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Constructive Characterization Theorems in Combinatorial Optimization

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Abstract

We give a survey on constructive characterization theorems and their applications in various fields of combinatorial optimization: edge- and vertex-connectivity problems, ear decompositions, and rigidity of graphs.

§ 1. Introduction

By constructive characterization we mean the following approach for describing a certain class of graphs, $\mathcal{H}$. A set of operations is given so that performing such an operation on a graph in $\mathcal{H}$ gives another graph in $\mathcal{H}$, and moreover, we can reach every graph in $\mathcal{H}$ by a sequence of such steps starting from a small initial set of basic instances in $\mathcal{H}$. For example, a graph is connected if and only if it can be obtained from a single vertex by adding edges between existing vertices or adding an edge between an old and a new vertex. The most important application of such a characterization is that it may give us a powerful proof technique: when proving a certain property for class $\mathcal{H}$, it is enough to show its validity for the basic instances and that it is preserved by the construction steps. Another kind of application is that a construction sequence itself may give us an NP-proof of the fact that a graph is in $\mathcal{H}$ and moreover it may give rise to a polynomial time algorithm for deciding whether a graph is in $\mathcal{H}$. Let us see two well-known examples for these kinds of applications.

A directed graph is called **strongly connected** if it contains a directed path between any two vertices. An undirected (directed) graph is called **k-edge-connected**,

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if by deleting any $k-1$ edges the remaining graph is still (strongly) connected. Note that 1-edge-connectedness simply means connectedness in the undirected case, and strongly connectedness in the directed case. By different versions of Menger's theorem we know that an undirected (directed) graph is k-edge-connected if and only if there are k edge-disjoint (directed) paths between any two vertices. The well known ear-decomposition gives a constructive characterization for 2-edge-connected graphs:

Proposition 1.1 ([36, Problem 6.28])**.** *An undirected graph is* 2*-edge-connected if and only if it can be built up from a single vertex by iteratively adding a new path whose endpoints are (possibly coincident) existing vertices.*

A natural question is when the edges of an undirected graph G can be oriented to get a strongly connected digraph. A trivial necessary condition is that G should be 2-edge-connected. Using the characterization above it is very easy to prove that this condition is sufficient as well: when adding a path, let us orient all its edges in one direction. In the later part of the paper we will give far reaching extensions of this characterization. At this point, however, let us consider another type of examples.

The topological ordering of acyclic directed graphs can be reformulated as a constructive characterization in the following way:

Proposition 1.2. *A directed graph is acyclic if and only if it can be built up from a single vertex by iteratively adding a new vertex with some edges from existing vertices to the new vertex.*

An undirected graph is called **chordal**, if any cycle of size at least 4 contains a chord, i.e. an edge between two vertices that are not adjacent on the cycle. The constructive characterization is as follows [19]:

Proposition 1.3. *A graph is chordal if and only if it can be built up from a single vertex by iteratively adding a new vertex and connecting it to some old vertices forming a clique.*

The importance of these characterizations is that such a construction is essentially the only known NP-proof of acyclicity and chordalness. Moreover, given a graph, we can give an efficient algorithm for deciding this property relying on the characterization. In the case of chordalness, in each step either we can find and leave out a vertex whose neighbours form a clique, or if no such vertex exists we can conclude that the original graph was not chordal.

At this point, we have to draw a distinction between "good" and "bad" characterizations. This will be explained using the following example due to Jüttner [30]:

Proposition 1.4. *A graph is Hamiltonian if and only if it can be built up from a single vertex by applying a sequence of the following operations: (i) add an edge; (ii) consider a vertex u with $d(u) = 2$ and two neighbours x and y ($x \neq y$). Replace u by two vertices u' and u'' with adjacent edges $u'u''$, xu' and yu''.*

The claim is straightforward: every cycle on n vertices can be built up by iterating step (ii), and by adding edges we can get any Hamiltonian graph this way. Albeit the similarity between Propositions 1.3 and 1.4, one has the impression that the former is a useful, but the latter a fairly useless claim. The reason comes from the algorithmic point of view: as we have seen, Proposition 1.3 gave us a decision algorithm, but the same does not hold in the case of Proposition 1.4. Given a chordal graph, we can reduce it by the reverse of the last step in a building sequence. However, even if we are ensured that a graph is Hamiltonian, there is no efficient way of deciding whether an edge can be deleted without destroying hamiltonicity.

There are graph families defined via a constructive characterization, for which the complexity status of membership testing is yet open: for example, the dual critical graphs. A graph is called **dual-critical** if it can be constructed from a single vertex by adding a new vertex to a graph in each step and connecting it to old vertices by an odd number of edges. It is known that a planar graph is dual-critical if and only if it is factor-critical. For the general case only a randomized algorithm is known based on computing the rank of a matrix with indeterminants [43].

A problem similar to the one concerning hamiltonicity may arise if we allow building sequences of exponential length (containing possibly operations as edge deletions and contractions). For these reasons in the rest of the paper we restrict ourselves to constructive characterizations where there is a polynomial time algorithm for either giving a building sequence or proving that the graph is not in $\mathcal{H}$. Nevertheless we have to remark that there are very important and deep characterizations not satisfying this property. For example, Hajós' construction [23] for non k-colorable graphs, or the Strong Perfect Graph Theorem [4] may be interpreted this way.

The rest of the paper is organized as follows. Proposition 1.1 will be the common starting point for the next three sections. In Section 2 we discuss characterizations for higher edge-connectivity and other types of edge-connectivity requirements. In Section 3 we discuss ear-decompositions from the aspect of the parity of the ears, while in Section 4 we consider vertex-connectivity problems. In Section 5 we survey some applications connected to constructive characterizations in combinatorial rigidity, and finally in Section 6 we list some open questions.

