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In-in formalism on tunneling background: Multidimensional quantum mechanics

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We reformulate quantum tunneling in a multidimensional system where the tunneling sector is nonlinearly coupled to oscillators. The WKB wave function is explicitly constructed under the assumption that the system was in the ground state before tunneling. We find that the quantum state after tunneling can be expressed in the language of the conventional in-in formalism. Some implications of the result to cosmology are discussed.

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I. INTRODUCTION

Quantum tunneling has been studied for a long time as one of the most exciting topics in various fields of science, from the study of the dynamics of atomic and molecular systems to condensed matter physics and field theory (see [1,2], and references therein). Regarding applications to cosmology, there is even a possibility that the universe was born via quantum tunneling [3]. Furthermore, the string theory landscape has been proposed as a possible setting of the early universe inflation [4]. In this framework, scalar fields are thought to tunnel among many false vacua (i.e. local minima of the potential) in the vast string theory potential landscape. The formulation of the false vacuum decay (i.e. the quantum tunneling from a false vacuum) in field theory was first considered in flat spacetimes [5,6], and was extended to include gravity in [7] (see [8] for the extension to multiple-field cases).

Multidimensional quantum tunneling has also been well studied [1], and is formulated by constructing the wave functions for quantum tunneling using the WKB method [9–12]. Field theoretic extension was developed in [13], and such formulation has been applied to the quantum fluctuations on a tunneling background. It was further extended to include gravity in [14]. As a result of these developments, it has been possible to calculate the quantum fluctuations in the universe after false vacuum decay [15–17].

All previous works on quantum tunneling neglect effects of nonlinear interactions. In other words, only free quantum field theory on a tunneling background has been considered so far. In light of the recent progress in observational cosmology, however, it is now important to study the observational consequences of nonlinear interactions. For example, the non-Gaussianity of the cosmological fluctuations is now a hot topic in cosmology [18–20]. It is clearly necessary to reformulate quantum field theory on a tunneling background with nonlinear interactions included, in order to calculate the non-Gaussianity in a universe undergoing quantum tunneling, as is motivated by the string landscape. Estimates for the non-Gaussianity in such a scenario have been calculated in the literature [21,22], but up to now there is no rigorous proof that the formulation used there is valid.

In this paper, we reformulate multidimensional quantum tunneling with nonlinear interactions, following the formulation by Yamamoto [12]. Although the formulation of the multidimensional system is interesting in itself, it can also be regarded as a first step towards the formulation of quantum field theory. We expect that extensions from multidimensional cases to field theory with gravitation are possible as before [12–14], but leave such issues to future studies.

As the simplest extension of the 1-dimensional case, we will study a 2-dimensional system in which the tunneling sector y is nonlinearly coupled to the oscillator η, as shown in Fig. 1. The restriction to a 2-dimensional system keeps calculations as simple as possible whilst still maintaining the essential features of multidimensional effects. The particle, originally positioned in the false vacuum at $(y_F, 0)$, moves to the nucleation point at $(y_N, 0)$ by quantum tunneling, and then rolls down classically, as shown in Fig. 1. For simplicity, in this paper we assume the classical path to be straight, and leave the consideration of curved paths to future work. Assuming that the potential is static, the wave function $\Psi(y, \eta)$ for such a particle is a solution of the time-independent Schrödinger equation. The boundary conditions for $\Psi(y, \eta)$ corresponding to the scenario outlined above are given as follows: $\Psi(y, \eta)$ should be an outgoing wave function outside the barrier, and $\Psi(y, \eta)$ should match the wave function for the quantum state before the quantum tunneling around the false vacuum.

Let us put a screen at y outside the barrier, and then prepare the above system many times and let the particles hit the screen. The particles hit the screen with different η each time, since $\Psi(y, \eta)$ is extended in the η direction. The statistical properties of η at y are given by the quantum expectation values with respect to $\Psi(y, \eta)$, defined as $\langle \eta^n \rangle_y \equiv \int d\eta \eta^n |\Psi(y, \eta)|^2$ where $n = 1, 2, 3, \dots$. In this paper, we obtain formulas for such quantities by constructing $\Psi(y, \eta)$ explicitly using the WKB method. If we define t as the time the particle takes to reach y from the

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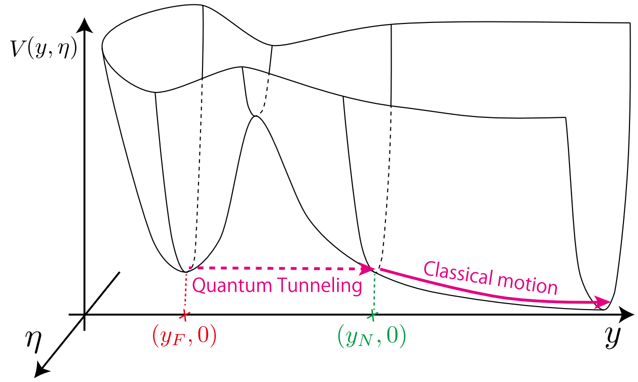

FIG. 1 (color online). The potential for the 2-dimensional system, where the tunneling sector y is nonlinearly coupled to the oscillator η. The particle moves from the false vacuum $(y_F, 0)$ to the nucleation point $(y_N, 0)$ by quantum tunneling, and rolls down classically from the nucleation point.

nucleation point, we can interpret $\Psi(y, \eta)$ as the t-dependent wave function with respect to η. Then, we find that our resulting formulas can be expressed in the language of the conventional in-in formalism [23,24]. Note that $\langle \eta^n \rangle_y$ at given y, or given t, can be regarded as the analogue of the n-point correlation functions at a given time in field theory, where the time is defined in terms of the value of the tunneling field.

This paper is organized as follows. In Sec. II, we obtain the expression for the quantum expectation value in the Schrödinger picture. In Sec. III, we move to the interaction picture, where the quantum expectation value is given in the in-in formalism form. In Sec. IV, we apply the formalism obtained in Sec. II and in Sec. III to a simple toy model for illustration purposes. Finally, we conclude in Sec. V.

