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# GEOMETRICAL CONDITIONS FOR DEFORMATION AND FRACTURE RELATED TO THE EARTHQUAKES (1)

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**Abstract**

The problem of the creation of the fractures system in the continua is discussed by the method of geometry. The concept of manifolds plays an important role for formulating the problems of general deformations including fractures. The stochastic nature of the fracturing is equivalent to the variety of the coordinate systems.

The conditions for the field equation are searched from two viewpoints, i.e., the St. Venant's compatibility conditions and the dislocation density. We find that these are concerned with conditions for integrability according to Poincare's lemma of differential forms.

## 1. Introduction

The dynamical mechanisms of crack propagation have been studied in detail with special interests to formulate the earthquake sources, e.g., Kostrov (1966), Ida and Aki (1973) and others. But their efforts are concentrated on the phenomena probably after the catastrophic fracturing, i.e. fault-making fracturing. They assumed the arrangement of fractures and the modes of displacement-discontinuties beforehand.

How do we formulate the state equation or the field equation if we assume that the state of media is in the isotropic heterogeneity. (Otsuka (1972)) For example there are problems of the spatial and temporal distribution of earthquakes and more fundamentally those of the criterion of fracturing. In order to answer these questions we make a start from a more general point of view.

Many researchers have made observations of fracturing phenomena (e.g. Schardin (1959)) The driving mechanisms for the fracture can be possibly pursued by various methods under assumptions but it can be said that main features of fractures are geometrically various at first sight. Hence we can first assume that conditions which determine its final state of deformed and fractured body are geometrical ones, especially differential geometrical ones, even if physical conditions remain unknown or can be introduced as parameters. Next we may assume that physical conditions are kept constant in varying the geometrical position of material point though it may not be regarded that the geometrical nature of deformed and fractured states is intrinsic in such phenomena.

The above reasoning will partly be justified by the fact that our observation apparatus in seismology are mostly scales, seismometers and others which measure the distances between points over the earth's surface, strains, the rate of displacements or strains, and the acceleration of the same quantities. We never insist on limiting the observations to the quantities of geometry, but thorough discussions of geometrical quantities of the observation field will be helpful to introducing the general observation theory in search of physical states.

There are researches for the yield point essentially in the same geometrical standpoint as above; Kondo (1950), Bilby et al. (1955) and others. Their basic idea is that the observation field spreads in a 3-dimensional Euclidean space and that the state of the material in a geometrical sense is characterized by a 3-dimensional manifold. It follows that our spatial observation domain is restricted to a narrow region, as is clear. In topological terminology it is distributed by the coordinate neighbourhood.

For example, if the dimension of observation field is three and the metric of space is Euclidean, the existence of an unstrained field is a priori assumed or at least realizable by means of virtual operations in a connected state. Definitions of terminology and relevant discussions will be found in later paragraphs. Introduction of a Euclidean space means the necessity of not only the idea of topology but that of metric which is the main tool of our observations. Local properties of observations can be represented by the local properties of the manifold, i.e., a Riemannian space, an anholonomic space and others.

## 2. The extension of the theory of elasticity

The continuum is assumed to be an elastic body which is characterized by the properties of the one to one correspondence between an applied force and deformation, i.e., non-hysteresis and non-retardation from the time of application of a force. Henceforth, the state of the medium is characterized under the standpoint of the above stated nature — manifold. We must make some assumptions that the medium has an unstrained state (undeformed state) in its initial state and that the initial manifold is a 3-dimensional Euclidean space in the large. Hence the Lagrangean parameters of the material point may be the Cartesian coordinates of a Euclidean space, for example. As deformation or transformation is locally observed, the quantities of deformation will be locally defined along the same line as the elasticity theory (Murnaghan(1951)).

