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Even Factors: Algorithms and Structure

By

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Abstract

Recent developments on even factors are presented. In a directed graph (digraph), a subset of edges is called an even factor if it forms a vertex-disjoint collection of directed cycles of even length and directed paths. The even factor problem is to find an even factor of maximum cardinality in a given digraph, which draws attention as a combinatorially tractable generalization of the non-bipartite matching problem. This problem is NP-hard, and solved in polynomial time for a certain class of digraphs, called odd-cycle-symmetric.

The independent even factor problem is a common generalization of the even factor and matroid intersection problems. In odd-cycle-symmetric digraphs, the independent even factor problem is polynomially solvable for general matroids. Also, the weighted version of the (independent) even factor problem is solved in polynomial time in odd-cycle-symmetric weighted digraphs, which are odd-cycle-symmetric digraphs accompanied by an edge-weight vector with a certain property.

In this paper, we exhibit that several important results on non-bipartite matching such as the Tutte-Berge formula, the TDI description and the Edmonds-Gallai decomposition extend to the even factor problem in odd-cycle-symmetric digraphs. Moreover, we show that for the independent even factor problem in odd-cycle-symmetric digraphs we can establish a min-max formula, a linear description with dual integrality and a decomposition theorem, which contain their counterparts in the matching problem and the matroid intersection problem. In particular, we focus on augmenting path algorithms for those problems, which commonly extends the classical algorithms for matching and matroid intersection. We also discuss the reasonableness of assuming the digraphs to be odd-cycle-symmetric.

§ 1. Introduction

The non-bipartite matching and matroid intersection problems are two celebrated combinatorial optimization problems which can be solved efficiently. Many elegant re-

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sults such as combinatorial algorithms [7, 9, 17] and totally dual integral (TDI) description [5, 8] are known. As a common framework of these two problems, Cunningham and Geelen [3] introduced the *independent path-matching problem* and showed a min-max formula and a TDI description, which commonly extend those for matching and matroid intersection. They also claimed that this problem can be solved in polynomial time by the ellipsoid method. Then, combinatorial approaches to path-matchings followed [10, 24, 25]. However, it was still unsettled whether we can establish a combinatorial algorithm for independent path-matchings.

In this context, Cunningham and Geelen [4] introduced a further generalization, *independent even factors*. Let (G, c) be a weighted digraph with $G = (V, E)$ and $c \in \mathbf{R}_+^E$, and let $\mathbf{M}^+$ and $\mathbf{M}^-$ be two matroids on V. An edge set $M \subseteq E$ is an *even factor* in G if M forms a vertex-disjoint collection of directed cycles of even length and directed paths. Remark that an odd-length path may be contained. An even factor M is *independent* if the set of vertices which have an leaving edge in M is an independent set in $\mathbf{M}^+$ and the set of vertices which have an entering edge in M is an independent set in $\mathbf{M}^-$. We consider the following problems.

The even factor problem (EFP): Finding an even factor M maximizing $|M|$.

The weighted even factor problem (WEFP): Finding an even factor M maximizing $c(M)$.

The independent even factor problem (IEFP): Finding an independent even factor M maximizing $|M|$.

The weighted independent even factor problem (WIEFP): Finding an independent even factor M maximizing $c(M)$.

Here, $c(M)$ denotes $\sum_{e \in M} c(e)$. It is easily seen that the EFP generalizes the nonbipartite matching problem and the IEFP commonly generalizes the EFP and the matroid intersection problem.

In [4], it is exhibited that the EFP is NP-hard in general and polynomially solvable in weakly symmetric digraphs, in which every edge e in any directed cycle has the reverse edge $\bar{e}$. For the EFP in weakly symmetric digraphs, they presented an extension of the Tutte-Berge formula and Edmonds-Gallai decomposition, and also presented a polynomial algorithm which extends the Tutte matrix for matchings. Moreover, they showed that the IEFP in weakly symmetric digraphs can be solved by calling an algorithm for the EFP polynomially many times. For the WEFP and WIEFP, they considered weakly symmetric weighted digraphs. A weighted digraph (G, c) is weakly symmetric if G is weakly symmetric and $c(e) = c(\bar{e})$ if $e, \bar{e} \in E$. They proposed a linear programming description of the even factors in weakly symmetric weighted digraphs which has dual integrality. They also proposed a primal-dual method for solving the WEFP which calls an algorithm for the EFP polynomially many times. For the WIEFP in weakly symmetric weighted digraphs, they gave a reduction to the valuated matroid intersection [18, 19], which calls an algorithm for the WEFP polynomially many times.

We remark here that the class of weakly symmetric weighted digraphs is broad enough to include the matching and matroid intersection problems. Cunningham and Geelen's approach [4] applies to a broader class of digraphs, called *odd-cycle-symmetric*. A digraph is odd-cycle-symmetric if every odd-length cycle (odd cycle) C has the reverse cycle $\bar{C}$. A weighted digraph (G, c) is odd-cycle-symmetric if G is odd-cycle-symmetric and $c(C) = c(\bar{C})$ for every odd cycle C. Note that a weakly symmetric (weighted) digraph is odd-cycle-symmetric. Király and Makai [15] presented a linear description of even factors in odd-cycle-symmetric weighted digraphs and proved its dual integrality. Harvey's algebraic matching algorithm [11] applies to the IEFP in an odd-cycle-symmetric digraph with two matroids linearly represented over the same field.

A combinatorial algorithm for the EFP in odd-cycle-symmetric digraphs had been open for several years since the introduction of the problem, and it was solved by Pap [22], who presented an augmenting path algorithm similar to Edmonds' matching algorithm [7]. Since then, the EFP is recognized as a combinatorially tractable generalization of the non-bipartite matching problem. Takazawa [26] extended Pap's algorithm to the WEFP by combining it with the weighted matching algorithm [6]. Iwata and Takazawa [14] extended Pap's algorithm to the IEFP by combining it with the matroid intersection algorithms [9, 17]. More recently, a combinatorial algorithm for the WIEFP is also proposed [27]. Even factors are not only a combinatorially tractable generalization of non-bipartite matching, but also compatible with matroids.

As above, most work on even factors is done for odd-cycle-symmetric digraphs. You may be worried whether assuming the digraphs to be odd-cycle-symmetric is reasonable. A characterization of odd-cycle-symmetric digraphs is given by Z. Király (see [15]). Also, Kobayashi and Takazawa [16] showed that the odd-cycle-symmetry of a digraph is a necessary and sufficient condition for the degree sequences of the even factors in the digraph to form a jump system, and the odd-cycle-symmetry of a weighted digraph is also a necessary and sufficient condition for the weighted even factors to induce an M-concave function on the jump system. This result connects the fields of even factors and discrete convexity analysis, and suggests that assuming that the digraphs to be odd-cycle-symmetric is reasonable and essential in considering an optimization problem of even factors.

