

DECOMPOSITIONS OF POLYHEDRAL PRODUCTS FOR SHIFTED COMPLEXES

KOUYEMON IRIYE AND DAISUKE KISHIMOTO

ABSTRACT. The conjecture of Bahri, Bendersky, Cohen and Gitler [BBCG] on wedge decompositions of polyhedral products of shifted complexes is settled affirmatively. As a corollary, it is proved that the homotopy type of the complement of a coordinate subspace arrangement associated with a shifted complex, tensored with \mathbb{R}^r for any $r \geq 1$, has the homotopy type of a wedge of spheres.

1. INTRODUCTION

Throughout the paper, spaces and maps mean compactly generated weak Hausdorff spaces having non-degenerate base points and base point preserving maps.

Let us begin with defining polyhedral products. Let K be an abstract simplicial complex on the index set $[m] = \{1, \dots, m\}$, where we assume that the empty set is a simplex of K for our convention. Let $(\underline{X}, \underline{A})$ be a collection of pairs of spaces indexed by $[m]$, say $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i \in [m]}$. For a simplex σ of K , we put

$$(\underline{X}, \underline{A})^\sigma = Y_1 \times \cdots \times Y_m, \quad \text{where } Y_i = \begin{cases} X_i & i \in \sigma \\ A_i & i \notin \sigma. \end{cases}$$

The polyhedral product (or the generalized moment-angle complex) of $(\underline{X}, \underline{A})$ with respect to K is defined as

$$\mathcal{Z}_K(\underline{X}, \underline{A}) = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma,$$

where the union is taken in $X_1 \times \cdots \times X_m$.

Polyhedral products (with respect to the boundary of a simplex) first appeared in the work of Porter [P] in which higher order Whitehead products are defined as the natural maps between certain polyhedral products. After this work, polyhedral products have been studied in homotopy theory along several directions. Recently, in the work of Davis and Januszkiewicz [DJ], the special polyhedral product $\mathcal{Z}_K(\underline{D}^2, \underline{S}^1)$, called the moment-angle complex of K , was found to play a fundamental role in their theory of, so-called, quasi-toric manifolds (cf. [BP]), which is a topological analogue of theory of toric varieties, where $(\underline{D}^2, \underline{S}^1)$ is the m -copies of (D^2, S^1) . Since then, many mathematicians have been studying polyhedral products in a variety of directions, not only in homotopy theory. See [Ba], [BBCG], [DO], [DS], [FT], [GT], [N], for example. In this paper, we are particularly interested in wedge decompositions of polyhedral products. Let us recall two results on wedge decompositions of polyhedral products; one is due to Grbić and Theriault [GT] and the other is Bahri, Bendersky, Cohen and Gitler [BBCG].

To state the result of Grbić and Theriault [GT], we introduce special simplicial complexes called shifted complexes.

Definition 1.1. An abstract simplicial complex K is called shifted if its vertex set is given a total order satisfying for any simplex $\sigma \in K$ and a vertex $v \in \sigma$, $(\sigma - v) \cup w$ is also a simplex of K whenever a vertex w satisfies $v < w$.

Remark 1.2. In the above definition of shifted complexes, the order of vertices is opposite to the one in [BBCG], [GT], which is convenient for us and is just a notational difference.

The most elementary examples of shifted complexes are skeleta of simplices. Other examples will be given in §5 below. We now state the result of Grbić and Theriault [GT].

Theorem 1.3 (Grbić and Theriault [GT]). *If K is a shifted complex, $\mathcal{Z}_K(\underline{D}^2, \underline{S}^1)$ has the homotopy type of a wedge of spheres.*

Remark 1.4. The proof of Grbić and Theriault [GT] heavily relies on the fact that S^1 has the classifying space, and then it cannot be applied to a general collection $(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i \in [m]}$.

A few years after the work of Grbić and Theriault [GT], Bahri, Bendersky, Cohen and Gitler [BBCG] gave another wedge decomposition of a suspension of a polyhedral product, which is a simple generalization of the standard homotopy equivalence $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$. Although they considered a general polyhedral product $\mathcal{Z}_K(\underline{X}, \underline{A})$, what we are interested in is the special polyhedral product $\mathcal{Z}_K(C\underline{X}, \underline{X})$, and then we here state the result for $\mathcal{Z}_K(C\underline{X}, \underline{X})$ only. Let us set notation. For a non-empty subset I of the vertex set of a simplicial complex K , let K_I denote the induced subcomplex on I (or the full subcomplex on I), that is, K_I is the maximum subcomplex of K whose vertex set is I . Let $|K|$ denote the geometric realization of K . For $\emptyset \neq J = \{j_1 < \dots < j_k\} \subset [m]$ and a collection of spaces $\underline{X} = \{X_i\}_{i \in [m]}$, we put $\widehat{X}^J = X_{j_1} \wedge \dots \wedge X_{j_k}$. We now state:

Theorem 1.5 (Bahri, Bendersky, Cohen and Gitler [BBCG]). *Let K be a simplicial complex on the index set $[m]$ and let $(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i \in [m]}$. Then there is a homotopy equivalence*

$$\Sigma \mathcal{Z}_K(C\underline{X}, \underline{X}) \simeq \Sigma \bigvee_{\emptyset \neq I \subset [m]} |K_I| * \widehat{X}^I.$$

Since shifted complexes have the homotopy types of wedges of spheres as we will see in §5 below, Theorem 1.3 seems to be a desuspension of Theorem 1.5 in the special case that $(C\underline{X}, \underline{X}) = (\underline{D}^2, \underline{S}^1)$. Supported by this observation, the following conjecture was posed in [BBCG].

Conjecture 1.6 (Bahri, Bendersky, Cohen and Gitler [BBCG]). *Let K be a simplicial complex on the index set $[m]$ and let $(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i \in [m]}$. If K is shifted, then there is a homotopy equivalence*

$$(1.1) \quad \mathcal{Z}_K(C\underline{X}, \underline{X}) \simeq \bigvee_{\emptyset \neq I \subset [m]} |K_I| * \widehat{X}^I.$$

The aim of this paper is to prove:

Theorem 1.7. *Conjecture 1.6 is true.*

Let us consider an application of Theorem 1.7 to coordinate subspace arrangements as in [GT]. Let \mathcal{A} be a subspace arrangement in \mathbb{R}^m which is a collection of vector subspaces of \mathbb{R}^m . Subspace arrangements have been studied by the interplay of technology in a wide area of mathematics including algebra, combinatorics, geometry and topology. Among other things, the topology of the complements of subspace arrangements plays a fundamental role in the investigation of subspace arrangements. Let $M(\mathcal{A})$ denote the complement $\mathbb{R}^m - \mathcal{A}$. Let us concentrate our discussion on the special subspace arrangements called the coordinate subspace arrangements. For $I \subset [m]$, we put

$$L_I = \{(x_1, \dots, x_m) \in (\mathbb{R}^r)^m \mid x_i = 0 \text{ for } i \notin I\}$$

which is called the coordinate subspace of $(\mathbb{R}^r)^m$, and a collection of such coordinate subspaces is called a coordinate subspace arrangement. One can assign a coordinate subspace arrangement to a simplicial complex K on the index set $[m]$ as

$$\mathcal{A}_K^r = \{L_\sigma \mid \sigma \text{ is a simplex of } K\}.$$

By definition, the complement $M(\mathcal{A}_K^r)$ is identified with the polyhedral product $\mathcal{Z}_K(\underline{\mathbb{R}}^r, \underline{\mathbb{R}}^r - 0)$. Through this identification, when K is a shifted complex, we apply Theorem 1.7 to $M(\mathcal{A}_K^r)$ and determine its homotopy type.

