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INFILTRATION INTO UNSATURATED MOIST SOIL

By

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Abstract

Infiltration into unsaturated moist soil is affected by the soil moisture content as well as the properties of soil and infiltration water. In this paper, one dimensional infiltration equation is solved assuming that the soil is the ideal soil and the vertical distribution of moisture content can be expressed by an approximate form of Smith's theory. The solution may be useful to evaluate the effect of physical factors of the system on the infiltration capacity and the total infiltration.

1. Introduction

A part of rainwater evaporates back to the atmosphere, other part flows out on the ground surface and the rest infiltrates into the ground recharging the groundwater. It is needless to say these phenomena are the essential important problem in the hydrology, especially in the water balance. In practice, technics of artificial recharge of groundwater are available where the level of groundwater must be kept at an economic level for the pumping. We call these technics "water spreading", in which there are two types, that is, surface type and underground type. In the former type, the water in ponds and ditches infiltrate into the ground naturally, and we must have the knowledges of the influence on the infiltration capacity by some physical factors to carry out these technics more effectively. The irrigation in the farm is the infiltration itself, and this is the very important problem in the agricalture.

The infiltration capacity varies by soil property, moisture content in the soil, temperature of the water and the soil, depth of groundwater level and depth of surface water. It is also known experimentally that the infiltration capacity varies by the time lapse from the beginning of the infiltration. M. R. Lewis and W. L. Powers [1937] have found a indefinite relation between the infiltration capacity and the depth of surface water, but L. Schiff [1953] has found on the Hesperia Sandy Loam, California, that the infiltration is directly proportional to the depth of the pond. J. H. Neal [1953] and A. L. Tisdall [1951] have, in their experimental study, shown that initial moisture content in the soil affects the infiltration capacity, that is, it is large to dry soil and small to wet soil.

We must finally know the functional relation between the infiltration capacity and the many parameters, i.e., soil property, depth of surface water, moisture content in the soil, depth of groundwater level, etc., for quantitative treatment of the infiltration problem. A. N. Kostiakov [1932] has proposed the next equation to show the functional relation between the total infiltration, i, and the time lapse, t,

$$i = Kt^{\alpha}, \tag{1}$$

where K and α are constants. Other well known infiltration equation is

$$f = f_{\infty}+(f_0-f_{\infty})\, e^{-\beta t}, \tag{2}$$

which has been proposed by W. Gardner and J. A. Widston [1921] and R. E. Horton [1939]. Symbols f, f_0 and f_{∞} are respectively the infiltration capacities at time t, initial and final stages, and β is a constant. These two equations have been made so as to fit the experimental data and the parameters in the equations have little physical meaning.

J. R. Philip [1954] and R. W. Stallman [1954] have introduced the infiltration equations which express the infiltration capacity by some of physical terms of the system. They have succeeded to clear the time variation of infiltration capacity and the relation between the infiltration and the depth of the surface water, but have not made allowances for the effects of the moisture content and finite depth of the groundwater level.

2. Differential equation of infiltration

In this paper we will discuss the one dimensional problem which corresponds to the infiltration of the rainwater on the wide plane and to that from the large pond, thus the water moves only downwards.

The soil is assumed as the ideal soil, that is, a heap of uniform small spheres. The porosity ranges from that of regular cubical packing, 0.476, to that of closed hexagonal packing, 0.260. Putting the origin at the ground surface and the z-axis to be downwards positive, we can write from the Darcy's law the filter velocity, v, as

$$v = \frac{k\rho g}{\mu}\cdot\frac{T+H+z}{z}, \tag{3}$$

where T is the capillary head difference across the wetting front, H the depth of surface water, k the permeability, ρ and μ are respectively the density and the viscosity of the water, and g is the gravity acceleration. When the soil is sturated, T becomes zero, then Eq. (3) is reduced to the common form for the downwards seepage through porous media.