This paper presents a more detailed description of the aforementioned results. In § 2, we introduce the EFP more formally and show how the theorems for non-bipartite matching are extended. In particular, we devote most part of this section to exhibiting a combinatorial algorithm for the EFP in odd-cycle-symmetric digraphs [22] and constructive proofs for these theorems. In §§ 3, 4 and 5, we deal with the WEFP, the IEFP and the WIEFP, respectively. In § 6, we discuss characterizations of odd-cycle-symmetric digraphs.

Let us close this section by presenting some notations which will be used in the following sections. Let $G = (V, E)$ be a digraph with vertex set V and edge set E. For $u, v \in V$, we denote an edge e from u to v by uv. The reverse edge of e is denoted by $\bar{e}$. The initial vertex and terminal vertex of e are respectively denoted by $\partial^+ e$ and $\partial^- e$. That is, $\bar{e} = vu$, $\partial^+ e = u$ and $\partial^- e = v$. Similarly, for $F \subseteq E$, define $\partial^+ F = \{v \mid v \in V, \exists e \in F, \partial^+ e = v\}$ and $\partial^- F = \{v \mid v \in V, \exists e \in F, \partial^- e = v\}$. For $U \subseteq V$, let $\delta^+ U = \{e \mid e \in E,\ \partial^+ e \in U\}$ and $\delta^- U = \{e \mid e \in E,\ \partial^- e \in U\}$. For $v \in V$, $\delta^+\{v\}$ and $\delta^-\{v\}$ are simply denoted by $\delta^+ v$ and $\delta^- v$, respectively. For $U \subseteq V$, the induced subgraph of U is $G[U] = (U, E[U])$, where $E[U] = \{e \mid e \in E, \partial^+ e \in U, \partial^- e \in U\}$. For $x \in \mathbf{R}^E$ and $F \subseteq E$, denote $x(F) = \sum_{e \in F} x(e)$. The numbers $|V|$ and $|E|$ are denoted by n and m, respectively, which will be used for displaying time-complexity of algorithms.

A subset of edges $\{e_1, \dots, e_k\}$ is said to be a path if $\partial^+ e_1$, $\partial^- e_1 = \partial^+ e_2$, $\partial^- e_2 = \partial^+ e_3, \dots, \partial^- e_{k-1} = \partial^+ e_k$ and $\partial^- e_k$ are distinct. A cycle is a subset of edges $\{e_1, \dots, e_k\}$ such that $\partial^- e_1 = \partial^+ e_2$, $\partial^- e_2 = \partial^+ e_3$, $\dots$, $\partial^- e_{k-1} = \partial^+ e_k$ and $\partial^- e_k = \partial^+ e_1$ are distinct. A path or a cycle $F = \{e_1, \dots, e_k\}$ is said to be odd if k is odd, and even if k is even. For F, $V(F)$ denotes the set of incident vertices $\bigcup_{i=1}^{k} \{\partial^+ e_i, \partial^- e_i\}$ and $\bar{F}$ denotes $\{\bar{e}_k, \dots, \bar{e}_1\}$.

For $U \subseteq V$, we denote by χ_U the characteristic vector of U, with $\chi_U(v) = 1$ for $v \in U$ and $\chi_U(v) = 0$ for $v \in V \setminus U$. For $u \in V$, we denote $\chi_{\{u\}}$ simply by χ_u.

A strongly connected component that has no edge entering from other components is called a source-component, and the number of source-components in $G[U]$ is denoted by $\mathrm{odd}^+(U)$. Similarly, a strongly connected component that has no edge leaving to other components is called a sink-component, and the number of sink-components in $G[U]$ is denoted by $\mathrm{odd}^-(U)$. For two vertex sets X^+ and X^-, we call (X^+, X^-) a *stable pair* if there is neither an edge e with $\partial^+ e \in X^+ \setminus X^-$ and $\partial^- e \in X^-$, nor an edge e with $\partial^+ e \in X^+$ and $\partial^- e \in X^- \setminus X^+$.

§ 2. The Even Factor Problem

First of all, let us exhibit the definition an even factor, the central object in this paper.

Definition 2.1 (Even factors [4]; see also [2]). Let $G = (V, E)$ be a digraph. A subset of edges $M \subseteq E$ is an *even factor* in G if it forms a vertex-disjoint collection of paths and even cycles.

Note that this definition implies that $|M \cap \delta^+ v| \leq 1$ and $|M \cap \delta^- v| \leq 1$ for an even factor M and a vertex $v \in V$, that is, $|M| = |\partial^+ M| = |\partial^- M|$.

The objective of the even factor problem (EFP) is to find an even factor M maximizing $|M|$ in a given digraph G. In the weighted even factor problem (WEFP), we are also given a weight vector $c \in \mathbf{R}_+^E$ and the objective is to find an even factor M maximizing $c(M)$ in the weighted digraph (G, c).

It is easy to see that even factors generalize non-bipartite matchings. Let $(\bar{G}, \bar{c})$ be a weighted undirected graph with $\bar{G} = (\bar{V}, \bar{E})$ and $\bar{c} \in \mathbf{R}_+^{\bar{E}}$ in which you are supposed to find a maximum-weight matching. Then, construct an instance (G, c) of the WEFP as follows: $G = (V, E)$, where $V = \bar{V}$; $E = \{uv, vu \mid u, v \in V \text{ are adjacent in } \bar{G}\}$; and $c(uv) = \bar{c}(\{u, v\})$, where $\{u, v\} \in \bar{E}$ is an edge connecting u and v. Observe that (G, c) has a maximum-weight even factor consisting of even cycles. By alternately picking up edges along these cycles, we obtain a vertex-disjoint set of edges, which corresponds to a maximum-weight matching in $(\bar{G}, \bar{c})$.

Unfortunately, the EFP is NP-hard. It is known, however, that several nice properties of the non-bipartite matching problem extend to the even factor problem in *odd-cycle-symmetric* digraphs.

Definition 2.2 (Odd-cycle-symmetric digraphs)**.**
A digraph G is *odd-cycle-symmetric* if any odd cycle C has the reverse cycle $\bar{C}$.

For the EFP in odd-cycle-symmetric digraphs, a min-max theorem which corresponds to the Tutte-Berge formula can be established [4, 22, 23].

Theorem 2.3. *Let $G = (V, E)$ be an odd-cycle-symmetric digraph. Then, it holds that*

$$(2.1) \quad \max\{|M| \mid M \text{ is an even factor in } G\} \\ = \min_{(X^+, X^-)} \{|V \setminus X^+| + |V \setminus X^-| + |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-)\},$$

where (X^+, X^-) runs over all stable pairs.

Observe that we obtain the Tutte-Berge formula by applying Theorem 2.3 to a digraph in which every edge has the reverse edge.

Here, let us show a constructive proof for Theorem 2.3 based on Pap's even factor algorithm [22]. First, let us prove the easier part of Theorem 2.3, max $\leq$ min.