Corollary 1.8. *If K is a shifted complex on the index set $[m]$, then $\mathcal{Z}_K(\underline{\mathbb{R}}^r, \underline{\mathbb{R}}^r - 0)$, and therefore $M(\mathcal{A}_K^r)$, has the homotopy type of a wedge of spheres.*

The organization of the paper is as follows. In §2, a sketch of the proof of Theorem 1.7 is given in order to clarify the crucial points and to motivate the construction below. In §3, we collect technical lemmas on pushouts which will be used in the following sections. In §4, the space \mathcal{Z}_K^m is introduced as a generalization of the polyhedral product $\mathcal{Z}_K(C\underline{X}, \underline{X})$. It is also proved that there are two pushouts involving \mathcal{Z}_K^m by which \mathcal{Z}_K^m turns out to be constructed inductively on m . In §5, the topology of shifted complexes is considered, by which the space \mathcal{W}_K^m is introduced in §6. In §6, it is also shown that there are two pushouts involving \mathcal{W}_K^m which are analogous to those involving \mathcal{Z}_K^m in §3. In §7, a stronger form of Theorem 1.7 is proved by collecting all the result obtained so far. Corollary 1.8 is also proved in this section.

The authors are grateful to the referee for useful advice and comments.

2. SKETCH OF THE PROOF

The idea of the proof of Theorem 1.7 is quite simple, but it needs particular constructions and arguments. So it may be helpful to clarify the crucial points of the proof by giving its rough sketch. Detailed constructions and arguments will be given in the following sections.

Let K be a simplicial complex on the index set $[k + 1, m] = \{k + 1, k + 2, \dots, m\}$ and let L be its subcomplex on $[\ell + 1, m]$. We fix a collection of spaces $\underline{X} = \{X_i\}_{i \in [m]}$. We first introduce the space \mathcal{Z}_K^m and the map $\rho_{K,L}^m : \mathcal{Z}_L^m \rightarrow \mathcal{Z}_K^m$. In the special case $k = 0$,

$$\mathcal{Z}_K^m = \mathcal{Z}_K(C\underline{X}, \underline{X}).$$

Suppose K, L are shifted. Then we also introduce the space \mathcal{W}_K^m and the map $\lambda_{K,L}^m : \mathcal{W}_L^m \rightarrow \mathcal{W}_K^m$ by using a property of shifted complexes. Likewise, if $k = 0$, by definition

$$\mathcal{W}_K^m \simeq \bigvee_{\emptyset \neq I \subset [m]} |K_I| * \widehat{X}^I.$$

Then our aim is to construct a homotopy equivalence $\epsilon_K^m : \mathcal{Z}_K^m \rightarrow \mathcal{W}_K^m$, which is done by induction on m as follows. Observe that the simplicial complex K is obtained as the union of the cone of the link of the vertex m , say L' , with the cone point m and the induced subcomplex $K' = K_{[k+1, m-1]}$, that is, K is obtained from a subcomplex with less vertices by adding a new vertex. This is the direction of our induction. It is proved that \mathcal{Z}_K^m is given as the pushout of

$$\mathcal{Z}_{K'}^{m-1} \times X_m \xleftarrow{\rho_{K',L'}^{m-1} \times 1} \mathcal{Z}_{L'}^{m-1} \times X_m \xrightarrow{1 \times \iota} \mathcal{Z}_{L'}^{m-1} \times CX_m,$$

where $\iota : X_m \rightarrow CX_m$ is the inclusion, and that \mathcal{W}_K^m is naturally homotopy equivalent to the pushout of

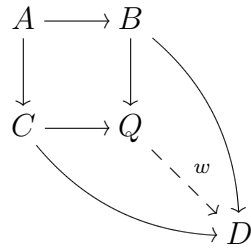
$$\mathcal{W}_{K'}^{m-1} \times X_m \xleftarrow{\lambda_{K',L'}^{m-1} \times 1} \mathcal{W}_{L'}^{m-1} \times X_m \xrightarrow{1 \times \iota} \mathcal{W}_{L'}^{m-1} \times CX_m.$$

Then, using these pushouts, the homotopy equivalences $\epsilon_{K'}^{m-1}$ and $\epsilon_{L'}^{m-1}$ are glued together to produce the homotopy equivalence ϵ_K^m . To do this, the naturality of $\epsilon_{K'}^{m-1}$ with respect to $\rho_{K',L'}^{m-1}$ and $\lambda_{K',L'}^{m-1}$ is necessary. Thus the induction must be proceeded by constructing ϵ_K^m as above together with showing the naturality

$$(2.1) \quad \epsilon_K^m \circ \rho_{K,L}^m = \lambda_{K,L}^m \circ \epsilon_L^m$$

for which constructions and discussions are elaborated.

Remark 2.1. We actually introduce the space \mathcal{W}_K^m in order to work only with strictly commutative diagrams instead of homotopy commutative diagrams. Let us here explain the reason why we restrict ourselves to work only with strictly commutative diagrams. Our reference for homotopy pushouts is [M]. Suppose there is the following diagram of solid arrows.



If the inner square $ABCQ$ is a homotopy pushout and the outer face $ABCD$ is homotopy commutative, then there exists a dotted arrow w making the whole diagram commute up to homotopy. But the problem is that (the homotopy class of) w depends on the commuting homotopy of the outer face $ABCD$. If we would like to show the naturality of the form (2.1) for w , its uniqueness is needed. To this end, all commuting homotopies are imposed to be constant, i.e. we work only with strictly commutative diagrams. Otherwise, we must keep track on all homotopies, including higher homotopies, which is impossible in general.

3. LEMMAS ON PUSHOUTS

In this section, before getting onto the main subject, we collect technical lemmas on pushouts which will be used below. Let us first consider the product of a pushout and a space. We will use this lemma without mentioning in what follows.

