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A NOTE ON APPLICATION OF FINITE-DIFFERENCE METHOD TO STABILITY ANALYSIS OF BOUNDARY LAYER FLOWS

By

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Abstract

An attempt is made to devise an efficient finite-difference method to solve an eigenvalue problem associated with hydrodynamic stability of boundary layer flows. Efficiencies of various systems of centered finite-difference with non-uniform grid distance are evaluated for plane Poiseuille flow, stability properties of which have been thoroughly investigated. Non-uniform finite differencing used here is expressed by a uniform finite-difference in the converted coordinate which results in higher resolution in the vicinity of the boundary surface. It is shown that there may be some optimal finite-differencing for obtaining accurate solutions efficiently.

1. Introduction

Some numerical methods have been developed to deal with problems of hydrodynamic stability during the last ten years. One of the most promising methods is to solve algebraic equations derived by approximating differential equations by a set of finite-difference equations. In fact, many investigators applied this method for studying the stability of various kinds of fluid flows, for instance, Thomas [1953], Kurtz and Crandall [1962], Deardorff [1963], Asai [1970a and b] to plane parallel flows, Lilly [1966] to Ekman flow, Eliasen [1954], Haltiner and Song [1962], Haltiner [1963], Arnason, Brown and Newburg [1967], Yanai and Nitta [1968] and Hirota [1968] to zonal flows in the earth's atmosphere and so forth. However, this method requires so many grid points, particularly for boundary layer flows, to reduce truncation errors in the solutions arising from both the finite grid distance and/or the finite location of the boundary that the method sometimes becomes ineffective. Even in such cases there may be some possible devices for obtaining solutions accurate enough without using a large number of grid points.

By postulating that the instability of boundary layer flows is centered in the vicinity of the boundary surface in which significant shear of the flow lies, we may be able to evolve an efficient and still simple finite-difference method with non-uniform grid spacing which specifies higher resolution near the boundary and lower resolution far from the boundary.

In this report the feasibility of applying the centered finite-difference method with non-uniform grid distance to stability analysis of boundary layer flows will be examined for plane Poiseuille flow, stability properties of which have been substantially studied by Lin [1944], Thomas [1953], Shen [1954] and others since Heisenberg's work [1924].

2. Governing equations

We shall confine our attention to stability of plane Poiseuille flow of homogeneous and incompressible fluid between two horizontal planes separated by the distance $2h$ as shown in Fig. 1. Let us introduce a reference length h and a reference

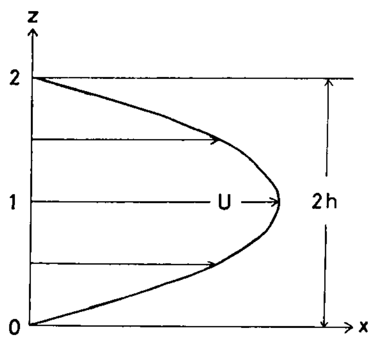

Fig. 1. The profile of the basic flow and the coordinate system.

velocity U which is the maximum velocity of the flow. We take a Cartesian coordinate system with the x axis directed parallel to the flow and the z axis upward. Nondimensionalizing all quantities in terms of the reference ones, we may express plane Poiseuille flow in the following form.

$$\bar{u}(z) = 1 - (z-1)^2, \tag{2.1}$$

where z is the dimensionless vertical coordinate and $\bar{u}$ is the dimensionless velocity of the flow.

Two-dimensional small amplitude perturbation which is of preferred mode (Squire [1933]) superimposed on the basic flow may be written in the form:

$$w = W(z) \exp(ikx + \sigma t),$$

where w is the vertical component of dimensionless perturbation velocity, k is the dimensionless wavenumber in the x direction and σ is the dimensionless frequency which is generally complex, so that σ is expressed by $\sigma = \sigma_r + i\,\sigma_i$ in which a positive value of σ_r denotes the amplification rate of an unstable perturbation. Then, the well-known Orr-Sommerfeld equation for W is derived in the following dimensionless form (e.g., Lin [1966]):

$$\frac{d^4W}{dz^4} - \{2k^2 + R(ik\bar{u} + \sigma)\} \frac{d^2W}{dz^2} + \left\{k^4 + k^2R(ik\bar{u} + \sigma) + ikR\frac{d^2\bar{u}}{dz^2}\right\} W = 0 , \tag{2.2}$$

where $R = Uh/\nu$ is the Reynolds number in which ν is the kinematic viscosity.