Let M be an even factor and (X^+, X^-) be a stable pair in G. Then, it holds that $M = M_1 \cup M_2 \cup M_3$, where

$$(2.2) \qquad M_1 = \left\{e \mid e \in M,\ \partial^+ e \in V \setminus X^+\right\},$$

$$(2.3) \qquad M_2 = \left\{e \mid e \in M,\ \partial^- e \in V \setminus X^-\right\},$$

$$(2.4) \qquad M_3 = \left\{e \mid e \in M,\ e \in E[X^+ \cap X^-]\right\}.$$

Remark that M_1 and M_2 are not necessarily disjoint. Now, we have

$$|M_1| \leq |V \setminus X^+|, \quad |M_2| \leq |V \setminus X^-|, \quad |M_3| \leq |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-).$$

Thus,

$$\begin{aligned} |M| &\leq |M_1| + |M_2| + |M_3| \\ &\leq |V \setminus X^+| + |V \setminus X^-| + |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-). \end{aligned} \tag{2.5}$$

Next, let us show that there exists an even factor M and a stable pair (X^+, X^-) that satisfy (2.5) by equality. The existence can be verified by looking at the following even factor algorithm.

Algorithm 2.4 (Even factor algorthm [22]).

Input: An odd-cycle-symmetric digraph $G = (V, E)$.

Output: A maximum even factor M in G.

Step 1: Set M to be an arbitrary even factor. (For instance, $M := \emptyset$.)

Step 2: Construct an auxiliary digraph $G_M = (V^*, E^*)$ as follows:

$$V^* = V^+ \cup V^-; \qquad E^* = \{u^+v^- \mid uv \in E \setminus M\} \cup \{v^-u^+ \mid uv \in M\};$$

where $V^+ = \{v^+ \mid v \in V\}$ and $V^- = \{v^- \mid v \in V\}$ are two copies of V. Also, define

$$S^+ = \{v^+ \mid v \in V \setminus \partial^+ M\}, \qquad S^- = \{v^- \mid v \in V \setminus \partial^- M\}.$$

For $F \subseteq E^*$, denote the subset of E that corresponds to F by $E(F)$.

Search a path P from a vertex in S^+ to a vertex in S^-. If such a path does not exist, then expand each pseudo-vertex in G and return M.

Step 3: If $M \triangle E(P)$ has no odd cycle, then update $M := M \triangle E(P)$, expand every pseudo-vertex in G and go to Step 2.

Step 4. Denote $P = \{e_1, e_2, \ldots, e_{2k+1}\}$ and define

$$P_i = \begin{cases} \emptyset & i = 0, \\ \{e_1, e_2, \ldots, e_{2i}\} & i = 1, \ldots, k, \\ P & i = k+1. \end{cases}$$

Let i^* be the minimum integer such that $M \triangle E(P_i)$ contains an odd cycle C. Then, update $M := M \triangle E(P_{i^*-1})$ and shrink C into a pseudo-vertex to obtain a new digraph G' and a new even factor M' in G'. Set $G := G'$ and $M := M'$, and then go to Step 2.

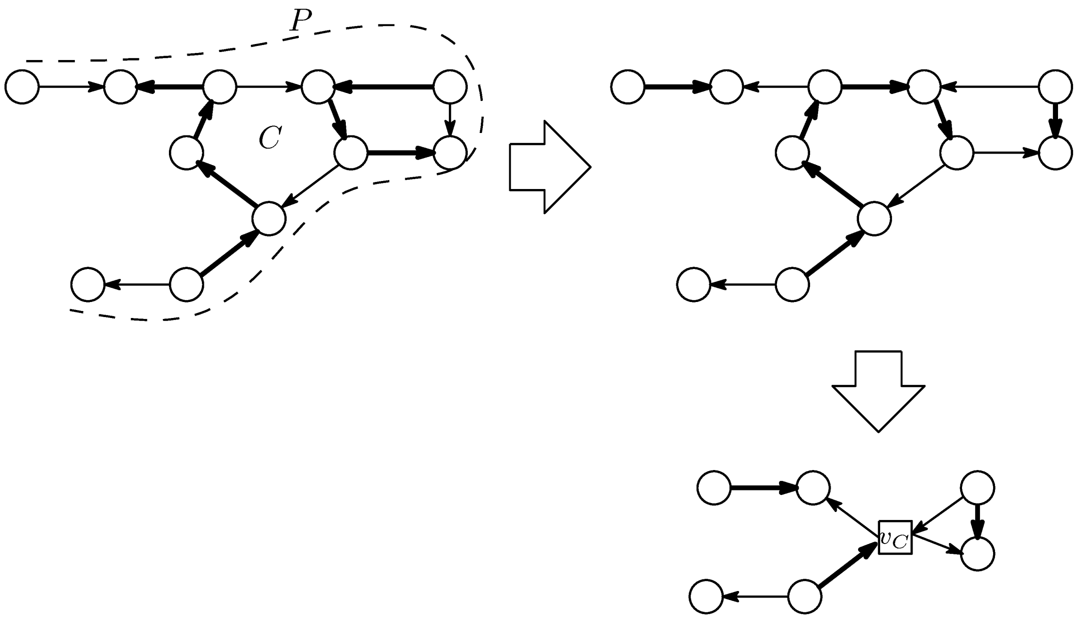

Figure 1. Shrinking of an odd cycle C (the bold edges belong to M).

Let us provide a detailed description of shrinking an odd cycle C into a pseudo-vertex v_C and expanding v_C to C. Shrinking of C consists of the following two operations: identifying the vertices in $V(C)$ to a single vertex v_C; and deleting the edges in $E[V(C)]$. The resulting digraph is denoted by G', and define M' to be edges in G' which correspond to the edges $M \setminus E[V(C)]$ in G. Before shrinking C, we update $M := M \triangle E(P_{i^*-1})$ so that $|M \cap C| = |C| - 1$. This implies that M' is an even factor in G' with $|M'| = |M| - |C| + 1$. An example of shrinking an odd cycle is shown in Figure 1.

Expanding of v_C is the reverse procedure of shrinking of C. The pseudo-vertex v_C is replaced by $G[V(C)]$ and an edge e incident to v_C connected to a vertex in $V(C)$ to which e was incident before shrinking C. An edge in M before expanding C remains to be in M after expanding v_C. Here, we have to take care that which edges in $E[V(C)]$ belong to M. In terms of $|M|$, we have to pick up $|C| - 1$ edges from $E[V(C)]$. Let e^+ denote the edge in $M \cap \delta^+ v_C$ and e^- denote the edge in $M \cap \delta^- v_C$, if they exist. In the expanded digraph, an even path P_C from $\partial^- e^-$ to $\partial^+ e^+$ is contained in $C \cup \bar{C}$. Also, pick up disjoint cycles of length two that cover $V(C) \setminus V(P_C)$. Adding the edges in P_C and these cycles to M, we have a new even factor in G with the desired size. An example of expanding of C is shown in Figure 2. The case where e^+ or e^- does not exist is easier.

This is a full description of the even factor algorithm. Observe that the time-complexity of the algorithm is $\mathrm{O}(n^4)$.