Lemma 3.1. *Suppose there is a pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow h \\ C & \xrightarrow{k} & D. \end{array}$$

Then the square

$$\begin{array}{ccc} A \times E & \xrightarrow{f \times 1} & B \times E \\ \downarrow g \times 1 & & \downarrow h \times 1 \\ C \times E & \xrightarrow{k \times 1} & D \times E \end{array}$$

is also a pushout.

Proof. Since spaces are compactly generated and weak Hausdorff as is remarked in the beginning of the paper, the exponential law holds as

$$\text{map}(X \times E, Y) \cong \text{map}(X, \text{map}(E, Y)),$$

i.e. the functor $- \times E$ is a left adjoint. Since left adjoint functors commute with colimits, the proof is completed. \square

The following two lemmas concern pushouts which are also homotopy pushouts. The first one computes special such pushouts.

Lemma 3.2. *Define Q as a pushout*

$$(3.1) \quad \begin{array}{ccc} A \times (B \vee C) & \xrightarrow{\iota \times (1 \vee 1)} & CA \times (B \vee C) \\ \downarrow 1 \times (1 \vee *) & & \downarrow \\ A \times (B \vee D) & \longrightarrow & Q, \end{array}$$

where $\iota : A \rightarrow CA$ is the inclusion. Then there is a homotopy equivalence

$$Q \xrightarrow{\simeq} B \vee \Sigma(A \wedge C) \vee (A \times D)$$

which is natural with respect to A, B, C, D .

Proof. Embed the pushout

$$\begin{array}{ccc} A & \xrightarrow{\iota} & CA \\ \parallel & & \parallel \\ A & \xrightarrow{\iota} & CA \end{array}$$

into (3.1). Taking the quotient of each corner, the resulting square

$$\begin{array}{ccc} A \times (B \vee C) & \xrightarrow{\iota \times (1 \vee 1)} & CA \times (B \vee C) \\ \downarrow 1 \times (1 \vee *) & & \downarrow \\ A \times (B \vee D) & \longrightarrow & Q/CA \end{array}$$

is also a pushout since colimits commute with colimits, where $X \times Y$ denotes the half smash product $X \times Y / X \times *$. Then since half smash products are distributive with respect to wedge sums as $X \times (Y \vee Z) = (X \times Y) \vee (X \times Z)$, we get

$$Q/CA = (CA \times B) \vee R \vee (A \times D),$$

where R is defined as the pushout

$$\begin{array}{ccc} A \times C & \xrightarrow{\iota \times 1} & CA \times C \\ \downarrow & & \downarrow \\ * & \longrightarrow & R. \end{array}$$

One easily sees that $R = \Sigma(A \wedge C)$ and then the projection

$$(3.2) \quad Q/CA \rightarrow B \vee \Sigma(A \wedge C) \vee (A \times D)$$

is a homotopy equivalence which is natural with respect to A, B, C, D .

Given neighborhoods of base points of B, C, D , one can define a neighborhood of CA in Q by restricting (3.1) to these neighborhoods. Moreover, if the neighborhoods of base points in B, C, D satisfy the conditions for NDR pairs, then so does the above neighborhood of CA in Q also. Then since B, C, D have non-degenerate base points, the inclusion $CA \rightarrow Q$ is a cofibration, implying the projection $Q \rightarrow Q/CA$ is a homotopy equivalence. The desired homotopy equivalence is the composite of this projection and (3.2). \square

The next lemma shows that one can produce a new homotopy equivalence by gluing together homotopy equivalences along pushouts which are also homotopy pushouts. The result is classical and well known so that the proof is skipped, see [Br].

Lemma 3.3. *Suppose there is a commutative diagram*

$$\begin{array}{ccccc} A_1 & \longleftarrow & A_2 & \xrightarrow{f} & A_3 \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 \\ B_1 & \longleftarrow & B_2 & \xrightarrow{g} & B_3 \end{array}$$

in which f, g are cofibrations and h_1, h_2, h_3 are homotopy equivalences. If A and B are pushouts of the first and the second rows, then the canonical map $A \rightarrow B$ is also a homotopy equivalence.

4. THE SPACE \mathcal{Z}_K^m

In this section, we introduce the space \mathcal{Z}_K^m and consider two pushouts involving \mathcal{Z}_K^m as is mentioned in §2.

From now on, the following notation is fixed.

- a collection of spaces $\{X_i\}_{i \in [m]}$
- a simplicial complex K on the index set $[k+1, m]$.
- subcomplexes $L \supset L'$ of K on the index sets $[\ell+1, m] \supset [\ell'+1, m]$

(Elements of $[k]$ are sometimes called the ghost vertices of K .)

If we put $(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i \in [k+1, m]}$, the polyhedral product $\mathcal{Z}_K(C\underline{X}, \underline{X})$ can be defined as above. Let us define the space \mathcal{Z}_K^m as

$$\mathcal{Z}_K^m = X^{[k]} \times \mathcal{Z}_K(C\underline{X}, \underline{X}),$$

where for $I = \{i_1 < \dots < i_k\} \subset [m]$, X^I denotes the product $X_{i_1} \times \dots \times X_{i_k}$. In particular, $\mathcal{Z}_\emptyset^m = X^{[m]}$ and for $k = 0$,

$$\mathcal{Z}_K^m = \mathcal{Z}_K(C\underline{X}, \underline{X}).$$

Let us denote the inclusion $\mathcal{Z}_L^m \rightarrow \mathcal{Z}_K^m$ by $\rho_{K,L}^m$. Then it holds that

$$\rho_{K,L'}^m = \rho_{K,L}^m \circ \rho_{L,L'}^m$$

where L' can be the empty set. If K, L are clear in the context, $\rho_{K,L}^m$ is abbreviated by ρ^m .

Let us consider pushouts involving \mathcal{Z}_K^m . We first set some notation. For a vertex v of K , let $\text{link}_K(v)$ and $\text{star}_K(v)$ denote the link and the star of v , respectively. The induced subcomplex $K_{[k+1, m]-v}$ is also denoted by $\text{rest}_K(v)$, following the notation of [GT].

