A Note On the Non-Equilibrium Formula of Artesian Water

Kyozo KIKKAWA*

The non-equilibrium formula on the artesian aquifer introduced by Theis has been displayed its valuable power on many problems involved in practical hydrology. Its theoretical foundations were also developed by Jacob,[1] Werner[2] and Nomitu.[3] So far as the author understand, it is based on the combination of the two fundamental equations. They are

$$q = -2\pi r T\frac{\partial \zeta}{\partial r} \tag{1}$$

$$\frac{\partial q}{\partial x} = -2\pi r S\frac{\partial \zeta}{\partial t} \tag{2}$$

where q is the amount of flow per unit time through any cross section of unit width of the aquifer, ζ is the drawdown of the piezometric level, t is the time since pumping began, r is the distance from the center of the pumped well and T and S are the coefficients of transmissibility and storage respectively. T and S are analysed as follows.

$$T = \mathrm{nc\,D} \tag{3}$$

$$S = \mathrm{nD}\rho\, g/E \tag{4}$$

where c is the transmission constant, n is the porosity, D is the average thickness of the aquifer, ρ is the density of water and g is the acceleration of gravity. E is defined as the modulus of compressibility, expressed by Jacob as $\left[\frac{1}{Ew} + \frac{b}{nE_s} + \frac{c}{E_c} + \frac{n_g}{np}\right]$,

In order to combine the equations (1) and (2), T and S have been assumed to be constant with respect to the change of time.

But, when it is considered that the equation (2) is based on the elastic volume change in accordance with the pressure change of artesian water, still more confirmation of the abovementioned assumption is desirable during progressive drawdowns are continued.

When, n and D in equations (3) and (4) are understood to be the values before pumping, which are assumed to be uniform being correspondent with the uniform artesian pressure, vaues of T and S at any time and any

* Geophysical Institute, Kyōto University

position since pumping began are expressed as the following T' and S'.

$$T' = c\ (n\mathrm{D} - S\zeta) = cnD\ (1 - \frac{\rho g}{E}\zeta)$$

$$S' = \frac{\rho g}{E}(nD - S\zeta) = \frac{nD\rho g}{E}(1 - \frac{\rho g}{E}\zeta)$$

putting $\frac{\rho g}{E}$ as α, we can gain

$$T' = T\ (1 - \alpha\zeta)$$

$$S' = S\ (1 - \alpha\zeta)$$

Substituting T' and S' respectively for T and S in equations (1) and (2),

$$\begin{cases} q = -2\pi\gamma T(1-\alpha\zeta)\dfrac{\partial\zeta}{\partial\gamma} = \dfrac{\pi T\gamma}{\alpha}\ \dfrac{\partial(1-\alpha\zeta)^2}{\partial\gamma} & (1') \\ \dfrac{\partial q}{\partial\gamma} = -2\pi\gamma S(1-\alpha\zeta)\dfrac{\partial\zeta}{\partial t} = \dfrac{\pi S\gamma}{\alpha}\ \dfrac{\partial(1-\alpha\zeta)^2}{\partial t} & (2') \end{cases}$$

$(1-\alpha\zeta)^2$ are put as Y. Then, by combining (1') and (2')

$$\frac{\partial^2 Y}{\partial\gamma^2} + \frac{1}{\gamma}\ \frac{\partial Y}{\partial\gamma} = \frac{S}{T}\ \frac{\partial Y}{\partial t} \tag{5}$$

$$\left.\begin{array}{lll} \text{Initial condition,} & t=0\,; & y=0 \quad \therefore\ Y=1 \\ \text{Boundary condition,} & t>0,\ r\to\infty\,; & y=0 \quad \therefore\ Y=1 \\ & r=r_0\,; & q=Q=\dfrac{\pi T}{\alpha}\left|\gamma\dfrac{\partial Y}{\partial\gamma}\right|_{\gamma=\gamma_0} \end{array}\right\} \tag{6}$$

where r_0 equals radius of well and Q is the pumping-rate being constant.

We can obtain the solution of equation (5) under conditions (6). That is

$$Y - 1 = -\frac{Q\alpha}{2\pi T}\int_u^\infty \frac{e^{-u}}{u}du$$

$$u = \frac{\gamma^2 S}{4Tt}$$

Then, $\quad \zeta - \alpha\zeta^2 = -Ei\ (-\gamma^2 S/4Tt) = W(r^2 S/4Tt)$

W is the so-called well-function.

The deviation from Theis' result is found to be $-\alpha\zeta^2$ in the left side. If the error of 10cm is caused by this difference in the drawdown of 10m., α must be larger than $10^{-3}\,\mathrm{m}^{-1}$. The values of α are known to be quite various but not found to exceed $10^{-8}\mathrm{m}^{-1}$. Therefore, it may be probable to conclude that this difference may be negligiblly small, in almost all cases, compared with other errors given in practical experiences.

References.

1) C. E. Jacob ; Trans. Amer. Geophys. Union. p. 783, 1941, Ⅲ.
2) P. Wilh Werner ; Ditto. p. 687, Vol. 27, Ⅴ 1946.
3) T. Nomitu and K. Yamasita ; "Tikyuturi" Vol. 7, nr. 1, p. 21. 1943. (in Japanese)

被圧地下水の非定常流動式に関する一考察（抄錄）

吉　川　恭　三

Theis の式として知られている被圧地下水層内の非定常流動式は被圧層の厚さが地下水圧の変化に伴つて弾性的に変化するという基礎的概念に立脚しているに拘らず、その数学的取扱にては層の厚さが常に一定に保たれているとしている。この一見矛盾した取扱から得られた結果の近似度を確める為、井戸揚水による周辺の地下水圧の変化に随つて層の厚さが変化することを考慮し、透水量係数及び貯蔵係数を時間、場所の函数として合理的な解を求めた。この結果と比較することにより、層の厚さ一定の仮定は之迄多くの場所で認められたような圧縮係数 $\frac{\rho g}{E}$ が $10^{-3}\mathrm{m}^{-1}$ より小な層に於ては実際的に殆ど誤差を伴わぬとしてよいことを明かにした。