Topology and its Applications 160 (2013) 1333–1344

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Hom complexes and hypergraph colorings ☆

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ARTICLE INFO

Article history:
Received 28 August 2012
Received in revised form 30 April 2013
Accepted 3 May 2013

MSC:
05C15
57M15

Keywords:
Hom-complex
Uniform hypergraph
Chromatic number

ABSTRACT

Babson and Kozlov (2006) [2] studied Hom-complexes of graphs with a focus on graph colorings. In this paper, we generalize Hom-complexes to r-uniform hypergraphs (with multiplicities) and study them mainly in connection with hypergraph colorings. We reinterpret a result of Alon, Frankl and Lovász (1986) [1] by Hom-complexes and show a hierarchy of known lower bounds for the chromatic numbers of r-uniform hypergraphs (with multiplicities) using Hom-complexes.

1. Introduction

1.1. Hom-*complexes of graphs*

Since Lovász solved the famous Kneser conjecture by relating the chromatic number of a given graph to connectivity of its neighborhood complex [7], it is a standard method to study combinatorial properties of graphs by relating them with topological properties of appropriately constructed polyhedral complexes from graphs. Then as is seen in [5], a plenty of complexes have been constructed from graphs. Among others, let us consider Hom-complexes which were first introduced by Lovász and studied further by Babson and Kozlov [2,10,11]. Compared to other complexes of graphs, the construction of Hom-complex $\mathrm{Hom}(G,H)$ for graphs G,H is quite natural; it is a *space* of maps from G to H. Moreover, some complexes of graphs concerning colorings are realized by special Hom-complexes [2,10] by which one can easily understand related construction. For example, a result of Lovász [7] can be reproved easily by using Hom-complexes as follows.

Let us start with a standard observation. Recall that an n-coloring of a graph G is a labeling of vertices of G by n colors in such a way that adjacent vertices have distinct colors. Then if K_n denotes the complete graph with n vertices, there is a one-to-one correspondence between n-colorings of G and homomorphisms of G into K_n. Suppose G admits an n-coloring. Then since the Hom-complex $\mathrm{Hom}(G,H)$ is natural with respect to G,H, there is a map

$$\mathrm{Hom}(T,G)\to\mathrm{Hom}(T,K_n) \tag{1}$$

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0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.05.002

for any graph T. Specialize T to the complete graph K_2 with 2 vertices. Then a natural C_2-action on K_2 yields C_2-actions on both $\mathrm{Hom}(T,G)$ and $\mathrm{Hom}(T,K_n)$, and furthermore, the map (1) is a C_2-map for $T=K_2$, where C_k denotes the cyclic group of order k. One can easily see that the C_2-actions are free and can also easily count the dimension of $\mathrm{Hom}(K_2,K_n)$ as $n-2$ by definition. Then it follows from the Borsuk–Ulam theorem that

$$\operatorname{conn}\mathrm{Hom}(K_2,G)\leqslant n-3,$$

where $\operatorname{conn}X$ denotes connectivity of a space X. Finally, since $\mathrm{Hom}(K_2,G)$ has the homotopy type of the neighborhood complex of G as in [2], we obtain the result of Lovász [7]. The point of this proof is that we can get C_2-actions and a C_2-map quite naturally, which is often the most difficult part of the above mentioned topological method for graphs.

1.2. Generalization to r-graphs

Let us now generalize graphs to r-uniform hypergraphs. Recall that an r-uniform hypergraph (or an r-graph, for short) G consists of the vertex set and the edge set which is a collection of r elements subsets of the vertex set. Then 2-graphs are simple graphs, for instance. Homomorphisms of r-graphs are obviously defined. In [11, Definition 9.24], Kozlov suggested a recipe to construct a *space* of a collection of maps between finite sets. Then one can define Hom-complexes for r-graphs as well. We would like to study colorings of r-graphs by using Hom-complexes as in the above case of graphs. Colorings of graphs are generalized to r-graphs as follows. An n-coloring of an r-graph is a labeling of vertices by n colors such that each edge contains more than 2 colors. Then for $r\geqslant 3$, colorings of r-graphs cannot be realized as homomorphisms. Then in order to study r-graph colorings by Hom-complexes, we must extend the category of r-graphs so that colorings become homomorphisms. If we extend the category of r-graphs to that of all hypergraphs, colorings become homomorphisms. However, this category is too big to control objects. So we need a much smaller extension of the category of r-graphs. For this purpose, we will consider r-graphs with multiplicities which were first introduced by Lange [6] in a different context. Then we will study colorings of r-graphs with multiplicities through Hom-complexes. More precisely, we will give a lower bound for the chromatic numbers of r-graphs with multiplicities using group actions on special Hom-complexes. Alon, Frankl and Lovász [1] defined certain simplicial complexes of r-graphs (without multiplicities) and gave a lower bound for the chromatic numbers by a rather tricky construction. We will show that these complexes are essentially the same as the above special Hom-complexes, and then we can interpret their construction in terms of Hom-complexes, which will make things clear. We will also consider Hom_+-complexes of r-graphs with multiplicities (cf. [11, Definition 20.1]) and show a hierarchy among lower bounds for the chromatic numbers.

1.3. Organization

The organization of the paper is as follows. In Section 2, we introduce r-graphs with multiplicities generalizing r-graphs by which we can study colorings of r-graphs as special homomorphisms. In Section 3, we recall a general construction of Hom-complexes of classes of maps between finite sets and then apply it to r-graphs with multiplicities. We show analogy of results of Babson and Kozlov [2] for Hom-complexes of r-graphs with multiplicities and give a lower bound for the chromatic number by special Hom-complexes. In Section 4, we show that the box-edge complexes of Alon, Frankl and Lovász [1] are realized by the above special Hom-complexes, by which we see that the above lower bound is the same as the one given by Alon, Frankl and Lovász [1]. In Section 5, we consider Hom_+-complexes of r-graphs with multiplicities and give another lower bound for the chromatic number. By comparing Hom-complexes and Hom_+-complexes, we show a hierarchy among the above two lower bounds.

2. *r*-Graphs with multiplicities

2.1. r-Graphs

Let us explain in detail why we introduce r-graphs with multiplicities. Recall that an r-uniform hypergraph (r-graph, for short) G is a pair of a finite set $V(G)$ and a collection $E(G)$ of r elements subsets of $V(G)$. $V(G)$ and $E(G)$ are respectively called the vertex set and the edge set of G. For r-graphs G,H, a homomorphism $f:G\to H$ is a map $f:V(G)\to V(H)$ satisfying $f_*(E(G))\subset E(H)$. Our objects are colorings of r-graphs. An n-coloring of an r-graph G is a map $c:V(G)\to[n]$ such that if $\{v_1,\dots,v_r\}\in E(G)$, $\{c(v_1),\dots,c(v_r)\}\subset[n]$ is not a singleton, where $[n]=\{1,2,\dots,n\}$. Then one sees that colorings cannot be realized by homomorphisms in general as in the case of graphs. Then generalizing r-graphs, we introduce r-graphs with multiplicities among which colorings are homomorphisms.