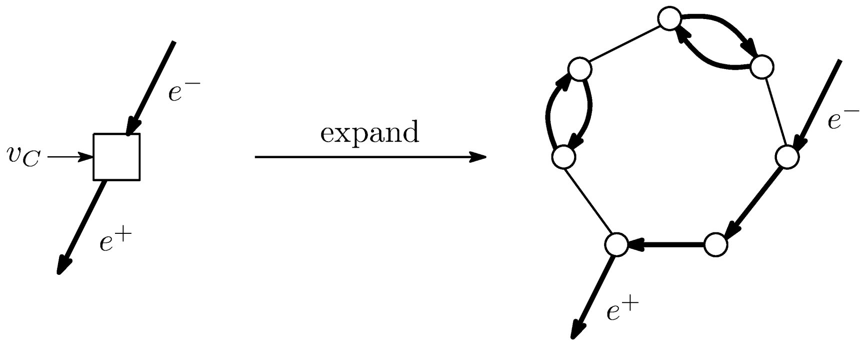

Figure 2. Expanding of an odd cycle C (the bold edges belong to M).

Theorem 2.5. *Algorithm 2.4 runs in* $\mathrm{O}(n^4)$ *time.*

Now, let us prove that the output M of Algorithm 2.4 is a maximum even factor by showing that there exists a stable pair (X^+, X^-) such that M and (X^+, X^-) satisfy (2.5) with equality.

Algorithmic proof for Theorem 2.3. Denote by G_M the digraph in which P was not found in Step 2. Let $R \subseteq V^*$ be the set of vertices to which G_M has a path from a vertex in S^+, $X^+ = \{v \mid v^+ \in R\}$, and $X^- = \{v \mid v^- \notin R\}$. Observe that (X^+, X^-) is a stable pair, and define $M_1, M_2, M_3 \subseteq M$ by (2.2)–(2.4). By the definition of R, we have that $M_1 \cap M_2 = \emptyset$, and $M_3 = \emptyset$. Moreover, $E[X^+ \cap X^-] = \emptyset$, which implies that $|X^+ \cap X^-| = \mathrm{odd}^+(X^+ \cap X^-)$. Thus, we have that (2.5) holds with equality for M and (X^+, X^-).

The argument above just assures that (2.1) holds in G, which may be obtained by shrinking odd cycles repeatedly. In what follows, we show that (2.1) holds in the original digraph by induction on the number of shrinking. Suppose that G is obtained by shrinking an odd cycle C in a digraph $\hat{G} = (\hat{V}, \hat{E})$. The proof gets completed if (2.1) holds for $\hat{G}$.

Associated with the stable pair (X^+, X^-) in G, define $\hat{X}, \hat{X} \subseteq \hat{V}$ by

$$(2.6) \qquad \hat{X}^+ = \begin{cases} X^+ & (v_C \notin X^+), \\ X^+ \cup V(C) & (v_C \in X^+), \end{cases} \qquad \hat{X}^- = \begin{cases} X^- & (v_C \notin X^-), \\ X^- \cup V(C) & (v_C \in X^-). \end{cases}$$

Now we prove that $(\hat{X}^+, \hat{X}^-)$ is a stable pair which certificates (2.1).

Firstly, one can easily see that $(\hat{X}^+, \hat{X}^-)$ forms a stable pair in $\hat{G}$, which follows from the fact that (X^+, X^-) is a stable pair in G.

Secondly, we estimate the value $|\hat{V} \setminus \hat{X}^+| + |\hat{V} \setminus \hat{X}^-| + |\hat{X}^+ \cap \hat{X}^-| - \mathrm{odd}^+(\hat{X}^+ \cap \hat{X}^-)$. Since v_C is the pseudo-vertex created in the latest shrinking, we have that $v_C \notin \partial^+ M$, in particular $v_C \in X^+$. (Hence, in (2.6) we do not have the case $v_C \notin X^+$.) Thus, $|\hat{V} \setminus \hat{X}^+| = |V \setminus X^+|$.

Assume $v_C \in X^+ \cap X^-$. Then, we have $|\hat{V} \setminus \hat{X}^-| = |V \setminus X^-|$. Moreover, $|\hat{X} \cap \hat{X}| - \mathrm{odd}^+(\hat{X}^+ \cap \hat{X}^-) = |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-) + |C| - 1$ follows from the following equations: $|\hat{X}^+ \cap \hat{X}^-| = |X^+ \cap X^-| + |C| - 1$; and $\mathrm{odd}^+(\hat{X}^+ \cap \hat{X}^-) = \mathrm{odd}(X^+ \cap X^-)$.

Assume $v_C \in X^+ \setminus X^-$. In this case, $|\hat{V} \setminus \hat{X}^-| = |V \setminus X^-| + |C| - 1$, and it is obvious that $|\hat{X} \cap \hat{X}| - \mathrm{odd}^+(\hat{X}^+ \cap \hat{X}^-) = |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-)$.

Therefore, in any case we have

$$\begin{aligned}(2.7)\quad & |\hat{V} \setminus \hat{X}^+| + |\hat{V} \setminus \hat{X}^-| + |\hat{X}^+ \cap \hat{X}^-| - \mathrm{odd}^+(\hat{X}^+ \cap \hat{X}^-) \\ & = |V \setminus X^+| + |V \setminus X^-| + |X^+ \cap X^-| - \mathrm{odd}^+(X^+ \cap X^-) + |C| - 1.\end{aligned}$$

Since $|M|$ increases by $|C| - 1$ by expanding C, we have that (2.5) holds with equality for M and $(\hat{X}^+, \hat{X}^-)$, and hence (2.1) holds for $\hat{G}$. □

The Edmonds-Gallai decomposition for the matching problem also extends to the EFP in odd-cycle-symmetric digraphs [4, 23]. Algorithm 2.4 provides not only a constructive proof for Theorem 2.3, but also such a decomposition theorem.

Let $G = (V, E)$ be an odd-cycle-symmetric digraph. As we have seen in the proof for Theorem 2.3, we obtain a stable pair (X^+, X^-) in G minimizing the right hand side (RHS) of (2.1) among all stable pairs, by applying Algorithm 2.4 to G. Besides, exchanging the roles of V^+ and V^- in the algorithm, we obtain another stable pair (Y^+, Y^-), which minimizes

$$(2.8)\qquad |V \setminus Y^+| + |V \setminus Y^-| + |Y^+ \cap Y^-| - \mathrm{odd}^-(Y^+ \cap Y^-)$$

among all stable pairs (Y^+, Y^-). For these stable pairs, we have the following theorem, whose proof is left to the reader.

Theorem 2.6 (See also [4, 23]). *For the stable pairs (X^+, X^-) and (Y^+, Y^-) defined above, set $V_D^+ = X^+$, $V_A^+ = V \setminus X^-$, $V_D^- = Y^-$, and $V_A^- = V \setminus Y^+$. Then, the following* 1–4 *hold.*

1. $V_D^+ = \{v \mid v \in V, \exists \textit{maximum even factor } M, v \notin \partial^+ M\}$, *and* $V_D^- = \{v \mid v \in V, \exists \textit{maximum even factor } M, v \notin \partial^- M\}$.

