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Hom complexes and hypergraph colorings

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ABSTRACT
Babson and Kozlov (2006) [2] studied Hom-complexes of graphs with a focus on graph colorings. In this paper, we generalize Hom-complexes to r-uniform hypergraphs (with multiplicities) and study them mainly in connection with hypergraph colorings. We reinterpret a result of Alon, Frankl and Lovász (1986) [1] by Hom-complexes and show a hierarchy of known lower bounds for the chromatic numbers of r-uniform hypergraphs (with multiplicities) using Hom-complexes.

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1. Introduction

1.1. Hom-complexes of graphs

Since Lovász solved the famous Kneser conjecture by relating the chromatic number of a given graph to connectivity of its neighborhood complex [7], it is a standard method to study combinatorial properties of graphs by relating them with topological properties of appropriately constructed polyhedral complexes from graphs. Then as is seen in [5], a plenty of complexes have been constructed from graphs. Among others, let us consider Hom-complexes which were first introduced by Lovász and studied further by Babson and Kozlov [2,10,11]. Compared to other complexes of graphs, the construction of Hom-complex Hom(G, H) for graphs G, H is quite natural; it is a space of maps from G to H. Moreover, some complexes of graphs concerning colorings are realized by special Hom-complexes [2,10] by which one can easily understand related construction. For example, a result of Lovász [7] can be reproved easily by using Hom-complexes as follows.

Let us start with a standard observation. Recall that an n-coloring of a graph G is a labeling of vertices of G by n colors in such a way that adjacent vertices have distinct colors. Then if \( K_n \) denotes the complete graph with \( n \) vertices, there is a one-to-one correspondence between n-colorings of G and homomorphisms of G into \( K_n \). Suppose G admits an n-coloring.

Then since the Hom-complex Hom(G, H) is natural with respect to G, H, there is a map

\[ \text{Hom}(T, G) \rightarrow \text{Hom}(T, K_n) \]  

(1)

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for any graph \( T \). Specialize \( T \) to the complete graph \( K_2 \) with 2 vertices. Then a natural \( C_2 \)-action on \( K_2 \) yields \( C_2 \)-actions on both \( \text{Hom}(T, G) \) and \( \text{Hom}(T, K_0) \), and furthermore, the map (1) is a \( C_2 \)-map for \( T = K_2 \), where \( C_k \) denotes the cyclic group of order \( k \). One can easily see that the \( C_2 \)-actions are free and can also easily count the dimension of \( \text{Hom}(K_2, K_0) \) as \( n - 2 \) by definition. Then it follows from the Borsuk–Ulam theorem that

\[
\text{conn} \text{Hom}(K_2, G) \leq n - 3,
\]

where \( \text{conn} X \) denotes connectivity of a space \( X \). Finally, since \( \text{Hom}(K_2, G) \) has the homotopy type of the neighborhood complex of \( G \) as in [2], we obtain the result of Lovász [7]. The point of this proof is that we can get \( C_2 \)-actions and a \( C_2 \)-map quite naturally, which is often the most difficult part of the above mentioned topological method for graphs.

1.2. Generalization to \( r \)-graphs

Let us now generalize graphs to \( r \)-uniform hypergraphs. Recall that an \( r \)-uniform hypergraph (or an \( r \)-graph, for short) \( G \) consists of the vertex set and the edge set which is a collection of \( r \) elements subsets of the vertex set. Then \( 2 \)-graphs are simple graphs, for instance. Homomorphisms of \( r \)-graphs are obviously defined. In [11, Definition 9.24], Kozlov suggested a recipe to construct a space of a collection of maps between finite sets. Then one can define \( \text{Hom} \)-complexes for \( r \)-graphs as well. We would like to study colorings of \( r \)-graphs by using \( \text{Hom} \)-complexes as in the above case of graphs. Colorings of graphs are generalized to \( r \)-graphs as follows. An \( n \)-coloring of an \( r \)-graph is a labeling of vertices by \( n \) colors such that each edge contains more than \( 2 \) colors. Then for \( r > 3 \), colorings of \( r \)-graphs cannot be realized as homomorphisms. Then in order to study \( r \)-graph colorings by \( \text{Hom} \)-complexes, we must extend the category of \( r \)-graphs so that colorings become homomorphisms. If we extend the category of \( r \)-graphs to that of all hypergraphs, colorings become homomorphisms. However, this category is too big to control objects. So we need a much smaller extension of the category of \( r \)-graphs. For this purpose, we will consider \( r \)-graphs with multiplicities which were first introduced by Lange [6] in a different context. Then we will study colorings of \( r \)-graphs with multiplicities through \( \text{Hom} \)-complexes. More precisely, we will give a lower bound for the chromatic numbers of \( r \)-graphs with multiplicities using group actions on special \( \text{Hom} \)-complexes. Alon, Frankl and Lovász [1] defined certain simplicial complexes of \( r \)-graphs (without multiplicities) and gave a lower bound for the chromatic numbers by a rather tricky construction. We will show that these complexes are essentially the same as the above special \( \text{Hom} \)-complexes, and then we can interpret their construction in terms of \( \text{Hom} \)-complexes, which will make things clear. We will also consider \( \text{Hom}_+ \)-complexes of \( r \)-graphs with multiplicities (cf. [11, Definition 20.1]) and show a hierarchy among lower bounds for the chromatic numbers.

1.3. Organization

The organization of the paper is as follows. In Section 2, we introduce \( r \)-graphs with multiplicities generalizing \( r \)-graphs by which we can study colorings of \( r \)-graphs as special homomorphisms. In Section 3, we recall a general construction of \( \text{Hom} \)-complexes of classes of maps between finite sets and then apply it to \( r \)-graphs with multiplicities. We show analogy of results of Babson and Kozlov [2] for \( \text{Hom} \)-complexes of \( r \)-graphs with multiplicities and give a lower bound for the chromatic number by special \( \text{Hom} \)-complexes. In Section 4, we show that the box-edge complexes of Alon, Frankl and Lovász [1] are realized by the above special \( \text{Hom} \)-complexes, by which we see that the above lower bound is the same as the one given by Alon, Frankl and Lovász [1]. In Section 5, we consider \( \text{Hom}_+ \)-complexes of \( r \)-graphs with multiplicities and give another lower bound for the chromatic number. By comparing \( \text{Hom} \)-complexes and \( \text{Hom}_+ \)-complexes, we show a hierarchy among the above two lower bounds.