Proposition 4.1. *There are two pushouts*

$$\begin{array}{ccc} \mathcal{Z}_{\text{link}_K(m)}^{m-1} \times X_m & \xrightarrow{1 \times \iota} & \mathcal{Z}_{\text{link}_K(m)}^{m-1} \times CX_m & \text{and} & X^{[m]} & \xrightarrow{\text{incl}} & X^{[k-1]} \times CX_k \times X^{[k+1, m]} \\ \downarrow \rho^{m-1 \times 1} & & \downarrow \text{incl} & & \downarrow \rho_{K, \emptyset} & & \downarrow \text{incl} \\ \mathcal{Z}_{\text{rest}_K(m)}^{m-1} \times X_m & \xrightarrow{\text{incl}} & \mathcal{Z}_K^m & & \mathcal{Z}_K^m & \xrightarrow{\rho_{K \sqcup k}^m} & \mathcal{Z}_{K \sqcup k}^m \end{array}$$

Proof. Observe that the pushout of simplicial complexes

$$(4.1) \quad \begin{array}{ccc} \text{link}_K(m) & \longrightarrow & \text{star}_K(m) \\ \downarrow & & \downarrow \\ \text{rest}_K(m) & \longrightarrow & K \end{array}$$

induces a pushout of spaces

$$(4.2) \quad \begin{array}{ccc} \mathcal{Z}_{\text{link}_K(m)}^{m-1} \times X_m & \xrightarrow{\text{incl}} & \mathcal{Z}_{\text{star}_K(m)}^m \\ \downarrow \rho^{m-1 \times 1} & & \downarrow \rho^m \\ \mathcal{Z}_{\text{rest}_K(m)}^{m-1} \times X_m & \xrightarrow{\text{incl}} & \mathcal{Z}_K^m \end{array}$$

Since every maximal simplex of $\text{star}_K(m)$ contains the vertex m , it holds that

$$\mathcal{Z}_{\text{star}_K(m)}^m = \mathcal{Z}_{\text{link}_K(m)}^{m-1} \times CX_m$$

and that the inclusion $\mathcal{Z}_{\text{link}_K(m)}^{m-1} \times X_m \rightarrow \mathcal{Z}_{\text{star}_K(m)}^m$ is identified with $1 \times \iota$. Then the first pushout is obtained.

The second pushout is induced from the pushout of simplicial complexes

$$\begin{array}{ccc} \emptyset & \longrightarrow & k \\ \downarrow & & \downarrow \\ K & \longrightarrow & K \sqcup k. \end{array}$$

□

5. TOPOLOGY OF SHIFTED COMPLEXES

Although combinatorial properties of shifted complexes have been studied extensively, their topological properties is rarely found in literature. Then in this section, we record elementary topological properties of shifted complexes. Using these topological properties of shifted complexes, the space \mathcal{W}_K^m will be defined in the next section.

To help understand shifted complexes, let us first give some examples.

Example 5.1. Any skeleton of a simplex is shifted by any order on vertices.

Example 5.2. The square graph is not shifted by any order on vertices.

Example 5.3. In the above two examples, the shiftiness does not depend on the order of vertices. However, by definition, the shiftiness depends on the order on vertices in general. Consider two 1-dimensional simplicial complexes on $[4]$ whose edge sets are

$$\{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\} \quad \text{and} \quad \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}.$$

These two simplicial complexes are isomorphic; a square with one diagonal edge. However, the former is shifted but not the latter.

Let us consider a subcomplex of a shifted complex. Of course, not every subcomplex of a shifted complex is a shifted complex. For example, the square graph is not shifted by any order on vertices as in Example 5.2 and it is a subcomplex of the 3-simplex which is shifted by any order on vertices. Notice that since every simplicial complex is a subcomplex of a simplex, the above third example implies that there is a subcomplex of a shifted complex which is not shifted by the induced order but is shifted by an alternative order. We are now interested in subcomplexes of a shifted complex which are shifted by the induced order. We here give two examples of such subcomplexes of a shifted complex.

Example 5.4. Any induced subcomplex of a shifted complex is shifted by the induced order. In particular, if K is shifted, for any vertex v of K , $\text{rest}_K(v)$ is shifted by the induced order.

Example 5.5. If K is shifted, $\text{star}_K(m)$ is shifted by the induced order. Then since $\text{link}_K(m)$ is an induced subcomplex of $\text{star}_K(m)$ and is included in $\text{rest}_K(m)$, $\text{link}_K(m)$ is a subcomplex of $\text{rest}_K(m)$ which is shifted by the induced order. Notice that if $v < m$, $\text{star}_K(v)$ may not be shifted by the induced order.

Let us start to consider the topology of shifted complexes. We first look at the connected components, by which the construction of the homotopy equivalence of Theorem 1.7 is divided into two cases; the connected component of the maximum vertex m of K and the remaining part.

Proposition 5.6. *If K is shifted and K_0 is the connected component of the vertex m , then it holds that for some $k_0 \in [k+1, m]$,*

$$V(K_0) = V(\text{star}_K(m)) = [k_0, m]$$

and that the remaining part $K - K_0$ is discrete, where $V(A)$ denotes the vertex set of a simplicial complex A .

Proof. If $K_0 = \{m\}$, the proposition is trivial. Suppose $K_0 \neq \{m\}$. Let k_0 be the minimum vertex of K_0 . Then k_0 is adjacent to some vertex, say v , implying that v is adjacent to m since $v \leq m$. This also implies that any $w \in [k_0, m]$ is adjacent to m . Then the first assertion is proved. Take any vertex u of $K - K_0$. If u is adjacent to some vertex, it is also adjacent to m as above, a contradiction. Then u is isolated, implying the second assertion. \square

We next give a convenient description of the homotopy types of shifted complexes, which enables us to work only with strictly commutative diagrams as is remarked in §2. Let us set notation. Put

$$\mathbf{m}(K) = \{\text{simplices of } \text{rest}_K(m) \text{ which are maximal in } K\}$$

and

$$\overline{K} = |K| / |\text{star}_K(m)|,$$

where $\mathbf{m}(K)$ is not a simplicial complex, just a collection of simplices. Notice that since $|\text{star}_K(m)|$ is contractible, the projection $|K| \rightarrow \overline{K}$ is a homotopy equivalence. Let $\alpha_{K,L}$ denote the map $\overline{L} \rightarrow \overline{K}$ induced from the inclusion $L \rightarrow K$. It is clear that

$$(5.1) \quad \alpha_{K,L'} = \alpha_{K,L} \circ \alpha_{L,L'}.$$

Proposition 5.7. *Suppose K, L are shifted. Then there is an identification*

$$\overline{K} = \bigvee_{\sigma \in \mathbf{m}(K)} S^{\dim \sigma}$$

through which the map $\alpha_{K,L}$ is identified with a wedge of the identity map of $\bigvee_{\sigma \in \mathbf{m}(L) \cap \mathbf{m}(K)} S^{\dim \sigma}$ and the constant map on the remaining summand.