2. *For every strongly connected component C in $G[V_D^+ \cap (V \setminus V_A^+)]$ or in $G[V_D^- \cap (V \setminus V_A^-)]$, $|V(C)|$ is odd.*

3. *If M is a maximum even factor, then the following* (a)–(c) *hold.*

   (a) *For every strongly connected component C in $G[V_D^+ \cap (V \setminus V_A^+)]$ or in $G[V_D^- \cap (V \setminus V_A^-)]$, $|M \cap E(C)| = |C| - 1$.*

(b) $V \setminus V_D^+ \subseteq \partial^+ M$, $V_A^+ \subseteq \partial^- M$, $V_A^- \subseteq \partial^+ M$, *and* $(V \setminus V_D^-) \subseteq \partial^- M$.

(c) *For a vertex* $v \in V_A^+ \cap \partial^- M$, *there exists a vertex* $u \in V_D^+$ *with* $uv \in M$, *and for a vertex* $v \in V_A^- \cap \partial^+ M$, *there exists a vertex* $u \in V_D^-$ *with* $vu \in M$.

4. *Any stable pair* (U^+, U^-) *minimizing the RHS of* (2.1) *satisfies that* $V_D^+ \subseteq U^+$ *and* $V_D^- \subseteq U^-$. *Similarly, any stable pair* (U^+, U^-) *minimizing* (2.8) *satisfies that* $V_D^+ \subseteq U^+$ *and* $V_D^- \subseteq U^-$.

§ 3. The Weighted Even Factor Problem

Let (G, c) be a weighted digraph with $G = (V, E)$ and $c \in \mathbf{R}_+^E$. This section considers the weighted even factor problem (WEFP), finding an even factor M maximizing $c(M)$.

The WEFP is NP-hard in general, since its special case, the EFP, is NP-hard. Here, we deal with odd-cycle-symmetric weighted digraphs, which are defined by extending Definition 2.2 to a weighted version.

Definition 3.1 (Odd-cycle-symmetric weighted digraphs). A weighted digraph (G, c) is *odd-cycle-symmetric* if any odd cycle C has the reverse cycle $\bar{C}$ such that $c(C) = c(\bar{C})$.

Define $\mathcal{U} = \{U \mid U \subseteq V,\ |U| \text{ is odd and } \geq 3\}$. The following is a linear relaxation of an integer program for the WEFP in (G, c).

$$
\begin{array}{llll}
\text{(P-EF)} & \text{max.} & \displaystyle\sum_{e \in E} c(e)x(e) & \\
& \text{sub. to} & x\left(\delta^+ v\right) \leq 1 & v \in V, \\
& & x\left(\delta^- v\right) \leq 1 & v \in V, \\
& & x\left(E[U]\right) \leq |U| - 1 & U \in \mathcal{U}, \\
& & x(e) \geq 0 & e \in E.
\end{array}
$$

Observe that a characteristic vector of an even factor in G is an integer feasible solution for (P-EF). The dual program of (P-EF) is given by

$$
\begin{array}{lll}
\text{(D-EF)} & \text{min.} & \displaystyle\sum_{v \in V} (y^+(v) + y^-(v)) + \sum_{U \in \mathcal{U}} (|U| - 1) z(U) \\
& \text{sub. to} & y^+(u) + y^-(v) + \displaystyle\sum_{\substack{U \in \mathcal{U}, \\ e \in E[U]}} z(U) \geq c(e) \quad e = uv \in E, \\
& & y^+(v) \geq 0 \quad v \in V, \\
& & y^-(v) \geq 0 \quad v \in V, \\
& & z(U) \geq 0 \quad U \in \mathcal{U}.
\end{array}
$$

For these two linear programs, Király and Makai [15] showed the following integrality theorem. We say that a set family $\mathcal{F}$ is *laminar* if $U_1 \subseteq U_2$, $U_2 \subseteq U_1$ or $U_1 \cap U_2 = \emptyset$ for all $U_1, U_2 \in \mathcal{F}$.

Theorem 3.2 ([15]). *If* (G, c) *is an odd-cycle-symmetric weighted digraph, then* (P-EF) *has an integral optimal solution. Moreover, if* (G, c) *is odd-cycle-symmetric and* c *is integer, then* (D-EF) *also has an integral optimal solution* (y^+, y^-, z) *such that* $\{U \mid z(U) > 0\}$ *is laminar.*

This theorem is an extension of the TDI theorem for non-bipartite matching [5]. Denote the polytope defined by the constraints in (P-EF) by P_{EF}, and that of (D-EF) by D_{EF}. Theorem 3.2 claims that if (G, c) is odd-cycle-symmetric then P_{EF} has an integer extreme point in the direction of the vector c. Moreover, D_{EF} also has an integer extreme point in the direction of the vector corresponding to the objective function of (D-EF) if (G, c) is odd-cycle-symmetric and c is integer. Remark that neither P_{EF} nor D_{EF} has an integer extreme point in all directions, which is expected by the NP-hardness of the EFP.

Several algorithm for the WEFP in odd-cycle-symmetric weighted digraphs are proposed [4, 26]. Cunningham and Geelen's [4] is a primal-dual method that calls an algorithm for the EFP polynomially many times. Incorporating Algorithm 2.4, their method obtains a combinatorial algorithm which runs in $\mathrm{O}(n^7)$ time. Takazawa's algorithm [26] is also a combinatorial and primal-dual algorithm. It finds integer optimal solutions for (P-EF) and (D-EF) by combining Algorithm 2.4 and the weighted matching algorithm [6]. The time-complexity of the algorithm is $\mathrm{O}(n^3 m)$.

§ 4. The Independent Even Factor Problem

Let $G = (V, E)$ be a digraph, and $\mathbf{M}^+$ and $\mathbf{M}^-$ be matroids on V whose independent set families are $\mathcal{I}^+$ and $\mathcal{I}^-$, respectively. In this paper, we indicate a matroid by a pair of its ground set and independent set family. So, $\mathbf{M}^+ = (V, \mathcal{I}^+)$ and $\mathbf{M}^- = (V, \mathcal{I}^-)$.

Denote the rank function and closure function of $\mathbf{M}^+$ by ρ^+ and cl^+, respectively. The fundamental circuit with respect to $I \in \mathcal{I}^+$ and $v \in V \setminus \mathrm{cl}^+(I)$ is denoted by $C^+(I \mid v)$. Their counterparts in $\mathbf{M}^-$ are denoted by ρ^-, cl^- and $C^-(I \mid v)$.

Definition 4.1 (Independent even factors). Let $G = (V, E)$ be a digraph, and $\mathbf{M}^+ = (V, \mathcal{I}^+)$ and $\mathbf{M}^- = (V, \mathcal{I}^-)$ be matroids. An edge set $M \subseteq E$ is an *independent even factor* in $(G, \mathbf{M}^+, \mathbf{M}^-)$ if M is an even factor in G such that $\partial^+ M \in \mathcal{I}^+$ and $\partial^- M \in \mathcal{I}^-$.