We assume the fixed and no-slip conditions at the upper and lower boundary planes at which both the vertical and horizontal components of perturbation velocity vanish, i.e.,

$$W = \frac{dW}{dz} = 0 \quad \text{at} \quad z = 0 \text{ and } z = 2. \tag{2.3}$$

Since (2. 1), (2. 2) and (2. 3) are symmetrical about $z=1$, solutions of (2. 2) under the conditions (2. 1) and (2. 3) must be either symmetrical or antisymmetrical about $z=1$. Therefore we consider only the lower half of the fluid layer and take up a symmetrical solution of the lowest mode which is most preferable. Then, instead of (2. 3) the boundary conditions required are

$$\left.\begin{aligned} W = \frac{dW}{dz} &= 0 \quad \text{at} \quad z = 0, \\ \text{and} \qquad \frac{dW}{dz} = \frac{d^3W}{dz^3} &= 0 \quad \text{at} \quad z = 1. \end{aligned}\right\} \tag{2.4}$$

3. Conversion of the coordinate and variable grid spacing

In a boundary layer flow, in general, significant variation of the velocity with height is confined to the vicinity of the boundary surface. Poiseuille flow between two parallel planes is one of the examples. By postulating that the energy source of all unstable disturbances is confined mostly to the layer of significant shear of the flow, we are able to design the finite differencing to represent this layer accurately without increasing number of grid points as a whole.

Now we introduce a new vertical coordinate, Z, defined as

$$Z = (1 - e^{-az}) / (1 - e^{-a}) , \tag{3.1}$$

where a is a constant parameter. We adopt a uniform grid spacing in the Z coordinate. Consequently a greater positive value of a specifies the grid spacing in such a way that higher resolution is achieved in the lower levels near the boundary. It is noticed in (3. 1) that the new coordinate, Z, tends to the original one, z, when a approaches zero.

With the use of (3. 1), (2. 1), (2. 2) and (2. 4) are transformed, respectively, into

$$\bar{u}(Z) = 1 - \left\{ \frac{1}{a} \ln\left(1 - \frac{Z}{b}\right) + 1 \right\}^2, \tag{3.2}$$

$$\begin{aligned} & a^4 (Z-b)^4 \frac{d^4 W}{dZ^4} + 6a^4 (Z-b)^3 \frac{d^3 W}{dZ^3} \\ & + a^2 (Z-b)^2 \{ 7a^2 - 2k^2 - R(ik\bar{u} + \sigma) \} \frac{d^2 W}{dZ^2} \\ & + a^2 (Z-b) \{ a^2 - 2k^2 - R(ik\bar{u} + \sigma) \} \frac{dW}{dZ} \\ & + \left[k^4 + k^2 R(ik\bar{u} + \sigma) + ikRa^2 \left\{ (Z-b)^2 \frac{d^2\bar{u}}{dZ^2} + (Z-b)\frac{d\bar{u}}{dZ} \right\} \right] W = 0, \end{aligned} \tag{3.3}$$

$$\left. \begin{aligned} & W = \frac{dW}{dZ} = 0 \qquad \text{at} \quad Z = 0, \\ \text{and} & \\ & \frac{dW}{dZ} = 0, \ \left\{ (Z-b)\frac{d^3}{dZ^3} + 3\frac{d^2}{dZ^2} \right\} W = 0 \qquad \text{at} \quad Z = 1, \end{aligned} \right\} \tag{3.4}$$

where $b = (1 - e^{-a})^{-1}$.

4. Finite-difference version

The linear differential equation (3. 3) and the boundary conditions (3. 4) will be transformed into a set of algebraic equations by approximating the derivatives of W with respect to Z by finite-differences. The fluid layer from $Z=0$ to $Z=1$ is divided vertically into n sublayers, each of which has equal thickness in the Z coordinate with the adjacent levels designated from 0 at $Z=0$ to n at $Z=1$.

Use of the centered difference approximations for the derivatives in (3. 3) at the lth level and (3. 4) results in the following set of homogeneous algebraic equations:

$$\begin{aligned} & (C_l + D_l) W_{l+2} + (-4C_l - 2D_l + E_l + F_l) W_{l+1} \\ & + (6C_l - 2E_l + G_l) W_l + (-4C_l + 2D_l + E_l - F_l) W_{l-1} \\ & + (C_l - D_l) W_{l-2} = 0, \end{aligned} \tag{4.1}$$

$$\left. \begin{aligned} & W_0 = 0, \ W_{-1} - W_1 = 0, \ W_{n-1} - W_{n+1} = 0, \\ & n(1-b) W_{n+2} + \{6 - 2n(1-b)\} W_{n+1} - 12 W_n + \{6 + 2n(1-b)\} W_{n-1} \\ & \qquad - n(1-b) W_{n-2} = 0, \end{aligned} \right\} \tag{4.2}$$