Proof. For any simplex σ of K , $\partial\sigma$ is included in $\text{star}_K(m)$ by the definition of shifted complexes. Then $|K| - |\text{star}_K(m)|$ is the disjoint union of the interior of maximal simplices of K which do not contain the vertex m , completing the proof. \square

Let us give an alternative description of \overline{K} which is convenient to construct \overline{K} inductively by the pushout corresponding to (4.1) as is seen in the proof of Proposition 6.1 below. Set

$$\begin{aligned} \mathbf{m}_1(K) &= \{\sigma \in K \mid m-1 \notin \sigma \text{ and } \sigma \cup (m-1) \in \mathbf{m}(K)\}, \\ \mathbf{m}_2(K) &= \{\tau \in \mathbf{m}(K) \mid m-1 \notin \tau\}. \end{aligned}$$

It will be useful in the proof of Proposition 6.1 below to write $\mathbf{m}_1(K)$ and $\mathbf{m}_2(K)$ by using $\mathbf{m}(\text{link}_K(m))$ and $\mathbf{m}(\text{rest}_K(m))$: if we put $\mathbf{m}_0(K) = \mathbf{m}(\text{link}_K(m)) \cap \mathbf{m}(\text{rest}_K(m))$, we have

$$\mathbf{m}_1(K) = \mathbf{m}(\text{link}_K(m)) - \mathbf{m}_0(K) \quad \text{and} \quad \mathbf{m}_2(K) = \mathbf{m}(\text{rest}_K(m)) - \mathbf{m}_0(K).$$

Proposition 5.8. *Suppose K, L are shifted. Then there is an identification*

$$\overline{K} = \bigvee_{\sigma \in \mathbf{m}_1(K)} \Sigma S^{\dim \sigma} \vee \bigvee_{\tau \in \mathbf{m}_2(K)} S^{\dim \tau}$$

through which the map $\alpha_{K,L}$ is identified with the restriction of $\Sigma \alpha_{\text{link}_K(m), \text{link}_L(m)} \vee \alpha_{\text{rest}_K(m), \text{rest}_L(m)}$.

Proof. By definition, we have

$$\mathbf{m}(K) = \mathbf{m}_2(K) \sqcup \{\sigma \cup (m-1) \mid \sigma \in \mathbf{m}_1(K)\}.$$

Then the proof is completed by Proposition 5.7. \square

We close this section with an observation about the dimension of simplices in $\mathbf{m}_2(K)$.

Proposition 5.9. *If K is connected, then $\dim \sigma > 0$ whenever $\sigma \in \mathbf{m}_2(K)$.*

Proof. By Proposition 5.6, we have $V(\text{link}_K(m)) = V(\text{rest}_K(m))$. Then if $\sigma \in \mathbf{m}(\text{rest}_K(m))$ satisfies $\dim \sigma = 0$, σ must belong to $\mathbf{m}(\text{link}_K(m))$ also. This completes the proof. \square

6. THE SPACE \mathcal{W}_K^m

In this section, we introduce the space \mathcal{W}_K^m and consider two pushouts involving \mathcal{W}_K^m which correspond to those in Proposition 4.1. The map $\theta_K^m(i)$ is also introduced which will be used to prove the naturality (2.1).

When K is shifted, define the space \mathcal{W}_K^m as

$$\mathcal{W}_K^m = X^{[k]} \times \bigvee_{\emptyset \neq I \subset [k+1, m]} \Sigma \overline{K}_I \wedge \widehat{X}^I.$$

Then since $\overline{K}_I \simeq |K_I|$, if $k = 0$,

$$\mathcal{W}_K^m \simeq \bigvee_{\emptyset \neq I \subset [m]} |K_I| * \widehat{X}^I.$$

Let δ_i denote the composite

$$(6.1) \quad X_i \rtimes \Sigma A \xrightarrow{1 \times \nabla} X_i \rtimes (\Sigma A \vee \Sigma A) = (X_i \rtimes \Sigma A) \vee (X_i \rtimes \Sigma A) \xrightarrow{\text{proj}} \Sigma A \vee (X_i \wedge \Sigma A)$$

which is a homotopy equivalence, where $X \rtimes Y$ is the half smash product $X \times Y / X \times *$ as above and ∇ is the comultiplication of ΣA . We now define the map

$$(6.2) \quad \pi(\ell, k) : X^{[\ell]} \times \Sigma A \rightarrow X^{[k]} \times \bigvee_{J \subset [k+1, \ell]} \Sigma A \wedge \widehat{X}^J$$

by applying $1 \times \delta_\ell \circ \pi_\ell, \dots, 1 \times \delta_{k+1} \circ \pi_{k+1}$ in turn, where $\pi : X_i \times Y \rightarrow X_i \rtimes Y$ is the projection. When K, L are shifted, the map $\lambda_{K,L}^m : \mathcal{W}_L^m \rightarrow \mathcal{W}_K^m$ is defined as the composite

$$\begin{aligned} \mathcal{W}_L^m &= X^{[\ell]} \times \bigvee_{\emptyset \neq I \subset [\ell+1, m]} \Sigma \overline{L}_I \wedge \widehat{X}^I \xrightarrow{\pi(\ell, k)} X^{[k]} \times \bigvee_{\substack{\emptyset \neq I \subset [\ell+1, m] \\ J \subset [k+1, \ell]}} \Sigma \overline{L}_I \wedge \widehat{X}^{I \cup J} \\ &\xrightarrow{1 \times \vee (\alpha_{K_I \cup J, L_I \wedge 1})} X^{[k]} \times \bigvee_{\emptyset \neq H \subset [k+1, m]} \Sigma \overline{K}_H \wedge \widehat{X}^H = \mathcal{W}_K^m. \end{aligned}$$

The definition of $\lambda_{K,L}^m$ is actually motivated by Lemma 3.2. Note, in particular, that the map $\lambda_{K,\emptyset}^m$ is the composite

$$\mathcal{W}_\emptyset^m = X^{[m]} \xrightarrow{\text{proj}} X^{[k]} \xrightarrow{\text{incl}} \mathcal{W}_K^m.$$

It follows from (5.1) that

$$(6.3) \quad \lambda_{K,L'}^m = \lambda_{K,L}^m \circ \lambda_{L,L'}^m,$$

where L' can be the empty set.

Let us consider two pushouts involving \mathcal{W}_K^m corresponding to those in Proposition 4.1. Let us first consider the left one in Proposition 4.1. When K is shifted, define $\overline{\mathcal{W}}_K^m$ as the pushout

$$(6.4) \quad \begin{array}{ccc} \mathcal{W}_{\text{link}_K(m)}^{m-1} \times X_m & \xrightarrow{1 \times \iota} & \mathcal{W}_{\text{link}_K(m)}^{m-1} \times CX_m \\ \downarrow \lambda^{m-1} \times 1 & & \downarrow \\ \mathcal{W}_{\text{rest}_K(m)}^{m-1} \times X_m & \longrightarrow & \overline{\mathcal{W}}_K^m, \end{array}$$

where we often abbreviate $\lambda_{K,L}^m$ by λ^m when K, L are clear in the context. If L is also shifted, we also define $\bar{\lambda}_{K,L}^m : \overline{\mathcal{W}}_L^m \rightarrow \overline{\mathcal{W}}_K^m$ as the induced map from a commutative diagram

$$\begin{array}{ccccc} \mathcal{W}_{\text{rest}_L(m)}^{m-1} \times X_m & \xleftarrow{\lambda^{m-1} \times 1} & \mathcal{W}_{\text{link}_L(m)}^{m-1} \times X_m & \xrightarrow{1 \times \iota} & \mathcal{W}_{\text{link}_L(m)}^{m-1} \times CX_m \\ \downarrow \lambda^{m-1} \times 1 & & \downarrow \lambda^{m-1} \times 1 & & \downarrow \lambda^{m-1} \times 1 \\ \mathcal{W}_{\text{rest}_K(m)}^{m-1} \times X_m & \xleftarrow{\lambda^{m-1} \times 1} & \mathcal{W}_{\text{link}_K(m)}^{m-1} \times X_m & \xrightarrow{1 \times \iota} & \mathcal{W}_{\text{link}_K(m)}^{m-1} \times CX_m. \end{array}$$

