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A NOTE ON THE CRITICAL TRACTIVE FORCE OF AN ALLUVIAL RIVER

By

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1. The grain size of the superficial bed sediment of an alluvial river takes a characteristic distribution as shown schematically in Fig. 1, which seems to be common for all alluvial rivers.

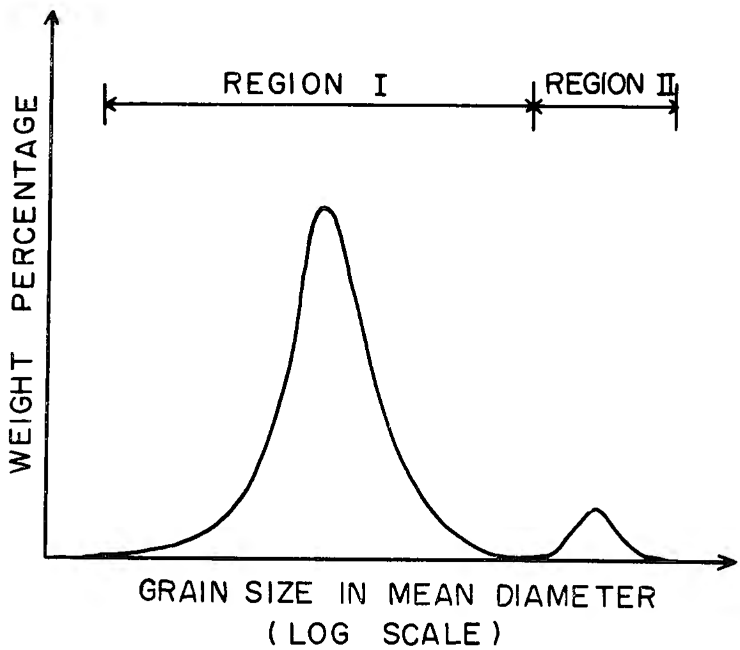

Fig. 1. Grain size distribution of the bed sediment of an alluvial river.

The sand grains of region I are supposed to be carried in suspension and deposited to the bottom, keeping equilibrium with those in suspension, while the sand grains of region II are carried by the processes of rolling and saltation along the bottom.

The shape of the sand grains of region I is usually angular and that of region II is rounded. Sand grains of intermediate size between regions I and II are lacking in many cases and occasionally region II disappears. It may be safely said that the main constituents of the bed sediment of an alluvial river belong to region I.

Many years ago, the present author (1941—1) derived theoretically the characteristic form of the grain size spectrum of region I, which is shown in Table 1 using $w/\sqrt{gjh}$ as nondimensional parameter (w : terminal velocity of sand grain in water, g : acceleration of gravity, j : energy slope, h : depth of river). Roughly speaking, the theoretical spectrum explains well the actual distribution.

Table 1. Theoretical percentage of grains in the bed sediment of region I.

$w/\sqrt{gjh}$	Percentage in bed sediment
0.05	0.004
0.10	0.03
0.15	0.18
0.20	1.06
0.25	6.89
0.30	42.11
0.35	23.09
0.40	13.20
0.45	7.37
0.50	4.20
0.60	1.31
0.70	0.41
0.80	0.13
0.90	0.04
1.00	0.01

The force ρgjh (ρ : density of water) is usually called tractive force and the tractive force which just picks up a grain of a given size is called critical tractive force for that grain and the correspoding $\sqrt{gjh}$ is called critical velocity. Table 1 suggests that in an alluvial river the tractive force is nearly equal to the critical tractive force for the largest grain of region I. From Table 1, however, we can not definitely say whether they are just equal or nearly equal to each other. The purpose of the present paper is to give some comments on the physical meaning of the relation $w_c/\sqrt{gjh}=1$? for the largest grain (w_c) belonging to region I which the author could not fully comprehend at that time.

2. A sand grain contained in running water is conceived to be mainly kept in suspension by the vertical eddy velocity w'. All the grains for which

$$|w| \leq \sqrt{\overline{(w')^2}} \tag{1}$$

will be suspended in water, where, w is the terminal velocity of a grain in water and the equality stands for the critical grain. Then, to clarify the nature of critical velocity for bed sediments, it is essential to get some relations between w' and some other characteristic quantities of a river.