2. \( r \)-Graphs with multiplicities

2.1. \( r \)-Graphs

Let us explain in detail why we introduce \( r \)-graphs with multiplicities. Recall that an \( r \)-uniform hypergraph (\( r \)-graph, for short) \( G \) is a pair of a finite set \( V(G) \) and a collection \( E(G) \) of \( r \) elements subsets of \( V(G) \). \( V(G) \) and \( E(G) \) are respectively called the vertex set and the edge set of \( G \). For \( r \)-graphs \( G, H \), a homomorphism \( f : G \rightarrow H \) is a map \( f : V(G) \rightarrow V(H) \) satisfying \( f_*(E(G)) \subseteq E(H) \). Our objects are colorings of \( r \)-graphs. An \( n \)-coloring of an \( r \)-graph \( G \) is a map \( c : V(G) \rightarrow [n] \) such that if \( \{v_1, \ldots, v_l\} \in E(G) \), \( c(v_1), \ldots, c(v_l) \subseteq [n] \) is not a singleton, where \( [n] = \{1, 2, \ldots, n\} \). Then one sees that colorings cannot be realized by homomorphisms in general as in the case of graphs. Then generalizing \( r \)-graphs, we introduce \( r \)-graphs with multiplicities among which colorings are homomorphisms.

2.2. \( r \)-Graphs with multiplicities

Recall that the \( n \)th symmetric product of a set \( V \) is defined as

\[
\text{SP}^n(V) = \underbrace{V \times \cdots \times V}_{n} / \Sigma_n,
\]
where the action of the symmetric group $\Sigma_n$ is given as $\sigma (v_1, \ldots, v_n) = (v_{\sigma(1)}, \ldots, v_{\sigma(n)})$ for $\sigma \in \Sigma_n$ and $v_1, \ldots, v_n \in V$. We denote an element of $SP^n(V)$ by $v_1 \cdots v_n$ for $v_1, \ldots, v_n \in V$. In [6], Lange used multisets which are naturally identified with elements of symmetric products. We now define $r$-graphs with multiplicities and their homomorphisms and colorings.

**Definition 1.**

1. An $r$-graph with multiplicities $G$ consists of a finite set $V(G)$ and a subset $E(G)$ of $SP^r(V(G)) \setminus \Delta$, where $\Delta = \{v \cdots v \in SP^r(V(G)) \setminus \Delta, V(G)$ and $E(G)$ are called the vertex set and the edge set of $G$, respectively.
2. Let $G, H$ be $r$-graphs with multiplicities. A homomorphism $f : G \to H$ is a map $g : V(G) \to V(H)$ satisfying $f_*(E(G)) \subset E(H)$.
3. An $n$-coloring of an $r$-graph with multiplicities $G$ is a map $c : V(G) \to [n]$ such that if $v_1 \cdots v_r \in E(G)$, $\{c(v_1), \ldots, c(v_r)\} \subset [n]$ is not a singleton.
4. The chromatic number $\chi(G)$ of an $r$-graph with multiplicities $G$ is the minimum integer $n$ such that $G$ admits an $n$-coloring.

**Remark 2.** If we allow $r$-graphs with multiplicities to have diagonal edges in $\Delta$, some $r$-graphs do not admit any colorings. Since we will study colorings, we have omitted diagonal edges from $r$-graphs with multiplicities.

Since $r$ elements subsets of a set $V$ may be regarded as elements of $SP^r(V) \setminus \Delta$, $r$-graphs with multiplicities and their homomorphisms and colorings include $r$-graphs and their homomorphisms and colorings.

Define an $r$-graph with multiplicities $k_n^{(r)}$ as the maximum $r$-graph with multiplicities with $n$ vertices. Namely,

$$V(k_n^{(r)}) = [n] \text{ and } E(k_n^{(r)}) = SP^r([n]) \setminus \Delta.$$

It is clear that there is the desired property as follows.

**Proposition 3.** There is a one-to-one correspondence between $n$-colorings of an $r$-graph with multiplicities $G$ and homomorphisms from $G$ to $k_n^{(r)}$.

### 3. Hom-complexes of $r$-graphs with multiplicities

**3.1. General Hom-complexes**

Let us first recall a recipe of general Hom-complexes suggested by Kozlov [11, Definition 9.24]. Let $S, T$ be finite sets. Then a map $S \to T$ is identified with an element of $T^S$. Let $\Delta^T$ be the simplex whose vertex set is $T$. Since $T^S$ is the vertex set of a direct product $\prod S \Delta^T$, a map $S \to T$ is identified with a vertex of $\prod S \Delta^T$. This simple observation leads us to the following definition of Hom-complexes which may be regarded as spaces of given maps between finite sets.

**Definition 4.** Let $S, T$ be finite sets and $C$ be a class of maps from $S$ to $T$. The Hom-complex $\text{Hom}^C(S, T)$ is the maximum subcomplex of $\prod S \Delta^T$ whose vertex set is $C$.

Let $S, T$ be finite sets and $C$ be a class of maps from $S$ to $T$. Given a map $f : T \to T'$ with $T'$ finite and a class $D$ of maps from $S$ to $T$. If $f_s(C) \subset D$, we can define a map of polyhedral complexes

$$f_* : \text{Hom}^C(S, T) \to \text{Hom}^D(S, T')$$

by sending $h \in C$ to $f \circ h \in D$. Dually, given a map $g : S' \to S$ and a class $E$ of maps from $S'$ to $T$ satisfying $g^*(C) \subset E$, we can also define a map of polyhedral complexes

$$g^* : \text{Hom}^C(S, T) \to \text{Hom}^E(S', T)$$

by sending $h \in C$ to $h \circ g \in E$.

By definition of the above induced maps, we have the following functoriality.

**Proposition 5.** Let $S, T$ be finite sets and $C$ be a class of maps from $S$ to $T$.

1. Let $T_1, T_2$ be finite sets and $D_1, D_2$ be classes of maps from $S$ to $T_1$ and $T_2$, respectively. If maps $f_1 : T \to T_1$ and $f_2 : T \to T_2$ satisfy $(f_1)_*(C) \subset D_1$ and $(f_2)_*(D_1) \subset D_2$, the induced maps on Hom-complexes satisfy

$$(f_2 \circ f_1)_* = (f_2)_* \circ (f_1)_*.$$
Let $S_1, S_2$ be finite sets and $\mathcal{E}_1, \mathcal{E}_2$ be classes of maps from $S_1$ and $S_2$ to $T$, respectively. If maps $g_1 : S_1 \to S$ and $g_2 : S_2 \to S_1$ satisfy $(g_1)^* (g_2^*) (\mathcal{E}_1) \subset \mathcal{E}_1$ and $(g_2)^* (\mathcal{E}_2) \subset \mathcal{E}_2$, the induced maps on $\text{Hom}$-complexes satisfy
\[ (g_2 \circ g_1)^* = (g_1)^* \circ (g_2)^*. \]

### 3.2. $\text{Hom}$-complexes of $r$-graphs with multiplicities

Let us return to $r$-graphs with multiplicities. Homomorphisms of $r$-graphs with multiplicities are maps between vertices satisfying certain properties. Since we are assuming vertex sets of $r$-graphs with multiplicities to be finite, we can apply the above general construction of $\text{Hom}$-complexes to $r$-graphs with multiplicities.