By (6.3) and the universality of pushouts, if L' is shifted, it holds that

$$\bar{\lambda}_{K,L'}^m = \bar{\lambda}_{K,L}^m \circ \bar{\lambda}_{L,L'}^m,$$

where L' can be the empty set.

Proposition 6.1. *Suppose K, L are connected and shifted. Then there is a homotopy equivalence*

$$\bar{\varphi}_K^m : \overline{\mathcal{W}}_K^m \xrightarrow{\simeq} \mathcal{W}_K^m$$

satisfying $\bar{\varphi}_K^m \circ \bar{\lambda}_{K,L}^m = \lambda_{K,L}^m \circ \bar{\varphi}_L^m$.

Proof. Note first that since K is connected, $V(\text{link}_K(m)) = V(\text{rest}_K(m)) = [k+1, m-1]$ by Proposition 5.6. Put

$$W_K(i) = \bigvee_{\emptyset \neq I \subset [k+1, m-1]} \bigvee_{\sigma \in \mathfrak{m}_i(K_{I \cup m})} \Sigma S^{\dim \sigma} \wedge \widehat{X}^I$$

for $i = 0, 1, 2$. Since $\text{link}_{K_{I \cup m}}(m) = (\text{link}_K(m))_I$ and $\text{rest}_{K_{I \cup m}}(m) = (\text{rest}_K(m))_I$ for $\emptyset \neq I \subset [k+1, m-1]$, there are identifications

$$\mathcal{W}_{\text{link}_K(m)}^{m-1} = X^{[k]} \times (W_K(0) \vee W_K(1)) \quad \text{and} \quad \mathcal{W}_{\text{rest}_K(m)}^{m-1} = X^{[k]} \times (W_K(0) \vee W_K(2)).$$

Through these identifications, one gets

$$\lambda_{\text{rest}_K(m), \text{link}_K(m)}^{m-1} = 1_{X^{[k]}} \times (1_{W_K(0)} \vee *)$$

by Proposition 5.7. Then Lemma 3.2 can be applied to the pushout (6.4) and hence a homotopy equivalence

$$(6.5) \quad \bar{\phi}_K^m : \overline{\mathcal{W}}_K^m \xrightarrow{\simeq} X^{[k]} \times (W_K(0) \vee \Sigma(W_K(1) \wedge X_m) \vee (W_K(2) \rtimes X_m))$$

is obtained, which is natural with respect to $W_K(i)$ for $i = 0, 1, 2$.

Since K is connected, it follows from Proposition 5.6 that $K_{I \cup m}$ is connected for any $\emptyset \neq I \in [k+1, m-1]$. Then by Proposition 5.9, one has $\dim \sigma > 0$ for any $\sigma \in \mathfrak{m}_2(K_{I \cup m})$, implying that $\Sigma^{-1}W_K(2) = \bigvee_{\emptyset \neq I \subset [k+1, m-1]} \bigvee_{\sigma \in \mathfrak{m}_2(K_{I \cup m})} S^{\dim \sigma} \wedge \widehat{X}^I$ is a suspension. Using this suspension parameter, one gets a homotopy equivalence

$$\Sigma^{-1}W_K(2) \rtimes X_m \xrightarrow{\simeq} \Sigma^{-1}W_K(2) \vee (\Sigma^{-1}W_K(2) \wedge X_m).$$

Hence one obtains a homotopy equivalence

$$\begin{aligned} X^{[k]} \times (W_K(0) \vee \Sigma(W_K(1) \wedge X_m) \vee (W_K(2) \rtimes X_m)) \\ \xrightarrow{\simeq} X^{[k]} \times (W_K(0) \vee \Sigma(W_K(1) \wedge X_m) \vee W_K(2) \vee (W_K(2) \wedge X_m)) \end{aligned}$$

which we denote by δ'_m .

By Proposition 5.7,

$$W_K(0) \vee W_K(2) = \bigvee_{\emptyset \neq I \subset [k+1, m-1]} \Sigma \overline{K}_I \wedge \widehat{X}^I$$

and by Proposition 5.8,

$$\Sigma(W_K(1) \wedge X_m) \vee (W_K(2) \wedge X_m) = \bigvee_{\emptyset \neq I \subset [k+1, m-1]} \Sigma \overline{K}_{I \cup m} \wedge \widehat{X}^{I \cup m}.$$

Then it follows that

$$(6.6) \quad \mathcal{W}_K^m = X^{[k]} \times (W_K(0) \vee \Sigma(W_K(1) \wedge X_m) \vee W_K(2) \vee (W_K(2) \wedge X_m)).$$

Thus we define the homotopy equivalence as

$$\bar{\varphi}_K^m = \delta'_m \circ \bar{\phi}_K^m.$$

Our remaining task is to show the naturality of $\bar{\varphi}_K^m$. By definition, the map $\lambda_{\text{link}_K(m), \text{link}_L(m)}^{m-1}$ is described as the composite

$$\begin{aligned} \mathcal{W}_{\text{link}_L(m)}^{m-1} &= X^{[\ell]} \times (W_L(0) \vee W_L(1)) \xrightarrow{\pi(\ell, k)} X^{[k]} \times \bigvee_{J \subset [k+1, \ell]} ((W_L(0) \wedge \widehat{X}^J) \vee (W_L(1) \wedge \widehat{X}^J)) \\ &\xrightarrow{1 \times (\alpha_0 \vee \alpha_1)} X^{[k]} \times (W_K(0) \vee W_K(1)) = \mathcal{W}_{\text{link}_K(m)}^{m-1}, \end{aligned}$$

where $\pi(\ell, k)$ is as in (6.2) and $\alpha_i : \bigvee_{J \subset [k+1, \ell]} W_L(i) \wedge \widehat{X}^J \rightarrow W_K(i)$ is given by the restriction of $\alpha_{(\text{link}_K(m))_{I \cup J}, (\text{link}_L(m))_I}$ for $\emptyset \neq I \subset [\ell+1, m]$. The same is true for $\lambda_{\text{rest}_K(m), \text{rest}_L(m)}^{m-1}$. Then if we define the map

$$\begin{aligned} \mu_{K, L}^m : X^{[\ell]} \times (W_L(0) \vee \Sigma(W_L(1) \wedge X_m) \vee (W_L(2) \rtimes X_m)) \\ \rightarrow X^{[k]} \times (W_K(0) \vee \Sigma(W_K(1) \wedge X_m) \vee (W_K(2) \rtimes X_m)) \end{aligned}$$

as the composite $(1 \times (\alpha_0 \vee \Sigma(\alpha_1 \wedge 1) \vee (\alpha_2 \times 1))) \circ \pi(\ell, k)$, then by the naturality of the homotopy equivalence of Lemma 3.2, it holds that

$$\bar{\phi}_K^m \circ \bar{\lambda}_{K,L}^m = \mu_{K,L}^m \circ \bar{\phi}_L^m.$$