The voids in the soil having, at the initial stage, moisture content, m, may be saturated immediately after the arrival of the wetting front, then the moisture content changes to the value of the porosity, P. Therefore, the downwards velocity of the wetting front may be written as the quotient of the filter velocity divided by the space which must be filled with the water, that is

$$\frac{dz}{dt} = \frac{v}{P-m} = \frac{k\rho g}{\mu(P-m)} \cdot \frac{T+H+z}{z} . \tag{4}$$

The infiltration capacity f is no more than the filter velocity, thus

$$f = K \cdot \frac{T+H+z}{z} , \tag{5}$$

where $K = k\rho g/\mu$ is a physical constant determined by the properties of the soil and the water, and T and m are generally a function of z and t. The total infiltration may be given as

$$i = \int_0^t f dt . \tag{6}$$

3. Moisture content in the soil

It has been experimentally known that the moisture content in the soil affects the infiltration, and that the vertical distribution of the moisture content relates to the groundwater level. Now we assume that the water in the soil is the capillary water only, because the quantity of the adhesive water is very little in comparison with the former. The vertical distribution of the soil moisture may be written by the theory of W. O. Smith [1933]. He divided the soil above groundwater level into three zones. In the lowest saturation zone, whole pores are filled with the water. In the next zone, small pores are filled with the water but the air remains in the large pores. Here, the meniscuses at the contacts of two soils combine each other and form irregular funicular masses of water, then he has named this zone "funicular zone". In the upper zone named "pendular zone", the meniscuses are retained individually at the contacts.

He expressed the upper end of the saturation zone h_C and that of the funicular zone h_D, in C.G.S. unit, as

$$h_C = \frac{\sigma}{\rho g r} \cdot \frac{(1-P)^{1/3}}{\{1.044-(1-P)^{1/3}\}} , \tag{7}$$

$$h_D = 6.452 \frac{\sigma}{\rho g r} , \tag{8}$$

where σ is the surface tension of the water and r the radius of the soil particle. If a point in the soil is shown by the height h above groundwater level, the soil moisture of this point can be given by his theory, that is, for $h_D > h > h_C$,

$$m=\frac{9(1-P)\sigma^2}{(\rho ghr)^2}\left\{q-\left[\frac{2\sigma}{\rho ghr}+\left(\frac{\sigma}{\rho ghr}\right)^2+1-q^2\right]^{1/2}\sin^{-1}\left[\frac{q}{1+\frac{\sigma}{\rho ghr}}\right]\right\}-\frac{(q-1)^2(q+2)}{3}, \quad (9)$$

where

$$q=\frac{2r+\Delta}{2r}=\frac{1}{2}\left[\frac{4\sqrt{2}\,\pi}{3(1-P)}\right]^{1/3}, \quad (10)$$

and for $h>h_D$,

$$m=\frac{3\pi}{4}\cdot\frac{\sigma^2[1+8.4(0.476-P)]}{(\rho ghr)^2}\left\{1-\left[\frac{2\sigma}{\rho ghr}+\left(\frac{\sigma}{\rho ghr}\right)^2\right]^{1/2}\sin^{-1}\left[\frac{1}{1+\frac{\sigma}{\rho ghr}}\right]\right\}. \quad (11)$$

Symbol Δ shows the hypothetical distance between two soil particles in the spaced hexagonal packing, and may be given as a function of the porosity and the soil radius as shown in Eq. (10).

As an example, putting $\sigma=72.75$ dyne/cm (for water at 20°C), $\rho=1$ g/cm^3, $g=980$ cm/s^2, and r of the order of 0.01 cm, we can evaluate that h_D-h_C for the ordinary porosity should be less than about 23 cm. Furthermore we can appreciate from the comparison of Eqs. (9), (10) and (11), that the vertical distribution of the moisture content in the funicular zone is not so differ from the extrapolation of that from the pendular zone. Thus we assume in this paper, that the vertical distribution of the moisture content for $h>h_C$ may be, when the groundwater level is sufficiently deep, shown approximately by the Eq. (11). But it has yet a complex form for the substitution into Eq. (4) and the integration, therefore, we will approximate further the Eq. (11) to more simple form by the following treatment.