II. FORMULATION: SCHRÖDINGER PICTURE

A. WKB analysis for 2-dimensional system

As mentioned in the Introduction, let us consider a 2-dimensional system. The Hamiltonian of the system is given by

$$\mathcal{H} = \frac{p_y^2}{2} + \frac{p_\eta^2}{2} + V(y, \eta), \tag{1}$$

where $V(y, \eta)$ has a false vacuum and nucleation point at $(y, \eta) = (y_F, 0)$ and $(y_N, 0)$, respectively, as shown in Fig. 1. The nucleation point is defined as the opposite end to the false vacuum on the tunneling path, which is the classical trajectory connecting the false vacuum and the region outside the potential barrier with minimum action. Separating $V(y, \eta)$ into the y part $V_{\text{tun}}(y)$ and the η part $V_\eta(y, \eta)$ as $V(y, \eta) = V_{\text{tun}}(y) + V_\eta(y, \eta)$, we assume for simplicity that $V_\eta(y, \eta)$ can be written as $V_\eta(y, \eta) = (\omega^2(y)/2)\eta^2 + V_{\text{int}}(y, \eta)$, where the nonlinear interaction term $V_{\text{int}}(y, \eta)$ consists of the cubic and higher order terms with respect to η. The vanishing of the linear term with respect to η in the potential guarantees that the tunneling path lies on the y axis. The inclusion of the nonlinear interaction term $V_{\text{int}}(y, \eta)$ is the essential new point in this paper, compared to the literature [15–17]. For later convenience, here we denote the y and η parts of the Hamiltonian as $\mathcal{H}_y = p_y^2/2 + V_{\text{tun}}(y)$ and $\mathcal{H}_\eta = p_\eta^2/2 + V_\eta(y, \eta)$, respectively.

In the system defined by Eq. (1), we consider the tunneling wave function $\Psi(y, \eta)$, which is a solution of the time-independent Schrödinger equation with eigenenergy E

$$\hat{\mathcal{H}}\Psi(y, \eta) = E\Psi(y, \eta). \tag{2}$$

Here, quantities with hat(ˆ) are operators, and $\hat{p}_y$ and $\hat{p}_\eta$ in $\hat{\mathcal{H}}$ are given by $(\hbar/i)(\partial/\partial y)$ and $(\hbar/i)(\partial/\partial \eta)$, respectively. In this paper, we concentrate on quantum tunneling from the quasi-ground-state, which is defined as the ground state for the potential expanded around the false vacuum. We can consider quantum tunneling from excited states, as in [12], but we leave such issues to future studies. As mentioned in the Introduction, $\Psi(y, \eta)$ should be an outgoing wave function outside the barrier.

We construct the tunneling wave function under the following assumptions:

(1) the WKB approximation is valid well inside and well outside the barrier,
(2) the coupling between the y and η directions is small,
(3) the region around the nucleation point where the WKB approximation breaks is narrow,
(4) the coupling between the y and η directions vanishes around the false vacuum.

We hope to return to more general cases, say, cases where assumptions (3) and/or (4) are relaxed, in future. If there was no coupling between the two directions [i.e. if $V_\eta(y, \eta)$ could be denoted as $V_\eta(\eta)$], the tunneling wave function $\Psi(y, \eta)$ would be given by the product of $\Psi_y(y)$ and $\Phi(\eta)$, where $\Psi_y(y)$ is the 1-dimensional tunneling wave function for $V_{\text{tun}}(y)$ and $\Phi(\eta)$ is the ground state for $V_\eta(\eta)$. In our case, however, we consider small but nonvanishing coupling, and thus we expand $\Psi(y, \eta)$ and E in Eq. (2) as

$$\Psi(y, \eta) = \Psi_y(y)\Phi(y, \eta), \qquad E = E_y + E_\eta. \tag{3}$$

Here, $\Psi_y(y)$ and E_y are, respectively, the wave function and energy of the 1-dimensional Schrödinger equation $\mathcal{H}_y\Psi_y(y) = E_y\Psi_y(y)$, which we will briefly discuss below. As a result of assumption (4), the quasi-ground-state is given by $\Psi_y(y)\Phi_F(\eta)$, where $\Phi_F(\eta)$ is the ground state for the η part of the potential around the false vacuum $V_F(\eta)(\equiv V_\eta(y_F, \eta))$. Here, by focusing on Eq. (2) around the false vacuum and denoting the ground state energy with respect to $V_F(\eta)$ as E_F, it can be seen that E_η is given by E_F.

As shown in Fig. 2, the tunneling path $y(\tau)$, or instanton, is a solution of the Euclidean equation of motion (EOM) $y''(\tau) - dV_{\text{tun}}/dy = 0$, where $'$ denotes the derivative

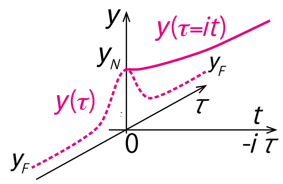

FIG. 2 (color online). The schematic picture of instanton $y(\tau)$ with the imaginary time τ (doted line) and its analytical continuation $y(t) \equiv y(\tau = it)$ with the real time t (solid line).

with respect to the imaginary, or Euclidean, time τ. The boundary conditions for $y(\tau)$ are given by $y(\pm\infty) = y_F$ and $y(0) = y_N$, where the freedom in choosing the origin of τ is fixed. Well inside the potential barrier, we rewrite the wave function as $\Psi_y(y) = e^{-S_y(y)/\hbar}$ with the Euclidean action $S_y(\tau)(= S_y(y(\tau)))$, and make the WKB expansion $S_y = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots$. Then, by solving the Schrödinger equation order by order and using the instanton $y(\tau)$, we can obtain $dS_0(y)/dy = y'(\tau)$, $S_1(y) = (1/2)\ln(dS_0/dy)$, and so on, where we take τ to be in the region $\tau \in (-\infty, 0)$. It is known that we can move from inside the barrier to outside the barrier by analytical continuation $\tau \to t = -i\tau$, where t is the real, or Lorentzian, time. After the analytical continuation, the instanton gives the classical motion of the particle $y(t) \equiv y(\tau = it)$, which starts rolling down from the nucleation point at $t = 0$, as shown in Figs. 1 and 2. Furthermore, the analytical continuation of the Euclidean action $S_y(t) \equiv S_y(\tau = it)$ gives the tunneling wave function $\Psi_y(y(t)) = e^{-S_y(t)/\hbar}$ well outside the barrier. In the following, we can use τ, t and y interchangeably.

Now, we will transform Eq. (2) inside the potential barrier. By substituting Eq. (3) with $E_\eta = E_F$ into Eq. (2) and using the 1-dimensional Schrödinger equation $\mathcal{H}_y\Psi_y(y) = E_y\Psi_y(y)$, we obtain

$$\hbar\frac{dS_y}{dy}\frac{\partial}{\partial y}\Phi(y,\eta) - \frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2}\Phi(y,\eta) + \hat{H}(y)\Phi(y,\eta) = 0, \tag{4}$$

where

$$\hat{H}(y) = \frac{\hat{p}_\eta^2}{2} + V_\eta(y,\eta) - E_F. \tag{5}$$

Here, we can neglect the second term in Eq. (4), since the y dependence of $\psi(y,\eta)$ is expected to be small as a result of assumption (2). By neglecting the second term in Eq. (4) and using the leading order relation in the WKB approximation $\hbar(dS_y/dy)(\partial/\partial y) \approx \hbar(\partial/\partial\tau)$, we can transform Eq. (4) into

$$-\hbar\frac{\partial}{\partial\tau}\Phi(\tau,\eta) = \hat{H}(\tau)\Phi(\tau,\eta). \tag{6}$$

This equation is of exactly the same form as the "time-dependent Schrödinger equation" with imaginary time τ, defined for $\tau \in (-\infty, 0)$.