The vertical eddy velocity w' is conceived intuitively to be related with tangential stress at the bottom of turbulent layer. The equation of motion of a linear river flow is given by

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau}{\partial z} \tag{2},$$

where u, t, x and z are usual notation, D/Dt is Lagrangian time derivative, p is hydrostatic pressure, τ is tangential Reynolds stress given by

$$\tau = -\rho\,\overline{u'w'} \tag{3},$$

u' being horizontal eddy velocity. Inserting the value of p into (2) and integrating from the bottom of the turbulent layer to the free surface, we get

$$\frac{\tau_0}{\rho} = gih - \int_0^h \frac{Du}{Dt}dz \tag{4},$$

or

$$\frac{\tau_0}{\rho} = gjh, \quad j = i - \frac{1}{gh}\int_0^h \frac{Du}{Dt}dz \tag{5},$$

where τ_0 is the bottom stress or tractive force, h is the depth of the river, i is the slope of the free surface and j is energy slope. In an alluvial river, at least in its middle and lower reaches,

$$j \cong i \tag{6}.$$

If we introduce friction velocity u^* defined by

$$\sqrt{\frac{\tau_0}{\rho}} \equiv u^* \tag{7},$$

there seems to exist an universal relation between u^* and $\sqrt{\overline{(w')^2}}$, *i. e.*

$$\sqrt{\overline{(w')^2}}/u^* = C \tag{8},$$

where C is the ratio of Kármán's mixing length (l_k) to Goldstein's one (l_g) (Munn, 1961), *i. e.*

$$C = l_k/l_g \tag{9}.$$

The ratio C depends upon Richardson number R_i and for neutral stability ($R_i=0$), C degenerates to a certain constant C_0 which seems to be universal as mentioned below.

Businger (1959) assumed

$$C_0=1 \tag{10},$$

which is the case when

$$u'=-w' \tag{11}.$$

Prandtl and many other authors assumed the relation (11), and all the theories of grain size spectrum of bed sediments of region I including the present author's cited above naturally lead to the relation

$$\frac{w_c}{\sqrt{gjh}}\left(=\frac{\sqrt{\overline{(w')^2}}}{u^*}\right)=1 \tag{12}.$$

But, whether C_0 is actually unity or not, is doubtful. In fact, Monin (1959) obtained in adiabatic air flow (*i. e.* in neutral stability)

$$C_0=0.86,$$

while Panofsky and McCormick (1960) report that in adiabatic flow

$$C_0=1.25 \tag{13}.$$

These are all derived from wind observations near the ground.

In the middle and lower Yangtze River, the grain size distribution of bed sediments takes very clearly the type of Fig. 1. Some examples and sampling locations are shown in Table 2 and Fig. 2 which are taken from the author's

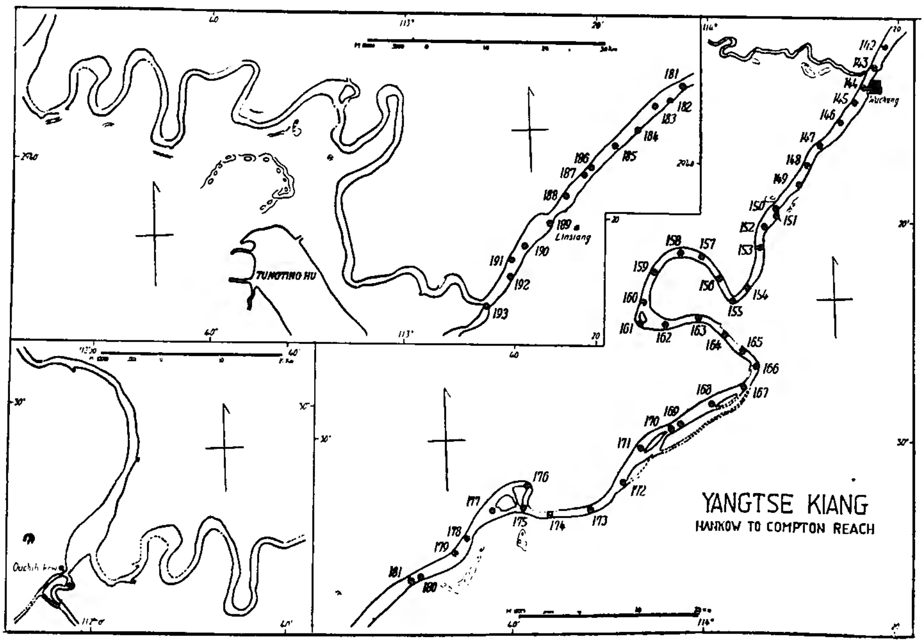

Fig. 2. Location of sampling of the bed sediments in Table 2.

Table 2. Percentage of grains in the bed sediment of the middle Yangtze River.