2.2. r-Graphs with multiplicities

Recall that the nth symmetric product of a set V is defined as

$$\mathrm{SP}^n(V)=\underbrace{V\times\cdots\times V}_{n}/\Sigma_n,$$

where the action of the symmetric group Σ_n is given as $\sigma(v_1,\dots,v_n)=(v_{\sigma(1)},\dots,v_{\sigma(n)})$ for $\sigma\in\Sigma_n$ and $v_1,\dots,v_n\in V$. We denote an element of $\mathrm{SP}^n(V)$ by $v_1\cdots v_n$ for $v_1,\dots,v_n\in V$. In [6], Lange used multisets which are naturally identified with elements of symmetric products. We now define r-graphs with multiplicities and their homomorphisms and colorings.

Definition 1.

(1) An r-graph with multiplicities G consists of a finite set $V(G)$ and a subset $E(G)$ of $\mathrm{SP}^r(V(G))\setminus\Delta$, where $\Delta=\{v\cdots v\in \mathrm{SP}^r(V(G))\}$. $V(G)$ and $E(G)$ are called the vertex set and the edge set of G, respectively.
(2) Let G,H be r-graphs with multiplicities. A homomorphism $f:G\to H$ is a map $g:V(G)\to V(H)$ satisfying $f_*(E(G))\subset E(H)$.
(3) An n-coloring of an r-graph with multiplicities G is a map $c:V(G)\to[n]$ such that if $v_1\cdots v_r\in E(G)$, $\{c(v_1),\dots,c(v_r)\}\subset[n]$ is not a singleton.
(4) The chromatic number $\chi(G)$ of an r-graph with multiplicities G is the minimum integer n such that G admits an n-coloring.

Remark 2. If we allow r-graphs with multiplicities to have diagonal edges in Δ, some r-graphs do not admit any colorings. Since we will study colorings, we have omitted diagonal edges from r-graphs with multiplicities.

Since r elements subsets of a set V may be regarded as elements of $\mathrm{SP}^r(V)\setminus\Delta$, r-graphs with multiplicities and their homomorphisms and colorings include r-graphs and their homomorphisms and colorings.

Define an r-graph with multiplicities $\mathcal{K}_n^{(r)}$ as the maximum r-graph with multiplicities with n vertices. Namely,

$$V\big(\mathcal{K}_n^{(r)}\big)=[n]\quad\text{and}\quad E\big(\mathcal{K}_n^{(r)}\big)=\mathrm{SP}^r\big([n]\big)\setminus\Delta.$$

It is clear that there is the desired property as follows.

Proposition 3. *There is a one-to-one correspondence between n-colorings of an r-graph with multiplicities G and homomorphisms from G to $\mathcal{K}_n^{(r)}$.*

3. Hom-complexes of r-graphs with multiplicities

3.1. General Hom-complexes

Let us first recall a recipe of general Hom-complexes suggested by Kozlov [11, Definition 9.24]. Let S,T be finite sets. Then a map $S\to T$ is identified with an element of T^S. Let Δ^T be the simplex whose vertex set is T. Since T^S is the vertex set of a direct product $\prod_S\Delta^T$, a map $S\to T$ is identified with a vertex of $\prod_S\Delta^T$. This simple observation leads us to the following definition of Hom-complexes which may be regarded as *spaces* of given maps between finite sets.

Definition 4. Let S,T be finite sets and $\mathcal{C}$ be a class of maps from S to T. The Hom-complex $\mathrm{Hom}^{\mathcal{C}}(S,T)$ is the maximum subcomplex of $\prod_S\Delta^T$ whose vertex set is $\mathcal{C}$.

Let S,T be finite sets and $\mathcal{C}$ be a class of maps from S to T. Given a map $f:T\to T'$ with T' finite and a class $\mathcal{D}$ of maps from S to T'. If $f_*(\mathcal{C})\subset\mathcal{D}$, we can define a map of polyhedral complexes

$$f_*:\mathrm{Hom}^{\mathcal{C}}(S,T)\to\mathrm{Hom}^{\mathcal{D}}\big(S,T'\big)$$

by sending $h\in\mathcal{C}$ to $f\circ h\in\mathcal{D}$. Dually, given a map $g:S'\to S$ and a class $\mathcal{E}$ of maps from S' to T satisfying $g^*(\mathcal{C})\subset\mathcal{E}$, we can also define a map of polyhedral complexes

$$g^*:\mathrm{Hom}^{\mathcal{C}}(S,T)\to\mathrm{Hom}^{\mathcal{E}}\big(S',T\big)$$

by sending $h\in\mathcal{E}$ to $h\circ g\in\mathcal{C}$.

By definition of the above induced maps, we have the following functoriality.

Proposition 5. *Let S,T be finite sets and $\mathcal{C}$ be a class of maps from S to T.*

(1) *Let T_1,T_2 be finite sets and $\mathcal{D}_1,\mathcal{D}_2$ be classes of maps from S to T_1 and T_2, respectively. If maps $f_1:T\to T_1$ and $f_2:T_1\to T_2$ satisfy $(f_1)_*(\mathcal{C})\subset\mathcal{D}_1$ and $(f_2)_*(\mathcal{D}_1)\subset\mathcal{D}_2$, the induced maps on Hom-complexes satisfy*

$$(f_2\circ f_1)_*=(f_2)_*\circ(f_1)_*.$$

(2) *Let S_1, S_2 be finite sets and $\mathcal{E}_1, \mathcal{E}_2$ be classes of maps from S_1 and S_2 to T, respectively. If maps $g_1 : S_1 \to S$ and $g_2 : S_2 \to S_1$ satisfy $(g_1)^*(\mathcal{C}) \subset \mathcal{E}_1$ and $(g_2)^*(\mathcal{E}_1) \subset \mathcal{E}_2$, the induced maps on* Hom*-complexes satisfy*

$$(g_2 \circ g_1)^* = (g_1)^* \circ (g_2)^*.$$

3.2. Hom-*complexes of r-graphs with multiplicities*

Let us return to r-graphs with multiplicities. Homomorphisms of r-graphs with multiplicities are maps between vertices satisfying certain properties. Since we are assuming vertex sets of r-graphs with multiplicities to be finite, we can apply the above general construction of Hom-complexes to r-graphs with multiplicities.

Definition 6. Let G, H be r-graphs with multiplicities and $\mathcal{C}$ be the set of homomorphisms from G to H. The Hom-complex $\mathrm{Hom}(G, H)$ is defined as $\mathrm{Hom}^{\mathcal{C}}(V(G), V(H))$.

By Proposition 5, we have the following.