§ 2. Edge-connectivity problems

One might wonder whether a construction similar to Proposition 1.1 might be given for higher edge-connectivity as well. The answer was given by Lovász in 1976:

Theorem 2.1 ([36, Problem 6.52]). *An undirected graph is $2k$-edge-connected if and only if it can be obtained from a single vertex by iteratively applying the following two operations:*

(i) add a new edge (possibly a loop),

(ii) subdivide k existing edges and identify the subdividing vertices.

The ear decomposition in Proposition 1.1 can be easily seen to be equivalent to the case $k = 1$. Later Mader gave a similar characterization for $2k + 1$-edge-connected graphs [38]. As for the $k = 1$ case, Theorem 2.1 immediately implies an orientation theorem, which is the weak version of Nash-Williams' theorem:

Theorem 2.2 ([40]). *An undirected graph has a k-edge-connected orientation if and only if it is $2k$-edge-connected.*

A directed counterpart of this theorem is due to Mader:

Theorem 2.3 ([39]). *A directed graph is k-edge-connected if and only if it can be obtained from a single vertex by iteratively applying the following two operations:*

(i) add a new edge (possibly a loop),

(ii) subdivide k existing edges and identify the subdividing vertices.

In this theorem and in Theorem 2.1 as well, operation (ii) will be called **pinching** k edges with z, where z is the vertex resulting from the identification. By pinching 0 edges we will simply mean the addition of a vertex. Note that using Theorem 2.2, Theorem 2.1 can easily be derived from Theorem 2.3.

In the proof of Theorem 2.3 an intrinsic tool is another deep theorem of Mader on splitting off. In a digraph $G = (V, E)$, the **splitting off** of the edges $e = xz$ and $f = zy$ is the operation of deleting e and f and adding the edge xy. Let $\rho(X) = \rho_E(X)$ and $\delta(X) = \delta_E(X)$ denote the in- and out-degrees of the set X, respectively; let $\rho(z)$ and $\delta(z)$ denote the in- and out degrees of the vertex z. If $\rho(z) = \delta(z)$, a **complete splitting at** z is a sequence of splitting off operations of all edges incident to z and finally removing z. We say that a digraph $G = (U + z, E)$ is k**-edge-connected in** U if there are k-edge-disjoint directed paths between any two vertices in U.

Theorem 2.4 (Mader [38]). *Let $G = (U + z, E)$ be a digraph which is k-edge-connected in U and $\rho(z) = \delta(z)$. Then there exists a complete splitting at z resulting in a k-edge-connected digraph.*

Now we give an outline of the proof. Given a k-edge-connected digraph $G = (V, E)$ if there is an edge uv so that $G - uv$ is still k-edge-connected, then leaving out uv is the inverse of operation (i) and thus the proof finishes by induction. Let us assume now that this is not the case, i.e. the graph is minimally k-edge-connected with respect to edge-inclusion. The theorem follows by combining Theorem 2.4 and a third theorem of Mader:

Theorem 2.5 ([37]). *In a minimally k-edge-connected digraph $G = (V, E)$, there exists a vertex z with $\rho(z) = \delta(z) = k$.*

We briefly outline the proof of this theorem by Frank in [13] as it contains some important ideas which will be used in the later theorems. Let us call a set $X \subseteq V$ **tight** if $\rho(X) = k$. As G is minimally k-edge-connected, every edge in E enters a tight set. It can be shown that given two crossing tight sets X and Y, $X \cap Y$ and $X \cup Y$ are tight as well. Using this lemma one can find a system $\mathcal{L}$ of tight sets so that every edge in E enters a member of $\mathcal{L}$, and $\mathcal{L}$ is cross-free, that is, any two members not containing each other are either disjoint or their union is V. A counting argument shows that $\mathcal{L}$ should contain for some z both $\{z\}$ and $V - \{z\}$ giving $\rho(z) = \delta(z) = k$.

From Theorem 2.3 one may also derive the constructive characterization of rooted k-edge-connected digraphs (see e.g. in [18]). A digraph $G = (V, E)$ is called **rooted k-edge-connected** if for a vertex $r_0 \in V$, there are k-edge-disjoint paths from r_0 to every vertex in $V - r_0$. Clearly, this is equivalent with $\rho(X) \geq k$ for every $X \subseteq V$, $r_0 \notin X$.

Theorem 2.6. *A directed graph is rooted k-edge-connected with a root $r_0 \in V$ if and only if it can be obtained from the single vertex r_0 by iteratively applying the following two operations.*

(i) add a new edge (possibly a loop),

(ii) pinch $0 \leq j \leq k-1$ edges with a vertex z and add $k - j$ new edges with head z.

An easy consequence is the important theorem of Edmonds on disjoint arborescences:

Theorem 2.7 ([7]). *A directed graph $G = (V, E)$ contains k edge disjoint spanning arborescences with root $r_0 \in V$ if and only if it is rooted k-edge-connected with root r_0.*

Similarly to Theorem 2.2, rooted k-edge-connected digraphs also have an undirected counterpart. An undirected graph is called **k-partition-connected**, if for every partition $\mathcal{P}$ of the vertex set, there are at least $k(t-1)$ edges between different classes of $\mathcal{P}$, where $t = |\mathcal{P}|$. Note that this is a property stronger than k-edge-connectivity.