The objective of the independent even factor problem (IEFP) is to find an independent even factor M maximizing $|M|$ for given $(G, \mathbf{M}^+, \mathbf{M}^-)$. In the weighted independent even factor problem (WIEFP), we are also given a weight vector $c \in \mathbf{R}_+^E$ and the objective is to find an independent even factor maximizing $c(M)$. Of course, the IEFP generalizes the EFP and the WIEFP generalizes the WEFP (consider a special case where $\mathbf{M}^+$ and $\mathbf{M}^-$ are free matroids). Moreover, the independent even factors also generalize matroid intersection.

Let $\mathbf{M}_1 = (V, \mathcal{I}_1)$ and $\mathbf{M}_2 = (V, \mathcal{I}_2)$ be matroids and $c \in \mathbf{R}_+^V$ a weight vector. You are supposed to find a common independent set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ maximizing $c(I)$. This problem is reduced to the WIEFP. Construct an associated bipartite digraph $G = (V_1, V_2; E)$ as follows. The vertex set V_1 and V_2 are copies of V, respectively, and the edge set is $E = \{v_1 v_2 \mid v \in V\}$, where $v_1 \in V_1$ (resp. $v_2 \in V_2$) denotes the copy of $v \in V$. On the edge set E, define a weight function c' by $c'(v_1 v_2) = c(v)$. Let $\mathbf{M}^+ = (V_1 \cup V_2, \mathcal{I}^+)$ be a matroid which is the direct sum of $\mathbf{M}_1$ and a free matroid on V_2, and let $\mathbf{M}^- = (V_1 \cup V_2, \mathcal{I}^-)$ be a matroid which is the direct sum of a free matroid on V_1 and $\mathbf{M}_2$. Then, it is easily seen that a subset $I \subseteq V$ belongs to $\mathcal{I}_1 \cap \mathcal{I}_2$ if and only if the edge set $E_I = \{(v_1, v_2) \mid v \in I\}$ is an independent even factor in $(G, \mathbf{M}^+, \mathbf{M}^-)$, and $c(I) = c'(E_I)$.

By the NP-hardness of the EFP, we know that the IEFP is also NP-hard. In odd-cycle-symmetric digraphs with general matroids, however, we can establish a min-max formula, decomposition theorem and a polynomial algorithm, which commonly extends those for matching and matroid intersection.

Theorem 4.2 ([14]). *Let $G = (V, E)$ be an odd-cycle-symmetric digraph and $\mathbf{M}^+ = (V, \mathcal{I}^+)$, $\mathbf{M}^- = (V, \mathcal{I}^-)$ be matroids. It holds that*

$$\begin{aligned} (4.1) \quad & \max\{|M| \mid M \text{ is an independent even factor in } (G, \mathbf{M}^+, \mathbf{M}^-)\} \\ & = \min_{(X^+, X^-)} \left\{\rho^+(V \setminus X^+) + \rho^-(V \setminus X^-) + \left|X^+ \cap X^-\right| - \mathrm{odd}^+(X^+ \cap X^-)\right\}, \end{aligned}$$

where (X^+, X^-) runs over all stable pairs.

Observe that this theorem is a common extension of Theorem 2.3 and a min-max formula for matroid intersection [8].

Several algorithms are proposed for solving the IEFP in odd-cycle-symmetric digraphs [4, 11, 14]. Among them, Iwata and Takazawa's one [14] is a common extension of Algorithm 2.4 and the classical matroid intersection algorithms [9, 17]. This algorithm provides a constructive proof for Theorem 4.2 and a decomposition theorem which commonly extends the Edmonds-Gallai decomposition for matching and the principal partition for matroid intersection (cf. [12, 13]).

Algorithm 4.3 (Independent even factor algorthm [14]).

Input: An odd-cycle-symmetric digraph $G = (V, E)$, and two matroids $\mathbf{M}^+ = (V, \mathcal{I}^+)$ and $\mathbf{M}^- = (V, \mathcal{I}^-)$.

Output: A maximum independent even factor M in $(G, \mathbf{M}^+, \mathbf{M}^-)$.

Step 1: Set M to be an arbitrary independent even factor. (For instance, $M := \emptyset$.)

Step 2: Construct an auxiliary digraph $G_M = (V^*, E^*)$ as follows:

$$\begin{aligned} V^* &= V^+ \cup V^-;\\ E^* &= \{u^+v^- \mid uv \in E \setminus M\} \cup \{v^-u^+ \mid uv \in M\}\\ &\quad \cup \{(u^+, v^+) \mid u \in \partial^+ M,\ v \in \mathrm{cl}^+(\partial^+ M) \setminus \partial^+ M,\ u \in C^+(\partial^+ M \mid v)\}\\ &\quad \cup \{(u^-, v^-) \mid u \in \mathrm{cl}^-(\partial^- M) \setminus \partial^- M,\ v \in \partial^- M,\ v \in C^-(\partial^- M \mid u)\}; \end{aligned}$$

where $V^+ = \{v^+ \mid v \in V\}$ and $V^- = \{v^- \mid v \in V\}$ are two copies of V. Also, define

$$S^+ = \{v^+ \mid v \in V \setminus \mathrm{cl}^+(\partial^+ M)\}, \qquad S^- = \{v^- \mid v \in V \setminus \mathrm{cl}^-(\partial^- M)\}.$$

For $F \subseteq E^*$, denote the subset of E that corresponds to F by $E(F)$.

Search a path P from a vertex in S^+ to a vertex in S^- with minimum number of edges. If such a path does not exist, then expand every pseudo-vertex in G and return M.

Step 3: If $M \triangle E(P)$ has no odd cycle, then update $M := M \triangle E(P)$, expand every pseudo-vertex in G and go to Step 2.

Step 4. Denote $E(P) = \{e_1, e_2, \ldots, e_{2k+1}\}$ and define

$$E(P_i) = \begin{cases} \emptyset & i = 0,\\ \{e_1, e_2, \ldots, e_{2i}\} & i = 1, \ldots, k,\\ E(P) & i = k+1. \end{cases}$$

Let i^* be the minimum integer such that $M \triangle E(P_i)$ contains an odd cycle C. Then, update $M := M \triangle E(P_{i^*-1})$ and shrink C into a pseudo-vertex to obtain a new digraph G' and a new independent even factor M' in G'. Set $G := G'$ and $M := M'$, and then go to Step 2.

One would see that this algorithm is a natural extension of Algorithm 2.4 which contains the idea of simultaneous exchangeability in matroids. The time-complexity of Algorithm 4.3 is $\mathrm{O}(n^4\gamma)$, where γ is the time for an independence test.

Remark. In Algorithm 4.3, a non-trivial operation appears in Step 4, shrinking an odd cycle C. When shrinking C, not only the digraph G but also the matroids $\mathbf{M}^+$ and $\mathbf{M}^-$ should be changed in order to maintain M to be an independent even factor in $(G, \mathbf{M}^+, \mathbf{M}^-)$. In [14], an operation *shrinking* of C for matroids is proposed, which constructs a matroid on $(V \setminus V(C)) \cup \{v_C\}$ from a matroid on V.

As Algorithm 2.4 provided a proof for Theorems 2.3 and 2.6, Algorithm 4.3 gives a proof for Theorem 4.2 and a decomposition theorem. First, Algorithm 4.3 finds a stable pair (X^+, X^-). This stable pair (X^+, X^-) and the output M can be proven to satisfy (4.1) with equality, and thus Theorem 4.2 follows.