Proposition 7. *Let* $\mathbf{Graph}^{(r)}$ *and* **Poly** *be the categories of r-graphs with multiplicities and polyhedral complexes, respectively. Then*

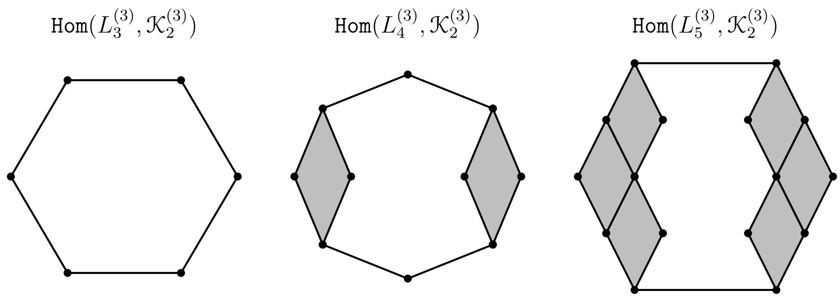
$$\left(\mathbf{Graph}^{(r)}\right)^{\mathrm{op}} \times \mathbf{Graph}^{(r)} \to \mathbf{Poly}, \quad (G, H) \mapsto \mathrm{Hom}(G, H)$$

is a functor.

By Proposition 3, the Hom-complex $\mathrm{Hom}(G, \mathcal{K}_n^{(r)})$ for an r-graph with multiplicities G is considered as a *space* of n-colorings of G. Then $\mathrm{Hom}(G, \mathcal{K}_n^{(r)})$ is especially important, and hence we here give some easy examples. Let $L_n^{(r)}$ denote the line r-graph with n vertices. Namely, $L_n^{(r)}$ is defined as

$$V\left(L_n^{(r)}\right) = [n] \quad \text{and} \quad E\left(L_n^{(r)}\right) = \left\{\{i, i+1, \ldots, i+r-1\} \,\middle|\, i = 1, \ldots, n-r+1\right\}.$$

Then $\mathrm{Hom}(L_n^{(3)}, \mathcal{K}_2^{(3)})$ for $n = 3, 4, 5$ are given as follows.



Note that $\mathrm{Hom}(L_n^{(3)}, \mathcal{K}_2^{(3)})$ for $n = 3, 4, 5$ have the same homotopy type. This will be justified below in a more general setting.

Let $C_n^{(r)}$ be the cyclic r-graph with n vertices. That is, $C_n^{(r)}$ is given as

$$V\left(C_n^{(r)}\right) = \mathbb{Z}/n \quad \text{and} \quad E\left(C_n^{(r)}\right) = \left\{\{i, i+1, \ldots, i+r-1\} \,\middle|\, i \in \mathbb{Z}/n\right\}.$$

Let us next consider $\mathrm{Hom}(C_n^{(3)}, \mathcal{K}_2^{(3)})$. Since $C_3^{(3)} = L_3^{(3)}$, $\mathrm{Hom}(C_3^{(3)}, \mathcal{K}_2^{(3)})$ is a hexagon. One can easily see that $\mathrm{Hom}(C_4^{(3)}, \mathcal{K}_2^{(3)})$ consists of discrete six points and that $\mathrm{Hom}(C_5^{(3)}, \mathcal{K}_2^{(3)})$ is the outer polygon of $\mathrm{Hom}(L_5^{(3)}, \mathcal{K}_2^{(3)})$. Then their homotopy types are not the same.

3.3. Lower bound for the chromatic number

By functoriality of $\mathrm{Hom}(G, H)$, group actions on G and H induce those on $\mathrm{Hom}(G, H)$. We next consider these group actions for special G. Let $K_n^{(r)}$ be the maximum r-graph with n vertices. Namely,

$$V\left(K_n^{(r)}\right) = [n] \quad \text{and} \quad E\left(K_n^{(r)}\right) = \mathrm{SP}^r\left([n]\right) \setminus \Delta_n,$$

where $\Delta_n = \{v_1 \cdots v_r \in \mathrm{SP}^r([n]) \mid v_i \neq v_j \text{ for } i \neq j\}$. Notice that by a cyclic permutation of vertices, the cyclic group C_n acts on $K_n^{(r)}$.

Lemma 8. *If r is a prime, the induced C_r-action on $\mathrm{Hom}(K_r^{(r)}, G)$ is free.*

Proof. Any face of $\mathrm{Hom}(K_r^{(r)}, G)$ is of the form $\Delta^{S_1} \times \cdots \times \Delta^{S_r}$ such that each $(v_1, \ldots, v_r) \in S_1 \times \cdots \times S_r$ satisfies $v_1 \cdots v_r \in E(G)$. Then in particular, there is no diagonal element $(v, \ldots, v)$ in $S_1 \times \cdots \times S_r$. Let g be a non-trivial element of C_r. Then by renumbering if necessary, we have

$$g \cdot (v_1, \ldots, v_r) = (v_r, v_1, \ldots, v_{r-1})$$

for $(v_1, \ldots, v_r) \in S_1 \times \cdots \times S_r$ since r is a prime. Suppose $g(\Delta^{S_1} \times \cdots \times \Delta^{S_r}) = \Delta^{S_1} \times \cdots \times \Delta^{S_r}$. Then we have observed that elements of S_1 belong to all $S_1, \ldots, S_r$, a contradiction. □

Using the index of the above free group action [8, Definition 6.2.3], we give a lower bound for the chromatic numbers of r-graphs with multiplicities. We set some notation. Let Γ be a non-trivial finite group (with the discrete topology). Let $E_n\Gamma$ be the join of $n+1$ copies of Γ on which Γ acts diagonally. Then this Γ-action on $E_n\Gamma$ is free and $E_n\Gamma$ has the homotopy type of a wedge of n-dimensional spheres. For a free Γ-complex X, the Γ-index of X is defined as

$$\mathrm{ind}_\Gamma X = \min\{n \mid \text{there is a } \Gamma\text{-map } X \to E_n\Gamma\}.$$

Let us list basic properties of $\mathrm{ind}_\Gamma X$.

Proposition 9. *Let Γ be a non-trivial finite group and let X, Y be free Γ-complexes.*

(1) *If there is a Γ-map from X to Y, we have*

$$\mathrm{ind}_\Gamma X \leqslant \mathrm{ind}_\Gamma Y.$$

(2) *The join $X * Y$ is a free Γ-space by the diagonal Γ-action for which it holds that*

$$\mathrm{ind}_\Gamma (X * Y) \leqslant \mathrm{ind}_\Gamma X + \mathrm{ind}_\Gamma Y + 1.$$

(3) *It holds that*

$$\mathrm{conn}\, X + 1 \leqslant \mathrm{ind}_\Gamma X \leqslant \dim X.$$

Proof. (1) follows from definition and (2) follows from the fact that $E_n\Gamma = E_m\Gamma * E_{n-m-1}\Gamma$. By the Borsuk–Ulam theorem due to Dold [3], we have $\mathrm{ind}_\Gamma E_n\Gamma = n$. Then (3) is shown by an easy obstruction argument. □

Put $B_n\Gamma = E_n\Gamma/\Gamma$, $B\Gamma = \bigcup_{n\geqslant 1} B_n\Gamma$ and $E\Gamma = \bigcup_{n\geqslant 1} E_n\Gamma$. Then the natural projection $E\Gamma \to B\Gamma$ is the well-known Milnor's universal principal Γ-bundle. Let $\varphi : X/\Gamma \to B\Gamma$ be the classifying map of a free Γ-complex X. Then it follows that $\mathrm{ind}_\Gamma X$ coincides with the minimum integer n such that φ factors through the inclusion $B_n\Gamma \to B\Gamma$, up to homotopy. By [4, Proposition 8.4], we obtain that $\mathrm{ind}_\Gamma X$ is equal to the LS-category of the classifying map φ, implying that there are a lot of quantities estimating $\mathrm{ind}_\Gamma X$ other than connectivity and dimension.