Theorem 2.8 (Frank [12]). *An undirected graph $G = (V, E)$ has a rooted k-edge-connected orientation with a root $r_0 \in V$ if and only if it is k-partition-connected.*

Note that a consequence is that if there is a rooted k-edge-connected orientation for some r_0, then there is also for any other root r_0'. This can also be seen directly: by reversing the orientation of k-edge-disjoint paths from r_0 to r_0'. From this orientation theorem and Edmonds' theorem we easily get the following result of Tutte:

Theorem 2.9 ([49]). *An undirected graph contains k edge-disjoint spanning trees if and only if it is k-partition-connected.*

We can also derive the following characterization from Theorems 2.6 and 2.8:

Theorem 2.10. *An undirected graph is k-partition-connected if and only if it can be obtained from a single vertex by iteratively applying of the following two operations.*

(i) *add a new edge,*

(ii) *pinch $0 \le j \le k-1$ edges with a new vertex z and add $k-j$ new edges incident to z.*

We will see some extensions of this theorem in Section 5. A natural common generalization of k-edge-connectivity and rooted k-edge-connectivity in digraphs is the following. We say that $G = (V, E)$ is **(k, ℓ)-edge-connected** for some integers $0 \le \ell \le k$, if G has a vertex r_0 such that for each vertex $z \ne r_0$, there exist k edge-disjoint paths from r_0 to z and ℓ edge-disjoint paths from z to r_0. Note that (k, k)-edge-connectivity coincides with k-edge-connectivity and $(k, 0)$-edge-connectivity means rooted k-edge-connectivity. By Theorem 2.7 it is equivalent to that there are k edge-disjoint spanning arborescences rooted in r_0 and also ℓ edge-disjoint spanning anti-arborescences (an anti-arborescence is a subgraph in which the re-orientation of all the edges gives an arborescence). Disjointness between arborescences and anti-arborescences is not required.

For undirected graphs a related concept is the following. An undirected graph is called (k, ℓ)-partition connected if for any partition of the vertices into $t \ge 2$ classes, there are at least $k(t-1) + \ell$ edges connecting distinct classes. The link between these concepts is the following generalization of Theorem 2.8 by Frank.

Theorem 2.11 ([12]). *For integers $0 \le \ell \le k$, an undirected graph G has a (k,ℓ)-edge-connected orientation if and only if G is (k,ℓ)-partition connected.*

The following recent theorem gives a common generalization of Theorems 2.3 and 2.6. It was formulated as conjecture by Frank [18]:

Theorem 2.12 ([31]). *For $0 \le \ell \le k-1$, a directed graph is (k,ℓ)-edge-connected with root $r_0 \in V$ if and only if it can be built up from the single vertex r_0 by the following two operations.*

(i) add a new edge,

(ii) for some i with $\ell \le i \le k-1$, pinch i existing edges with a new vertex z, and add $k-i$ new edges entering z and leaving existing vertices.

Using Theorem 2.11, the following is a consequence:

Theorem 2.13. *For $0 \le \ell \le k-1$, an undirected graph is (k,ℓ)-partition-connected if and only if it can be built up from a single vertex by the following two operations.*

(i) add a new edge,

(ii) for some i with $\ell \le i \le k-1$, pinch i existing edges with a new vertex z, and add $k-i$ new edges between z and some existing vertices.

Besides $\ell = 0$ and $\ell = k$, the following special cases of Theorem 2.12 were known beforehand. $\ell = 1$ was shown by Frank and Szegő [18], and the case $\ell = k-1$ was proved by Frank and Király [16]. In this theorem and also in Theorem 2.3, it is straightforward that all graphs constructed by operations (i) and (ii) are (k,ℓ)-edge-connected, the nontrivial part will be the opposite direction. The reverse operation of (i) is removing an edge, thus we may focus our attention to minimally (k,ℓ)-edge-connected digraphs in the sense that removing any edge would destroy (k,ℓ)-edge-connectivity.

In the proof of Theorem 2.3 the splitting off theorem has been used. However, for the cases $\ell = 1$ and $\ell = k-1$ a nontrivial generalization of Mader's theorem is needed, which is due to Frank. We say that the digraph $G = (U+z, E)$ is **(k,ℓ)-edge-connected in** U, if for a root vertex $r_0 \in U$, we require k-edge-disjoint paths between every two vertices in $U - r_0$ and ℓ paths from every vertex in $U - r_0$ to r_0.

Theorem 2.14 ([11]). *Let $G = (U+z, E)$ be a digraph which is (k,ℓ)-edge-connected in U and $\rho(z) = \delta(z)$. Then there exists a complete splitting at z resulting in a (k,ℓ)-edge-connected digraph.*

The characterization for general ℓ is more difficult than that for $\ell = 0$ and $\ell = k$ for the following reason. In the latter cases, it was enough to find a vertex satisfying certain conditions for the in- and outdegrees, and one could always perform the splitting off operations at such a vertex. However, even for $\ell = 1$ or $\ell = k-1$ the conditions for the degree do not suffice and the arguments become slightly more complicated.

Now we sketch the proof for $\ell = k-1$ by Frank and Király [16]. Consider a minimally $(k, k-1)$-edge-connected digraph. To apply the reverse operation of (ii) at a vertex z, a necessary condition is $\rho(z) = k$, $\delta(z) = k-1$. We will call such vertices **special**. If we manage to find a special vertex z and an edge uz such that $G - uz$ is $(k, k-1)$-edge-connected in $U = V - z$, then Theorem 2.14 may be applied in $G' = (U + z, E - uz)$ and it gives a $(k, k-1)$-edge-connected digraph G'' on U. Then we can apply (ii) to G'' to get G by pinching the $k-1$ edges resulted by the splitting off with z and finally adding the edge uz.

However, not every special vertex z admits an edge uz as above (it is already non-trivial to prove the existence of a special vertex). The proof uses an indirect argument: assume that every edge in $xy \in E$ satisfies one of the following. Either y is not special and $G - xy$ is not $(k, k-1)$-edge-connected or y is special, but $G - xy$ is not $(k, k-1)$-edge-connected in $V - y$. One can define a notion of tight sets so that each edge will be blocked by a tight set. Then the uncrossing method can be used for these tight sets to derive a final contradiction.

The proof of the general case in Theorem 2.12 needs to handle certain difficulties which did not occur for $\ell = k-1$. Starting from a minimally (k, ℓ)-edge-connected digraph, we call a vertex z **special** if according to its degrees it might be the result of operation (ii), that is, if $\ell \le \delta(z) \le k-1$ and $\rho(z) = k$. We say that a subset F of edges entering a special vertex z is **locally admissible** in z, if $G - F$ is (k, ℓ)-edge-connected in $V - z$. F will be called **sufficient** at z if $|F| = k - \delta(z)$. If we can find a sufficient locally admissible F, then Theorem 2.14 can be applied for $G - F$ and z and the proof finishes as for $\ell = k-1$.