where

$$C_l = n^4 a^4 (Z_l - b)^4,$$
$$D_l = 3n^3 a^4 (Z_l - b)^3,$$
$$E_l = n^2 a^2 (Z_l - b)^2 \{7a^2 - 2k^2 - R(ik\bar{u}_l + \sigma)\},$$
$$F_l = \frac{1}{2} na^2 (Z_l - b) \{a^2 - 2k^2 - R(ik\bar{u}_l + \sigma)\},$$
$$G_l = k^4 + k^2 R(ik\bar{u}_l + \sigma) + ikRa^2\left\{(Z_l-b)^2 \left(\frac{d^2\bar{u}}{dZ^2}\right)_l + (Z_l-b)\left(\frac{d\bar{u}}{dZ}\right)_l\right\},$$
$$Z_l = l n^{-1}.$$

The variable W_l denotes those at the lth level.

The system of finite-difference equations (4. 1) and (4. 2) constitutes a set of simultaneous linear equations in n unknown variables, i.e., W_l at $l = 1, 2, \ldots\ldots, n$. This yields the so-called frequency equation for the frequency σ (e.g., Asai [1970a]). A series of computations were made for a number of different values of a, R and k.

5. Results

(1) *Uniform grid spacing*

First of all, accuracies of finite-difference method having uniform grid distance are examined for different numbers of subdivision, n, for the case of $R=10^4$ and $k=1$ which was investigated by Thomas [1953]. Figure 2 shows variations of calculated values of the amplification rate σ_r and the phase velocity c $(=-\sigma_i/k)$ of unstable perturbation with the value of n. The solution obtained by Thomas [1953] which may be regarded as an exact one is indicated by the dash-dotted line. The

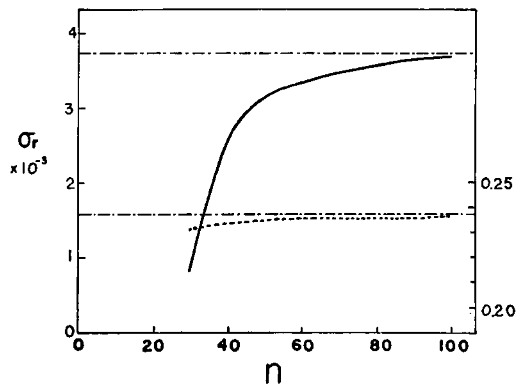

Fig. 2. Calculated values of the amplification rate σ_r (solid line) and the phase velocity c (broken line) for various numbers of subdivision n for $R=10^4$ and $k=1$. Thomas' solution is indicated by dash-dotted line.

numerical solutions obtained here approach consistently the exact solution with an increasing value of n. It appears that more than 100 of subdivision are required to retain the errors within 1%.

(2) *Non-uniform grid spacing*

In order to see efficiency of the non-uniform finite differencing variations of calculated values of the amplification rate and the phase velocity with the value of a are shown in Fig. 3. As the value of a is either smaller or larger than a certain value, the calculated solution diverges from the exact one. It is thus evident that there exist an optimum for a to yield the most accurate solution for a fixed number of subdivision.

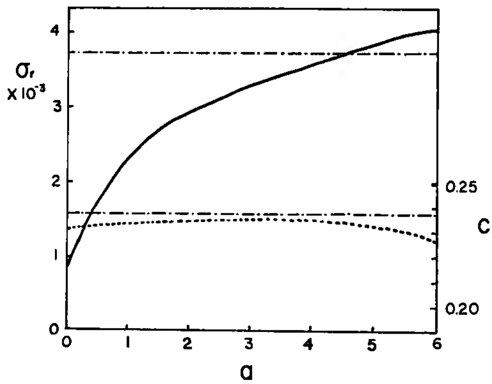

Fig. 3. Variations of amplification rate (solid line) and phase velocity (broken line) with the value of a for $n=30$. The others are same as in Fig. 2.