We now give a lower bound for the chromatic number of r-graphs with multiplicities.

Theorem 10. *Let G be an r-graph with multiplicities. If r is a prime, there holds*

$$\chi(G) \geqslant \frac{\mathrm{ind}_{C_r} \mathrm{Hom}(K_r^{(r)}, G) + 1}{r-1} + 1.$$

Proof. By Lemma 8, $\mathrm{Hom}(K_r^{(r)}, H)$ is a free C_r-complex for any r-graph with multiplicities H. Suppose there is an n-coloring of G, or equivalently, a homomorphism $f : G \to \mathcal{K}_n^{(r)}$. By Proposition 7, the induced map $f_* : \mathrm{Hom}(K_r^{(r)}, G) \to \mathrm{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is a C_r-map, implying

$$\mathrm{ind}_{C_r} \mathrm{Hom}\big(K_r^{(r)}, G\big) \leqslant \dim \mathrm{Hom}\big(K_r^{(r)}, \mathcal{K}_n^{(r)}\big)$$

by Proposition 9. We then count the dimension of $\mathrm{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$. Any face of $\mathrm{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is given as $\Delta^{S_1} \times \cdots \times \Delta^{S_r}$ such that $S_1, \ldots, S_r \subset [n]$ and $S_1 \cap \cdots \cap S_r = \emptyset$. The maximum of $|S_1| + \cdots + |S_r|$ is $nr - n$ and then the dimension of $\mathrm{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is $nr - n - r = (r-1)(n-1) - 1$, completing the proof. □

By Proposition 9, we obtain the following.

Corollary 11. *Let G be an r-graph with multiplicities. If r is a prime, we have*

$$\chi(G) \geqslant \frac{\mathrm{conn}\, \mathrm{Hom}(K_r^{(r)}, G) + 2}{r-1} + 1.$$

3.4. Homotopy lemmas

Let us recall three lemmas from [2, Proposition 3.2] and [11, Theorems 15.24, 15.28] which will be used below. We first set some notation. Let P be a poset. We denote the order complex of P by $\Delta(P)$. That is, $\Delta(P)$ is a simplicial complex whose n-simplices are chains in P of length $n+1$. For $p \in P$, let

$$P_{\leqslant p} = \{q \in P \mid q \leqslant p\} \quad \text{and} \quad P_{\geqslant p} = \{q \in P \mid q \geqslant p\}.$$

We first state the famous Quillen fiber lemma.

Lemma 12. *([11, Theorem 15.28]) Let $\varphi : P \to Q$ be a poset map between finite posets. If $\Delta(\varphi^{-1}(Q_{\leqslant q}))$ is contractible for any $q \in Q$, then $\Delta(\varphi) : \Delta(P) \to \Delta(Q)$ is a homotopy equivalence.*

We next state a variant of the Quillen fiber lemma proved in [2, Proposition 3.2].

Lemma 13. *([2, Proposition 3.2]) For a poset map $\varphi : P \to Q$ between finite posets, suppose the following conditions.*

(1) *$\Delta(\varphi^{-1}(q))$ is contractible for any $q \in Q$.*
(2) *For any $q \in Q$ and $p \in \varphi^{-1}(Q_{\geqslant q})$, the poset $\varphi^{-1}(q) \cap P_{\leqslant p}$ has the maximum.*

Then $\Delta(\varphi) : \Delta(P) \to \Delta(Q)$ is a homotopy equivalence.

Finally, we recall the generalized nerve lemma which is frequently used in combinatorial algebraic topology. Let $\mathcal{A}$ be a covering of a space X by non-empty subspaces $A_1, \dots, A_n$. Then we associate to $\mathcal{A}$ a poset whose elements are non-empty intersections of $A_1, \dots, A_n$ and the order is defined by inclusions. The nerve of $\mathcal{A}$ is by definition the order complex of this poset associated to $\mathcal{A}$.

Lemma 14. *([11, Theorem 15.24]) Let $\mathcal{A}$ be a covering of a polyhedral complex K by non-empty subcomplexes $A_1, \dots, A_n$. Suppose that for any $i_1 < \dots < i_t$, there exists k such that $A_{i_1} \cap \dots \cap A_{i_t}$ is either empty or $(k-t+1)$-connected. Then K is k-connected if and only if so is the nerve of $\mathcal{A}$.*

3.5. Homotopy type of $\mathrm{Hom}(K_m^{(r)}, \mathcal{K}_n^{(r)})$

As is mentioned above, for an r-graph with multiplicities G, the Hom-complex $\mathrm{Hom}(G, \mathcal{K}_n^{(r)})$ is especially important. Then we determine the homotopy type of this Hom-complex in the special case $G = K_m^{(r)}$.

For $\boldsymbol{t} = (t_1, \dots, t_n)$ with non-negative integers $t_1, \dots, t_n$, we define a polyhedral complex $\Delta^m(\boldsymbol{t})$ as the subcomplex of $\underbrace{\Delta^{[n]} \times \dots \times \Delta^{[n]}}_{m}$ whose faces are $\Delta^{S_1} \times \dots \times \Delta^{S_m}$ such that $|\{k \in [m] \mid i \in S_k\}| \leqslant t_i$ for $i = 1, \dots, n$. Note that if $\boldsymbol{t}' = (t'_1, \dots, t'_n)$ satisfies $t_k, t'_k \geqslant m$ for some k and $t_i = t'_i$ for $i \neq k$, then $\Delta^m(\boldsymbol{t}) = \Delta^m(\boldsymbol{t}')$. As in the proof of Theorem 10, for $\boldsymbol{s} = (r-1, \dots, r-1) \in [m]^n$, we have

$$\Delta^m(\boldsymbol{s}) = \mathrm{Hom}\big(K_m^{(r)}, \mathcal{K}_n^{(r)}\big).$$

We determine the homotopy type of $\Delta^m(\boldsymbol{t})$ and, consequently, the homotopy type of $\mathrm{Hom}(K_m^{(r)}, \mathcal{K}_n^{(r)})$. For a polyhedral complex K, let $\mathcal{F}(K)$ denote the face poset of K.

Theorem 15. *For $\boldsymbol{t} = (t_1, \dots, t_n)$ with $0 \leqslant t_i \leqslant m$, $\Delta^m(\boldsymbol{t})$ has the homotopy type of a wedge of $(t_1 + \dots + t_n - m)$-dimensional spheres.*

Proof. Put $|\boldsymbol{t}| = t_1 + \dots + t_n$. As in the proof of Theorem 10, one can easily deduce that the dimension of $\Delta^m(\boldsymbol{t})$ is $|\boldsymbol{t}| - m$. Then we only have to show that $\Delta^m(\boldsymbol{t})$ is $(|\boldsymbol{t}| - m - 1)$-connected.