**Definition 6.** Let $G, H$ be $r$-graphs with multiplicities and $C$ be the set of homomorphisms from $G$ to $H$. The $\text{Hom}$-complex $\text{Hom}(G, H)$ is defined as $\text{Hom}^C(V(G), V(H))$.

By Proposition 5, we have the following.

**Proposition 7.** Let $\text{Graph}^{(r)}$ and $\text{Poly}$ be the categories of $r$-graphs with multiplicities and polyhedral complexes, respectively. Then
\[ (\text{Graph}^{(r)})^{\text{op}} \times \text{Graph}^{(r)} \to \text{Poly}, \quad (G, H) \mapsto \text{Hom}(G, H) \]

is a functor.

By Proposition 3, the $\text{Hom}$-complex $\text{Hom}(G, K_n^{(r)})$ for an $r$-graph with multiplicities $G$ is considered as a space of $n$-colorings of $G$. Then $\text{Hom}(G, K_n^{(r)})$ is especially important, and hence we here give some easy examples. Let $L_n^{(r)}$ denote the line $r$-graph with $n$ vertices. Namely, $L_n^{(r)}$ is defined as
\[ V(L_n^{(r)}) = [n] \quad \text{and} \quad E(L_n^{(r)}) = \{(i, i + 1, \ldots, i + r - 1) \mid i = 1, \ldots, n - r + 1\}. \]

Then $\text{Hom}(L_n^{(r)}, K_2^{(3)})$ for $n = 3, 4, 5$ are given as follows.

Note that $\text{Hom}(L_n^{(3)}, K_2^{(3)})$ for $n = 3, 4, 5$ have the same homotopy type. This will be justified below in a more general setting.

Let $C_n^{(r)}$ be the cyclic $r$-graph with $n$ vertices. That is, $C_n^{(r)}$ is given as
\[ V(C_n^{(r)}) = \mathbb{Z}/n \quad \text{and} \quad E(C_n^{(r)}) = \{(i, i + 1, \ldots, i + r - 1) \mid i \in \mathbb{Z}/n\}. \]

Let us next consider $\text{Hom}(C_n^{(3)}, K_2^{(3)})$. Since $C_n^{(3)} = r_3^{(3)}$, $\text{Hom}(C_n^{(3)}, K_2^{(3)})$ is a hexagon. One can easily see that $\text{Hom}(r_3^{(3)}, K_2^{(3)})$ consists of discrete six points and that $\text{Hom}(r_3^{(3)}, K_2^{(3)})$ is the outer polygon of $\text{Hom}(L_5^{(3)}, K_2^{(3)})$. Then their homotopy types are not the same.

### 3.3. Lower bound for the chromatic number

By functoriality of $\text{Hom}(G, H)$, group actions on $G$ and $H$ induce those on $\text{Hom}(G, H)$. We next consider these group actions for special $G$. Let $K_n^{(r)}$ be the maximum $r$-graph with $n$ vertices. Namely,
\[ V(K_n^{(r)}) = [n] \quad \text{and} \quad E(K_n^{(r)}) = \text{SP}^r([n]) \setminus \Delta_n, \]
where $\Delta_n = \{v_1 \cdots v_r \in \text{SP}^r([n]) \mid v_i \neq v_j \text{ for } i \neq j\}$. Notice that by a cyclic permutation of vertices, the cyclic group $C_n$ acts on $K_n^{(r)}$. 
Lemma 8. If $r$ is a prime, the induced $C_r$-action on $\text{Hom}(K^{(r)}, G)$ is free.

Proof. Any face of $\text{Hom}(K^{(r)}, G)$ is of the form $\Delta_{S_1} \times \cdots \times \Delta_{S_r}$ such that each $(v_1, \ldots, v_r) \in S_1 \times \cdots \times S_r$ satisfies $v_1 \cdots v_r \in E(G)$. Then in particular, there is no diagonal element $(v_1, \ldots, v)$ in $S_1 \times \cdots \times S_r$. Let $g$ be a non-trivial element of $C_r$. Then by renumbering if necessary, we have

$$g \cdot (v_1, \ldots, v_r) = (v_r, v_1, \ldots, v_{r-1})$$

for $(v_1, \ldots, v_r) \in S_1 \times \cdots \times S_r$ since $r$ is a prime. Suppose $g(\Delta_{S_1} \times \cdots \times \Delta_{S_r}) = \Delta_{S_1} \times \cdots \times \Delta_{S_r}$. Then we have observed that elements of $S_1$ belong to all $S_1, \ldots, S_r$, a contradiction. \hfill $\square$

Using the index of the above free group action [8, Definition 6.2.3], we give a lower bound for the chromatic numbers of $r$-graphs with multiplicities. We set some notation. Let $\Gamma$ be a non-trivial finite group (with the discrete topology). Let $E_n\Gamma$ be the join of $n + 1$ copies of $\Gamma$ on which $\Gamma$ acts diagonally. Then this $\Gamma$-action on $E_n\Gamma$ is free and $E_n\Gamma$ has the homotopy type of a wedge of $n$-dimensional spheres. For a free $\Gamma$-complex $X$, the $\Gamma$-index of $X$ is defined as

$$\text{ind}_{\Gamma} X = \min\{n \mid \text{there is a } \Gamma\text{-map } X \to E_n\Gamma\}.$$ 

Let us list basic properties of $\text{ind}_{\Gamma} X$.

Proposition 9. Let $\Gamma$ be a non-trivial finite group and let $X, Y$ be free $\Gamma$-complexes.

1. If there is a $\Gamma$-map from $X$ to $Y$, we have

$$\text{ind}_{\Gamma} X \leq \text{ind}_{\Gamma} Y.$$ 

2. The join $X \star Y$ is a free $\Gamma$-space by the diagonal $\Gamma$-action for which it holds that

$$\text{ind}_{\Gamma} (X \star Y) \leq \text{ind}_{\Gamma} X + \text{ind}_{\Gamma} Y + 1.$$ 

3. It holds that

$$\text{conn} X + 1 \leq \text{ind}_{\Gamma} X \leq \text{dim} X.$$ 

Proof. (1) follows from definition and (2) follows from the fact that $E_n\Gamma = E_m\Gamma \ast E_{n-m-1}\Gamma$. By the Borsuk–Ulam theorem due to Dold [3], we have $\text{ind}_{\Gamma} E_n\Gamma = n$. Then (3) is shown by an easy obstruction argument. \hfill $\square$

Put $B_n\Gamma = E_n\Gamma / \Gamma$, $B\Gamma = \bigcup_{n \geq 1} B_n\Gamma$ and $E\Gamma = \bigcup_{n \geq 1} E_n\Gamma$. Then the natural projection $E\Gamma \to B\Gamma$ is the well-known Milnor’s universal principal $\Gamma$-bundle. Let $\varphi : X / \Gamma \to B\Gamma$ be the classifying map of a free $\Gamma$-complex $X$. Then it follows that $\text{ind}_{\Gamma} X$ coincides with the minimum integer $n$ such that $\varphi$ factors through the inclusion $B_n\Gamma \to B\Gamma$, up to homotopy. By [4, Proposition 8.4], we obtain that $\text{ind}_{\Gamma} X$ is equal to the LS-category of the classifying map $\varphi$, implying that there are a lot of quantities estimating $\text{ind}_{\Gamma} X$ other than connectivity and dimension.