Thus our aim is to find a special vertex z and a sufficient locally admissible set F at z. It is easy to characterize the maximal size of a locally admissible set for a given special z which can be smaller than $k - \delta(z)$. In fact, it turns out that the locally admissible sets in z form a matroid, which allows us to find a sufficient locally admissible set with an efficient algorithm – once it is ensured that one exists. However, in proving the existence of a sufficient locally admissible set the locally admissible sets belonging to different special vertices should be handled together, resulting in a very technical and complicated argument.

The importance of the $(k, k-1)$-edge-connected case investigated by Frank and Király is due to the following nice application. An important open question is the following: given an undirected graph $G = (V, E)$ and a subset of vertices $T \subseteq V$, when does G admit a strongly connected orientation so that the vertices with odd in-degree are exactly those in T. A trivial necessary condition is that $|T|+|E|$ should be even, but no necessary and sufficient condition is known. If we ask that when does this property hold for every subset T satisfying $|T| + |E|$ even, the question can be answered not only for strongly connectedness but for higher connectivity as well:

Theorem 2.15 (Frank, Király [16]). *For an undirected graph $G = (V, E)$, the following are equivalent:*

(1) G has a k-edge-connected T-odd orientation for every $T \subseteq V$ with $|T| + |E|$ even.

(2) G is $(k + 1, k)$-partition connected.

(3) G can be built up from a single vertex by a sequence of (i) adding new edges, and (ii) pinching k existing edges with a vertex z and add one more edge from an existing vertex to z.

A particularly nice thing about this theorem is that at first sight it is neither clear if property (1) is in NP, nor if it is in co-NP. Property (2) gives a co-NP certificate as given a deficient partition it is easy to find a T not admitting a T-odd orientation, and (3) gives an NP-certificate, since using the construction sequence it is easy to give a T odd orientation for any T with $|T| + |E|$ odd.

This application has motivated the investigation of $(k, k-1)$-partition-connected graphs. No generalization of this theorem is known for general (k, ℓ)-partition-connected graphs.

§ 3. Ear-decompositions with parity

Let us turn back to Proposition 1.1. It is often called ear-decomposition: by ears we mean the paths added during the construction. A remarkable feature is that we have a great liberty in choosing a decomposition sequence. In particular, one can show that an arbitrary 2-edge-connected subgraph of the 2-edge-connected $G = (V, E)$ can be extended to an ear-decomposition of G. (This is an important difference with higher connectivity and vertex connectivity, where only a very restricted class of construction sequences can be extended to obtain a certain graph; in some cases there is only a unique sequence.)

It is also straightforward to see that the number of ears added in any decomposition is $|E| - |V| + 1$. The number of even length ears may vary in the different decompositions: one might ask the minimum number $\varphi(G)$ of even ears in a decomposition. The motivation for this is Lovász' following theorem. A graph $G = (V, E)$ is called **factor-critical** if for arbitrary $v \in V$, $G - v$ has a perfect matching (for example, an odd cycle is factor-critical).

Theorem 3.1 ([35]). *A graph G is factor-critical if and only if it has an odd ear-decomposition, that is, an ear decomposition consisting of only odd ears, or equivalently, $\varphi(G) = 0$.*

Graphs with $\varphi(G) = 1$ were described by Hetyei [26] as the matching-covered graphs, i.e. graphs for which every edge is contained in a perfect matching. A theorem describing the exact value of $\varphi(G)$ was given by Frank, connecting this quantity with T-joins. Given a subset T of V with even cardinality, a T**-join** is a subgraph of G so that the odd-degree vertices are exactly those in T. Given T, we might be interested in the minimum size of a T-join. Furthermore, let $\mu(G)$ denote the minimum size T-join over all even subsets T in the graph G. Frank showed that this is equal to the maximum size of a subset J for which $|C \cap J| \le |C|/2$ holds for every circuit C of G.

Theorem 3.2 ([14]). *In a 2-edge-connected graph G, $\varphi(G) = 2\mu(G) - |V| + 1$.*

§ 4. Vertex-connectivity problems

We say that an undirected (directed) graph is k**-vertex-connected**, if by deleting any $k - 1$ vertices the remaining graph is still (strongly) connected, and the graph has at least $k + 1$ vertices. Analogous versions of Menger's theorem say that k-vertex-connectivity is equivalent with having k internally vertex disjoint (directed) paths between any two vertices. One might wonder if for vertex connectivity we also have as nice characterizations as for edge connectivity. A promising sign is that for 2-vertex-connectivity we have a well-known analogue of Proposition 1.1:

Proposition 4.1. *An undirected graph is 2-vertex-connected if and only if it can be built up from a circuit by iteratively adding new paths whose endpoints are distinct old vertices.*

However, vertex connectivity in general seems to be significantly more difficult to handle and no characterization analogous to Theorem 2.1 is known for general k. In fact, no nice characterization is known even for $k = 4$. On the other hand, for $k = 3$ we have four distinct characterizations with applications for planar graphs and independent spanning trees.

Theorem 4.2 (Tutte [50]). *A graph is 3-vertex-connected if and only if it can be built up from the complete graph K_4 by the following two steps:*

(i) *add a new edge parallel with an existing edge, and*

(ii) *take a vertex* z *with* $d(z) \geq 4$*, and replace it by an edge* $z'z''$*. Partition the edges incident to* z *into two groups both of size at least* 2*. Replace every edge* xz *in the first group by* xz' *and every edge* yz *in the second group by* yz''*.*

Let us call step (ii) the **vertex splitting**. A particularly nice application of this theorem is that it leads to a simple proof of Kuratowski's theorem.