No. of sample	Distance above Woosung in km	U. S. Bureau of Soils classification																Specific gravity
		Medium gravel		Fine gravel		Coarse sand		Medium sand		Fine sand			Very fine sand		Silt		Clay	
		Size of opening in mm																
		4.699	2.362	1.397	0.991	0.833	0.417	0.351	0.246	0.208	0.147	0.104	0.074	0.05	0.02	0.005	<0.005	
177	1192.1			0.2 —	— —	0.1 —	0.2 —	0.2 —	2.8 —	17.7 —	41.4 —	29.1 0.35	13.1 0.26	4.16 0.17	0.14 —	0.01 —		2.71
178	1197.6				0.7 —	0.3 —	1.3 —	0.4 —	4.5 —	5.1 —	51.0 —	33.5 0.50	3.1 0.29	0.38 0.11	0.01 —			2.74
179	1202.6			0.2 —	0.2 —	0.1 —	1.3 —	2.6 —	43.6 —	29.6 —	17.4 —	2.6 0.08	1.2 0.08	0.81 0.04	0.04 —			2.71
180	1207.6			0.1 —	0.3 —	0.2 —	5.7 —	12.1 —	43.1 —	13.3 —	18.6 —	5.5 0.19	0.6 0.11	0.16 0.12	0.01 —			2.72
181	1209.6				0.1 —	0.1 —	0.6 —	0.4 —	2.1 —	3.8 —	60.6 —	27.3 0.79	3.1 0.58	0.49 0.10	0.02 —	0.01 —		2.77
182	1213.8						0.1 —	0.1 —	2.3 —	28.8 —	58.4 —	8.8 0.15	1.0 0.03	0.31 0.02	0.01 —			2.69
183	1216.3					0.1 —	0.1 —	0.2 —	0.4 —	0.5 —	2.6 —	50.1 0.38	37.4 0.53	6.83 0.89	0.13 —	0.02 —		2.76
184	1222.0			0.2 —	— —	— —	0.3 —	0.4 —	6.2 —	35.5 —	45.2 —	9.5 0.14	2.0 0.04	0.56 0.03	0.03 —	0.01 —		2.62
185	1226.5						0.1 —	0.1 —	1.1 —	3.2 —	50.0 —	36.6 0.53	16.8 0.25	1.21 0.10	0.04 —			2.75
186	1232.0		0.8 —	0.6 —	1.5 —	0.6 —	5.2 —	3.1 —	11.3 —	20.8 —	38.3 —	13.5 0.23	3.2 0.08	0.69 0.02	0.03 —	0.01 —		2.67
188	1240.5		1.0 —	1.6 —	3.0 —	1.2 —	8.3 —	3.1 —	11.0 —	15.1 —	38.3 —	13.9 0.10	2.7 0.07	0.52 0.04	0.03 —			2.73
189	1245.5								0.1 —	0.5 —	10.7 —	59.2 0.34	23.3 0.26	5.28 0.20	0.13 —	0.01 —		2.68
190	1251.0			0.7 —	0.6 —	0.3 —	1.6 —	0.5 —	1.9 —	5.0 —	31.7 —	43.9 0.34	8.8 0.16	4.44 0.06	0.17 —	0001 —		2.67
191	1254.0					0.1 —	0.1 —	0.1 —	0.2 —	0.5 —	4.8 —	28.7 0.14	48.2 0.33	16.08 0.51	0.35 —	0.01 —		2.72
192	1258.0						0.1 —	— —	0.2 —	0.3 —	1.7 —	39.6 0.17	46.3 0.35	10.84 0.18	0.27 —	0.01 —		2.67
193	1265.5									1.8 —	1.2 —	0.3 —	1.1 —	13.7 —	12.0 —	0.2 —	69.8 —	2.60

paper (Hayami, 1941—2). As seen in Table 2, the largest grains of region I is about 0.80~0.95 mm in mean diameter*. And their terminal velocity in water is about 8.0~9.5 cm/sec. In the middle reaches of the Yangtze River 1192~1265 km above Woosung of Table 2, the value of $\sqrt{gjh}$ is about 7 in

* The shape of grains of the bed sediment is angular and in these examples the ratio of maximum span to minimum span is about 1.3 : 1.0. One half of the grains passes through the openning of a shieve with minimum span and the other half passes with maximum span. Minimum span ×1.1 is here called mean diameter.

high water (h=25m, $i=2\times10^{-5}$). Taking these values, we get

$$\frac{w_c}{\sqrt{gjh}}=1.14\sim1.36=1.25 \text{ (in mean)} \qquad (14).$$

The mean value agrees well with the value obtained by Panofsky and McCormick. When we consider the difference of medium, one being air and the other being water, the agreement is rather astonishing and seems to suggest the universal value 1.25 of the constant C_0 near the bottom of turbulent layer.

If we assume the turbulence is homogeneous and isotropic $(\overline{(u')^2}=\overline{(w')^2})$, we see from equations (3) and (7) that $(u^*)^2/\overline{(w')^2}$—cousequently gjh/w_c^2 also—represents nothing but the correlation coefficient between u' and w'. The supposition $C_0=1.25$ means, therefore, that this correlation coefficient universally takes the value 0.64 near the bottom of turbulent layer in neutral stability.

This paper is dedicated to the late Prof. Dr. Kinzō Seno, mourning his unexpected death and recalling his numerous works of deep insight in hydrology.

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