If we suppose that the coordinates of the medium or the Lagrangean parameters are $x^i$, $i=1,2,3$, then the line element $ds$ of the deformed body can be written in a quadratic differential form $ds^2=g_{ij}\,dx^i dx^j$ where $g_{ij}$ is the fundamental metric tensor. We mention here the possibility of the virtual operation. If the fundamental metric tensor $g_{ij}$ cannot be reduced to $\delta_{ij}$ (Kronecker's delta $i=j$ $\delta_{ij}=1$; $i\neq j$, $\delta_{ij}=0$) by means of local affine transformations of neighbourhoods of each point of a deformed body, which formally means the inversion of the strained state to the initial unstrained state, it might follow that the medium cut into small pieces which have the properties of local Euclidean metric will never be connected in a prescribed accuracy in the large. St. Venant's compatibility conditions of the infinitesimal elasticity theory is a key concept to the above proposition of non-uniqueness of correspondence between the initial state and the final state.

We present some analogical examples in a 2-dimensional plane or shell. A flat plane corresponds to the Euclidean initial state in the 2-dimension. After an application of force, a flat plane is bent into a 3-dimensional Euclidean space, that is, a 2-dimensional manifold is realizable in a 3-dimensional Euclidean space in a topologically connected state. But how do we immerse the 3-dimensional Riemannian manifold in a 3-dimensional Euclidean space when our real medium spread in 3 directions is deformed by the virtual operation? Of cource, our observation field is limited to a 3-dimensional Euclidean space in a static problem. From this point of view, we can assume that the infinite increment of deformation is impossible, as is in harmony with the experimetal data that the material strength is finite.

We generally call an undeformed initial state a natural state and a deformed final state an Eulerian state. We may introduce an intermediate state which is not realizable in a 3-dimensional Euclidean space but in a higher than 3-dimensional Euclidean space, and which satisfies "logically" the one to one correspondence between an applied force and deformation. Metaphorically the geometrical conditions for the non-unique final state are prepared in an intermediate state. The possible realization of several states will be controlled by stochastic processes and predicted only statistically, which might be the intrinsic nature of fracturing processes.

Next we may extend the dimension of the intial state to larger dimension than three of a Euclidean space and generalize the geometrical state including the physical state. Dynamical problems including the time parameter which will extend the degree of freedom of general deformation have not yet been solved. It is due to the mathematical difficulties and a lack of knowledge of the true situation, that is, the recognition of an intermediate state. According to the above stated difficulties, it is supposed that for the time being the time parameter is neglected and that the spatial parameters are essential. Briefly to sum up, the introduction of properties of a non-Euclidean space and the stochastic nature of deformation and fracture is necessarily postulated.

From now on, we simplify the problems of deformation and fracture to take some examples.

We assume that the deformed body is a differentiable manifold with affine connections and that it is locally designated $x^i$, $i=1,2,3$. As the transformation to the general curvilinear coordinates is easily performed, the coordinates are supposed to be rectangular Cartesian coordinates. Some terminology relevant to the discussions is cleared by the expression of geometry as follows. A deformed body represented by a manifold is a space connected in sequence by locally Euclidean spaces which are called tangent spaces. Hence the connection coefficients are necessarily introduced.

Following the theory of 2-dimensional surfaces, if we may represent the tangent space by the differential form, we develop the deformed body into the 3-dimensional Euclidean space.
The tangent space $L$ is

$$L = dx^i \frac{\partial}{\partial x^i} = e_i\, dx^i, \quad L = e_\alpha \omega^\alpha.$$

$$\omega^\alpha = O_i^\alpha dx^i, \quad O_i^\alpha e_\alpha = e_i\,. \tag{1}$$

where $e_\alpha$ is the moving frame.
(Einstein's convention is assumed over whole expressions)

Next the connection coefficients are introduced

$$de_\alpha = \omega_\alpha^\beta e_\beta, \quad \omega_\alpha^\beta = \Gamma^\beta_{\alpha\gamma}\omega^\gamma. \tag{2}$$

where it is assumed that, if we may consider the intermediate state, the connection defined here is equivalent to the postulate to neglect the normal direction of a manifold which is not immersed in the 3-dimensional Euclidean space. The exterior differential of $L$ is followed

$$dL = (d\omega^\beta - \omega_\alpha^\beta \omega^\alpha)\, e_\beta \tag{3}$$

As there is no reason to assume that $dL=0$ (4), we write $dL=T^\beta e_\beta$ (5) where we may call $T^\beta$ a torsion tensor. Generally the torsion tensor will be written in the form of the second order differential form. If the component of $T^\beta$ is $T^\beta{}_{\alpha\gamma}$, then