Let us now check the consistency of the approximation used to derive Eq. (6), by estimating the size of the second term in Eq. (4). To next-to-leading order in the WKB approximation, the coefficient of the second term in Eq. (4) can be approximated as

$$\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2} \approx -\frac{\hbar y''}{y'^3}\hbar\frac{\partial}{\partial\tau} + \frac{1}{2y'^2}\left(\hbar\frac{\partial}{\partial\tau}\right)^2. \tag{7}$$

Here, $\hbar y''/y'^3 \approx 2(dS_1(y)/dy)/(dS_0(y)/dy)$ and $(\hbar\partial/\partial\tau)$ can be estimated as $O(\hbar\omega)$ using Eq. (6). Thus, when the first and second operators on the rhs act on $\Phi(\tau,\eta)$, they give terms that are suppressed, under WKB approximation, by factors of $O((dS_1(y)/dy)/(dS_0(y)/dy))$ and $O(\hbar\omega/y'^2)$ relative to other terms in Eq. (4), respectively.

It may be useful to make a comment on the WKB expansion used above. Strictly speaking, this expansion is not merely a expansion in $\hbar$ where η is considered to be $O(\hbar^{1/2})$, as was done in [12]. In such an expansion, the nonlinear interaction terms would not appear in Eq. (6), since the nonlinear interaction terms would become higher order in $\hbar$ [e.g. η^3 term would become $O(\hbar^{3/2})$]. Rather, here we have expanded equations based on the fact that the classical part of the wave function $S_0(y)$ dominates over quantum effects, which makes it possible to consistently take into account the effect of nonlinear interaction terms in Eq. (6).

We can also transform Eq. (2) outside the barrier, following similar arguments to those outlined above but with the real time t instead of the imaginary time τ. As a result of the analytical continuation $\tau \to t = -i\tau$, we obtain

$$i\hbar\frac{\partial}{\partial t}\Phi(t,\eta) = \hat{H}(t)\Phi(t,\eta), \tag{8}$$

which is the "time-dependent Schrödinger equation" with real time t, defined for $t \in (0,\infty)$. For later convenience, let us recall that the original 2-dimensional wave function $\Psi(y,\eta)$ is denoted as

$$\Psi(y,\eta) = \exp[-S(t)/\hbar]\Phi(t,\eta), \tag{9}$$

where $y(= y(\tau = it))$ is inside and outside the potential barrier for $t \in (+i\infty, 0)$ and $t \in (0,\infty)$, respectively.

Around the false vacuum or the nucleation point, where the WKB approximation is not valid, we determine Φ using matching conditions. Thanks to assumptions (3) and (4), the matching conditions are given in a simple way. Firstly, the matching condition at $y = y_N$ is given by

$$\lim_{\tau\to -0}\Phi(\tau,\eta) = \lim_{t\to +0}\Phi(t,\eta), \tag{10}$$

since $\Psi(y,\eta) = \Psi_y(y)\Phi(y,\eta)$ on both sides of y_N should have the same value at y_N. Here, we can use Eq. (6) and (8) until very close to y_N thanks to assumption (3). Secondly, the matching condition at $y = y_F$ is given by

$$\lim_{\tau\to -\infty}\Phi(\tau,\eta) = \Phi_F(\eta), \tag{11}$$

since the wave function is assumed to match the quasi-ground-state around the false vacuum, which is given by $\Psi_y(y)\Phi_F(\eta)$ due to assumption (4), as mentioned below Eq. (3).

B. Expectation values of operators

We will obtain the tunneling wave function by solving Eq. (6) and (8) with the matching condition Eq. (10) and (11). For notational simplicity, we introduce bra-ket notation, where Eq. (8) is written as

$$i\hbar\frac{\partial}{\partial t}|\Phi(t)\rangle = \hat{H}(t)|\Phi(t)\rangle, \tag{12}$$

with

$$\langle\eta|\Phi(t)\rangle = \Phi(t, \eta). \tag{13}$$

The formal solution to Eq. (12) is given by

$$|\Phi(t)\rangle = P\left(\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} H(t')dt'\right]\right)|\Phi(t_0)\rangle, \tag{14}$$

where $0 < t_0 < t$ and the path ordering operator P orders operators according to their order along the integration path. From now on, we omit ˆ over operators for brevity. Similarly, the formal solution to Eq. (6) is given by

$$|\Phi(\tau)\rangle = P\left(\exp\left[-\frac{1}{\hbar}\int_{-i\tau_0}^{-i\tau} H(\tau')d\tau'\right]\right)|\Phi(\tau_0)\rangle, \tag{15}$$

for $\tau_0 < \tau < 0$. The expressions in Eqs. (14) and (15) are not valid at the nucleation point, where the WKB approximation breaks down. However, thanks to the matching condition given by Eq. (10), which can be written in bra-ket notation as $|\Phi(\tau = -0)\rangle = |\Phi(t = +0)\rangle$, we can connect the two expressions at the nucleation point as

$$\begin{aligned}|\Phi(t)\rangle &= P\left(\exp\left[-\frac{i}{\hbar}\int_{0}^{t} H(t')dt'\right]\right)|\Phi(0)\rangle \\ &= P\left(\exp\left[-\frac{i}{\hbar}\int_{-i\tau_0\to 0\to t} H(t')dt'\right]\right)|\Phi(-i\tau_0)\rangle,\end{aligned} \tag{16}$$

where $\int_{-i\tau_0\to 0\to t} dt' = \int_0^t dt' + \int_{-i\tau_0}^0 dt'$.