Theorem 4.3 ([33]). *A graph* G *is planar if and only if it contains no subdivided* K_5 *or* $K_{3,3}$*.*

The following proof was given by Thomassen [48]. The first important observation is that it is enough to prove the statement for 3-vertex-connected graphs as a minimal possible counterexample can be shown to be 3-vertex-connected. We start from a planar embedding of the graph K_4 and modify this embedding in each step according to the building sequence given by the theorem. One can show that in step (ii) the edges incident to the vertex z' are consecutive at z in the planar drawing, otherwise a K_5 or $K_{3,3}$ minor could be found. This argument leads to an even stronger statement: a simple planar graph has a planar embedding with straight-line edges and convex faces.

A problem with Theorem 4.2 is that it might happen that a construction sequence of a 3-connected simple graph may contain non-simple graphs in the intermediate steps. Wheels of size at least 5 are such an example, where a wheel is a cycle plus an extra vertex connected to all vertices of the cycle. By the following stronger theorem, starting from the wheels we can achieve any simple 3-connected graph by preserving simplicity:

Theorem 4.4 (Tutte [50]). *A simple graph is 3-vertex-connected if and only if it can be built up from a wheel by the following operations:*

(i) *add a new edge between two non-adjacent vertices, and*

(ii) *perform a vertex splitting.*

We also present another kind of characterization by Barnette and Grünbaum [1] and Titov [47]:

Theorem 4.5. *A graph is 3-vertex-connected if and only if it can be built up from* K_4 *by the following operations:*

(i) *add a new edge,*

(ii) *pick two non-parallel edges, subdivide them and connect the subdividing vertices by a new edge,*

(iii) subdivide an edge and connect the subdividing vertex to a vertex different from the endpoints of the edge picked.

Using this theorem Barnette and Grünbaum have given a short proof of a theorem of Steinitz stating that the 1-skeletons of the 3-dimensional polytopes are exactly the 3-connected planar graphs.

Now we present a different kind of characterization, which will be a natural generalization of Proposition 4.1 by giving an ear-decomposition for 3-vertex-connected graphs. Before presenting this, we show an application of Proposition 4.1. We say that m, not necessarily edge disjoint spanning trees are **independent** in the graph $G=(V,E)$ if for some specified root $s\in V$, for every $v\in V-s$ the m paths from s to v in the spanning trees are internally vertex disjoint. First we show how Proposition 4.1 allows us to prove the existence of 2 independent trees in a 2-vertex-connected graph, then we present the ear decompositions for 3-vertex-connected graphs which will imply the existence of 3 independent trees. The first is due to Itai and Rodeh [27] while the second to Cheriyan and Maheshwari [3].

We say that $g:V\to\{1,\dots,n\}$ with $n=|V|$ is an **s-t-numbering** of the 2-vertex connected graph $G=(V,E)$ for some edge $st\in E$, if $g(s)=1$, $g(t)=n$ and furthermore each vertex $v\in V-\{s,t\}$ has neighbours u,w with $g(u)<g(v)$ and $g(w)>g(v)$. Given an $s-t$ numbering, the existence of two independent trees with root s is straightforward. For each $v\in V-\{s,t\}$ let us include the edge uv in the first and vw in the second spanning tree with u,w as above, and let us also add the edge st to the second tree. Given an ear decomposition of G as in Proposition 4.1 with the edge st being on the first cycle, it is easy to construct an s-t-numbering so that the neighbours v,w of u with smaller and larger g values will be its neighbours on the ear first containing u. Note that for an arbitrary edge st we can find an ear decomposition with the first cycle containing st, thus this way we can find two independent trees with arbitrary root.

For 3-vertex-connected graphs, Cheriyan and Maheshwari show the existence of a special ear decomposition called nonseparating ear decomposition. For the ear decomposition $P_0\cup P_1\cup\ldots\cup P_k$ of $G=(V,E)$, let V_i denote the vertex set of the first i ears and let $\bar{G}_i=G[V-V_i]$. We say that an ear is trivial if it is a single edge, otherwise it is nontrivial. For an edge st and a vertex u, we say that this is a **nonseparating ear decomposition** through the edge st avoiding vertex u if (i) u is a neighbour of t, and $d(u),d(t)\geq 3$; (ii) P_0 contains st and u is the only internal vertex of the last nontrivial ear; and (iii) for each $0\leq i\leq m$, $\bar{G}_i$ is connected and each internal vertex of P_i has a neighbour in $\bar{G}_i$.

Theorem 4.6 (Cheriyan, Maheshwari [3])**.** *$G=(V,E)$ is a 3-vertex-connected graph if and only if it has a nonseparating ear decomposition through st avoiding u for some st and u.*

The proof uses Theorem 4.4. Now we show how this theorem can be applied for proving the existence of 3 independent trees. Consider an st-numbering g as above, and let $p(x)$ denote the index of the first ear through x. Every $v \in V - \{s, t, u\}$ can be shown to have three neighbours x, y and z with $p(v) < p(x)$, $p(v) \geq p(y)$ and $g(v) > g(y)$; and $p(v) \geq p(z)$, $g(v) < g(z)$. One may construct the three trees using these three neighbours as parents.

Finally we mention that Kriesell [32] has given a characterization of 3-vertex-connected triangle-free graphs.

§ 5. Rigidity

Rigidity theory analyzes the possible deformations of a bar-and-joint framework in the plane or in higher dimensions. Some of these geometric questions lead to purely combinatorial problems where constructive characterization proves to be an efficient tool. In this section, we present some of these applications for two- and three-dimensional rigidity. About basic results in rigidity see for example [21] and [51].

§ 5.1. Generic rigidity in two dimensions

By a two dimensional **bar-and-joint framework** we mean a pair (G, p), where p is a mapping of the vertices of a graph $G = (V, E)$ into the plane. The edges of the graph are imagined to be rigid bars, which can rotate around the vertices. How can one describe the deformations of such a setup?