For $F \subset [n]$, put $\boldsymbol{t} - F = (t'_1, \dots, t'_n)$ such that

$$t'_i = \begin{cases} \max\{t_i - 1, 0\} & i \in F, \\ t_i & i \notin F. \end{cases}$$

Then if $F \subset F' \subset [n]$, we have $\Delta^\ell(\boldsymbol{t} - F) \supset \Delta^\ell(\boldsymbol{t} - F')$. We also have that if $|\boldsymbol{t}| - |F| - \ell < 0$ for $F \subset [n]$, $\Delta^\ell(\boldsymbol{t} - F) = \emptyset$. We now define a functor

$$\rho : \mathcal{F}\big(\mathrm{sk}_{|\boldsymbol{t}|-m} \Delta^{[n]}\big)^{\mathrm{op}} \to \mathbf{Poly}$$

by $\rho(F) = \Delta^{m-1}(\boldsymbol{t} - F)$ and inclusions $\Delta^{m-1}(\boldsymbol{t} - F) \supset \Delta^{m-1}(\boldsymbol{t} - F')$ for $F \subset F' \in \mathcal{F}(\mathrm{sk}_{|\boldsymbol{t}|-m}\Delta^{[n]})$, where $\mathrm{sk}_k K$ denotes the k-skeleton of a polyhedral complex K. By definition, $\Delta^m(\boldsymbol{t})$ is the union of $\Delta^F \times \Delta^{m-1}(\boldsymbol{t} - F)$ for all non-empty $F \in \mathcal{F}(\mathrm{sk}_{|\boldsymbol{t}|-m}\Delta^{[n]})$. Namely, we have

$$\Delta^m(\boldsymbol{t}) = \operatorname{hocolim} \rho.$$

Since ρ maps every arrow to a cofibration, we get a homotopy equivalence

$$\operatorname{hocolim} \rho \xrightarrow{\simeq} \operatorname{colim} \rho.$$

See [11, Theorem 15.19]. Notice that $\operatorname{colim}\rho$ is covered by subcomplexes $\Delta^{m-1}(\boldsymbol{t} - \{i\})$ for $i \in [n]$ and that $\Delta^{m-1}(\boldsymbol{t} - F) \cap \Delta^{m-1}(\boldsymbol{t} - F') = \Delta^{m-1}(\boldsymbol{t} - F \cup F')$ for $F, F' \in \mathcal{F}(\mathrm{sk}_{|\boldsymbol{t}|-m}\Delta^{[n]})$.

If $|\boldsymbol{t}| = m$, $\Delta^m(\boldsymbol{t})$ is a discrete finite set. Apply Lemma 14 to the above covering of $\operatorname{colim}\rho$ inductively on $|\boldsymbol{t}| - m$. Thus we obtain the desired result. $\square$

Corollary 16. $\mathtt{Hom}(K_m^{(r)}, \mathcal{K}_n^{(r)})$ *has the homotopy type of a wedge of* $((r-1)n - m)$*-dimensional spheres.*

3.6. *Vertex deletion and* Hom*-complexes*

In [2, Proposition 5.1], a relation between vertex deletion of G and the homotopy type of $\mathtt{Hom}(G, H)$ is considered when G, H are graphs. We prove analogy for r-graphs with multiplicities here by a quite similar way. In [2, Proposition 5.1], a condition for vertex deletion is given by a neighborhood of a vertex. As for graphs, a neighborhood of a vertex v is considered as both the set of vertices adjacent to v and the set of edges with the end v. As for r-graphs with multiplicities, these two sets cannot be identified for $r \geqslant 3$, and then we define two kinds of neighborhoods of vertices.

Let G be an r-graph with multiplicities. For a vertex v of G, we define $\mathrm{N}(v)$ as the set of $v_1 \cdots v_s \in \mathrm{SP}^s(V(G))$ for some $1 \leqslant s \leqslant r-1$ satisfying $\underbrace{v \cdots v}_{r-s} v_1 \cdots v_s \in E(G)$ and $v_1, \ldots, v_s \neq v$. For $v_1 \cdots v_s \in \mathrm{SP}^s(V(G))$ with $1 \leqslant s \leqslant r-1$, we also define $\check{\mathrm{N}}(v_1 \cdots v_s)$ as the set of vertices w of G satisfying $\underbrace{w \cdots w}_{r-s} v_1 \cdots v_s \in E(G)$.

For a vertex v of G, let $G \setminus v$ denote the maximum r-subgraph with multiplicities of G whose vertex set is $V(G) \setminus v$. We now state our result.

Theorem 17. *Let G, H be an r-graph with multiplicities. Suppose that there are vertices u, v of G satisfying $\mathrm{N}(u) \supset \mathrm{N}(v)$. Then the inclusion $i : G \setminus v \to G$ induces a homotopy equivalence*

$$i^* : \mathtt{Hom}(G, H) \xrightarrow{\simeq} \mathtt{Hom}(G \setminus v, H).$$

Proof. As is mentioned above, the proof is quite analogous to [2, Proposition 5.1]. Note that any face of $\mathtt{Hom}(T, H)$ for an r-graph with multiplicities T is identified with a map $V(T) \to 2^{V(H)} \setminus \emptyset$. For $\eta \in \mathcal{F}(\mathtt{Hom}(G \setminus v, H))$, the fiber $\mathcal{F}(i^*)^{-1}(\eta)$ is the set of $\tau \in \mathcal{F}(\mathtt{Hom}(G, H))$ satisfying

$$\tau|_{V(G)\setminus v} = \eta.$$

Since

$$\bigcap_{v_1 \cdots v_s \in \mathrm{N}(v)} \bigcap_{(w_1, \ldots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s)} \check{\mathrm{N}}(w_1 \cdots w_s) \supset \bigcap_{v_1 \cdots v_s \in \mathrm{N}(u)} \bigcap_{(w_1, \ldots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s)} \check{\mathrm{N}}(w_1 \cdots w_s) \supset \eta(u) \neq \emptyset,$$

we can define $\nu \in \mathcal{F}(i^*)^{-1}(\eta)$ by

$$\nu(v) = \bigcap_{v_1 \cdots v_s \in \mathrm{N}(v)} \bigcap_{(w_1, \ldots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s)} \check{\mathrm{N}}(w_1 \cdots w_s)$$

and $\nu|_{V(G)\setminus v} = \eta$. By definition, ν is the maximum of $\mathcal{F}(i^*)^{-1}(\eta)$, and thus in particular, the order complex $\Delta(\mathcal{F}(i^*)^{-1}(\eta))$ is contractible.