Eq. (11) can be transformed into the product of a parameter and a function of height from the groundwater level as

$$m=A\cdot\Phi, \quad (12)$$

$$A=\frac{3\pi}{4}B^2\Big[1+8.4(0.476-P)\Big], \quad (13)$$

$$\Phi=\frac{1}{(hr)^2}\left\{1-\left[\frac{2B}{hr}+\left(\frac{B}{hr}\right)^2\right]^{1/2}\sin^{-1}\frac{hr}{hr+B}\right\}, \quad (14)$$

where $B=\sigma/\rho g$. It is very difficult to approximate analytically the right side of Eq. (14) to a simple form, but we can find the numerical approximate relation by the graphical method, that is,

$$\Phi\fallingdotseq\frac{0.786}{(hr+0.2)^2}, \quad (15)$$

for which we may find fairly satisfactory degree of the approximation.

4. Capillary head difference at the wetting front

When the soil is perfectly dry, the form of meniscus at the wetting front may be decided by the geometry of the soil only. Then, the capillary head difference is written in the condition of complete wetting as

$$T = \frac{2\sigma}{\rho g R}, \tag{16}$$

where R is the radius of the meniscus and varies in some range when the wetting front progresses. But, supposing the homogeneous porous media as the soil, we can take a constant R_0 as a mean value of R, then T becomes a constant for the dry soil, thus

$$T = T_0 \equiv \frac{2\sigma}{\rho g R_0}. \tag{17}$$

While, the matter is not simple for the unsaturated moist soil, and there may be two or three ideas for the estimation of the capillary head difference at the wetting front. The first idea is that, so far as the soil is unsaturated, the form of the meniscus at the wetting front depends only on the geometry of the soil, regardless of the water content which has been preserved in the soil. In this case, Eq. (17) is also available.

The second idea is that the capillary head difference at the wetting front varies with the moisture content. This accords with the experimental fact that the infiltration capacity is large for the dry soil and small for the wet soil. As described in the former section, the moisture content in the soil depends on the height from the groundwater level, then we must take T as a function of z.

The pressure at an inner point of the water held at a contact point of two soil particles is less than the outer atmospheric pressure, and the relation

$$p = \frac{2\sigma}{R}$$

holds between the pressure difference p and the radius of the meniscus. As the moisture content decrease, R becomes small and p grows large. The distribution of this pressure difference may be regarded as the distribution of the absorbing power of water into the soil, thus introducing the capillary potential Ψ, we get

$$\Psi = -\frac{p}{\rho} = -\frac{2\sigma}{\rho R} = -gh. \tag{18}$$

Then, from Eqs. (16) and (18), denoting the depth of the groundwater level as D, it follows that

$$T = h = D - z. \tag{19}$$

The relation between T and m is, from Eqs. (12), (15) and (19),

$$T \fallingdotseq \frac{1}{r}\left(\sqrt{\frac{0.786\,A}{m}} - 0.2\right). \tag{20}$$

When the ratio of the moisture content to the pore space is written by θ, that is $\theta = m/P$, the capillary potential Ψ is given as

$$\Psi \doteqdot -\frac{g}{r}\left(\sqrt{\frac{0.786\,A}{P\theta}}-0.2\right), \tag{21}$$

which coincides qualitatively with the experimental results by L. A. Richards [1928]. From the Eq. (19), it follows as a natural consequence that when h increases infinitely T becomes larger also infinitely. But this result may be contradictory to the experimental result.

The third idea is a compromise between the first and the second. That is, Eq. (17) is available in the upper region where the moisture content is negligible, and Eq. (19) is valid in the lower region. If we write the boundary between these two regions as h_E, the third idea may be arranged as follows:

$$\left.\begin{array}{lll} T = T_0 & \text{for} & h \geq h_E\,, \\ T = h & \text{for} & h_E > h > 0\,. \end{array}\right\} \tag{22}$$

In the saturation zone, the meniscus vanishes but the capillary head difference acting at the implicit wetting front can also be written as $T=h$. These three cases are diagrammatized in Fig. 1.