We now give a lower bound for the chromatic number of $r$-graphs with multiplicities.

Theorem 10. Let $G$ be an $r$-graph with multiplicities. If $r$ is a prime, there holds

$$\chi(G) \geq \frac{\text{ind}_{C_r} \text{Hom}(K^{(r)}, G) + 1}{r - 1} + 1.$$ 

Proof. By Lemma 8, $\text{Hom}(K^{(r)}, H)$ is a free $C_r$-complex for any $r$-graph with multiplicities $H$. Suppose there is an $n$-coloring of $G$, or equivalently, a homomorphism $f : G \to K_n^{(r)}$. By Proposition 7, the induced map $f_* : \text{Hom}(K^{(r)}, G) \to \text{Hom}(K^{(r)}, K_n^{(r)})$ is a $C_r$-map, implying

$$\text{ind}_{C_r} \text{Hom}(K^{(r)}, G) \leq \text{dim} \text{Hom}(K^{(r)}, K_n^{(r)})$$

by Proposition 9. We then count the dimension of $\text{Hom}(K^{(r)}, K_n^{(r)})$. Any face of $\text{Hom}(K^{(r)}, K_n^{(r)})$ is given as $\Delta_{S_1} \times \cdots \times \Delta_{S_r}$ such that $S_1, \ldots, S_r \subseteq [n]$ and $S_1 \cap \cdots \cap S_r = \emptyset$. The maximum of $|S_1| + \cdots + |S_r|$ is $nr - n$ and then the dimension of $\text{Hom}(K^{(r)}, K_n^{(r)})$ is $nr - n - r = (r - 1)(n - 1) - 1$, completing the proof. \hfill $\square$

By Proposition 9, we obtain the following.

Corollary 11. Let $G$ be an $r$-graph with multiplicities. If $r$ is a prime, we have

$$\chi(G) \geq \frac{\text{conn} \text{Hom}(K^{(r)}, G) + 2}{r - 1} + 1.$$
3.4. Homotopy lemmas

Let us recall three lemmas from [2, Proposition 3.2] and [11, Theorems 15.24, 15.28] which will be used below. We first set some notation. Let $P$ be a poset. We denote the order complex of $P$ by $\Delta(P)$. That is, $\Delta(P)$ is a simplicial complex whose $n$-simplices are chains in $P$ of length $n + 1$. For $p \in P$, let

$$P_{\leq p} = \{ q \in P \mid q \leq p \} \quad \text{and} \quad P_{\geq p} = \{ q \in P \mid q \geq p \}.$$  

We first state the famous Quillen fiber lemma.

**Lemma 12.** ([11, Theorem 15.28]) Let $\varphi : P \to Q$ be a poset map between finite posets. If $\Delta(\varphi^{-1}(Q \leq q))$ is contractible for any $q \in Q$, then $\Delta(\varphi) : \Delta(P) \to \Delta(Q)$ is a homotopy equivalence.

We next state a variant of the Quillen fiber lemma proved in [2, Proposition 3.2].

**Lemma 13.** ([2, Proposition 3.2]) For a poset map $\varphi : P \to Q$ between finite posets, suppose the following conditions.

1. $\Delta(\varphi^{-1}(q))$ is contractible for any $q \in Q$.
2. For any $q \in Q$ and $p \in \varphi^{-1}(Q \geq q)$, the poset $\varphi^{-1}(q) \cap P_{\leq p}$ has the maximum.

Then $\Delta(\varphi) : \Delta(P) \to \Delta(Q)$ is a homotopy equivalence.

Finally, we recall the generalized nerve lemma which is frequently used in combinatorial algebraic topology. Let $\mathcal{A}$ be a covering of a space $X$ by non-empty subspaces $A_1, \ldots, A_n$. Then we associate to $\mathcal{A}$ a poset whose elements are non-empty intersections of $A_1, \ldots, A_n$ and the order is defined by inclusions. The nerve of $\mathcal{A}$ is by definition the order complex of this poset associated to $\mathcal{A}$.

**Lemma 14.** ([11, Theorem 15.24]) Let $\mathcal{A}$ be a covering of a polyhedral complex $K$ by non-empty subcomplexes $A_1, \ldots, A_n$. Suppose that for any $i_1 < \cdots < i_t$, there exists $k$ such that $A_{i_1} \cap \cdots \cap A_{i_t}$ is either empty or $(k-t+1)$-connected. Then $K$ is $k$-connected if and only if so is the nerve of $\mathcal{A}$.

3.5. Homotopy type of $\text{Hom}(K_m^{(r)}, K_n^{(r)})$

As is mentioned above, for an $r$-graph with multiplicities $G$, the hom-complex $\text{Hom}(G, K_m^{(r)})$ is especially important. Then we determine the homotopy type of this hom-complex in the special case $G = K_m^{(r)}$.

For $t = (t_1, \ldots, t_n)$ with non-negative integers $t_1, \ldots, t_n$, we define a polyhedral complex $\Delta^m(t)$ as the subcomplex of $\Delta[m] \times \cdots \times \Delta[n]$ whose faces are $\Delta^{S_1} \times \cdots \times \Delta^{S_m}$ such that $|\{k \in [m] \mid i \in S_k\}| \leq t_i$ for $i = 1, \ldots, n$. Note that if $t' = (t'_1, \ldots, t'_n)$ satisfies $t_k, t'_k \geq m$ for some $k$ and $t_i = t'_i$ for $i \neq k$, then $\Delta^m(t) = \Delta^m(t')$. As in the proof of Theorem 10, for $s = (r-1, \ldots, r-1) \in [m]^n$, we have

$$\Delta^m(s) = \text{Hom}(K_m^{(r)}, K_n^{(r)}).$$

We determine the homotopy type of $\Delta^m(t)$ and, consequently, the homotopy type of $\text{Hom}(K_m^{(r)}, K_n^{(r)})$. For a polyhedral complex $K$, let $\mathcal{F}(K)$ denote the face poset of $K$.

**Theorem 15.** For $t = (t_1, \ldots, t_n)$ with $0 \leq t_i \leq m$, $\Delta^m(t)$ has the homotopy type of a wedge of $(t_1 + \cdots + t_n - m)$-dimensional spheres.