$$T^\beta = T^\beta{}_{\alpha\gamma}\omega^\gamma\omega^\alpha \tag{6}$$

where

$$T^\beta{}_{\alpha\gamma} = -T^\beta{}_{\gamma\alpha} \tag{7}$$

In the following discussion, we consider the second derivative of affine bases

$$dde_\alpha = (d\omega_\alpha^\gamma - \omega_\alpha^\beta \omega_\beta^\gamma)\, e_\gamma \tag{8}$$

Since there is not any probable reason to assume that $\quad d^2e_\alpha = 0 \quad$ (9),
we replace the right hand side of (9) by $\quad d^2e_\alpha = R_\alpha^\gamma e_\gamma \quad$ (10)
where $R_\alpha^\gamma$ is called a curvature tensor. Generally the curvature tensor $R_\alpha^\gamma$ will be written in the form of the second order differential form. If the component of $R_\alpha^\gamma$ is $R^\gamma{}_{\alpha\delta\varepsilon}$, then

$$R_\alpha^\gamma = R^\gamma{}_{\alpha\delta\varepsilon}\omega^\varepsilon\omega^\delta \tag{11}$$

For later convenience, the nature of the torsion and the curvature is fully described as follows. We take a small parallelogram ABCD in the deformed body represented by a manifold. This may be what we call lattice in the crystallography when it is undeformed.

There are two paths along which we develop point C from point A, i.e. the paths passing point B and D. The breakage of the connection at point C means

$$dL \neq 0 \tag{4$'$}$$

and the discrepancy in affine bases at point C means $\quad dde_\alpha \neq 0 \quad$ (9)′

The discrepancy at point C is represented by $(\Delta x)^\alpha$ when AB is $\underset{1}{dx^i}$ and AD $\underset{2}{dx^i}$,

$$(\Delta x)^\alpha = \underset{21}{d\omega^\alpha} - \underset{12}{d\omega^\alpha} = \frac{\partial O_i^\alpha}{\partial x_j}(\underset{2}{dx^j}\underset{1}{dx^i} - \underset{1}{dx^j}\underset{2}{dx^i}) + O_i^\alpha(\underset{21}{ddx^i} - \underset{12}{ddx^i}) \tag{12}$$

Since we can suppose generally $\underset{21}{ddx^i} = \underset{12}{ddx^i}$,

$$(\Delta x)^\alpha = \frac{\partial O_i^\alpha}{\partial x^j}(\underset{2}{dx^j}\underset{1}{dx^i} - \underset{1}{dx^j}\underset{2}{dx^i}) \tag{13}$$

From the expression (2),

$$d\omega^\beta - \omega_\alpha^\beta \omega^\alpha = \left(\frac{\partial O_i^\beta}{\partial x^j} - \Gamma^\beta_{\gamma\alpha} O_i^\gamma O_j^\alpha\right) dx^j dx^i \tag{14}$$

From the expressions (6) and (14),

$$\frac{\partial O_i^\beta}{\partial x^j} - \Gamma^\beta_{\gamma\alpha} O_i^\gamma O_j^\alpha = T^\beta{}_{\gamma\delta} O_i^\gamma O_j^\delta \tag{15}$$

Then the expression (13) is replaced by

$$(\Delta x)^\alpha = -(\Gamma^\alpha_{\gamma\beta} O_i^\gamma O_j^\beta + T^\alpha{}_{\gamma\delta} O_i^\gamma O_j^\delta)\, \underset{1}{dx^i}\underset{2}{dx^j} \tag{16}$$

This is just the definition of Burgers' vector of crystal dislocation. Hence we have arrived at the idea of dislocation from the formulae (2) and (4)′, where needless to say the dislocation expressed by the formula (16) means the dislocation density.