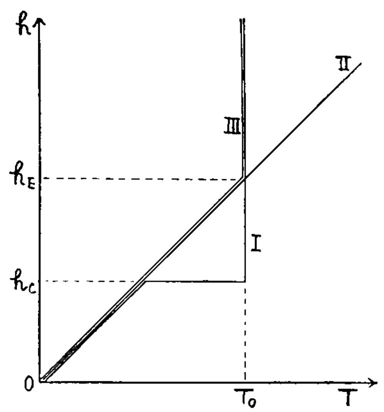

Fig. 1. Schematic diagram of the three cases of the capillary head difference (T) depending on the height (h) from the groundwater level.

We have now no theoretical ground to discuss merits and demerits of above mentioned three ideas of the capillary head difference at the wetting front, and we can only choose more proper idea for the each practical case.

5. Solution of the infiltration equation

The time changes of the infiltration capacity, the total infiltration and the downwards velocity of the wetting front can be known from the solution of the infiltration equation, which may be divided into three cases correspondingly to the three ideas of the capillary head difference.

Case I

From Eqs. (4), (12), (13), (15) and (17), it follows

$$\frac{dz}{dt} = \frac{K[r(D-z)+a]^2}{P\{[r(D-z)+a]^2-b\}} \cdot \frac{T_0+H+z}{z}, \tag{23}$$

and a solution for the condition, $z=0$ at $t=0$, is

$$t = \frac{P}{K}\Big\{z+(T_0+H)\log\left|\frac{T_0+H}{T_0+H+z}\right|+\frac{b(T_0+H)}{[r(T_0+H+D)+a]^2} \times\log\left|\frac{(T_0+H+z)(rD+a)}{[r(D-z)+a](T_0+H)}\right|-\frac{bz}{[r(T_0+H+D)+a][r(D-z)+a]}\Big\}. \tag{24}$$

where $a \equiv 0.2$, $b \equiv 0.786\, A/P$. The infiltration capacity and the total infiltration are given respectively as follows,

$$\left.\begin{aligned} f &= K\cdot\frac{T_0+H+z}{z}, \\ i &= \int_0^t f\,dt = \int_0^z (P-m)\,dz = P\Big\{z-\frac{bz}{[r(D-z)+a](rD+a)}\Big\}. \end{aligned}\right\} \tag{25}$$

When the front arrives at the supper end of the saturation zone, z becomes $D-h_C$, then

$$\left.\begin{aligned} t_1 &= \frac{P}{K}\Big\{D-h_C+(T_0+H)\log\left|\frac{T_0+H}{T_0+H+D-h_C}\right|+\frac{b(T_0+H)}{[r(T_0+H+D)+a]^2} \\ &\quad\times\log\left|\frac{(T_0+H+D-h_C)(rD+a)}{(rh_c+a)(T_0+H)}\right|-\frac{b(D-h_C)}{[r(T_0+H+D)+a](rh_c+a)}\Big\}, \\ f_1 &= K\cdot\frac{T_0+H+D-h_C}{D-h_C}, \\ i_1 &= P\Big[D-h_C-\frac{b(D-h_C)}{(rh_C+a)(rD+a)}\Big]. \end{aligned}\right\} \tag{26}$$

In the saturation zone, i.e. for $D>z>D-h_C$, it may be supposed that the whole water in the pore moves, then we may rather be able to put $m=0$ and the phenomena may be equivalent to the simple capillary phenomena and the relation (19) available. Thus the differential equation is reduced to

$$\frac{dz}{dt} = \frac{K}{P}\cdot\frac{D+H}{z}, \tag{27}$$

and the solution for the condition $z=D-h_C$ at $t=t_1$, is

$$\left.\begin{aligned} t &= t_1+\frac{P}{2K(D+H)}\Big[z^2-(D-h_C)^2\Big], \\ &\text{and} \\ f &= K\cdot\frac{h+H+z}{z} = K\cdot\frac{D+H}{z}, \\ i &= \int_0^{t_1} K\frac{T_0+H+z}{z}dt+\int_{t_1}^t K\frac{D+H}{z}dt = P\Big[z-\frac{b(D-h_C)}{(rh_C+a)(rD+a)}\Big]. \end{aligned}\right\} \tag{28}$$