The matching around the false vacuum is given as follows. We consider a wave function which matches the quasi-ground-state around the false vacuum. The ket $|\Omega_F\rangle$ corresponding to the quasi-ground-state $\Phi_F(\eta)$ can be given by

$$|\Omega_F\rangle = \lim_{T\to\infty} e^{-\frac{1}{\hbar}H_F T}|\Phi\rangle, \tag{17}$$

where $H_F \equiv H(+i\infty)$ and $|\Phi\rangle$ is arbitrary as long as it is not orthogonal to $|\Omega_F\rangle$. We don't need to care about the overall normalization of $|\Omega_F\rangle$, since it will be canceled in the calculations of quantum expectation values, as will be seen below. In deriving Eq. (17), we use the fact that the ground state has $H_F = 0$ while excited states have $H_F > 0$, which comes from the definition of $H(y)$ in Eq. (5). From assumption (4), there exists a τ_0 such that for $\tau < \tau_0$ we can approximate $H(\tau)$ and $|\Phi(-i\tau)\rangle$ as H_F and $|\Omega_F\rangle$, respectively. Thus, using Eqs. (16) and (17), the state evolving from $|\Omega_F\rangle$ at $t = -i\tau_0$ is given by

$$\begin{aligned}|\Phi(t)\rangle &= P\left(\exp\left[-\frac{i}{\hbar}\int_{-i\tau_0\to 0\to t} H(t')dt'\right]\right)\lim_{T\to\infty} e^{-\frac{1}{\hbar}H_F T}|\Phi\rangle \\ &= P\left(\exp\left[-\frac{i}{\hbar}\int_{+i\infty\to 0\to t} H(t')dt'\right]\right)|\Phi\rangle.\end{aligned} \tag{18}$$

Now we are able to evaluate the quantum expectation values. For an operator $\mathcal{O}$ with respect to η (i.e. some function of η and p_η), the quantum expectation value at given $y(= y(t))$ outside the barrier is given by

$$\langle\mathcal{O}\rangle_y = \frac{\int_{-\infty}^{\infty} d\eta\Psi^*(y, \eta)\mathcal{O}\Psi(y, \eta)}{\int_{-\infty}^{\infty} d\eta|\Psi(y, \eta)|^2} = \frac{\langle\Phi(t)|\mathcal{O}|\Phi(t)\rangle}{\langle\Phi(t)|\Phi(t)\rangle}. \tag{19}$$

To derive the second line, we use Eq. (9) and cancel the factors $e^{-\frac{1}{\hbar}S(t)}$ appearing both in numerator and denominator. Taking the Hermitian conjugate of Eq. (18), we obtain

$$\begin{aligned}\langle\Phi(t)| &= (|\Phi(t)\rangle)^\dagger \\ &= \langle\Phi|P\left(\exp\left[-\frac{i}{\hbar}\int_{t\to 0\to -i\infty} H(t')dt'\right]\right),\end{aligned} \tag{20}$$

where $H(t^*) = H(t)$ since $H(y)$ given in Eq. (5) depends only on y and $y(-\tau) = y(\tau)$ due to the Euclidean time inversion symmetry of the instanton. By substituting Eqs. (18) and (20) into Eq. (19), we obtain the resulting formula for the quantum expectation values in the Schrödinger picture

$$\langle\mathcal{O}\rangle_y = \frac{\langle\Phi|P(\mathcal{O}\exp[-\frac{i}{\hbar}\int_C H(t')dt'])|\Phi\rangle}{\langle\Phi|P(\exp[-\frac{i}{\hbar}\int_C H(t')dt'])|\Phi\rangle}, \tag{21}$$

where

$$C\colon\ +i\infty \to 0 \to t \to 0 \to -i\infty \tag{22}$$

is the time integration path, as shown in Fig. 3, and $\mathcal{O}$ is ordered by P as if it is defined at t. In the denominator of

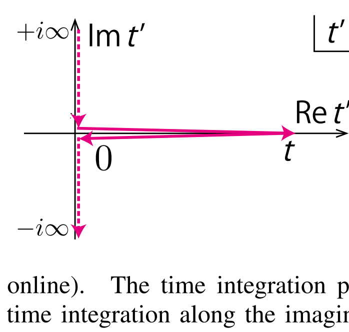

FIG. 3 (color online). The time integration path C given by Eq. (22). The time integration along the imaginary axis (doted line) corresponds to the evolution of the quantum state during tunneling, and along the real axis (solid line) corresponds to the evolution after tunneling.

Eq. (21), we can deform the integration path from C to $i\infty \to -i\infty$ using $P(\exp[-\frac{i}{\hbar}\int_{0\to t\to 0} H(t')dt']) = 1$. If $|\Phi\rangle$ was chosen to be orthogonal to $|\Omega_F\rangle$, we could obtain the quantum expectation values for quantum tunneling from an excited state, as studied in [12]. We leave such issues to future studies.

III. FORMULATION: INTERACTION PICTURE

A. Relation between interaction and Schrödinger pictures

Since the expression given in Eq. (21) is difficult to evaluate directly, in this section we will move from the Schrödinger picture formulation to the interaction picture one. This can be accomplished almost in the same way as usual, but taking into account the nonunitarity of the evolution operator for the imaginary part of the integration path. The interaction picture formulation may be helpful when considering the multidimensional tunneling system in the context of quantum field theory, where the interaction picture is employed.

First of all, we introduce the evolution operator

$$\begin{aligned} U(t_2, t_1) &= P\left(\exp\left[-\frac{i}{\hbar}\int_{t_1}^{t_2} H(t)dt\right]\right) \\ &\equiv 1 + (-i)\int_{t_1}^{t_2} H(t')dt' + (-i)^2 \\ &\quad \times \int_{t_1}^{t_2} dt' \int_{t_1}^{t'} dt'' H(t')H(t'') + \cdots, \end{aligned} \tag{23}$$

where t_1 and t_2 are on the path C given by Eq. (22). The inverse operator for $U(t_2, t_1)$ is given by

$$(U(t_2, t_1))^{-1} = U(t_1, t_2), \tag{24}$$

which can be confirmed by explicit calculation of $(U(t_2, t_1)(U(t_2, t_1))^{-1}$ using Eq. (23). The combination rule

$$U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1), \tag{25}$$

is satisfied as usual. It should be noted that $U(t_2, t_1)$ is not generally a unitary operator since the path C includes the imaginary part, and that $U(t_2, t_1)$ satisfies the relation $U(t_2, t_1)^\dagger = U(t_2^*, t_1^*)^{-1}$.

To find interaction picture expression, we expand the full Hamiltonian given in Eq. (5) as $H(t) = H_0(t) + H_{\rm int}(t)$, where the free part $H_0(t)$ and the interaction part $H_{\rm int}(t)$ are given, respectively, by

$$H_0(t) = \frac{p_\eta^2}{2} + \frac{\omega^2(t)}{2}\eta^2 - E_F, \qquad H_{\rm int}(t) = V_{\rm int}(y(t), \eta). \tag{26}$$

Using $H_0(t)$, we can define the annihilation and creation operators at each t, respectively, as

$$\begin{aligned} a_t &= \sqrt{\frac{2\omega(t)}{\hbar}}\eta + i\sqrt{\frac{2}{\hbar\omega(t)}}p_\eta, \\ a_t^\dagger &= \sqrt{\frac{2\omega(t)}{\hbar}}\eta - i\sqrt{\frac{2}{\hbar\omega(t)}}p_\eta, \end{aligned} \tag{27}$$

where a_t and $a_t^\dagger$ satisfy the usual commutation relation. The eigenstates with respect to $H_0(t)$ can be defined with a_t and $a_t^\dagger$ as

$$|n_t\rangle = \frac{1}{\sqrt{n!}}(a_t^\dagger)^n|0_t\rangle, \qquad a_t|0_t\rangle = 0, \tag{28}$$

where they satisfy

$$H_0(t)|n_t\rangle = E_{n_t}|n_t\rangle, \quad E_{n_t} = \hbar\omega(t)\left(n + \frac{1}{2}\right) - E_F. \tag{29}$$