More formally, suppose we have a bar in the plane with coordinate vectors of the endpoints $p(u)$ and $p(v)$. If both u and v start to move in the direction of vectors m_u and m_v, respectively, then $(p(u)-p(v))^T(m(u)-m(v)) = 0$ holds. By an **infinitesimal motion** of the framework we mean a function $m : V \to \mathbb{R}^2$ so that the previous equality holds for every edge uv of G.

The **rigidity matrix** of a framework is a matrix M of size $|E| \times 2|V|$, where for each edge $uv \in E$, in the row corresponding to uv, the entries in the two columns corresponding to the vertices u and v contain the coordinates of $p(u) - p(v)$ and $p(v) - p(u)$, respectively, and the remaining entries are zeros. It is easy to see that m is an infinitesimal motion if and only if m is in the nullspace of M. The set of trivial infinitesimal motions form a 3-dimensional subspace of the nullspace, hence the rank of the matrix, $r(M)$ is at most $2|V| - 3$.

We call a mapping of a graph **generic**, if the coordinates of the vertices are algebraically independent over $\mathbb{Q}$. It can be shown that whenever a graph has a framework with $r(M) = 2|V| - 3$, then this holds for every generic mapping as well. In this case, the graph G is called **generically rigid**.

Which graphs are minimal to this property in the sense that none of the edges can be deleted so that the graph remains generically rigid? Clearly, $|E| = 2|V| - 3$ must hold. It is also easy to see that a similar equation must also hold for every induced subgraph: $i_G(X) \leq 2|X| - 3$ for every $X \subseteq V, |X| \geq 2$, where $i_G(X)$ is the number of edges spanned by X. Surprisingly, this is enough: the following theorem summarizes the results of Henneberg [25], Laman [34] and Tay and Whiteley [46].

Theorem 5.1. *For a graph G, the following are equivalent:*

(1) G is minimally generically rigid;

(2) $|E| = 2|V| - 3$ and $i_G(X) \leq 2|X| - 3$ for every $X \subseteq V, |X| \geq 2$; and

(3) G can be built up from a single edge by iteratively applying the following operations:

- *(i) add a new vertex z and connect it to two different existing vertices x and y,*
- *(ii) subdivide an edge uv with a new vertex z and add a new edge between z and an existing vertex w different from u and v.*

Such graphs are called **Laman graphs**. Operations (i) and (ii) were introduced by Henneberg back in 1911, thus (3) is called the Henneberg construction of the graph. Sometimes step (i) is referred as 0-extension and step (ii) as 1-extension.

§ 5.2. Planar Laman graphs

Besides the Henneberg construction, planar Laman graphs have another constructive characterization, which uses a single operation, different from those seen before. Let xy be an edge of a planar graph and $E_1, E_2, \{xy\}$ a partition of the edges incident to x such that the sets are consecutive in the planar embedding. Substitute x with the vertices x_1 and x_2, so that the edges in E_i are adjacent to x_i, $i = 1, 2$. Replace the edge xy by x_1y, x_2y and x_1x_2. This operation is called **planar vertex splitting**.

Theorem 5.2 (Fekete, Jordán, Whiteley [9]). *A planar graph is Laman if and only if it can be built up from a single edge by a sequence of planar vertex splittings.*

The inverse operation of vertex splitting is the contraction of two vertices in a triangle. In contrast with the Henneberg construction, where the inverse operations can be applied to any vertex with degree two or three, here not every triangle has two contractable vertices. Still, this construction can be very useful. For instance, it leads to another proof of the following characterization of planar Laman graphs.

A **pseudo-triangle** is a simple planar polygon with exactly three convex angles. A **pseudo-triangulation** is a realization of a planar graph with straight edges, pseudotriangle inner faces and a convex outer face. If all vertices have a concave angle, the pseudo-triangulation is **pointed**. Streinu proved that if a planar graph has a pointed pseudo-triangulation, then it is Laman [42]. The reverse of this theorem was proved by Haas et al. [22]. These theorems together give the following characterization.

Theorem 5.3. *A planar graph G is Laman if and only if it has a pointed pseudo-triangulation.*

The vertex splitting operation can also be used to build up other families of graphs, closely related to the planar Laman graphs. A **Laman-plus-one-graph** is a Laman graph with an extra edge.

Theorem 5.4 ([9]). *The planar Laman-plus-one-graphs are exactly the graphs generated from K_4 by planar vertex splittings.*

Interestingly, the planar dual of a vertex splitting is almost exactly the same as operation (ii) in the Henneberg construction, except that the new edges may be incident to the same vertices. Which graphs can one get using this weaker operation? Not surprisingly, the answer is the planar duals of planar Laman graphs. A graph is called **co-Laman**, if $|E| = 2|V| - 1$ and $i_G(X) \leq 2|X| - 2$ for every $\emptyset \neq X \subseteq V$. Fekete, Jordán and Whiteley also proved the following theorem:

Theorem 5.5 ([9]). *A graph is planar co-Laman if and only if it can be consructed from a loop by iterating the following operation: subdivide an edge uv with a new vertex z and add an edge between z and an existing vertex (possibly u or v).*

§ 5.3. Sparse Graphs

Minimally generically rigid graphs in two dimensions turned out to be a class of graphs with linear number of edges 'everywhere' in the graph. This property can be generalized in the following sense. A graph $G = (V, E)$ is called **$[k, \ell]$-sparse**, if $i_G(X) \leq k|X| - \ell$ for every $X \subseteq V, |X| \geq \ell/k$. If we have equality for V that is, $|E| = k|V| - \ell$ holds then G is called a **$[k, \ell]$-graph**. In particular, the Laman graphs are the $[2, 3]$-graphs. For another well-known case, let us consider the following theorem of Nash-Williams on covering by spanning trees, which can be regarded as a counterpart of Theorem 2.9 by Tutte on packing.