When the front arrives at the groundwater level, for which $z=D$, then

$$\left.\begin{aligned} t_2 &= t_1+\frac{Phc(2D-hc)}{2K(D+H)}, \\ f_2 &= K\frac{D+H}{D}, \\ i_2 &= P\left[D-\frac{b(D-hc)}{(rhc+a)(rD+a)}\right]. \end{aligned}\right\} \tag{29}$$

After arriving at the groundwater level,

$$\begin{aligned} i &= \int_0^{t_1} K\frac{T_0+H+z}{z}dt+\int_{t_1}^{t_2} K\frac{D+H}{z}dt+\int_{t_2}^{t} K\frac{D+H}{D}dt \\ &= P\left[D-\frac{b(D-hc)}{(rhc+a)(rD+a)}\right]+K\left(\frac{D+H}{D}\right)(t-t_2). \end{aligned} \tag{30}$$

The time variations of the infiltration capacity and the total infiltration are plotted in Figs. 2 and 3 for several values of T_0 and H. Fig. 2 corresponds to the extreme case in which the capillary effect is negligibly small, and Fig. 3 corresponds to the case in which the depth of the surface water is negligible. In Fig. 3, notable breaks are found on the curves of the infiltration capacity and the total infiltration correspondently with the upper end of the saturation zone. Such a notable break has been discovered experimentally by Y. Kira [1956] as shown in Fig. 4.

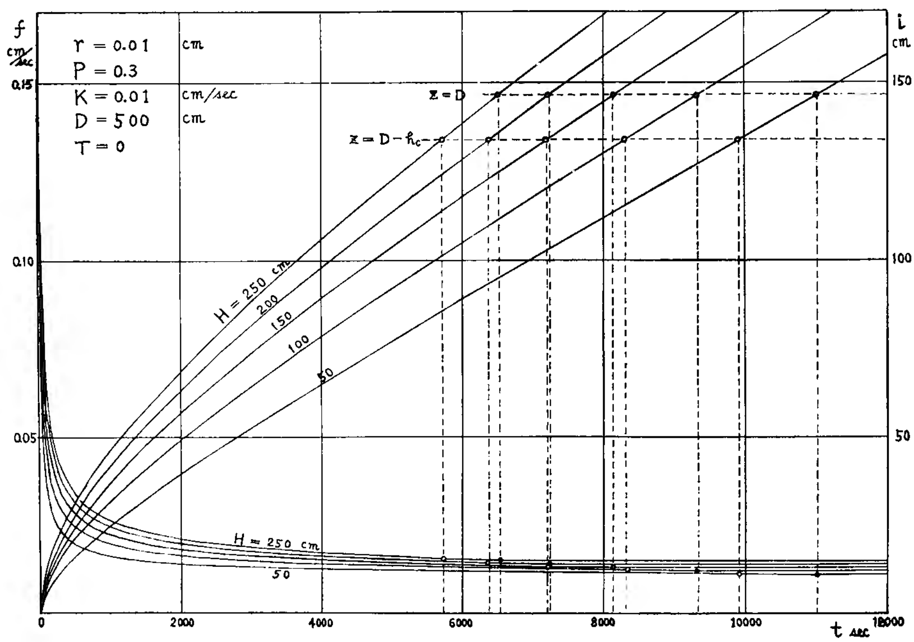

Fig. 2. The infiltration capacities and the total infiltrations as functions of the time in Case I-a. Capillary head difference is taken as a constant (zero) but depth of surface water varies as a parameter. White and black points on the curves show the arrival points at the upper end of saturation zone and at the groundwater level respectively.