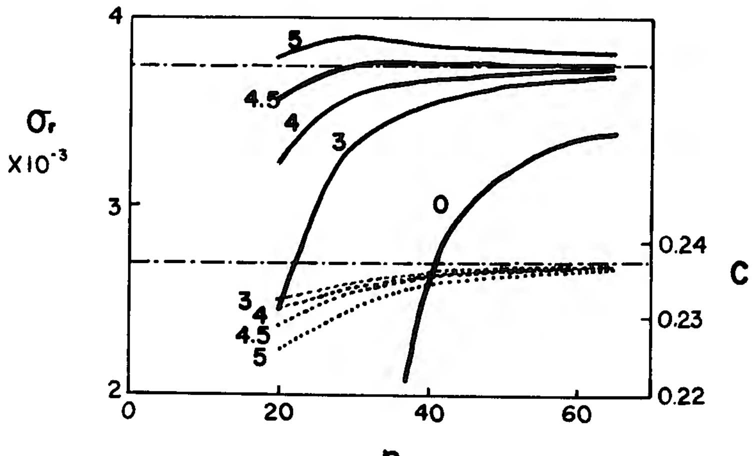

Fig. 4. Dependence of calculated values of σ_r (solid line) and c (broken line) on the number of subdivision n for different values of a. Numeral attached to each line denotes the value of a.

An inspection of Fig. 3 indicates that $a \doteq 4$ is the optimum for the present case.

Now let us examine dependence of calculated solutions on n for different values of a, which is shown in Fig. 4. Highly accurate solution whose error is within 1% can be expected even for $n=30$ when $a=4.5$. This implies that the non-uniform finite differencing may be much efficient in obtaining an accurate solution compared with the conventional uniform one.

(3) *Implication of optimal value of a*

We should note here some features of an optimal non-uniform grid system. Figure 5 shows the vertical profiles of the real part (W_r) and the imaginary part (W_i) of the eigenfunction solution W in the converted coordinate Z for different values of a. Since W_i is negligibly small compared with W_r, the profile of W is almost linear with respect to Z for $a=4.5$ which is the optimum yielding the most efficient finite differencing.

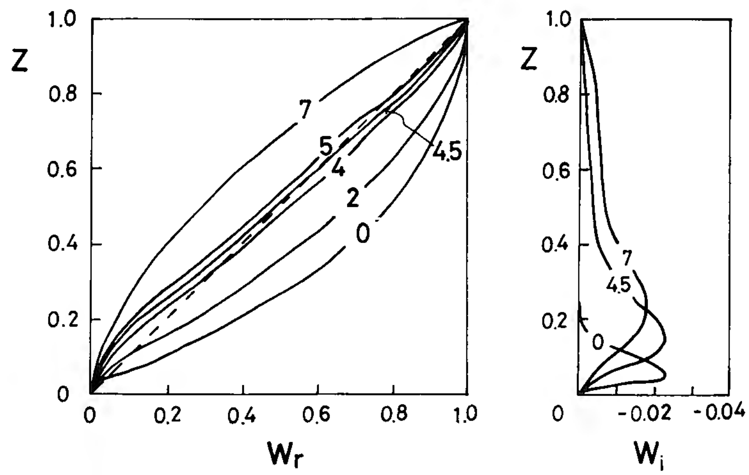

Fig. 5. Vertical profiles of the real part (W_r) and the imaginary part (W_i) of eigenfunction solution (W) for different values of a which are denoted by numerals attached to the curves. The linear profile is denoted by the broken line for comparison.

It must be remembered that replacement of derivatives by the centered differences used here necessitates neglecting additional terms of higher derivatives than the respective derivatives by two or more orders. This means that terms neglected in approximating the differential equation (3. 3) by the difference equation (4. 1) consist of higher derivatives than the second derivative. Therefore an optimal grid system of centered finite differencing will be obtained by choosing a value of a in such a way that dependence of W on Z is close to being linear or quadratic.

6. Conclusion

In order to design an optimal centered finite differencing applicable to stability analysis of boundary layer flows we examined the efficiency of the finite differences of non-uniform grid spacing for plane Poiseuille flow whose stability properties are well known.

First, a new coordinate Z is introduced following the conversion defined as

$$Z=(1-e^{-az})/(1-e^{-a}),$$

and the conventional centered finite differencing with a uniform grid spacing is applied to the new coordinate. Then examination in efficiency of the finite-difference method is made by varying the parameter a which is assumed positive. This non-uniform differencing specifies higher resolution near the boundary and lower resolution far from the boundary. An extent of non-uniformity of grid spacing depends upon the value of a.

It is thus found that there exists an optimum value of a to yield the most efficient scheme of centered differencing. For instance, a is about 4.5 for $R=10^4$ and $k=1$, and a small change of the optimum can be expected for different values of R and k. It is noticed that the optimum is obtained by choosing the value of a in such a way that dependence of eigenfunction solution on Z is close to be linear or quadratic.

The finite-difference method proposed here may be applicable to stability analysis of the other types of boundary layer flow. The results obtained for Ekman layer flow will be reported in a separate paper.

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