**Proof.** Put $|t| = t_1 + \cdots + t_n$. As in the proof of Theorem 10, one can easily deduce that the dimension of $\Delta^m(t)$ is $|t| - m$. Then we only have to show that $\Delta^m(t)$ is $(|t| - m - 1)$-connected.

For $F \subset [n]$, put $t - F = (t'_1, \ldots, t'_n)$ such that

$$t'_i = \begin{cases} \max(t_i - 1, 0) & i \in F, \\ t_i & i \notin F. \end{cases}$$

Then if $F \subset F' \subset [n]$, we have $\Delta^{t - F} \supset \Delta^{t - F'}$. We also have that if $|t| - |F| - \ell < 0$ for $F \subset [n]$, $\Delta^{t - F} = \emptyset$. We now define a functor

$$\rho : \mathcal{F}(sk_{|t|-m}[n])^{op} \to \text{Poly}.$$
by $\rho(F) = \Delta^{m-1}(t - F)$ and inclusions $\Delta^{m-1}(t - F) \supset \Delta^{m-1}(t - F')$ for $F \subset F' \in F(\sk_k \Delta^{|n|})$, where $\sk_k K$ denotes the $k$-skeleton of a polyhedral complex $K$. By definition, $\Delta^m(t)$ is the union of $\Delta^F \times \Delta^{m-1}(t - F)$ for all non-empty $F \in F(\sk_k \Delta^{|n|})$. Namely, we have

$$\Delta^m(t) = \text{hocolim} \, \rho.$$ 

Since $\rho$ maps every arrow to a cofibration, we get a homotopy equivalence

$$\text{hocolim} \, \rho \cong \colim \, \rho.$$ 

See [11, Theorem 15.19]. Notice that $\colim \, \rho$ is covered by subcomplexes $\Delta^{m-1}(t - [i])$ for $i \in [n]$ and that $\Delta^{m-1}(t - F) \cap \Delta^{m-1}(t - F') = \Delta^{m-1}(t - F \cup F')$ for $F, F' \in F(\sk_k \Delta^{|n|})$.

If $|t| = m$, $\Delta^m(t)$ is a discrete finite set. Apply Lemma 14 to the above covering of $\colim \, \rho$ inductively on $|t| - m$. Thus we obtain the desired result. $\square$

**Corollary 16.** $\text{Hom}(K_m^{(r)}, K_n^{(r)})$ has the homotopy type of a wedge of $((r - 1)n - m)$-dimensional spheres.

### 3.6. Vertex deletion and $\text{Hom}$-complexes

In [2, Proposition 5.1], a relation between vertex deletion of $G$ and the homotopy type of $\text{Hom}(G, H)$ is considered when $G, H$ are graphs. We prove analogy for $r$-graphs with multiplicities here by a quite similar way. In [2, Proposition 5.1], a condition for vertex deletion is given by a neighborhood of a vertex. As for graphs, a neighborhood of a vertex $v$ is considered as both the set of vertices adjacent to $v$ and the set of edges with the end $v$. As for $r$-graphs with multiplicities, these two sets cannot be identified for $r \geq 3$, and then we define two kinds of neighborhoods of vertices.

Let $G$ be an $r$-graph with multiplicities. For a vertex $v$ of $G$, we define $N(v)$ as the set of $v_1 \cdots v_s \in \SP^r(V(G))$ for some $1 \leq s \leq r - 1$ satisfying $\bigcap_{i=1}^s v_i \in E(G)$ and $v_1, \ldots, v_s \neq v$. For $v_1 \cdots v_s \in \SP^r(V(G))$ with $1 \leq s \leq r - 1$, we also define

$$\tilde{N}(v_1 \cdots v_s)$$

as the set of vertices $w$ of $G$ satisfying $w_1 \cdots w_s \in E(G)$.

For a vertex $v$ of $G$, let $G \setminus v$ denote the maximum $r$-subgraph with multiplicities of $G$ whose vertex set is $V(G) \setminus v$. We now state our result.

**Theorem 17.** Let $G, H$ be an $r$-graph with multiplicities. Suppose that there are vertices $u, v$ of $G$ satisfying $N(u) \supset N(v)$. Then the inclusion $i : G \setminus v \to G$ induces a homotopy equivalence

$$i^* : \text{Hom}(G, H) \cong \text{Hom}(G \setminus v, H).$$

**Proof.** As is mentioned above, the proof is quite analogous to [2, Proposition 5.1]. Note that any face of $\text{Hom}(T, H)$ for an $r$-graph with multiplicities $T$ is identified with a map $V(T) \to 2^{|V(H)| \setminus \emptyset}$. For $\eta \in F(\text{Hom}(G \setminus v, H))$, the fiber $F(i^*)^{-1}(\eta)$ is the set of $\tau \in F(\text{Hom}(G, H))$ satisfying

$$\tau|_{V(G) \setminus v} = \eta.$$ 

Since

$$\bigcup_{v_1 \cdots v_s \in \tilde{N}(v)} (w_1, \ldots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s) \supset \bigcup_{v_1 \cdots v_s \in N(u)} (w_1, \ldots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s) \supset \eta(u) \neq \emptyset,$$

we can define $v \in F(i^*)^{-1}(\eta)$ by

$$v(v) = \bigcup_{v_1 \cdots v_s \in \tilde{N}(v)} (w_1, \ldots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s)$$

and $v|_{V(G) \setminus v} = \eta$. By definition, $v$ is the maximum of $F(i^*)^{-1}(\eta)$, and thus in particular, the order complex $\Delta(F(i^*)^{-1}(\eta))$ is contractible.

Choose $\tau \in F(\text{Hom}(G, H))$ and $\eta \in F(\text{Hom}(G \setminus v, H))$ satisfying $\tau(w) \supset \eta(w)$ for $w \in V(G) \setminus v$. Observe that $F(i^*)^{-1}(\eta) \cap F(\text{Hom}(G, H))_{\tau|_{V(G) \setminus v}}$ consists of $\sigma \in F(\text{Hom}(G, H))$ satisfying $\sigma(v) \subset \tau(v)$ and $\sigma|_{V(G) \setminus v} = \eta$. Then it has the maximum $\mu$ such that $\mu(v) = \tau(v)$ and $\mu|_{V(G) \setminus v} = \eta$. We have seen that Lemma 13 can be applied to $F(i^*) : F(\text{Hom}(G, H)) \to F(\text{Hom}(G \setminus v, H))$, which completes the proof. $\square$
We generalize the above observation on the homotopy type of $\Hom(L^{(3)}_n, K^{(3)}_2)$.