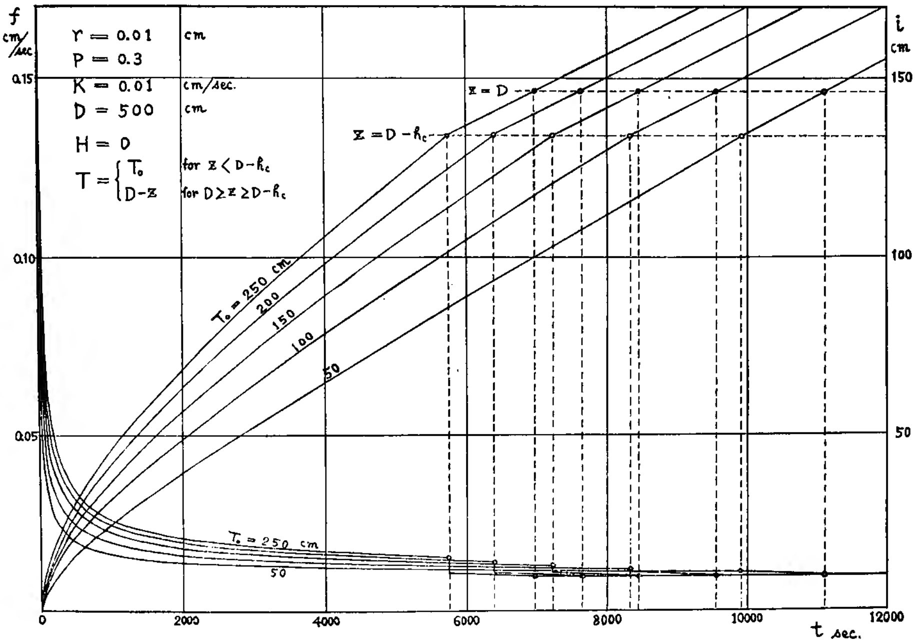

Fig. 3. The infiltration capacities and the total infiltrations as functions of the time in Case I-b. Capillary head difference varies as a parameter, while depth of surface water is taken as a constant (zero).

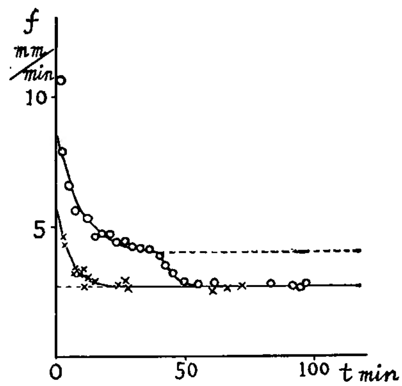

Fig. 4. An experimental result of the infiltration capacity curves, where a notable break appears. (after Y. Kira, [1956])

Case II

From the second idea, the differential equation may be written as

$$\frac{dz}{dt}=\frac{K[r(D-z)+a]^2}{P\{[r(D-z)+a]^2-b\}}\cdot\frac{D+H}{z}, \tag{31}$$

and in the upper zone above the saturation, the solution for the condition, $z=0$ at $t=0$, is

$$
\left.\begin{aligned}
t &= \frac{P}{K(D+H)}\left\{\frac{z^2}{2}-\frac{bz}{r[r(D-z)+a]}+\frac{b}{r^2}\log\left|\frac{rD+a}{r(D-z)+a}\right|\right\}. \\
f &= K\cdot\frac{D+H}{z}. \\
i &= P\left\{z-\frac{bz}{[r(D-z)+a](rD+a)}\right\}.
\end{aligned}\right\} \quad (32)
$$

and

At the upper end of the saturation zone,

$$
\left.\begin{aligned}
t_1 &= \frac{P}{K(D+H)}\left\{\frac{(D-h_C)^2}{2}-\frac{b(D-h_C)}{r(rh_C+a)}+\frac{b}{r^2}\log\left|\frac{rD+a}{rh_C+a}\right|\right\}. \\
f_1 &= K\cdot\frac{D+H}{D-h_C}. \\
i_1 &= P\left[D-h_C-\frac{b(D-h_C)}{(rh_C+a)(rD+a)}\right].
\end{aligned}\right\} \quad (33)
$$

In and under the saturation zone, the expression should be the same as in the Case I. The time variations of the infiltration capacity and the total infiltration are plotted in Fig. 5 for various depth of the surface water.