Next, by exchanging the roles of $\mathbf{M}^+$ and $\mathbf{M}^-$, we obtain another stable pair (Y^+, Y^-), which minimizes

$$\rho^+(V \setminus Y^+) + \rho^-(V \setminus Y^-) + \left|Y^+ \cap Y^-\right| - \mathrm{odd}^-(Y^+ \cap Y^-). \tag{4.2}$$

For these two stable pairs, the following theorem holds. Observe that this theorem commonly extends Theorem 2.6 and the principal partition for matroid intersection.

Theorem 4.4 ([14]). *For the stable pairs (X^+, X^-) and (Y^+, Y^-) defined above, set $V_D^+ = X^+$, $V_A^+ = V \setminus X^-$, $V_D^- = Y^-$, and $V_A^- = V \setminus Y^+$. Then, the following* 1–4 *hold.*

1. $V_D^+ = \{v \mid v \in V,\ \exists\text{maximum independent even factor } M, v \notin \mathrm{cl}^+(\partial^+ M)\}$, *and* $V_D^- = \{v \mid v \in V,\ \exists\text{maximum independent even factor } M, v \notin \mathrm{cl}^-(\partial^- M)\}$.
2. *For every strongly connected component C in $G[V_D^+ \cap (V \setminus V_A^+)]$ or in $G[V_D^- \cap (V \setminus V_A^-)]$, $|V(C)|$ is odd and $V(C) \in \mathcal{I}^+ \cap \mathcal{I}^-$.*
3. *If M is a maximum independent even factor, then the following* (a)–(c) *hold.*
   (a) *For every strongly connected component C in $G[V_D^+ \cap (V \setminus V_A^+)]$ or in $G[V_D^- \cap (V \setminus V_A^-)]$, $|M \cap E(C)| = |C| - 1$.*
   (b) $|(V \setminus V_D^+) \cap \partial^+ M| = \rho^+(V \setminus V_D^+)$, $|V_A^+ \cap \partial^- M| = \rho^-(V_A^+)$, $|V_A^- \cap \partial^+ M| = \rho^+(V_A^-)$, *and* $|(V \setminus V_D^-) \cap \partial^- M| = \rho^-(V \setminus V_D^-)$.
   (c) *For a vertex $v \in V_A^+ \cap \partial^- M$, there exists a vertex $u \in V_D^+$ with $uv \in M$, and for a vertex $v \in V_A^- \cap \partial^+ M$, there exists a vertex $u \in V_D^-$ with $vu \in M$.*
4. *Any stable pair (U^+, U^-) minimizing the RHS of* (4.1) *satisfies that $V_D^+ \subseteq U^+$ and $V_D^- \subseteq U^-$. Similarly, any stable pair (U^+, U^-) minimizing* (4.2) *satisfies that $V_D^+ \subseteq U^+$ and $V_D^- \subseteq U^-$.*

§ 5. The Weighted Independent Even Factor Problem

An instance of the weighted independent even factor problem (WIEFP) is denoted by $(G, c, \mathbf{M}^+, \mathbf{M}^-)$, where $G = (V, E)$, $c \in \mathbf{R}_+^E$, $\mathbf{M}^+ = (V, \mathcal{I}^+)$ and $\mathbf{M}^- = (V, \mathcal{I}^-)$. The objective of the WIEFP is to find an independent even factor in $(G, \mathbf{M}^+, \mathbf{M}^-)$ maximizing $c(M)$.

The WIEFP in odd-cycle-symmetric weighted digraphs is polynomially solvable [4, 27]. The algorithm in [4] reduces the problem to valuated matroid intersection, and it calls an algorithm for the WEFP polynomially many times. Incorporating the algorithm in [26], this algorithm archives the time-complexity $\mathrm{O}(n^6 m + n^3\gamma)$, where γ is the time for an independence test.

On the other hand, the algorithm in [27] combines the algorithm in [26] and Algorithm 4.3 and runs in $\mathrm{O}(n^5 + n^4\gamma)$. It is a primal-dual algorithm which finds integer optimal solutions for the following linear programs:

$$\begin{array}{llll}
\text{(P-IEF)} & \text{max.} & \displaystyle\sum_{e\in E} c(e)x(e) & \\
& \text{sub. to} & x\left(\delta^+ U\right) \le \rho^+(U) & U \subseteq V, \\
& & x\left(\delta^- U\right) \le \rho^-(U) & U \subseteq V, \\
& & x\left(E[U]\right) \le |U| - 1 & U \in \mathcal{U}, \\
& & x(e) \ge 0 & e \in E.
\end{array}$$

$$\begin{array}{lll}
\text{(D-IEF)} & \text{min.} & \displaystyle\sum_{U\subseteq V} \left(\rho^+(U)y^+(U) + \rho^-(U)y^-(U)\right) + \sum_{U\in\mathcal{U}} (|U|-1)z(U) \\
& \text{sub. to} & \displaystyle\sum_{\substack{U\subseteq V,\\ U\ni\partial^+ e}} y^+(U) + \sum_{\substack{U\subseteq V,\\ U\ni\partial^- e}} y^-(U) + \sum_{\substack{U\in\mathcal{U},\\ e\in E[U]}} z(U) \ge c(e) \quad e \in E, \\
& & y^+(U) \ge 0 \quad U \subseteq V, \\
& & y^-(U) \ge 0 \quad U \subseteq V, \\
& & z(U) \ge 0 \quad U \in \mathcal{U}.
\end{array}$$

Thus, the algorithm constructively proves the following integrality theorem, which corresponds to the TDI theorems for matching [5] and matroid intersection [9]. We say that a set family $\mathcal{F}$ is *laminar* if $U_1 \subseteq U_2$, $U_2 \subseteq U_1$ or $U_1 \cap U_2 = \emptyset$ for all $U_1, U_2 \in \mathcal{F}$. Also, we say that $\mathcal{F}$ is *nested* if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$ for all $U_1, U_2 \in \mathcal{F}$.

Theorem 5.1 ([27]). *For an instance* $(G, c, \mathbf{M}^+, \mathbf{M}^-)$ *of the WIEFP,* (P-IEF) *has an integral optimal solution if* (G, c) *is odd-cycle-symmetric. Moreover, if* (G, c) *is odd-cycle-symmetric and* c *is integer,* (D-IEF) *also has an integral optimal solution* (y^+, y^-, z) *such that* $\{U \mid y^+(U) > 0\}$ *and* $\{U \mid y^-(U) > 0\}$ *are nested and* $\{U \mid z(U) > 0\}$ *is laminar.*

§ 6. Characterization of Odd-Cycle-Symmetric Digraphs

In the preceding sections, we dealt with (independent) even factors in odd-cycle-symmetric digraphs. We viewed that in odd-cycle-symmetric digraphs,

- the non-bipartite matching and matroid intersection problems can be reduced,
- the EFP, WEFP, IEFP and WIEFP can be solved in polynomial time, and
- important theorems of non-bipartite matching and matroid intersection extend.