When $H_0(t)$ explicitly depends on time, a_t and $|0_t\rangle$ also become time dependent. a_t at the times $t = t_1$ and $t = t_2$ are related by a Bogolubov transformation, and $|0_{t_1}\rangle$ and $|0_{t_2}\rangle$ are annihilated by a_{t_1} and a_{t_2}, respectively. The evolution operator for the free Hamiltonian $H_0(t)$ is given by

$$U^{(0)}(t_2, t_1) = P\left(\exp\left[-\frac{i}{\hbar}\int_{t_1}^{t_2} H_0(t)dt\right]\right). \tag{30}$$

Interaction picture operators $\mathcal{O}_I(t)$ are defined by

$$\mathcal{O}_I(t) \equiv U^{(0)}(0, t)\mathcal{O}U^{(0)}(t, 0), \tag{31}$$

where $\mathcal{O}$ are Schrödinger picture operators. In the interaction picture, states are evolved with the evolution operator for $H_I(t)$, given by

$$U_I(t_2, t_1) = P\left(\exp\left[-\frac{i}{\hbar}\int_{t_1}^{t_2} H_I(t)dt\right]\right), \tag{32}$$

where the interaction Hamiltonian $H_I(t)$ is defined as

$$H_I(t) \equiv H_{\rm int}(\eta_I(t), t). \tag{33}$$

For any t_1 and t_2 on C given by Eq. (22), we can rewire $U_I(t_2, t_1)$ in terms of $U(t_2, t_1)$ from Eq. (23) and $U^{(0)}(t_2, t_1)$ from Eq. (30) as

$$U_I(t_2, t_1) = U(t_2, t_1)U^{(0)}(t_1, t_2) = U^{(0)}(t_1, t_2)U(t_2, t_1), \tag{34}$$

which can be confirmed by explicit calculation.

To describe $\eta_I(t)$ and $p_{\eta I}(t)$ in a simple way, we introduce a positive frequency function $u(t)$ and a negative frequency function $v(t)$. They are defined as solutions to the linearized EOM,

$$\ddot{u}(t) = -\omega^2(t)u(t), \qquad \ddot{v}(t) = -\omega^2(t)v(t), \tag{35}$$

which are complex conjugate to each other when t is real;

$$u^*(t) = v(t) \quad \text{for real } t, \tag{36}$$

and satisfy Klein-Goldon (KG) normalization,

$$u(t)\dot{v}(t) - \dot{u}(t)v(t) = i\hbar. \tag{37}$$

Here, a dot denotes the derivative with respect to t. When t is imaginary, since we define $u(t)$ and $v(t)$ by analytical continuation from real t, Eqs. (35) and (37) still hold but Eq. (36) is no longer true. It should be noted that the freedom in choosing $u(t)$ corresponds to the freedom to make an arbitrary Bogolubov transformation.

Using $u(t)$ and $v(t)$, we can define the annihilation operator a and the creation operator $a^\dagger$, respectively, as

$$a = -\frac{i}{\hbar}(\eta_I(t)\dot{v}(t) - p_{\eta I}(t)v(t)),$$
$$a^\dagger = \frac{i}{\hbar}(\eta_I(t)\dot{u}(t) - p_{\eta I}(t)u(t)). \tag{38}$$

We will see below that the operators defined in Eq. (38) are time independent and Hermitian conjugate to each other. Firstly, it can be explicitly shown that these operators are time independent by differentiating a and $a^\dagger$ in Eq. (38) with respect to t and using Eq. (35) and the evolution equations for $\eta_I(t)$ and $p_{\eta I}(t)$,

$$\dot{\eta}_I(t) = \frac{1}{i\hbar}[\eta_I(t), H_0(t)] = p_{\eta I}(t),$$
$$\dot{p}_{\eta I}(t) = \frac{1}{i\hbar}[p_{\eta I}(t), H_0(t)] = -\omega^2(t)\eta_I(t). \tag{39}$$

Since Eqs. (35) and (39) are valid not only for real t but also for imaginary t, Eq. (38) can be used even when t is imaginary. Secondly, by considering Eq. (38) when t is real and using Eq. (36) and the Hermiticity of $\eta_I(t)$ and $p_{\eta I}(t)$, it is clear that a and $a^\dagger$ defined in Eq. (38) are Hermitian conjugate to each other. Using Eqs. (37) and (38), $\eta_I(t)$ and $p_{\eta I}(t)$ can be written, respectively, as

$$\eta_I(t) = au(t) + a^\dagger v(t), \quad p_{\eta I}(t) = a\dot{u}(t) + a^\dagger \dot{v}(t). \tag{40}$$

It should be noted that Eq. (40) is valid not only for real t but also for imaginary t.

B. In-in formalism along complex path

For later convenience, we introduce the state $|\Phi_N\rangle$, which is the state at the nucleation point when nonlinear interactions are switched off. By taking the limit $t \to \pm i\infty$ in Eqs. (26)–(29), we define ω_F, a_F, $|n_F\rangle$, H_{0_F} and E_{n_F}. Using those asymptotic quantities, $|\Phi_N\rangle$ is obtained as

$$|\Phi_N\rangle = \lim_{T\to\infty} e^{E_{0_F}T} U^{(0)}(0, iT)|0_F\rangle, \tag{41}$$

where the normalization factor $e^{E_{0_F}T}$ is introduced to make the expression finite and constant in the limit $T \to \infty$. As a result of the explicit t dependence of the free Hamiltonian $H_0(t)$, $|\Phi_N\rangle$ is not proportional to $|0_F\rangle$ in general. The difference between $|\Phi_N\rangle$ and $|0_F\rangle$ is determined by solving the EOMs for the positive and negative frequency functions given in Eq. (35).[1]

As will be confirmed below, the annihilation operator a that annihilates $|\Phi_N\rangle$ is associated with $u(t)$ and $v(t)$ defined with the boundary conditions

$$u(t) \xrightarrow{t\to -i\infty} e^{-i\omega_F t}, \qquad v(t) \xrightarrow{t\to +i\infty} e^{i\omega_F t}, \tag{42}$$

up to constant factors determined by the KG normalization. Note that $u(t)$ and $v(t)$ satisfy the conditions for positive and negative frequency functions given by Eqs. (36) and (37). The corresponding annihilation operator is defined by substituting $v(t)$ given by Eq. (42) into Eq. (38), and can be rewritten as

$$\begin{aligned} a &= -\frac{i}{\hbar} U^{(0)}(0, t)(\eta\dot{v}(t) - p_\eta v(t))U^{(0)}(t, 0) \\ &\propto \lim_{T\to\infty} e^{-\omega_F T} U^{(0)}(0, iT) a_F U^{(0)}(iT, 0). \end{aligned} \tag{43}$$