On the other hand, by Proposition 5.8, $\lambda_{K,L}^m$ is given as the composite $(1 \times (\alpha_0 \vee \Sigma(\alpha_1 \wedge 1) \vee \alpha_2 \vee (\alpha_2 \wedge 1))) \circ \pi(\ell, k)$. Since the suspension parameters used by δ'_m above and by δ_i in the definition of $\mu_{K,L}^m$ are distinct, they commute, implying

$$(6.7) \quad \delta'_m \circ \mu_{K,L}^m = \lambda_{K,L}^m \circ \delta'_m.$$

Thus, summarizing, we have obtained

$$\bar{\varphi}_K^m \circ \bar{\lambda}_{K,L}^m = \delta'_m \circ \bar{\phi}_K^m \circ \bar{\lambda}_{K,L}^m = \delta'_m \circ \mu_{K,L}^m \circ \bar{\phi}_L^m = \lambda_{K,L}^m \circ \delta'_m \circ \bar{\phi}_L^m = \lambda_{K,L}^m \circ \bar{\varphi}_L^m$$

which is the desired naturality. \square

Let us next consider the pushout involving \mathcal{W}_K^m corresponding to the right one in Proposition 4.1. When K is shifted, define $\widehat{\mathcal{W}}_{K \sqcup k}^m$ as the pushout

$$(6.8) \quad \begin{array}{ccc} X^{[m]} & \xrightarrow{\text{incl}} & X^{[k-1]} \times CX_k \times X^{[k+1,m]} \\ \downarrow \lambda_{K,\emptyset}^m & & \downarrow \\ \mathcal{W}_K^m & \longrightarrow & \widehat{\mathcal{W}}_{K \sqcup k}^m. \end{array}$$

When L is also shifted and $\ell = k$, the map $\hat{\lambda}_{K \sqcup k, L \sqcup k}^m : \widehat{\mathcal{W}}_{L \sqcup k}^m \rightarrow \widehat{\mathcal{W}}_{K \sqcup k}^m$ is defined as the induced map from a commutative diagram

$$\begin{array}{ccccc} \mathcal{W}_L^m & \xleftarrow{\lambda_{L,\emptyset}^m} & X^{[m]} & \xrightarrow{\text{incl}} & X^{[k-1]} \times CX_k \times X^{[k+1,m]} \\ \downarrow \lambda_{K,L}^m & & \parallel & & \parallel \\ \mathcal{W}_K^m & \xleftarrow{\lambda_{K,\emptyset}^m} & X^{[m]} & \xrightarrow{\text{incl}} & X^{[k-1]} \times CX_k \times X^{[k+1,m]}. \end{array}$$

Then, analogously to $\bar{\lambda}_{K,L}^m$ above, if L' is also shifted and $\ell' = k$, it holds that

$$\hat{\lambda}_{K \sqcup k, L' \sqcup k}^m = \hat{\lambda}_{K \sqcup k, L \sqcup k}^m \circ \hat{\lambda}_{L \sqcup k, L' \sqcup k}^m.$$

Proposition 6.2. *If K, L are shifted and $k = \ell$, there is a homotopy equivalence*

$$\hat{\varphi}_{K \sqcup k}^m : \widehat{\mathcal{W}}_{K \sqcup k}^m \xrightarrow{\simeq} \mathcal{W}_{K \sqcup k}^m$$

satisfying $\hat{\varphi}_{K \sqcup k}^m \circ \hat{\lambda}_{K \sqcup k, L \sqcup k}^m = \lambda_{K \sqcup k, L \sqcup k}^m \circ \hat{\varphi}_{L \sqcup k}^m$. Moreover, the composite

$$\mathcal{W}_K^m \xrightarrow{\nu} \widehat{\mathcal{W}}_{K \sqcup k}^m \xrightarrow{\hat{\varphi}_{K \sqcup k}^m} \mathcal{W}_{K \sqcup k}^m$$

coincides with the map $\lambda_{K \sqcup k, K}^m$, where ν is the bottom arrow of (6.8).

Proof. Let us first define a homotopy equivalence $\hat{\delta}_i$ as the composite

$$\Sigma(A \times X_i) \xrightarrow{(1 \times \nabla) \circ \nabla} \bigvee^3 \Sigma(A \times X_i) \xrightarrow{\text{proj}} \Sigma A \vee \Sigma X_i \vee \Sigma(A \wedge X_i),$$

where ∇ is the comultiplication. Using $\hat{\delta}_i$, let us next define a homotopy equivalence

$$(6.9) \quad \delta^m(i) : \Sigma X_i \wedge X^{[i+1,m]} \xrightarrow{\cong} \bigvee_{\emptyset \neq I \subset [i+1,m]} \Sigma(\hat{I} \sqcup i) \wedge \widehat{X}^{I \sqcup i},$$

where \hat{I} denotes the maximum of I and the base point of the two points set $\hat{I} \sqcup i$ is \hat{I} . Applying $\hat{\delta}_m, \hat{\delta}_{m-1}, \dots, \hat{\delta}_{i+2}$ to $\Sigma X_i \wedge X^{[i+1,m]}$ in turn, one gets a homotopy equivalence

$$\Sigma X_i \wedge X^{[i+1,m]} \xrightarrow{\cong} \bigvee_{\emptyset \neq I \subset [i+1,m]} \Sigma \widehat{X}^{I \sqcup i}.$$

By identifying $\Sigma \widehat{X}^{I \sqcup i}$ with $\Sigma(\hat{I} \sqcup i) \wedge \widehat{X}^{I \sqcup i}$, $\delta^m(i)$ is defined.