Now, in the last section of this paper, we discuss how broad the class of odd-cycle-symmetric digraphs is. The following is a characterization of odd-cycle-symmetric digraphs by Z. Király (see [15] for details). A *block* is a strongly connected component whose underlying graph is biconnected. A digraph is said to be *bipartite* if its underlying graph is bipartite.

Theorem 6.1. *A block of an odd-cycle-symmetric digraph is either bipartite or symmetric. That is, in a block of an odd-cycle-symmetric digraph, no odd cycle exists or every edge has the reverse edge.*

A characterization of odd-cycle-symmetric weighted digraphs immediately follows Theorem 6.1.

Theorem 6.2. *Let (G, c) be an odd-cycle-symmetric weighted digraph. Then, $c(C) = c(\bar{C})$ holds for any cycle C in a non-bipartite block of G.*

Remark. By these characterizations, we can test in linear time whether a given (weighted) digraph is odd-cycle-symmetric.

Another characterization of odd-cycle-symmetric digraphs is proposed [16], which is from a view point of discrete convex analysis [20]. We devote the rest of this section to presenting this characterization.

Let V be a finite set. For $x = (x(v)), y = (y(v)) \in \mathbf{R}^V$, define

$$[x, y] = \{z \mid z \in \mathbf{R}^V, \min(x(v), y(v)) \leq z(v) \leq \max(x(v), y(v)), \forall v \in V\}.$$

For $x, y \in \mathbf{Z}^V$, a vector $s \in \mathbf{Z}^V$ is called an *(x, y)-increment* if $s = \chi_u$ or $s = -\chi_u$ for some $u \in V$ and $x + s \in [x, y]$.

Definition 6.3 (Jump systems [1]). A non-empty set $J \subseteq \mathbf{Z}^V$ is said to be a *jump system* if it satisfies the following exchange axiom:

For any $x, y \in J$ and for any (x, y)-increment s with $x + s \notin J$, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$.

A set $J \subseteq \mathbf{Z}^V$ is a *constant-parity system* if $x(V) - y(V)$ is even for any $x, y \in J$. J.F. Geelen characterized constant-parity jump systems by a stronger exchange axiom (see [21] for details).

Theorem 6.4. *A non-empty set J is a constant-parity jump system if and only if it satisfies the following exchange axiom:*

> **(EXC).** *For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$ and $y - s - t \in J$.*

An *M-concave* (*M-convex*) *function on a constant-parity jump system* of [21] is a quantitative extension of a jump system.

Definition 6.5 (M-concave functions on a constant-parity jump system [21])**.** For $J \subseteq \mathbf{Z}^V$, we call $f : J \to \mathbf{R}$ an *M-concave function on a constant-parity jump system* if it satisfies the following:

> **(M-EXC).** For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$, $y - s - t \in J$, and $f(x) + f(y) \leq f(x + s + t) + f(y - s - t)$.

Observe that (M-EXC) implies that J satisfies (EXC), i.e., J is a constant-parity jump system.

A jump system and an M-concave function on a constant-parity jump system appear in many combinatorial optimization problems which can be solved efficiently. For instance, in undirected graphs, it is known that the degree sequences of all subgraphs and those of all matchings are jump systems. Formally, for an undirected graph G with vertex set V and edge set E, the degree sequence $d_F \in \mathbf{Z}^E$ of $F \subseteq E$ is defined by

$$d_F(v) = |\{e \mid e \in F,\ e \text{ is incident to } v\}|.$$

Define $J_{\mathrm{SG}}(G) \subseteq \mathbf{Z}^V$ and $J_{\mathrm{M}}(G) \subseteq \{0,1\}^V$ by

$$\begin{aligned} J_{\mathrm{SG}}(G) &= \{d_F \mid F \subseteq E\}, \\ J_{\mathrm{M}}(G) &= \{d_M \mid M \text{ is a matching in } G\}. \end{aligned}$$

Then, both $J_{\mathrm{SG}}(G)$ and $J_{\mathrm{M}}(G)$ are constant-parity jump systems, provided that G has no loops.

In a weighted undirected graph (G, c), M-concave functions on J_{SG} and J_{M} also naturally arise. Define $f_{\mathrm{SG}} : J_{\mathrm{SG}} \to \mathbf{R}$ and $f_{\mathrm{M}} : J_{\mathrm{M}} \to \mathbf{R}$ by

$$\begin{aligned} f_{\mathrm{SG}}(x) &= \max\{c(F) \mid F \subseteq E,\ d_F = x\}, \\ f_{\mathrm{M}}(x) &= \max\{c(M) \mid M \text{ is a matching},\ d_M = x\}. \end{aligned}$$

Then, f_{SG} and f_{M} are M-concave functions on J_{SG} and J_{M}, respectively.

Kobayashi and Takazawa [16] considered whether an analogous statement holds for even factors. To begin with, let us introduce the degree sequence in digraphs. Let $G = (V, E)$ be a digraph with vertex set V and edge set E. Make two copies V^+ and V^- of V. The copy of $v \in V$ in V^+ (resp. in V^-) is denoted by v^+ (resp. v^-).

Definition 6.6 (Degree sequences in digraphs)**.** For a digraph $G = (V, E)$ and its edge set $F \subseteq E$, the *degree sequence* of F is a vector $d_F \in \mathbf{Z}^{V^+ \cup V^-}$ defined by

$$d_F(v^+) = |F \cap \delta^+ v|,\ d_F(v^-) = |F \cap \delta^- v|\ (v \in V).$$

Let $J_{\mathrm{EF}}(G) \subseteq \mathbf{Z}^{V^+ \cup V^-}$ be the set of the degree sequences of all even factors in G. That is,

$$J_{\mathrm{EF}}(G) = \{d_M \mid M \text{ is an even factor in } G\}.$$

By the definition of even factors, one would easily see that $J_{\mathrm{EF}}(G) \subseteq \{0,1\}^{V^+ \cup V^-}$ and $J_{\mathrm{EF}}(G)$ is a constant-parity system. Then, the following theorem holds.

Theorem 6.7 ([16])**.** *The set $J_{\mathrm{EF}}(G)$ is a constant-parity jump system if and only if G is odd-cycle-symmetric.*

Moreover, this relation is extended to a weighted version. For a weighted digraph (G, c), define $f_{\mathrm{EF}} : J_{\mathrm{EF}}(G) \to \mathbf{R}$ by

$$f_{\mathrm{EF}}(x) = \max\left\{ c(M) \;\middle|\; M \text{ is an even factor, } d_M = x \right\}$$

for $x \in J_{\mathrm{EF}}(G)$. As an extension of Theorem 6.7, we have the following theorem.

Theorem 6.8 ([16])**.** *The function f_{EF} is an M-concave function on a constant-parity jump system if and only if (G, c) is an odd-cycle-symmetric weighted digraph.*

Theorems 6.7 and 6.8 exhibit necessary and sufficient conditions for the even factors to have a matroidal structure. These theorems suggest that the assumption of the odd-cycle-symmetry is reasonable and essential in dealing with optimization problems on even factors.

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