In deriving the first equality we used Eqs. (31) and (38) and the t independence of a, and in deriving the second we used Eq. (27) in the limit $t \to i\infty$ along with Eq. (42). Then, using Eqs. (41) and (43), we can explicitly show that

$$\begin{aligned} a|\Phi_N\rangle &\propto \lim_{T\to\infty} e^{(E_{0_F}-\omega_F)T} U^{(0)}(0, iT) a_F U^{(0)}(iT, 0) \\ &\quad \times U^{(0)}(0, iT)|0_F\rangle = 0, \end{aligned} \tag{44}$$

as we stated above.

Now we will move from the Schrödinger picture to the interaction picture. By inserting $U^{(0)}(t_1, t_2)U^{(0)}(t_2, t_1) = 1$ into Eq. (21) many times, and using Eqs. (31) and (41), we obtain

$$\begin{aligned} \langle \mathcal{O}\rangle_y &= \frac{\begin{array}{c}\langle\Phi|U^{(0)}(-i\infty, 0)U^{(0)}(0, -i\infty)U(-i\infty, 0)U(0, t)U^{(0)}(t, 0)U^{(0)}(0, t) \\ \times\, \mathcal{O}U^{(0)}(t, 0)U^{(0)}(0, t)U(t, 0)U(0, i\infty)U^{(0)}(i\infty, 0)U^{(0)}(0, i\infty)|\Phi\rangle\end{array}}{\langle\Phi|U^{(0)}(-i\infty, 0)U^{(0)}(0, -i\infty)U(-i\infty, 0)U(0, i\infty)U^{(0)}(i\infty, 0)U^{(0)}(0, i\infty)|\Phi\rangle} \\ &= \frac{\langle\Phi_N|U_I(-i\infty, t)\mathcal{O}_I(t)U_I(t, i\infty)|\Phi_N\rangle}{\langle\Phi_N|U_I(-i\infty, i\infty)|\Phi_N\rangle}, \end{aligned} \tag{45}$$

where the overall factors appearing in both numerator and denominator cancel each other. To make the correspondence between this result and that of the conventional in-in formalism [23,24] clearer, we can rewrite Eq. (45) as

[1] The effect of the explicit t dependence of $H_0(t)$ was determined by directly solving the Schrödinger equation in [12]. For the correspondence between this work and [12], see the Appendix.

$$\langle \mathcal{O}(t) \rangle = \frac{\langle P(\mathcal{O}_I(t) \exp[-\frac{i}{\hbar}\int_C H_I(t')dt'])\rangle^{(N)}}{\langle P(\exp[-\frac{i}{\hbar}\int_C H_I(t')dt'])\rangle^{(N)}}, \quad (46)$$

where the time integration path C is given by Eq. (22), $\langle \mathcal{O}(t) \rangle \equiv \langle \mathcal{O} \rangle_y$, and $\langle \rangle^{(N)}$ is defined as $\langle \mathcal{O} \rangle^{(N)} \equiv \langle \Phi_N | \mathcal{O} | \Phi_N \rangle / \langle \Phi_N | \Phi_N \rangle$. We can deform the integration path in the denominator from C to $i\infty \rightarrow -i\infty$ using $P(\exp[-\frac{i}{\hbar}\int_{0\rightarrow t\rightarrow 0} H_I(t')dt']) = 1$.

Since the annihilation operator a annihilates $|\Phi_N\rangle$, Wick's theorem can be used to evaluate Eq. (46) as usual. The N-point correlation function $\langle P(\eta_I(t_1)\eta_I(t_2)\ldots \eta(t_N))\rangle^{(N)}$ vanishes when N is odd, but is given by

$$\begin{aligned} &\langle P(\eta_I(t_1)\eta_I(t_2)\ldots \eta(t_N))\rangle^{(N)} \\ &\quad = \sum_{\text{set of pairs}} \prod_{\text{pairs}} \langle P(\eta_I(t_i)\eta_I(t_j))\rangle^{(N)}, \end{aligned} \quad (47)$$

when N is even. Here, the 2-point correlation function $\langle P(\eta_I(t_1)\eta_I(t_2))\rangle^{(N)}$ can be evaluated as

$$\begin{aligned} &\langle P(\eta_I(t_1)\eta_I(t_2))\rangle^{(N)} \\ &\quad = \begin{cases} u(t_1)v(t_2) & \text{when } t_1 \text{ precedes } t_2 \text{ along } C, \\ u(t_2)v(t_1) & \text{when } t_2 \text{ precedes } t_1 \text{ along } C, \end{cases} \end{aligned} \quad (48)$$

where $u(t)$ and $v(t)$ are given by Eq. (42).

Before closing this section, let us summarize what we have found. The expression given in Eq. (46) is in the same form as the conventional in-in formalism, which is often used in quantum field theory calculations involving interactions [23,24]. However, the time integration path C: $i\infty \rightarrow 0 \rightarrow t \rightarrow 0 \rightarrow -i\infty$ is different from the usual case, where the time integration path is $t_0 \rightarrow t \rightarrow t_0$ when the initial state is given at an initial time t_0 or $-\infty(1 - i\epsilon) \rightarrow t \rightarrow -\infty(1 + i\epsilon)$ when the initial state is given in the past infinity. In our case, the quasi-ground-state is chosen as the initial state of the false vacuum, and the corresponding time is given as $t = \pm i\infty$ using the instanton $y(\tau)$ defined with Euclidean time $\tau = it$. In Eq. (46), the imaginary part of the integration path C corresponds to the evolution inside the barrier, or during tunneling, while the real part corresponds to the evolution outside the barrier, or after tunneling.