**Corollary 18.** Let $I_n^{(r)}$ be the line $r$-graph with $n$ vertices as above, and let $G$ be an $r$-graph with multiplicities. Then for $n \geq r$, we have

$$\Hom(I_n^{(r)}, G) \simeq \Hom(I_n^{(r)}, G).$$

**Proof.** If $n > r$, we have $\mathbb{N}(n) \subset \mathbb{N}(n-r)$. Then by Theorem 17, it holds that $\Hom(I_n^{(r)}, G) \simeq \Hom(I_n^{(r)}, G)$. Thus the result follows by induction on $n$. \hfill $\square$

4. Relation between box-edge complexes and $\Hom$-complexes

4.1. Box-edge complexes

Let $G$ be an $r$-graph (without multiplicities). In [1], Alon, Frankl and Lovász introduced a simplicial complex $B_{\text{edge}}(G)$ with a $C_r$-action which we call the box-edge complex of $G$, where we follow the name and the notation of [9]. By an ad-hoc and tricky construction concerning $B_{\text{edge}}(G)$, they gave a lower bound for the chromatic number of $G$. We will show that this construction is realized by special $\Hom$-complexes of $r$-graphs with multiplicities, by which we can reprove and interpret a result of Alon, Frankl and Lovász [1, Proposition 2.1] in a quite natural way.

Let $\pi : V^n \rightarrow \text{SP}^n(V)$ denote the projection for a set $V$. Originally, the box-edge complexes were defined only for $r$-graphs (without multiplicities). However, their definition can be applied to $r$-graphs with multiplicities straightforwardly.

**Definition 19.** Let $G$ be an $r$-graph with multiplicities. The box-edge complex of $G$ is an abstract simplicial complex defined as

$$B_{\text{edge}}(G) = \{ F \subset V(G)^r \mid \pi(F) \subset E(G) \},$$

on which the cyclic group $C_r$ acts as the restriction of the permutation action on $V(G)^r$.

Notice here that as is shown in [1, Proposition 2.1], if $r$ is a prime, the $C_r$-action on $B_{\text{edge}}(G)$ is free.

4.2. Result of Alon, Frankl and Lovász

We prove that the box-edge complex $B_{\text{edge}}(G)$ is given by a special $\Hom$-complex.

**Theorem 20.** For an $r$-graph with multiplicities $G$, there is a $C_r$-map

$$B_{\text{edge}}(G) \rightarrow \Hom(K^{(r)}_r, G)$$

which is a homotopy equivalence. In particular, if $r$ is a prime, it is a $C_r$-homotopy equivalence.

**Proof.** The face poset of $\Hom(K_r^{(r)}, G)$ is given as

$$\mathcal{F}(\Hom(K_r^{(r)}, G)) = \{ F_1 \times \cdots \times F_r \mid F_1, \ldots, F_r \subset V(G) \text{ and } \pi(F_1 \times \cdots \times F_r) \subset E(G) \},$$

where the order is given by inclusions. Then as the face poset of $B_{\text{edge}}(G)$ is given in (2), we can define a map

$$\varphi : \mathcal{F}(B_{\text{edge}}(G)) \rightarrow \mathcal{F}(\Hom(K_r^{(r)}, G)), \quad F \mapsto \pi_1(F) \times \cdots \times \pi_r(F),$$

where $\pi_i : V(G)^r \rightarrow V(G)$ is the $i$th projection. Then by definition, $\varphi$ is a $C_r$-map and hence so is $\Delta(\varphi)$.

Take any $F_1 \times \cdots \times F_r \in \mathcal{F}(\Hom(K_r^{(r)}, G))$. Then the post $\varphi^{-1}(\Hom(K_r^{(r)}, G)_{\leq F_1 \times \cdots \times F_r})$ has the maximum $F_1 \times \cdots \times F_r$, implying that $\Delta(\varphi^{-1}(\Hom(K_r^{(r)}, G)_{\leq F_1 \times \cdots \times F_r}))$ is contractible. Thus by Lemma 12, $\Delta(\varphi)$ is a homotopy equivalence. The desired map is the composite

$$B_{\text{edge}}(G) \xrightarrow{\cong} \Delta(\mathcal{F}(B_{\text{edge}}(G))) \xrightarrow{\Delta(\varphi)} \Delta(\mathcal{F}(\Hom(K_r^{(r)}, G))) \xrightarrow{\cong} \Hom(K_r^{(r)}, G),$$

where the first and the last arrows are the natural homeomorphisms between polyhedral complexes and their barycentric subdivision. Therefore we have established the first assertion. Then the $C_r$-action on $\Hom(K_r^{(r)}, G)$ is free by Lemma 8. Moreover, the $C_r$-action on $B_{\text{edge}}(G)$ is also free as is noted above. Thus the second assertion follows from the first one. \hfill $\square$

**Remark 21.** Recently, Thansri [14, Corollary 4.9] showed that $B_{\text{edge}}(G)$ and $\Hom(K_r^{(r)}, G)$ has the same $\Sigma_r$-equivariant simple homotopy type for an $r$-graph (without multiplicities) $G$. 

By Corollary 11, we obtain a result of Alon, Frankl and Lovász [1, Proposition 2.1].

**Corollary 22.** Let $G$ be an $r$-graph with multiplicities. If $r$ is a prime, we have

$$
\chi(G) \geq \frac{\text{conn} \mathcal{B}_{\text{edge}}(G) + 2}{r - 1} + 1.
$$

Alon, Frankl and Lovász [1, §3] proved Corollary 22 by constructing a map from $\mathcal{B}_{\text{edge}}(G)$ into a Euclidean space with a certain $C_r$-action, which seems quite ad-hoc and tricky. Using $\text{Hom}$-complexes, this construction will turn out to be the induced map between $\text{Hom}$-complexes from a given coloring.

Let $M_{r,n}(\mathbb{R})$ be the space of $r \times n$ real matrices. We let $C_r$ act on $M_{r,n}(\mathbb{R})$ as the cyclic permutation of rows. Let $Y$ be a subspace of $M_{r,n}(\mathbb{R})$ consisting of matrices $(a_{ij})$ satisfying

$$
\sum_{i=1}^{r} a_{ik} = 0, \quad \sum_{j=1}^{n} a_{ij} = 0 \quad \text{and} \quad \sum_{i,j} a_{ij}^2 \neq 0
$$

for $k = 1, \ldots, n$ and $\ell = 1, \ldots, r$. Then $Y$ is also a $C_r$-subspace of $M_{r,n}(\mathbb{R})$. Let $G$ be an $r$-graph with multiplicities which admits an $n$-coloring, say $c$. Alon, Frankl and Lovász [1, §3] defined a $C_r$-map

$$
\tilde{c}: \mathcal{B}_{\text{edge}}(G) \to M_{r,n}(\mathbb{R})
$$

by sending a vertex $(v_1, \ldots, v_r)$ of $\mathcal{B}_{\text{edge}}(G)$ to a matrix $\sum_{i=1}^{r} (E_{i,c(i)} - E_{i,c(i)+1})$, where $E_{i,j}$ is the matrix whose $(i, j)$ entry is 1 and other entries are 0 and $E_{i,n+1}$ means $E_{i,1}$. They showed that $\tilde{c}$ has its image in $Y$ and applied a special generalization of the Borsuk–Ulam theorem to obtain Corollary 22.