Theorem 5.6 ([41]). *An undirected graph $G = (V, E)$ can be covered by k forests if and only if it is $[k, k]$-sparse. In particular, E can be partitioned to k spanning trees if and only if G is a $[k, k]$-graph.*

Characterizations for $[k, k]$-sparse graphs and $[k, k]$-graphs were first investigated by Nash-Williams and formulated later explicitly by Tay.

Theorem 5.7 ([46]). *An undirected graph G is $[k, k]$-sparse if and only if it can be built up from a single vertex by iteratively applying the following operations:*

(i) add a new vertex z with at most k incident edges;

(ii) pinch $0 \le j \le k-1$ edges with a new vertex z and add $k-j$ new edges incident to z.

G is a $[k, k]$-graph if and only if it can be obtained from a single vertex by iteratively applying only step (ii).

Note that the addition of a vertex with exactly k edges falls under both (i) and (ii). Step (ii) here is identical with step (ii) in Theorem 2.10. In fact, Theorem 5.7 can be easily derived from Theorem 2.10. First of all, it is clear that if a graph can be covered by k forests or partitioned to k spanning trees, then the construction steps preserve this property. For the other direction, a counting argument shows the existence of a vertex z of degree at most $2k-1$. If $d(z) \le k$ then the inverse of (i) can be performed. Otherwise, we may add edges to G not incident to z to make G k-partition-connected. It can be shown that in Theorem 2.10, the inverse of (ii) can be performed at any vertex of degree at most $2k-1$. (This is related to the fact that in Theorem 2.3, the inverse of (ii) can be performed at any vertex of both in- and out-degree k). We are done by performing the same step in G.

We can see that this characterization is closely related to the one we had in Section 2, so one might wonder if a general characterization is known for $[k, \ell]$-sparse graphs. The answer is unfortunately no. In the following, we survey the known special cases. It turns out that already the case $\ell = k+1$ solved by Frank and Szegő [18] is much harder than the case $k = \ell$. Its importance is that it generalizes the Henneberg-Laman characterization for $[2, 3]$-graphs.

Theorem 5.8. *A graph $G = (V, E)$ is $[k, k+1]$-sparse if and only if it can be built up from a single vertex by using the following operations:*

(i) add a new vertex z with at most k edges incident to it, so that no k parallel edges arise,

(ii) pinch $0 \le j \le k-1$ edges with a new vertex z and add $k-j$ new edges incident to z, so that the resulting graph does not contain k parallel edges.

Let K_2^{k-1} denote the graph on two vertices with $k-1$ parallel edges ($k \ge 2$). G is a $[k, k+1]$-graph if and only if it can be obtained from K_2^{k-1} by using only operation (ii).

To prove Theorem 5.7 it sufficed to find a vertex with the right degree (at most $2k-1$) and at such a vertex we can always perform the inverse of step (ii). This is not true in the case of Theorem 5.8: there are examples for all $k \geq 3$ showing that an arbitrary vertex with degree at most $2k-1$ cannot be considered as the result of operation (ii). Recall that we have seen an analogous situation at Theorem 2.12. The main difference is that in the proof of Theorem 2.12 it was easy to characterize when a vertex admits the inverse of step (ii), and the hard part was to prove the existence of such a vertex. On the contrary, here the more difficult task is to characterize vertices where the inverse of (ii) may be preformed, and then an easier global counting argument shows that such a vertex really exists.

One might wonder if this theorem may be generalized further, for arbitrary ℓ. Fekete and Szegő have given the characterization for $0 \leq \ell \leq k$. In a graph $G=(V,E)$, for $i,j,k \geq 0$, $0 \leq i+j \leq k$ let $K(k,i,j)$ denote the following operation: pinch j edges of the graph with a new vertex z, add i loops incident to z and $k-i-j$ edges between z and old vertices. Let P_ℓ denote the graph with a single vertex and ℓ loops.

Theorem 5.9 ([10]). *For $1 \leq \ell \leq k$ a graph is a $[k,\ell]$-graph if and only if it can be obtained from $P_{k-\ell}$ by iteratively applying $K(i,j,k)$ operations with $i+j \leq k-1$, $i,j \geq 0$, $i \leq k-\ell$.*

If $\ell = 0$, the graph is a $[k,0]$-graph if and only if it can be built up from P_k by iteratively applying $K(k,i,j)$ operations with $i+j \leq k$, $i,j \geq 0$.

For this theorem Fekete [8] gave a simpler proof using Theorem 2.4 and the following orientation theorem for $[k,\ell]$-graphs.

Theorem 5.10 ([8]). *For $0 \leq \ell \leq k$ a graph is a $[k,\ell]$-graph if and only if for some vertex r_0 there is an r_0-rooted ℓ-edge-connected orientation so that every vertex in $V-r_0$ has in-degree k, and r_0 has in-degree $k-\ell$.*

A conjecture generalizing Theorem 5.8 for $\ell > k$ is the following:

Conjecture 5.11 ([45], Conjecture 8.2). *For $k < \ell \leq \frac{3k}{2}$, a graph $G=(V,E)$ is $[k,\ell]$-sparse if and only if it can be built up from a single vertex by using the following operations:*

(i) add a vertex z with at most k edges incident to it so that no $2k-\ell+1$ parallel edges arise,

(ii) pinch $1 \leq j \leq k$ edges with a new vertex z and add $k-j$ edges to z, so that the resulting graph does not contain $2k-\ell+1$ parallel edges.

Although most of the elements of the proof of Theorem 5.8 carry over for this conjecture, it turned out to be false for $\frac{4k+2}{3}k \leq l \leq \frac{3}{2}k$ ([45], Section 8.4). However, it is still open for $k < l < \frac{4k+2}{3}$.