Choose $\tau \in \mathcal{F}(\mathtt{Hom}(G, H))$ and $\eta \in \mathcal{F}(\mathtt{Hom}(G \setminus v, H))$ satisfying $\tau(w) \supset \eta(w)$ for $w \in V(G) \setminus v$. Observe that $\mathcal{F}(i^*)^{-1}(\eta) \cap \mathcal{F}(\mathtt{Hom}(G, H))_{\leqslant \tau}$ consists of $\sigma \in \mathcal{F}(\mathtt{Hom}(G, H))$ satisfying $\sigma(v) \subset \tau(v)$ and $\sigma|_{V(G)\setminus v} = \eta$. Then it has the maximum μ such that $\mu(v) = \tau(v)$ and $\mu|_{V(G)\setminus v} = \eta$. We have seen that Lemma 13 can be applied to $\mathcal{F}(i^*) : \mathcal{F}(\mathtt{Hom}(G, H)) \to \mathcal{F}(\mathtt{Hom}(G \setminus v, H))$, which completes the proof. $\square$

We generalize the above observation on the homotopy type of $\mathrm{Hom}(L_n^{(3)}, \mathcal{K}_2^{(3)})$.

Corollary 18. *Let $L_n^{(r)}$ be the line r-graph with n vertices as above, and let G be an r-graph with multiplicities. Then for $n \geqslant r$, we have*

$$\mathrm{Hom}\big(L_n^{(r)}, G\big) \simeq \mathrm{Hom}\big(L_r^{(r)}, G\big).$$

Proof. If $n > r$, we have $\mathrm{N}(n) \subset \mathrm{N}(n-r)$. Then by Theorem 17, it holds that $\mathrm{Hom}(L_n^{(r)}, G) \simeq \mathrm{Hom}(L_{n-1}^{(r)}, G)$. Thus the result follows by induction on n. $\square$

4. Relation between box-edge complexes and Hom-complexes

4.1. Box-edge complexes

Let G be an r-graph (without multiplicities). In [1], Alon, Frankl and Lovász introduced a simplicial complex $\mathrm{B}_{\mathrm{edge}}(G)$ with a C_r-action which we call the box-edge complex of G, where we follow the name and the notation of [9]. By an ad-hoc and tricky construction concerning $\mathrm{B}_{\mathrm{edge}}(G)$, they gave a lower bound for the chromatic number of G. We will show that this construction is realized by special Hom-complexes of r-graphs with multiplicities, by which we can reprove and interpret a result of Alon, Frankl and Lovász [1, Proposition 2.1] in a quite natural way.

Let $\pi : V^n \to \mathrm{SP}^n(V)$ denote the projection for a set V. Originally, the box-edge complexes were defined only for r-graphs (without multiplicities). However, their definition can be applied to r-graphs with multiplicities straightforwardly.

Definition 19. Let G be an r-graph with multiplicities. The box-edge complex of G is an abstract simplicial complex defined as

$$\mathrm{B}_{\mathrm{edge}}(G) = \big\{F \subset V(G)^r \,\big|\, \pi(F) \subset E(G)\big\} \tag{2}$$

on which the cyclic group C_r acts as the restriction of the permutation action on $V(G)^r$.

Notice here that as is shown in [1, Proposition 2.1], if r is a prime, the C_r-action on $\mathrm{B}_{\mathrm{edge}}(G)$ is free.

4.2. Result of Alon, Frankl and Lovász

We prove that the box-edge complex $\mathrm{B}_{\mathrm{edge}}(G)$ is given by a special Hom-complex.

Theorem 20. *For an r-graph with multiplicities G, there is a C_r-map*

$$\mathrm{B}_{\mathrm{edge}}(G) \to \mathrm{Hom}\big(K_r^{(r)}, G\big)$$

which is a homotopy equivalence. In particular, if r is a prime, it is a C_r-homotopy equivalence.

Proof. The face poset of $\mathrm{Hom}(K_r^{(r)}, G)$ is given as

$$\mathcal{F}\big(\mathrm{Hom}\big(K_r^{(r)}, G\big)\big) = \big\{F_1 \times \cdots \times F_r \,\big|\, F_1, \ldots, F_r \subset V(G) \text{ and } \pi(F_1 \times \cdots \times F_r) \subset E(G)\big\}, \tag{3}$$

where the order is given by inclusions. Then as the face poset of $\mathrm{B}_{\mathrm{edge}}(G)$ is given in (2), we can define a map

$$\varphi : \mathcal{F}\big(\mathrm{B}_{\mathrm{edge}}(G)\big) \to \mathcal{F}\big(\mathrm{Hom}\big(K_r^{(r)}, G\big)\big), \qquad F \mapsto \pi_1(F) \times \cdots \times \pi_r(F),$$

where $\pi_i : V(G)^r \to V(G)$ is the ith projection. Then by definition, φ is a C_r-map and hence so is $\Delta(\varphi)$.

Take any $F_1 \times \cdots \times F_r \in \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$. Then the poset $\varphi^{-1}(\mathrm{Hom}(K_r^{(r)}, G)_{\leqslant F_1 \times \cdots \times F_r})$ has the maximum $F_1 \times \cdots \times F_r$, implying that $\Delta(\varphi^{-1}(\mathrm{Hom}(K_r^{(r)}, G)_{\leqslant F_1 \times \cdots \times F_r}))$ is contractible. Thus by Lemma 12, $\Delta(\varphi)$ is a homotopy equivalence. The desired map is the composite

$$\mathrm{B}_{\mathrm{edge}}(G) \xrightarrow{\cong} \Delta\big(\mathcal{F}\big(\mathrm{B}_{\mathrm{edge}}(G)\big)\big) \xrightarrow{\Delta(\varphi)} \Delta\big(\mathcal{F}\big(\mathrm{Hom}\big(K_r^{(r)}, G\big)\big)\big) \xrightarrow{\cong} \mathrm{Hom}\big(K_r^{(r)}, G\big),$$

where the first and the last arrows are the natural homeomorphisms between polyhedral complexes and their barycentric subdivision. Therefore we have established the first assertion. Suppose r is a prime. Then the C_r-action on $\mathrm{Hom}(K_r^{(r)}, G)$ is free by Lemma 8. Moreover, the C_r-action on $\mathrm{B}_{\mathrm{edge}}(G)$ is also free as is noted above. Thus the second assertion follows from the first one. $\square$

Remark 21. Recently, Thansri [14, Corollary 4.9] showed that $\mathrm{B}_{\mathrm{edge}}(G)$ and $\mathrm{Hom}(K_r^{(r)}, G)$ has the same Σ_r-equivariant simple homotopy type for an r-graph (without multiplicities) G.

By Corollary 11, we obtain a result of Alon, Frankl and Lovász [1, Proposition 2.1].

Corollary 22. *Let G be an r-graph with multiplicities. If r is a prime, we have*

$$\chi(G) \geqslant \frac{\operatorname{conn} \mathrm{B}_{\text{edge}}(G)+2}{r-1}+1.$$

Alon, Frankl and Lovász [1, §3] proved Corollary 22 by constructing a map from $\mathrm{B}_{\text{edge}}(G)$ into a Euclidean space with a certain C_r-action, which seems quite ad-hoc and tricky. Using $\mathtt{Hom}$-complexes, this construction will turn out to be the induced map between $\mathtt{Hom}$-complexes from a given coloring.