We now define a map $g: \text{Hom}(K_r^{(r)}, K_n^{(r)}) \to M_{r,n}(\mathbb{R})$ by sending a vertex $(i_1, \ldots, i_r) \in [n]^r$ to a matrix $\sum_{k=1}^{r} (E_{k,i_k} - E_{k, i_{k+1}})$. Then one can easily see that $g$ is a $C_r$-map and has is image in $Y$. By definition, we have the following.

**Proposition 23.** Let $G$ be an $r$-graph with multiplicities which has an $n$-coloring $c$. Then there is a commutative diagram

$$
\text{Hom}(K_r^{(r)}), G) \xrightarrow{c} \text{Hom}(K_r^{(r)}, K_n^{(r)}) \xrightarrow{g} M_{r,n}(\mathbb{R})
$$

where the left vertical arrow is as in Theorem 20.

We close this section by remarking that the complex of an $r$-graph (without multiplicities) introduced by Kříž [12, §4] is the barycentric subdivision of $\text{Hom}(K_r^{(r)}, G)$ and then essentially the same as the box-edge complex $\mathcal{B}_{\text{edge}}(G)$.

5. **$\text{Hom}_+^+$-complexes and colorings**

5.1. **General $\text{Hom}_+^+$-complexes**

In [11, Definition 20.1], $\text{Hom}_+^+$-complexes of graphs were introduced which are variants of $\text{Hom}$-complexes. As in the case of $\text{Hom}$-complexes, we can give a general recipe for $\text{Hom}_+^+$-complexes of partial maps between finite sets and will apply it to $r$-graphs with multiplicities.

Let $S, T$ be finite sets. A partial map from $S$ to $T$ is a map from a non-empty subset of $S$ into $T$. Then a partial map from $S$ to $T$ is identified with an element of

$$(T \cup \{\emptyset\})^S \setminus (\emptyset, \ldots, \emptyset).$$

Let $K, L$ be abstract simplicial complexes. Recall that the join $K \ast L$ is an abstract simplicial complex whose simplices are of the form $(\sigma, \tau)$ where $\sigma \in K$, $\tau \in L$ and either $\sigma$ or $\tau$ is not empty. Then a partial map from $S$ to $T$ is identified with a vertex of the join $*_S \Delta^T$. Analogously to $\text{Hom}$-complexes, we are led to the following definition.

**Definition 24.** Let $S, T$ be finite sets and $C$ be a class of partial maps from $S$ to $T$. The $\text{Hom}_+^+$-complex $\text{Hom}_+^C(S, T)$ is defined as the maximum subcomplex of $*_S \Delta^T$ whose vertex set is $C$.

Analogously to $\text{Hom}$-complexes, we can define induced maps between $\text{Hom}_+^+$-complexes under certain conditions and see that these induced maps satisfy naturality corresponding to **Proposition 5**.
5.2. \(\text{Hom}_+\)-complexes of \(r\)-graphs with multiplicities

Let \(G, H\) be \(r\)-graphs with multiplicities. A partial homomorphism from \(G\) to \(H\) is a map from a subset \(V\) of \(V(G)\) into \(V(H)\) which is a homomorphism from the maximum \(r\)-subgraph with multiplicities of \(G\) whose vertex set is \(V\) into \(H\). We now define \(\text{Hom}_+\)-complexes of \(r\)-graphs with multiplicities.

**Definition 25.** Let \(G, H\) be \(r\)-graphs with multiplicities. The \(\text{Hom}_+\)-complex \(\text{Hom}_+(G, H)\) is defined as \(\text{Hom}^C_r(V(G), V(H))\) for the set \(C\) of all partial homomorphisms from \(G\) to \(H\).

Similarly to Proposition 7, we have the following.

**Proposition 26.** Let \(\text{Graph}^{(r)}\) and \(\text{Poly}\) be the categories of \(r\)-graphs with multiplicities and polyhedral complexes, respectively. Then

\[(\text{Graph}^{(r)})^{\text{op}} \times \text{Graph}^{(r)} \to \text{Poly}, \quad (G, H) \mapsto \text{Hom}_+(G, H)\]

is a functor.

Then as in the case of \(\text{Hom}\)-complexes, we can construct group actions on \(\text{Hom}_+\)-complexes by those on \(r\)-graphs with multiplicities. For instance, the natural \(C_r\)-action on \(K^{(r)}_r\) induces a \(C_r\)-action on \(\text{Hom}_+(K^{(r)}_r, G)\) for an \(r\)-graph with multiplicities \(G\). Analogously to Lemma 8, we can prove the following.

**Lemma 27.** Let \(G\) be an \(r\)-graph with multiplicities. If \(r\) is a prime, the \(C_r\)-action on \(\text{Hom}_+(K^{(r)}_r, G)\) is free.

Using this \(C_r\)-action, we obtain a lower bound for the chromatic numbers.

**Theorem 28.** Let \(G\) be an \(r\)-graph with multiplicities. If \(r\) is a prime, it holds that

\[\chi(G) \geq \frac{\text{ind}_{C_r} \text{Hom}_+(K^{(r)}_r, G) + 1}{r - 1}\]

**Proof.** Note that the dimension of \(\text{Hom}_+(K^{(r)}_r, K^{(r)}_m)\) is \(nr - n - 1 = (r - 1)n - 1\). Then the result follow quite similarly to Theorem 10. \(\Box\)

**Corollary 29.** Let \(G\) be an \(r\)-graph with multiplicities. If \(r\) is a prime, we have

\[\chi(G) \geq \frac{\text{conn} \text{Hom}_+(K^{(r)}_r, G) + 2}{r - 1}\]

In [6, p. 5] Lange defined a complex \(B_0(G)\) for an \(r\)-graph with multiplicities and gave a lower bound for the chromatic number of \(G\) by using \(B_0(G)\). By definition, \(B_0(G)\) coincides with \(\text{Hom}_+(K^{(r)}_r, G)\) and a lower bound in Theorem 28 is the same as the one given by Lange.

As in Section 3, let us consider the homotopy type of \(\text{Hom}_+(K^{(r)}_r, K^{(r)}_m)\). In the case of \(\text{Hom}_+\)-complexes, one can describe \(\text{Hom}_+(K^{(r)}_m, K^{(r)}_n)\) explicitly by Sarkaria’s formula [13, (2.2)] as follows.

**Theorem 30.** ([13, (2.2)]) We have

\[\text{Hom}_+(K^{(r)}_m, K^{(r)}_n) \cong \ast_{\Delta^{[m]}}\ast_{K^{(r)}_{r-2}}\Delta^{[n]}\]

In particular, \(\text{Hom}_+(K^{(r)}_m, K^{(r)}_n)\) has the homotopy type of a wedge of \(\binom{m-1}{r-1}\) copies of \((r - 1)n - 1\)-dimensional spheres.