§ 5.4. 3-dimensional rigidity

In Section 5.1 we presented some basic definitions for rigidity in two dimensions. All of these notions can be easily extended to higher dimensions. For example, in d dimension the size of the rigidity matrix is $|E| \times d|V|$ and a graph is generically rigid if it has a framework for which $r(M) = d|V| - \binom{d+1}{2}$. This gives again a necessary condition for a graph to be minimally rigid in d-dimensions: $i_G(X) \leq d|V| - \binom{d+1}{2}$ for all $|X| \geq d+2$, and $|E| = d|V| - \binom{d+1}{2}$. However, this is not the same as G being $[d, \frac{d+1}{2}]$-sparse since the condition has to hold only for large subsets $|X|$. We have seen that in two dimensions these conditions were sufficient, however, this fails already for $d = 3$. A three-dimensional counterexample is the so-called double-banana graph (see [51]). We call a graph **Laman in 3-dimensions**, if $E = 3|V| - 6$ and $i_G(X) \leq 3|X| - 6$ for every $X \subseteq V$, $|X| \geq 3$. The double-banana graph is Laman but not generically rigid. Still, there are graph families for which being Laman is sufficient. For instance, a triangulated planar graph is Laman and Gluck [20] has proved that it is generically rigid.

What can we say about generically rigid graphs in general? There is a general constructive characterization theorem for all minimally generically rigid graphs. The characterization uses three operations. A (3-dimensional) **0-extension** is the addition of a vertex and joining it to three different existing vertices. A **1-extension** is the deletion of an edge, joining its endpoints and two further vertices to a new vertex z. Finally, a **2-extension** consists of deleting two edges, joining their endpoints and one or two further vertices to a new vertex z.

Theorem 5.12 ([46]). *Minimally generically rigid graphs in $\mathbb{R}$ can be constructed from a triangle using $0, 1$ and 2-extensions.*

The first two operations are known to preserve rigidity, however, for 2-extension it has not yet been proved nor disproved. One has to remark that even if it were true, Theorem 5.12 would not necessarily give a good characterization in the sense we discussed in the Introduction: given a minimally generically rigid graph, the complexity status of finding a construction sequence is yet open.

§ 5.5. Global rigidity

By the definition of generic rigidity, two triangles with an edge in common are generically rigid in 2 dimensions. However, the same length of bars can be drawn into the plane by 'folding' the triangles at their common edge. The aim of the following definition is to exclude this kind of deformation. For a graph G, two frameworks (G, p) and (G, q) are **equivalent** if $|p(u) - p(v)| = |q(u) - q(v)|$ holds for every edge uv of G. If the previous equality holds for every pair of vertices, we call the frameworks **congruent**.

A framework (G, p) is **globally rigid** if it is congruent with any equivalent framework. We would like to characterize when a framework is globally rigid. In case of generic frameworks, the following theorem gives necessary conditions in any dimensions. A graph is called **redundantly rigid**, if it remains generically rigid after deleting any of its edges.

Theorem 5.13 (Hendrickson [24]). *If a generic framework (G, p) is globally rigid in d-dimensions, then either G is the complete graph K_i for some $1 \leq i \leq d+1$ or G is $(d+1)$-connected and redundantly rigid.*

Hendrickson conjectured that these conditions are sufficient. For $d > 2$ it turned out to be false [5], however, it is true for $d = 2$.

Theorem 5.14 (Jackson, Jordán [28]). *For a graph $G = (V, E), |V| \geq 4$, a generic framework (G, p) is globally rigid in two dimensions if and only if G is 3-connected and redundantly rigid.*

The proof uses the following theorem:

Theorem 5.15 ([28]). *A graph is 3-connected and redundantly rigid in $\mathbb{R}$ if and only if it can be built up from K_4 by adding edges and applying 1-extensions.*

For the other direction, Connelly [6] proved that if a graph G can be built up from K_4 by adding edges to the graph and applying 1-extensions, then every generic framework of G is globally rigid. Note that it also follows that global rigidity is a generic property.

§ 6. Open problems

In Theorem 2.12, the proof consists of finding a vertex z in a minimally (k, ℓ)-edge-connected digraph so that the reverse of operation (ii) can be performed at z. For $k = \ell$ it is known that there are in fact at least two such vertices (see [37]).

Problem 6.1. *Prove or disprove that two (or more) such vertices exist for other values of ℓ.*

Problem 6.2 (Király). *(k, ℓ)-edge-connectivity may be defined the same way for fractional values of k and ℓ. Find a constructive characterization of (k, ℓ)-edge-connected graphs for fractional k and ℓ, for example, if k and ℓ are both half-integers.*

A possible extension of Theorem 2.12 can be the following.

Problem 6.3 (Frank). *Given integers j and k with $j < k$, find a constructive characterization of the graphs $G = (V, E)$ with some vertex $r_1 \in V$ so that $\delta(r_1) = j$ and $\rho(X) \geq k$ for every $\emptyset \neq X \subsetneq V - r_1$.*

Note that if all edges leaving r_1 enter the same vertex $r_0 \neq r_1$, then $G - r_1$ is $(k, k-j)$-edge-connected with root r_0.

Problem 6.4. *Prove or disprove Conjecture 5.11 for $k + 1 < \ell < \frac{4k+2}{3}$.*

Problem 6.5 (Frank). *Find a constructive characterization for strongly connected or in general, k-edge-connected directed hypergraphs, or r-uniform directed hypergraphs.*

See [17] and [2] for some results concerning directed hypergraphs. For the next problem, in a digraph by a directed cut we mean the set of edges entering a vertex set Z so that the out-degree of Z is zero.

Problem 6.6 (Frank [15]). *Find a constructive characterization of digraphs for which every directed cut is of size at least k.*

The motivation of finding such characterization is that it might be useful for proving Woodall's conjecture:

Conjecture 6.7 (Woodall). *If every directed cut of a digraph has at least k edges, then the edge-set can be partitioned into k parts so that each part has at least one edge from every directed cut.*

In the above problems, we were interested in finding a characterization of a given class of graphs. An opposite type of question is when a class of graphs is given by a characterization, but no other kind of description is known. The following is a question of this type. (Recall the definition of dual critical graphs in the Introduction.)

Problem 6.8. *Design an algorithm to decide if a graph is dual-critical.*

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