Let $\mathrm{M}_{r,n}(\mathbb{R})$ be the space of $r \times n$ real matrices. We let C_r act on $\mathrm{M}_{r,n}(\mathbb{R})$ as the cyclic permutation of rows. Let Y be a subspace of $\mathrm{M}_{r,n}(\mathbb{R})$ consisting of matrices (a_{ij}) satisfying

$$\sum_{i=1}^{r} a_{ik}=0, \qquad \sum_{j=1}^{n} a_{\ell j}=0 \quad \text{and} \quad \sum_{i,j} a_{ij}^2 \neq 0$$

for $k=1,\dots,n$ and $\ell=1,\dots,r$. Then Y is also a C_r-subspace of $\mathrm{M}_{r,n}(\mathbb{R})$. Let G be an r-graph with multiplicities which admits an n-coloring, say c. Alon, Frankl and Lovász [1, §3] defined a C_r-map

$$\bar{c}: \mathrm{B}_{\text{edge}}(G) \to \mathrm{M}_{r,n}(\mathbb{R})$$

by sending a vertex $(v_1,\dots,v_r)$ of $\mathrm{B}_{\text{edge}}(G)$ to a matrix $\sum_{i=1}^{r}(E_{i,c(i)}-E_{i,c(i)+1})$, where $E_{i,j}$ is the matrix whose (i,j) entry is 1 and other entries are 0 and $E_{i,n+1}$ means $E_{i,1}$. They showed that $\bar{c}$ has its image in Y and applied a special generalization of the Borsuk–Ulam theorem to obtain Corollary 22.

We now define a map $g: \mathtt{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)}) \to \mathrm{M}_{r,n}(\mathbb{R})$ by sending a vertex $(i_1,\dots,i_r) \in [n]^r$ to a matrix $\sum_{k=1}^{r}(E_{k,i_k} - E_{k,i_k+1})$. Then one can easily see that g is a C_r-map and has image in Y. By definition, we have the following.

Proposition 23. *Let G be an r-graph with multiplicities which has an n-coloring c. Then there is a commutative diagram*

$$\begin{tikzcd}
\mathtt{Hom}(K_r^{(r)}, G) \arrow[r, "c_*"] \arrow[d, "\simeq"] & \mathtt{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)}) \arrow[d, "g"] \\
\mathrm{B}_{\text{edge}}(G) \arrow[r, "\bar{c}"] & Y
\end{tikzcd}$$

where the left vertical arrow is as in Theorem 20.

We close this section by remarking that the complex of an r-graph (without multiplicities) introduced by Kříž [12, §4] is the barycentric subdivision of $\mathtt{Hom}(K_r^{(r)}, G)$ and then essentially the same as the box-edge complex $\mathrm{B}_{\text{edge}}(G)$.

5. $\mathtt{Hom}_+$-complexes and colorings

5.1. General $\mathtt{Hom}_+$-complexes

In [11, Definition 20.1], $\mathtt{Hom}_+$-complexes of graphs were introduced which are variants of $\mathtt{Hom}$-complexes. As in the case of $\mathtt{Hom}$-complexes, we can give a general recipe for $\mathtt{Hom}_+$-complexes of partial maps between finite sets and will apply it to r-graphs with multiplicities.

Let S, T be finite sets. A partial map from S to T is a map from a non-empty subset of S into T. Then a partial map from S to T is identified with an element of

$$\left(T \cup \{\emptyset\}\right)^S \setminus (\emptyset,\dots,\emptyset).$$

Let K, L be abstract simplicial complexes. Recall that the join $K * L$ is an abstract simplicial complex whose simplices are of the form (σ,τ) where $\sigma \in K$, $\tau \in L$ and either σ or τ is not empty. Then a partial map from S to T is identified with a vertex of the join $*_{s\in S}\Delta^T$. Analogously to $\mathtt{Hom}$-complexes, we are led to the following definition.

Definition 24. Let S, T be finite sets and $\mathcal{C}$ be a class of partial maps from S to T. The $\mathtt{Hom}_+$-complex $\mathtt{Hom}_+^{\mathcal{C}}(S,T)$ is defined as the maximum subcomplex of $*_{s\in S}\Delta^T$ whose vertex set is $\mathcal{C}$.

Analogously to $\mathtt{Hom}$-complexes, we can define induced maps between $\mathtt{Hom}_+$-complexes under certain conditions and see that these induced maps satisfy naturality corresponding to Proposition 5.

5.2. Hom_+-complexes of r-graphs with multiplicities

Let G, H be r-graphs with multiplicities. A partial homomorphism from G to H is a map from a subset V of $V(G)$ into $V(H)$ which is a homomorphism from the maximum r-subgraph with multiplicities of G whose vertex set is V into H. We now define Hom_+-complexes of r-graphs with multiplicities.

Definition 25. Let G, H be r-graphs with multiplicities. The Hom_+-complex $\mathrm{Hom}_+(G, H)$ is defined as $\mathrm{Hom}_+^{\mathcal{C}}(V(G), V(H))$ for the set $\mathcal{C}$ of all partial homomorphisms from G to H.

Similarly to Proposition 7, we have the following.

Proposition 26. *Let* $\mathbf{Graph}^{(r)}$ *and* **Poly** *be the categories of r-graphs with multiplicities and polyhedral complexes, respectively. Then*

$$\left(\mathbf{Graph}^{(r)}\right)^{\mathrm{op}} \times \mathbf{Graph}^{(r)} \to \mathbf{Poly}, \qquad (G, H) \mapsto \mathrm{Hom}_+(G, H)$$

is a functor.

Then as in the case of Hom-complexes, we can construct group actions on Hom_+-complexes by those on r-graphs with multiplicities. For instance, the natural C_r-action on $K_r^{(r)}$ induces a C_r-action on $\mathrm{Hom}_+(K_r^{(r)}, G)$ for an r-graph with multiplicities G. Analogously to Lemma 8, we can prove the following.

Lemma 27. *Let G be an r-graph with multiplicities. If r is a prime, the C_r-action on $\mathrm{Hom}_+(K_r^{(r)}, G)$ is free.*

Using this C_r-action, we obtain a lower bound for the chromatic numbers.

Theorem 28. *Let G be an r-graph with multiplicities. If r is a prime, it holds that*

$$\chi(G) \geqslant \frac{\mathrm{ind}_{C_r} \mathrm{Hom}_+(K_r^{(r)}, G) + 1}{r-1}.$$

Proof. Note that the dimension of $\mathrm{Hom}_+(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is $nr - n - 1 = (r-1)n - 1$. Then the result follow quite similarly to Theorem 10. $\square$

Corollary 29. *Let G be an r-graph with multiplicities. If r is a prime, we have*

$$\chi(G) \geqslant \frac{\mathrm{conn}\,\mathrm{Hom}_+(K_r^{(r)}, G) + 2}{r-1}.$$

In [6, p. 5], Lange defined a complex $\mathrm{B}_0(G)$ for an r-graph with multiplicities and gave a lower bound for the chromatic number of G by using $\mathrm{B}_0(G)$. By definition, $\mathrm{B}_0(G)$ coincides with $\mathrm{Hom}_+(K_r^{(r)}, G)$ and a lower bound in Theorem 28 is the same as the one given by Lange.