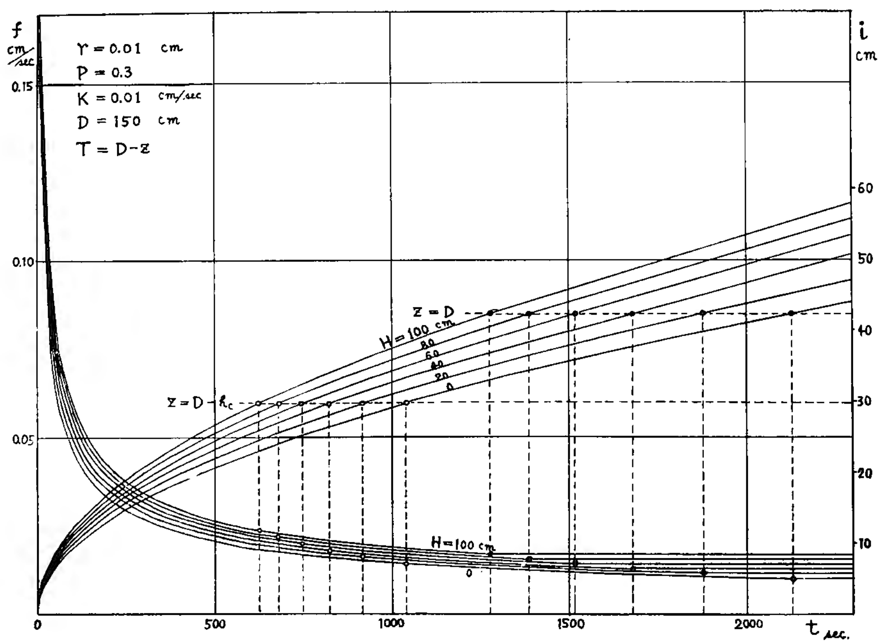

Fig. 5. The infiltration capacities and the total infiltrations as functions of the time in Case II. Capillary head difference is taken to be proportional to height from the groundwater level, and depth of surface water varies as a parameter.

Case III

Above the saturation zone, when $z<D-h_E$ the expressions should be no difference

from that of Case I. Then, for $z=D-h_E$, Eq. (24) is reduced to

$$t_E = \frac{P}{K}\left\{D-h_E+(T_0+H)\log\left|\frac{T_0+H}{T_0+H+D-h_E}\right|+\frac{b(T_0+H)}{[r(T_0+H+D)+a]^2}\right.$$
$$\left.\times\log\left|\frac{(T_0+H+D-h_E)(rD+a)}{(rh_E+a)(T_0+H)}\right|-\frac{b(D-h_E)}{(rh_E+a)[r(T_0+H+D)+a]}\right\}. \quad (34)$$

When $z>D-h_E$, solving Eq. (31) for the condition $t=t_E$ at $z=D-h$, itfollo ws that

$$t = t_E+\frac{P}{K(D+H)}\left\{\frac{(D-z-h_E)(h_E-z-D)}{2}-\frac{b(rD+a)(h_E-D+z)}{r[r(D-z)+a](rh_E+a)}\right.$$
$$\left.+\frac{b}{r^2}\log\left|\frac{rh_E+a}{r(D-z)+a}\right|\right\}. \quad (35)$$

and, f and i can be shown by the expressions in Eq. (32).

At the the upper end of the saturation zone, putting $z=D-h_C$,

$$t_1 = t_E+\frac{P}{K(D+H)}\left\{\frac{(h_C-h_E)(h_E-2D+h_C)}{2}-\frac{b(rD+a)(h_E-h_C)}{r(rh_C+a)(rh_E+a)}+\frac{b}{r^2}\log\left|\frac{rh_E+a}{rh_C+a}\right|\right\}, \quad (36)$$

and, f_1 and i_1 can be shown by the expressions in Eq. (33).