We give exposition for the curvature with making use of the same parallelogram.

$$(\Delta e)_\alpha = (\underset{21}{dd} - \underset{12}{dd})\, e_\alpha$$

$$\underset{21}{d\omega_\alpha^\beta} = (\underset{2}{d}\Gamma^\beta_{\alpha\gamma})\, \underset{1}{\omega^\gamma} + \Gamma^\beta_{\alpha\gamma}\, (\underset{21}{d\omega^\gamma})$$

$$\underset{2}{de_\beta} = \Gamma^\varepsilon_{\beta\delta} \underset{2}{\omega^\delta} e_\varepsilon\,.$$

Hence $$(\Delta e)_\alpha = \frac{1}{2}\left(\frac{\partial \Gamma^\varepsilon{}_{\alpha\gamma}}{\omega^\delta} - \frac{\partial \Gamma^\varepsilon{}_{\alpha\delta}}{\omega^\gamma} + \Gamma^\beta_{\alpha\gamma}\Gamma^\varepsilon_{\beta\delta} - \Gamma^\beta_{\alpha\delta}\Gamma^\varepsilon_{\beta\gamma}\right)(\underset{2}{\omega^\delta}\underset{1}{\omega^\gamma} - \underset{1}{\omega^\delta}\underset{2}{\omega^\gamma})\, e_\varepsilon \tag{17}$$

Here the curvature can be represented by the exterior product form (11).

$$R^\varepsilon{}_{\alpha\gamma\delta} = \frac{\partial \Gamma^\varepsilon{}_{\alpha\gamma}}{\omega^\delta} - \frac{\partial \Gamma^\varepsilon{}_{\alpha\delta}}{\omega^\gamma} + \Gamma^\beta_{\alpha\gamma}\Gamma^\varepsilon_{\beta\delta} - \Gamma^\beta_{\alpha\delta}\Gamma^\varepsilon_{\beta\gamma} \tag{18}$$

where

$$R^\varepsilon{}_{\alpha\gamma\delta} = -R^\varepsilon{}_{\alpha\delta\gamma} \tag{19}$$

The St. Venant's compatibility condition is a good illustration of curvature tensor with a metric of deformed body. For simplicity we may assume that $dL=0$ which is so called "without torsion". This corresponds to the assumption that the deformed body should be a Riemannian manifold with a symmetric connection. The fundamental metric tensor $g_{ij}$ is defined by an inner product of affine bases of a tangent space as follows.

$$(e_i, e_j) = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij} \tag{20}$$

where this quadratic form is positive definite as usual. The line element $ds$ is defined on the tangent space $L = dx^i e_i$

$$ds^2 = (L, L) = g_{ij} dx^i dx^j \tag{21}$$

The above introduced metric enables us to define the quantities of deformation, i.e., the strains.

$$\varepsilon_{ij} = \frac{1}{2}(g_{ij} - \delta_{ij}) \tag{22}$$

Similarly for the developed line element,

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(g_{\alpha\beta} - \delta_{\alpha\beta}) \tag{22$'$}$$

where

$$g_{\alpha\beta} = g_{ij} O^i_\alpha O^j_\beta$$

If it is supposed that the relation between the connection coefficients $\Gamma^k_{ij}$ that $\Gamma^k_{ij} = \Gamma^k_{ji}$ (23) (without torsion), then according to the relation that

$$dg_{ij} = (de_i, e_j) + (e_i, de_j) \tag{23}$$

we can replace $\Gamma^k_{ij} = \left\{ {k \atop ij} \right\}$,

where $$\left\{ {k \atop ij} \right\} = \frac{1}{2} g^{ke} (\partial g_{ej}/\partial x^i + \partial g_{ie}/\partial x^j - \partial g_{ij}/\partial x^e) \tag{24}$$

The curvature tensor for the deformed body should be expressed by

$$R^l{}_{jkl} = \frac{\partial}{\partial x^l}\left\{ {i \atop jk} \right\} - \frac{\partial}{\partial x^k}\left\{ {i \atop jl} \right\} + \left\{ {i \atop ml} \right\}\left\{ {m \atop jk} \right\} - \left\{ {i \atop mk} \right\}\left\{ {m \atop jl} \right\}$$