IV. APPLICATION TO TOY MODEL

A. Toy model

For illustration purposes, we explicitly apply the formalism obtained above to a simple toy model. We assume that the instanton is given by

$$y(\tau) \approx \begin{cases} y_F & (-\infty < \tau < -\tau_W), \qquad (\tau_W < \tau < \infty), \\ y_N & (-\tau_W < \tau < +\tau_W), \\ > y_N & (0 < t(= -i\tau) < \infty), \end{cases} \quad (49)$$

where τ_W $(0 < \tau_W)$ is the wall size of the thin-wall instanton, and that the potential $V_\eta(y, \eta)$ is given by

$$V_\eta(y, \eta) = \frac{\omega^2}{2}\eta^2 + \tilde{\lambda}(y)\eta^3, \quad (50)$$

where the y-dependent coupling constant $\tilde{\lambda}(y)$ is assumed to be effective only inside the potential barrier (i.e. $y_F < y < y_N$). By substituting Eqs. (49) and (50) into Eq. (26), $H(\tau) = H_0 + H_{\text{int}}(\tau)$ can be written as

$$\begin{aligned} H_0 &= \frac{p^2}{2} + \frac{\omega^2}{2}\eta^2 - \frac{\hbar\omega}{2}, \\ H_{\text{int}}(\tau) &\approx \lambda\delta(\tau - \tau_W)\eta^3 + \lambda\delta(\tau + \tau_W)\eta^3, \end{aligned} \quad (51)$$

where $\delta(x)$ is Dirac's delta function and $\lambda = \int_{-\tau_W - 0}^{-\tau_W + 0} \tilde{\lambda}(\bar{y}(\tau))d\tau$. Here, the eigenenergy of the quasi-ground-state is given by $E_F = \hbar\omega/2$, since $H_{\text{int}}(\tau)$ vanishes around the false vacuum and the quasi-ground-state is the ground state for H_0. In the following, we denote the ground state and the annihilation operator associated with H_0 as $|0\rangle$ and a, respectively. We will calculate $\langle \eta \rangle_y$, or $\langle \eta(t) \rangle$, using both the the Schrödinger and interaction picture expressions, given in Eqs. (21) and (46), respectively. Although $\langle \eta(t) \rangle = 0$ in the free theory calculation, we obtain $\langle \eta(t) \rangle(t) \neq 0$ as a result of the effect of nonlinear interaction.

B. Calculation in Schrödinger picture

To evaluate Eq. (21), we obtain $|\Phi(t)\rangle$ using Eq. (18). The evolution of the ground state $|0\rangle$ defined at the false vacuum ($t' = +i\infty$) to behind the wall ($t' = -i(-\tau_W - 0)$) is trivial since $H(t')$ is simply given by H_0 in this region, and we obtain

$$|\Phi(-i(-\tau_W - 0))\rangle = |0\rangle. \quad (52)$$

Using Eq. (51), the evolution of the state across the wall [i.e. $t' = -i(-\tau_W - 0) \rightarrow -i(-\tau_W + 0)$] is given by

$$|\Phi(-\tau_W + 0)\rangle = e^{-\frac{\lambda}{\hbar}\eta^3}|\Phi(-\tau_W - 0)\rangle. \quad (53)$$

Since $H(t')$ is again simply H_0 from in front of the wall ($t' = -i(-\tau_W + 0)$) to outside the barrier ($t' = t$), the evolution of the state between them is given by

$$|\Phi(t)\rangle = \exp\left[-\frac{i}{\hbar}H_0(t - i\tau_W)\right]|\Phi(-i(-\tau_W + 0))\rangle. \quad (54)$$

By combining Eqs. (52)–(54) we obtain, to first order in λ,

$$|\Phi(t)\rangle \approx \exp\left[-\frac{i}{\hbar}H_0(t - i\tau_W)\right]\left(1 - \frac{\lambda}{\hbar}\eta^3\right)|0\rangle, \quad (55)$$

and its Hermitian conjugate is given by

$$\langle \Phi(t)| \approx \langle 0|\left(1 - \frac{\lambda}{\hbar}\eta^3\right)\exp\left[\frac{i}{\hbar}H_0(t + i\tau_W)\right]. \quad (56)$$

By substituting Eqs. (55) and (56) into Eq. (21) we obtain, to leading order in λ,

$$\begin{aligned}\langle\eta(t)\rangle &\approx -\frac{\lambda}{\hbar}\left\langle 0\,\middle|\,\eta\exp\left[-\frac{i}{\hbar}H_0(t-i\tau_W)\right]\eta^3\right. \\ &\quad\left. + \eta^3\exp\left[\frac{i}{\hbar}H_0(t+i\tau_W)\right]\eta\,\middle|\,0\right\rangle \\ &= -\frac{3\hbar\lambda}{2\omega^2}\cos(\omega t)e^{-\omega\tau_W}. \end{aligned} \tag{57}$$

To obtain the second line, we used $[a, a^\dagger] = 1$, $H_0|0\rangle = 0$, $[H_0, a] = -\hbar\omega$, $[H_0, a^\dagger] = \hbar\omega$ and $\eta = (\hbar/2\omega)^{1/2}(a + a^\dagger)$.

C. Calculation in interaction picture

Since H_0 is independent of t, Eq. (35) can be easily solved. $u(t)$ and $v(t)$ defined with the boundary conditions in Eq. (42) are given, respectively, by

$$u(t) = \sqrt{\frac{\hbar}{2\omega}}e^{-i\omega t}, \qquad v(t) = \sqrt{\frac{\hbar}{2\omega}}e^{i\omega t}. \tag{58}$$

By using $H_{\rm int}(\tau)$ given in Eq. (51) along with Eq. (32), we obtain, to first order in λ,

$$\exp\left[-\frac{i}{\hbar}\int_C H_I(t')dt'\right] \approx 1 - \frac{\lambda}{\hbar}\eta_I^3(i\tau_W) - \frac{\lambda}{\hbar}\eta_I^3(-i\tau_W). \tag{59}$$

By substituting Eq. (59) into Eq. (45) we obtain, to leading order in λ,

$$\begin{aligned}\langle\eta(t)\rangle &\approx -\frac{\lambda}{\hbar}\langle\eta_I(t)\eta_I^3(i\tau_W) + \eta_I^3(-i\tau_W)\eta_I(t)\rangle^{(N)} \\ &= -\frac{3\hbar\lambda}{2\omega^2}\cos(\omega t)e^{-\omega\tau_W}, \end{aligned} \tag{60}$$

which is in agreement with Eq. (57), as it should be. To obtain the second line, we used Wick's theorem, as in Eq. (47). Here, for example, $\langle\eta_I(t_1)\eta_I^3(t_2)\rangle^{(N)}$ can be evaluated as

$$\begin{aligned}\langle\eta_I(t_1)\eta_I^3(t_2)\rangle^{(N)} &= 3\langle\eta_I(t_1)\eta_I(t_2)\rangle^{(N)}\langle\eta_I^2(t_2)\rangle^{(N)} \\ &= 3u(t_1)u(t_2)v^2(t_2). \end{aligned} \tag{61}$$

V. CONCLUSION

We have studied a 2-dimensional tunneling system, where the tunneling sector y is nonlinearly coupled to an oscillator η. Assuming the system is initially in a quasi-ground-state at the false vacuum, the 2-dimensional tunneling wave function $\psi(y, \eta)$ has been constructed using the WKB method. We have considered the effect of nonlinear interactions, which has not been studied in the context of multidimensional tunneling systems before, to our knowledge.