5.3. Hierarchy of lower bounds for the chromatic number

Let \(G\) be an \(r\)-graph with multiplicities. We have obtained so far two kinds of lower bounds for the chromatic number of \(G\), one is given by \(\text{Hom}(K^{(r)}_r, G)\) in Theorem 10 and the other is given by \(\text{Hom}_+(K^{(r)}_r, G)\) in Theorem 28. We have also seen that these lower bounds are related to formerly known ones [1, Proposition 2.1], [6, Theorem 3]. We describe \(\text{Hom}_+(K^{(r)}_r, G)\) by using \(\text{Hom}(K^{(r)}_r, G)\) and then get an inequality between the above lower bounds.
Theorem 31. For an r-graph with multiplicities G, there is a $C_r$-map

$$\text{Hom}_+(K_r^{(r)}, G) \to \partial \Delta^{[r]} \ast \text{Hom}(K_r^{(r)}, G)$$

which is a homotopy equivalence, where $C_r$ acts diagonally on $\partial \Delta^{[r]} \ast \text{Hom}(K_r^{(r)}, G)$.

Proof. Let $P, Q$ be finite posets. Recall that the join $P \ast Q$ is a poset whose underlying set is $P \sqcup Q$ and order is defined as $x < y$ if either $x, y \in P$ with $x < y$, $x, y \in Q$ with $x < y$ or $x \in P, y \in Q$. Then it follows that

$$\Delta(P \ast Q) = \Delta(P) \ast \Delta(Q).$$

Note that the face poset of $\text{Hom}_+(K_r^{(r)}, G)$ is the disjoint union of $\mathcal{F}(\text{Hom}(K_r^{(r)}, G))$ in (3) and

$$\begin{cases} F_1 \times \cdots \times F_r \mid F_1, \ldots, F_r \subset V(G), \ F_i = \emptyset \text{ for some } i \text{ and } \bigcup_{i=1}^r F_i \neq \emptyset \end{cases},$$

where the order is given by inclusions and $F_1 \times \cdots \times F_n$ with $F_{i_1}, \ldots, F_{i_k} \neq \emptyset$ and $F_j = \emptyset$ for $j \neq i_1, \ldots, i_k$ means $F_{i_1} \times \cdots \times F_{i_k}$.

We then define a poset map

$$\varphi: \mathcal{F}(\text{Hom}_+(K_r^{(r)}, G)) \to \mathcal{F}(\partial \Delta^{[r]} \ast \text{Hom}(K_r^{(r)}, G))$$

as

$$\varphi(F_1 \times \cdots \times F_r) = \begin{cases} \{i_1, \ldots, i_k\} \in \mathcal{F}(\partial \Delta^{[r]}) & \bigcup_{i \neq i_1, \ldots, i_k} F_i = \emptyset \text{ and } F_{i_1}, \ldots, F_{i_k} \neq \emptyset, \\ F_1 \times \cdots \times F_r \in \mathcal{F}(\text{Hom}_+(K_r^{(r)}, G)) & F_1, \ldots, F_r \neq \emptyset. \end{cases}$$

By definition, $\varphi$ is a $C_r$-map. For $F_1 \times \cdots \times F_r \in \mathcal{F}(\text{Hom}(K_r^{(r)}, G)) \subset \mathcal{F}(\partial \Delta^{[r]} \ast \mathcal{F}(\text{Hom}(K_r^{(r)}, G)))$, $\varphi^{-1}(\mathcal{F}(\partial \Delta^{[r]} \ast \mathcal{F}(\text{Hom}(K_r^{(r)}, G))))$ has the maximum $F_1 \times \cdots \times F_r$. For $\{i_1, \ldots, i_k\} \in \partial \Delta^{[r]} \subset \mathcal{F}(\partial \Delta^{[r]} \ast \mathcal{F}(\text{Hom}(K_r^{(r)}, G)))$, $\varphi^{-1}(\mathcal{F}(\partial \Delta^{[r]} \ast \mathcal{F}(\text{Hom}(K_r^{(r)}, G))))$ has the maximum

$$F_1 \times \cdots \times F_r, \quad F_{i_1} = \cdots = F_{i_k} = [n] \quad \text{and} \quad \bigcup_{i \neq i_1, \ldots, i_k} F_i = \emptyset.$$

Then for any $x \in \mathcal{F}(\partial \Delta^{[r]} \ast \mathcal{F}(\text{Hom}(K_r^{(r)}, G)))$, $\Delta(\mathcal{F}(\partial \Delta^{[r]} \ast \mathcal{F}(\text{Hom}(K_r^{(r)}, G))))$ is contractible, and it follows from Lemma 12 that $\Delta(\varphi)$ is a homotopy equivalence. Thus the composite

$$\text{Hom}_+(K_r^{(r)}, G) \xrightarrow{\cong} \Delta(\mathcal{F}(\text{Hom}_+(K_r^{(r)}, G))) \xrightarrow{\Delta(\varphi)} \Delta(\mathcal{F}(\partial \Delta^{[r]} \ast \mathcal{F}(\text{Hom}(K_r^{(r)}, G))))$$

$$= \Delta(\mathcal{F}(\partial \Delta^{[r]})) \ast \Delta(\mathcal{F}(\text{Hom}(K_r^{(r)}, G))) \xrightarrow{\cong} \partial \Delta^{[r]} \ast \text{Hom}(K_r^{(r)}, G)$$

is the desired homotopy equivalence, where the first and the last arrows are the natural homeomorphisms. \(\square\)

Corollary 32. Let G be an r-graph with multiplicities. If r is a prime, there holds

$$\chi(G) \geq \frac{\text{ind}_{C_r} \text{Hom}(K_r^{(r)}, G) + 1}{r - 1} + 1 \geq \frac{\text{ind}_{C_r} \text{Hom}_+(K_r^{(r)}, G) + 1}{r - 1} \geq \frac{\text{ind}_{C_r} \text{Hom}(K_r^{(r)}, G) + 1}{r - 1} \geq \frac{\text{ind}_{C_r} \text{Hom}(K_r^{(r)}, G) + 1}{r - 1} - 1.$$

Proof. The first inequality follows from Theorem 10 and the second from Proposition 9 and Theorem 31. As in the proof of Theorem 31, $\mathcal{F}(\text{Hom}(K_r^{(r)}, G))$ is a subposet of $\mathcal{F}(\text{Hom}_+(K_r^{(r)}, G))$ including the $C_r$-actions. Then there is a $C_r$-map $\text{Hom}(K_r^{(r)}, G) \to \text{Hom}_+(K_r^{(r)}, G)$, implying the third inequality by Proposition 9. The fourth inequality follows from Proposition 9 and the last equality from Theorem 31. \(\square\)

References