In and under the saturation zone, the expressions should be the same as that of Case I and Case II. Case III is diagrammatized in Fig. 6 where the notable

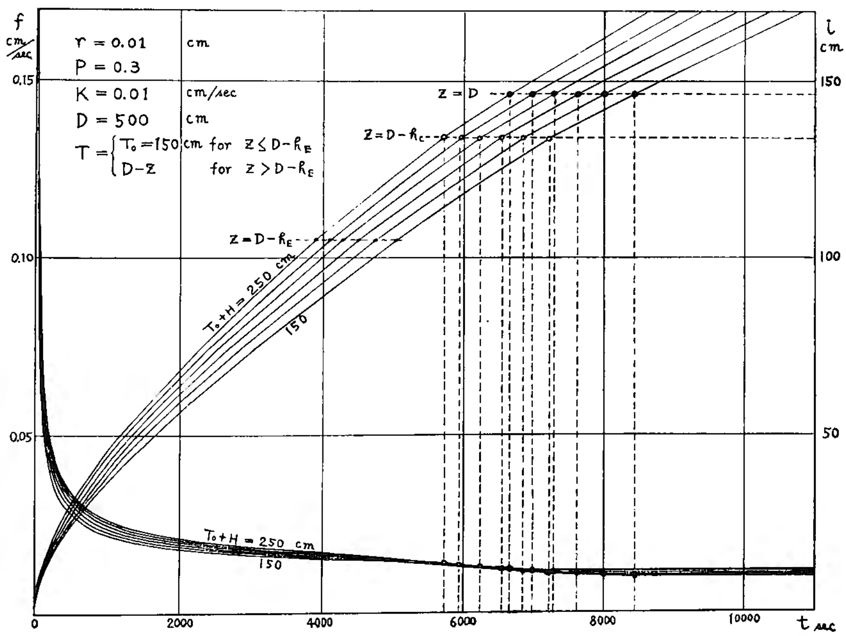

Fig. 6 The infiltration capacities and the total infiltrations as functions of the time in Case III. Capillary head difference is taken as a constant above a boundary, below which to be proportional to height from the groundwater level, and depth of surface water varies as a parameter. The above mentioned boundary is shown by small points.

breaks are also shown on the infiltration capacity curves, and they should be caused by the discontinuity of the capillary head difference.

As the special case, if the soil is perfectly dry, relations $D \to \infty$, $m \to 0$ and $T = T_0$ hold, then

$$z + (T_0 + H) \log \left| \frac{T_0 + H}{T_0 + H + z} \right| = \frac{Kt}{P}, \tag{37}$$

from which we can get the similar equations of J. R. Philip [1954], R. W. Stallman [1954], W. H. Green and G. A. Ampt [1911]. They are all special cases which are included in our theory.

References

Gardner, W. and J. A. Widston, 1921; Movement of soil moisture, Soil Science, 11, 215.

Green, W. H. and G. A. Ampt, 1911; Flow of air and water through soils, J. Agr. Research, 4,1.

Horton, R. E., 1939; Approach toward a physical interpretation of infiltration capacity, Soil Sci. Soc. Amer. Proc., 5, 399.

Kira, Y., M. Mizunoe and H. Toyoda, 1956; Hatachi-kangai no kenkyu (in Japanese), Miscellaneous Pub. Nat. Inst. Agr. Sci., F, 1.

Kostiakov, A. N., 1932; On the dynamics of the coefficient of water-percolation in soils and on the necessity for studying it from a dynamic point of view for purposes of amelioration, Trans. 6th Com. International Soc. Soil Sci., A, 17.

Lewis, M. R. and W. L. Powers, 1937; Study of factors affecting infiltration, Soil Sci. Soc. Amer. Pro., 3, 334.

Neal, J. H., 1938; Effect of the degree of slope and rainfall characteristics on runoff and soil erosion, Univ. Missouri Research Bull., 280.

Philip, J. R., 1954; An infiltration equation with physical significance, Soil Science, 77, 153.

Richards, L. A., 1928; The usefulness of capillary potential to soil-moisture and plant investigation, J. Agr. Research, 37, 719.

Schiff, L., 1953; The effect of surface head on infiltration rates based on the performance of ring infiltrometers and ponds, Trans. Amer. Geophys. Union, 34, 257.

Smith, W. O., 1933; The final distribution of retained liquid in an ideal uniform soil, Physics, 4, 425.

Stallman, R. W., 1954; Discussion of "The effect of surface head on infiltration rates based on the performance of ring infiltrometers and ponds", Trans. Amer. Geophys. Union, 35, 171.

Tisdall, A. L., 1951; Antecedent soil moisture and its relation to infiltration, Australian J. Agr. Research, 2, 342.