We have determined the quantum expectation values with respect to the η direction at a given y outside the barrier. We first introduced a Schrödinger picture formulation to obtain Eq. (21) in Sec. II, and then moved to an interaction picture formulation in Sec. III to obtain Eq. (46). The resulting formula given in Eq. (46) is of the same form as the conventional in-in formalism, which is often used in quantum field theory calculations with interactions [23,24]. However, the time integration path is modified to the one consisting of an imaginary part in addition to a real part.

The difference in the integration path for the usual case and the quantum tunneling case can be understood as follows. In the usual case, an initial state is given at some finite past $t = t_0$ or the infinite past $t = -\infty$, both of which are defined on the real axis. However, in the case of quantum tunneling, the initial state is given at the false vacuum, where the corresponding time is $t = \pm i\infty$. In our case, the imaginary part of the integration path corresponds to the evolution of the quantum state during tunneling, while the real part corresponds to the evolution after the quantum tunneling.

In this paper, the formulation has been done in a multidimensional quantum mechanical system. In order to apply it to cosmology, we need to extend the formulation to field theory, with gravitational effects included. Such an extension has been done in the case without interactions in [12–14], and we expect similar extension to be possible in the case with interactions. Although a full derivation is now under investigation, one might naively expect that the integration path will also consist of an imaginary part corresponding to the evolution during quantum tunneling, and real part corresponding to the evolution after quantum tunneling. Calculations assuming this naive expectation to be true have already been performed in the literature [21,22].

Observable effects resulting from nonlinear interactions, such as the non-Gaussianity of cosmological fluctuations, are now recognized as powerful tools to probe the early universe. It is therefore important for us to be able to determine such features that may result from models involving quantum tunneling, which are motivated by the string landscape.

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APPENDIX: POSITIVE FREQUENCY FUNCTION AND WAVE FUNCTION

In this appendix, we will illustrate the the relation between the positive frequency function $u(t)$ used in this work and its corresponding wave function $\psi(\eta, t)$ used in the literature [12]. We will employ the 1-dimensional harmonic oscillator with Hamiltonian

$$H = \frac{p_\eta^2}{2} + \frac{\omega^2}{2}\eta^2, \tag{A1}$$

as an example. We will also see how the freedom in choosing $u(t)$ and $v(t)$ is related to the Bogolubov transformation.

As usual, the ground state $|0\rangle$ and the corresponding annihilation operator a are given by

$$a = \sqrt{\frac{\omega}{2\hbar}}\eta + i\frac{1}{\sqrt{2\hbar\omega}}p_\eta, \qquad a|0\rangle = 0, \tag{A2}$$

where the Hamiltonian can be rewritten as $H = \hbar\omega(a^\dagger a + \frac{1}{2})$ and the commutation relation is given by $[a, a^\dagger] = 1$. The Bogolubov transformed vacuum state $|\tilde{0}\rangle$ and corresponding annihilation operator $\tilde{a}$ are constructed as

$$\tilde{a} = \alpha a + \beta a^\dagger, \qquad \tilde{a}|\tilde{0}\rangle = 0, \tag{A3}$$

where α and β satisfy $|\alpha|^2 - |\beta|^2 = 1$. Here, $\tilde{a}$ satisfies the commutation relation $[\tilde{a}, \tilde{a}^\dagger] = 1$ but is nothing to do with the Hamiltonian.

In the Heisenberg picture, operators are defined as $\mathcal{O}_H(t) = e^{\frac{i}{\hbar}Ht}\mathcal{O}e^{-\frac{i}{\hbar}Ht}$, where operators with and without subscript H correspond to Heisenberg and Schrödinger operators, respectively. The positive frequency functions $u(t)$, which are solutions to the EOM $\ddot{u}(t) = -\omega^2 u(t)$ and satisfy the Klein-Gordon normalization $u\dot{u}^* - \dot{u}u^* = i\hbar$, define the corresponding annihilation operators a_u by

$$a_u = \frac{1}{i}(\eta_H(t)\dot{u}^*(t) - p_{\eta H}(t)u^*(t)). \tag{A4}$$

The positive frequency function $u_0(t) = \sqrt{\hbar/2\omega}e^{-i\omega t}$ gives the annihilation operator a of the ground state defined in Eq. (A2), while $\tilde{u}(t) = \alpha^* u_0(t) - \beta u_0^*(t)$ gives $\tilde{a}$ of the Bogolubov transformed vacuum state defined in Eq. (A3).

We will explicitly construct the wave function $\psi_u(\eta) = \langle\eta|0_u\rangle$ where $|0_u\rangle$ satisfies $a_u|0_u\rangle = 0$. Using Eq. (A4), $\mathcal{O}_H(t) = e^{\frac{i}{\hbar}Ht}\mathcal{O}e^{-\frac{i}{\hbar}Ht}$ and $p_\eta = -i\hbar(\partial/\partial\eta)$, we can rewrite $a_u|0_u\rangle = 0$ in terms of the wave function as

$$\left(i\hbar u^*(t)\frac{\partial}{\partial\eta} + \dot{u}^*(t)\eta\right)e^{-\frac{i}{\hbar}Ht}\psi_u(\eta, t) = 0, \tag{A5}$$

where $H = -(\hbar^2/2)(\partial^2/\partial\eta^2) + (\omega^2/2)\eta^2$.

For the ground state, the positive frequency function is given by $u_0(t)$ and $H = \hbar\omega/2$. By solving Eq. (A5), we obtain, neglecting an imaginary phase,

$$\psi_0(\eta) = \sqrt{\frac{\omega}{\pi\hbar}}\exp\left[-\frac{\omega\eta^2}{2\hbar}\right], \tag{A6}$$

where $\psi_0(\eta)$ is the well known ground state wave function for the harmonic oscillator, as expected. Here we choose the overall normalization such that $\int d\eta|\psi_0(\eta, t)|^2 = 1$.

For the Bogolubov transformed vacuum state, the positive frequency function is given by $\tilde{u}(t)$. Using the Hermiticity of H and solving Eq. (A5), we obtain, neglecting an imaginary phase,

$$\tilde{\psi}(\eta, t) = e^{\frac{i}{\hbar}Ht}\left(\sqrt{\frac{1}{\pi\hbar}\frac{\dot{\tilde{u}}^*(t)}{\tilde{u}^*(t)}}\exp\left[\frac{i}{2\hbar}\frac{\dot{\tilde{u}}^*(t)}{\tilde{u}^*(t)}\eta^2\right]\right), \tag{A7}$$

where we again choose the overall normalization such that $\int d\eta|\tilde{\psi}(\eta, t)|^2 = 1$.

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