subject to the initial data $u(0,x) = \varphi(x)$. Our main concern is to investigate the problem of global existence of small H^s-solutions. We focus our attention on radially symmetric solutions to the pure-power nonlinear Schrödinger equation (5.1), and we intend to explore the subject concerning the global existence of small solutions when p is somewhat smaller than the L^2-critical power $1 + (4/n)$ and the radially symmetric initial data φ is in the scale-critical homogeneous Sobolev space.

Let $U(t) := e^{it\Delta}$ as before, and let $\dot{H}^s_2(\mathbb{R}^n)$ denote the homogeneous Sobolev space $\{ v \in \mathcal{S}'(\mathbb{R}^n) : |D_x|^s v \in L^2(\mathbb{R}^n) \}$ for $-n/2 < s < n/2$. For $n \geq 3$ and p with $4/(n+1) < p - 1 < 4/n$ it is possible to pick out the exponent $q_0 > 2$ satisfying

$$\max\left(\frac{1}{p}, \frac{2}{p-1} - \frac{n-1}{2}\right) < \frac{2}{q_0} < \frac{2}{p-1} - \frac{n+1}{2p}.$$

See Proposition 3.1 of [9] for the proof. We set

$$\alpha_0 := \frac{n+2}{q_0} - \frac{2}{p-1}.$$

Using the weighted estimates (3.1) and (4.3), we can prove

Theorem 5.1. *Suppose that $n \geq 3$, $4/(n+1) < p - 1 < 4/n$, $\lambda \in \mathbb{C}$. Set $\sigma_0 = (n/2) - (2/(p-1))$. There exists a positive constant δ depending on n, p, λ such that if radially symmetric data $\varphi \in \dot{H}^{\sigma_0}_2(\mathbb{R}^n)$ is small so that $\| |D_x|^{\sigma_0} \varphi \|_{L^2(\mathbb{R}^n)} \leq \delta$, then the integral equation*

$$u(t) = U(t)\varphi - i\lambda \int_0^t U(t-\tau)|u(\tau)|^{p-1}u(\tau)d\tau \tag{5.2}$$

has a unique radially symmetric solution $u \in C(\mathbb{R}; \dot{H}^{\sigma_0}_2(\mathbb{R}^n))$ satisfying

$$\sup_{t\in\mathbb{R}} \| |D_x|^{\sigma_0} u(t, \cdot)\|_{L^2(\mathbb{R}^n)} + \| |x|^{-\alpha_0} u\|_{L^{q_0}(\mathbb{R}\times\mathbb{R}^n)} \leq C\delta \tag{5.3}$$

for a suitable constant $C > 0$.

Notice that $-1/2 < \sigma_0 < 0$ for $4/(n+1) < p - 1 < 4/n$. Hence the crucial first step toward showing this theorem is to deal with the initial data having only the negative-order differentiability. The Strichartz-type estimates (1.4) manifest the gain of integrability exponents for the free solutions with initial data in $L^2(\mathbb{R}^n)$. They have played an essential role in the development of the local and global (in time) H^s-theory for $s \geq 0$ (see Cazenave and Weissler [2], Kato [13], Pecher [21], Nakamura and Ozawa [19], and Nakanishi and Ozawa [20]), together with space-time estimates of solutions to inhomogeneous equations as well as elaborate estimates of products of functions in fractional-order Sobolev or Besov spaces. For the purpose of carrying out the contraction-mapping argument in the present setting we hence notice that the Strichartz-type estimates (3.1), which manifest the gain of derivatives as well as integrability exponents, will be helpful in the first step. This is the case, and the estimate (3.1) plays a crucial role in the first step of the proof.

In the next step toward the proof we need space-time estimates for the inhomogeneous Schrödinger equation. Fortunately, by virtue of the Christ-Kiselev lemma [4] along with the TT^* argument, we have obtained the weighted estimates (4.3) of radially symmetric solutions to the inhomogeneous equation directly from the weighted estimates for the free Schrödinger equation. Making use of these weighted estimates (3.1) and (4.3), we can carry out the contraction-mapping argument to show the theorem. We refer the readers to [9] for the details of the proof.

We end this paper by noting that the method using (3.1) and (4.3) does not seem to allow one to get a result similar to Theorem 5.1 for $n = 2$ and $4/3 < p - 1 < 2$, though it is expected to hold.

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