As in Section 3, let us consider the homotopy type of $\mathrm{Hom}_+(K_m^{(r)}, \mathcal{K}_n^{(r)})$. In the case of Hom_+-complexes, one can describe $\mathrm{Hom}_+(K_m^{(r)}, \mathcal{K}_n^{(r)})$ explicitly by Sarkaria's formula [13, (2.2)] as follows.

Theorem 30. *([13, (2.2)]) We have*

$$\mathrm{Hom}_+\left(K_m^{(r)}, \mathcal{K}_n^{(r)}\right) \cong *_n \mathrm{sk}_{r-2} \Delta^{[m]}.$$

In particular, $\mathrm{Hom}_+(K_m^{(r)}, \mathcal{K}_n^{(r)})$ has the homotopy type of a wedge of $\binom{m-1}{r-1}^n$ copies of $((r-1)n-1)$-dimensional spheres.

5.3. Hierarchy of lower bounds for the chromatic number

Let G be an r-graph with multiplicities. We have obtained so far two kinds of lower bounds for the chromatic number of G, one is given by $\mathrm{Hom}(K_r^{(r)}, G)$ in Theorem 10 and the other is given by $\mathrm{Hom}_+(K_r^{(r)}, G)$ in Theorem 28. We have also seen that these lower bounds are related to formerly known ones [1, Proposition 2.1], [6, Theorem 3]. We describe $\mathrm{Hom}_+(K_r^{(r)}, G)$ by using $\mathrm{Hom}(K_r^{(r)}, G)$ and then get an inequality between the above lower bounds.

Theorem 31. *For an r-graph with multiplicities G, there is a C_r-map*

$$\mathrm{Hom}_+\big(K_r^{(r)}, G\big) \to \partial\Delta^{[r]} * \mathrm{Hom}\big(K_r^{(r)}, G\big)$$

which is a homotopy equivalence, where C_r acts diagonally on $\partial\Delta^{[r]} * \mathrm{Hom}(K_r^{(r)}, G)$.

Proof. Let P, Q be finite posets. Recall that the join $P * Q$ is a poset whose underlying set is $P \sqcup Q$ and order is defined as $x < y$ if either $x, y \in P$ with $x < y$, $x, y \in Q$ with $x < y$ or $x \in P$, $y \in Q$. Then it follows that

$$\Delta(P * Q) = \Delta(P) * \Delta(Q).$$

Note that the face poset of $\mathrm{Hom}_+(K_r^{(r)}, G)$ is the disjoint union of $\mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$ in (3) and

$$\left\{F_1 \times \cdots \times F_r \;\middle|\; F_1, \ldots, F_r \subset V(G),\ F_i = \emptyset \text{ for some } i \text{ and } \bigcup_{i=1}^{r} F_i \neq \emptyset\right\},$$

where the order is given by inclusions and $F_1 \times \cdots \times F_n$ with $F_{i_1}, \ldots, F_{i_k} \neq \emptyset$ and $F_j = \emptyset$ for $j \neq i_1, \ldots, i_k$ means $F_{i_1} \times \cdots \times F_{i_k}$. We then define a poset map

$$\varphi : \mathcal{F}\big(\mathrm{Hom}_+\big(K_r^{(r)}, G\big)\big) \to \mathcal{F}\big(\partial\Delta^{[r]}\big) * \mathcal{F}\big(\mathrm{Hom}\big(K_r^{(r)}, G\big)\big)$$

as

$$\varphi(F_1 \times \cdots \times F_r) = \begin{cases} \{i_1, \ldots, i_k\} \in \mathcal{F}(\partial\Delta^{[r]}) & \bigcup_{i \neq i_1, \ldots, i_k} F_i = \emptyset \text{ and } F_{i_1}, \ldots, F_{i_k} \neq \emptyset, \\ F_1 \times \cdots \times F_r \in \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G)) & F_1, \ldots, F_r \neq \emptyset. \end{cases}$$

By definition, φ is a C_r-map. For $F_1 \times \cdots \times F_r \in \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G)) \subset \mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$, $\varphi^{-1}((\mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G)))_{\leqslant F_1 \times \cdots \times F_r})$ has the maximum $F_1 \times \cdots \times F_r$. For $\{i_1, \ldots, i_k\} \in \partial\Delta^{[r]} \subset \mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$, $\varphi^{-1}((\mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G)))_{\leqslant \{i_1, \ldots, i_k\}})$ has the maximum

$$F_1 \times \cdots \times F_r, \qquad F_{i_1} = \cdots = F_{i_k} = [n] \quad \text{and} \quad \bigcup_{i \neq i_1, \ldots, i_k} F_i = \emptyset.$$

Then for any $x \in \mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$, $\Delta((\mathcal{F}(\partial\Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G)))_{\leqslant x})$ is contractible, and it follows from Lemma 12 that $\Delta(\varphi)$ is a homotopy equivalence. Thus the composite

$$\begin{aligned}\mathrm{Hom}_+\big(K_r^{(r)}, G\big) &\xrightarrow{\cong} \Delta\big(\mathcal{F}\big(\mathrm{Hom}_+\big(K_r^{(r)}, G\big)\big)\big) \xrightarrow{\Delta(\varphi)} \Delta\big(\mathcal{F}\big(\partial\Delta^{[r]}\big) * \mathcal{F}\big(\mathrm{Hom}\big(K_r^{(r)}, G\big)\big)\big) \\ &= \Delta\big(\mathcal{F}\big(\partial\Delta^{[r]}\big)\big) * \Delta\big(\mathcal{F}\big(\mathrm{Hom}\big(K_r^{(r)}, G\big)\big)\big) \xrightarrow{\cong} \partial\Delta^{[r]} * \mathrm{Hom}\big(K_r^{(r)}, G\big)\end{aligned}$$

is the desired homotopy equivalence, where the first and the last arrows are the natural homeomorphisms. □

Corollary 32. *Let G be an r-graph with multiplicities. If r is a prime, there holds*

$$\begin{aligned}\chi(G) &\geqslant \frac{\operatorname{ind}_{C_r} \mathrm{Hom}(K_r^{(r)}, G) + 1}{r-1} + 1 \geqslant \frac{\operatorname{ind}_{C_r} \mathrm{Hom}_+(K_r^{(r)}, G) + 1}{r-1} \geqslant \frac{\operatorname{ind}_{C_r} \mathrm{Hom}(K_r^{(r)}, G) + 1}{r-1} \\ &\geqslant \frac{\operatorname{conn} \mathrm{Hom}(K_r^{(r)}, G) + 1}{r-1} = \frac{\operatorname{conn} \mathrm{Hom}_+(K_r^{(r)}, G) + 1}{r-1} - 1.\end{aligned}$$

Proof. The first inequality follows from Theorem 10 and the second from Proposition 9 and Theorem 31. As in the proof of Theorem 31, $\mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$ is a subposet of $\mathcal{F}(\mathrm{Hom}_+(K_r^{(r)}, G))$ including the C_r-actions. Then there is a C_r-map $\mathrm{Hom}(K_r^{(r)}, G) \to \mathrm{Hom}_+(K_r^{(r)}, G)$, implying the third inequality by Proposition 9. The fourth inequality follows from Proposition 9 and the last equality from Theorem 31. □

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