For instance, we caluculate these quantities in the case of a 2-dimensional problem. As the other components are disappered, the only component is

$$R_{1212} = g_{1l} R^l{}_{212} \tag{25}$$

Substituting $\varepsilon_{ij}$ for $g_{ij}$ using the relation that $g_{ij} = 2\varepsilon_{ij} + \delta_{ij}$ we easily derive the St. Venant's compatibility condition for a 2-dimensional problem.

$$R_{1212} = \frac{\partial^2 \varepsilon_{22}}{(\partial x^1)^2} + \frac{\partial^2 \varepsilon_{11}}{(\partial x^2)^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x^1 \partial x^2} \tag{25$'$}$$

disregarding the product higher than 2. If $R_{1212} = 0$ (26), we can conclude that the

deformed body is equivalent to a Euclidean space locally. Therefore the St. Venant's condition insists that a deformed body is locally Euclidean regardless of the deformation. A more elementary exposition will be as follows.
The differential form

$$d\varphi = \left( \frac{\partial \varepsilon_{12}}{\partial x^1} - \frac{\partial \varepsilon_{11}}{\partial x^2} \right) dx^1 + \left( \frac{\partial \varepsilon_{22}}{\partial x^1} - \frac{\partial \varepsilon_{12}}{\partial x^2} \right) dx^2 \tag{27}$$

is completely integrable only when the relation (26) is satisfied (Frobenius's theorem for the system of Pfaffian equations) According to this first integral of (27), we can define the rotation $\varphi$ and the displacements $u^i$.

$$\begin{aligned} du^1 &= \varepsilon_{11} dx^1 + (\varepsilon_{12} - \varphi)\, dx^2 \\ du^2 &= (\varepsilon_{12} + \varphi)\, dx^1 + \varepsilon_{22} dx^2 \end{aligned} \tag{28}$$

We summarize the above derived relation with using more compact expressions.

$$\begin{aligned} \omega &= (\omega^\alpha) \\ e &= (e_\alpha) \\ \Omega &= (\omega^\alpha_\beta) \\ T &= (T^\alpha) \\ R &= (R^\alpha_\beta) \end{aligned} \tag{29}$$

where $\omega$, $e$, $\Omega$, $T$ and $R$ express the differential in a tangent space, the bases of a tangent space, the connection coefficients, the torsion and the curvature respectively. The tangent space (vectorial differential form) $L = \omega e$ (30) can be exterior-differentiated with only using the connection formula $de = \Omega e$ (31) that is postulated.

$$dL = (d\omega - \omega\Omega)\, e = Te \tag{32}$$

The exterior derivative of $de$ is

$$dde = (d\Omega - \Omega^2)\, e = Re \tag{33}$$

From Frobenius's theorem the assumption of $T=0$ (34) is equivalent to the condition of the complete integrability of the differential form (30) and $R=0$ (35), the differential form (31), as has already been proved in (16) and (17). We can also get the relations between the torsion $T$ and the curvature $R$ from the higher derivatives of $T$.

$$dT = -T\Omega + \omega R \tag{36}$$

This relation shows that the curvature $R$ should be concerned with the space density of point defects in view of the physical meaning of the torsion.

## 3. Conclusion

From preceding discussions the deformation is generally conditioned by both the torsion and the curvature. The condition of torsion=0 and curvature=0 is equivalent to the assumptions of the classical theory of elasticity. Under these conditions we have proved the uniqueness of rotations and displacements particularly in the case of 2-dimension. From the classical theory we can extend to the following three cases that torsion$\neq$0 or curvature$\neq$0. In the first place, the case that torsion$\neq$0 and curvature=0 is the Burgers' original definition of dislocations which has been used for the analysis of seismic source functions. The case that torsion=0 and curvature$\neq$0 is the problem of proper strain, especially in thermoelasticity. The most general case will be that of torsion$\neq$0 and curvature$\neq$0, although some ambiguity still remains in physical images concerning the curvature, which must be made more clear in future. The case should be the problems of fracture rather than those of deformation and we conclude that the "fracture" should be determined by the constraint of torsion=0 and curvature=0 quantitatively.

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