Applying Lemma 3.2, there is a natural homotopy equivalence

$$\hat{\phi}_{K \sqcup k}^m : \widehat{W}_{K \sqcup k}^m \xrightarrow{\cong} X^{[k-1]} \times (\Sigma(X_k \wedge X^{[k+1,m]}) \vee (X_k \times \bigvee_{\emptyset \neq I \subset [k+1,m]} \Sigma \overline{K}_I \wedge \widehat{X}^I)).$$

Consider the homotopy equivalence

$$\begin{aligned} 1 \times (\delta^m(k) \vee \delta_k) : X^{[k-1]} \times (\Sigma(X_k \wedge X^{[k+1,m]}) \vee (X_k \times \bigvee_{\emptyset \neq I \subset [k+1,m]} \Sigma \overline{K}_I \wedge \widehat{X}^I)) \\ \xrightarrow{\cong} X^{[k-1]} \times \bigvee_{\emptyset \neq I \subset [k+1,m]} (\Sigma(\overline{K}_I \wedge \widehat{X}^I) \vee \Sigma(\overline{K}_I \vee (\hat{I} \sqcup k)) \wedge \widehat{X}^{I \sqcup k}). \end{aligned}$$

Since $\overline{K}_I \vee (\hat{I} \sqcup k) = \overline{K}_{I \sqcup k}$, the right hand side is identified with $\mathcal{W}_{K \sqcup k}^m$, and then we define the homotopy equivalence as

$$\hat{\varphi}_{K \sqcup k}^m = (1 \times (\delta^m(k) \vee \delta_k)) \circ \hat{\phi}_{K \sqcup k}^m.$$

We next prove the naturality of $\hat{\varphi}_{K \sqcup k}^m$. By the naturality of $\hat{\phi}_{K \sqcup k}^m$,

$$(1 \times (1 \vee (1 \times \lambda_{K,L}^m))) \circ \hat{\phi}_{L \sqcup k}^m = \hat{\phi}_{K \sqcup k}^m \circ \hat{\lambda}_{K \sqcup k, L \sqcup k}^m.$$

On the other hand, it clearly holds that

$$(1 \times (\delta^m(k) \vee \delta_k)) \circ (1 \times (1 \vee (1 \times \lambda_{K,L}^m))) = \lambda_{K \sqcup k, L \sqcup k}^m \circ (1 \times (\delta^m(k) \vee \delta_k)).$$

Then, combining the above two equalities together with the definition of $\hat{\varphi}_{K \sqcup k}^m$, the naturality of $\hat{\varphi}_{K \sqcup k}^m$ is proved.

The last assertion follows from the construction of $\hat{\varphi}_{K \sqcup k}^m$ and of the homotopy equivalence of Lemma 3.2. Thus the proof is completed. \square

Suppose K is shifted. Let us further investigate the pushout (6.8) by describing its right arrow $X^{[k-1]} \times CX_k \times X^{[k+1,m]} \rightarrow \widehat{W}_{K \sqcup k}^m$ which we denote by ϑ_K . For $i \in [k+1, m-1]$, define the map $\theta_K^m(i) : X^{[i-1]} \times CX_i \times X^{[i+1,m]} \rightarrow \mathcal{W}_K^m$ as

$$\theta_K^m(i) = \lambda_{K, [i,m]}^m \circ \hat{\varphi}_{[i,m]}^m \circ \vartheta_{[i+1,m]}$$

which will be used to prove the naturality (2.1) in the next section, where $[j, m]$ means the discrete simplicial complex on the index set $[j, m]$. Define $\theta_K^m(m)$ also as the composite

$$X^{[m-1]} \times CX_m \xrightarrow{\text{proj}} X^{[k]} \xrightarrow{\text{incl}} \mathcal{W}_K^m.$$

We here list some properties of $\theta_K^m(i)$.

Proposition 6.3. *If K, L are shifted, the map $\theta_K^m(i)$ has the following properties.*

- (1) For $i \in [k+1, m]$, $\theta_K^m(i)|_{X^{[m]}} = \lambda_{K, \emptyset}^m$.
- (2) For $i \in [\ell+1, m]$, $\theta_K^m(i) = \lambda_{K, L}^m \circ \theta_L^m(i)$.
- (3) The composite

$$X^{[k-1]} \times CX_k \times X^{[k+1, m]} \xrightarrow{\vartheta_K} \widehat{W}_{K \sqcup k}^m \xrightarrow{\hat{\varphi}_{K \sqcup k}^m} \mathcal{W}_{K \sqcup k}^m$$

coincides with $\theta_{K \sqcup k}^m(k)$.

Proof. (1) and (2) immediately follow from the definition of $\theta_K^m(i)$. (3) follows from the naturality of $\hat{\varphi}_K^m$ in Proposition 6.2. \square

Let us consider another naturality of $\hat{\varphi}_K^m$. When L is also shifted and $k < \ell$, the map $\nu_{K, L \sqcup \ell}^m : \widehat{W}_{L \sqcup \ell}^m \rightarrow \mathcal{W}_K^m$ is defined as the induced map from a commutative diagram

$$\begin{array}{ccccc} \mathcal{W}_L^m & \xleftarrow{\lambda_{L, \emptyset}^m} & X^{[m]} & \xrightarrow{\text{incl}} & X^{[\ell-1]} \times CX_\ell \times X^{[\ell+1, m]} \\ \downarrow \lambda_{K, L}^m & & \downarrow \lambda_{K, \emptyset}^m & & \downarrow \theta_K^m(\ell) \\ \mathcal{W}_K^m & \xlongequal{\quad} & \mathcal{W}_K^m & \xlongequal{\quad} & \mathcal{W}_K^m \end{array}$$

By Proposition 6.3 (2), if L, L' are shifted and $k \leq \ell < \ell'$, then

$$(6.10) \quad \nu_{K, L' \sqcup \ell'}^m = \lambda_{K, L}^m \circ \nu_{L, L' \sqcup \ell'}^m.$$

Corollary 6.4. *If K, L are shifted and $k < \ell$, then $\nu_{K, L \sqcup \ell}^m = \lambda_{K, L \sqcup \ell}^m \circ \hat{\varphi}_{L \sqcup \ell}^m$.*

Proof. By Proposition 6.3 (3), we have $\nu_{L \sqcup \ell, L \sqcup \ell}^m = \hat{\varphi}_{L \sqcup \ell}^m$. Then by (6.10), it holds that

$$\nu_{K, L \sqcup \ell}^m = \lambda_{K, L \sqcup \ell}^m \circ \nu_{L \sqcup \ell, L \sqcup \ell}^m = \lambda_{K, L \sqcup \ell}^m \circ \hat{\varphi}_{L \sqcup \ell}^m,$$

completing the proof. \square

Let us next consider the relation of $\theta_K^m(i)$ and the pushout (6.4). To this end, we give an explicit description of $\theta_K^m(i)$. Put

$$\mathcal{S}^m(i, j) = X^{[j-1]} \times \bigvee_{\substack{\emptyset \neq I \subset [i+1, m] \\ J \subset [j, i-1]}} \Sigma(\hat{I} \sqcup i) \wedge \widehat{X}^{I \sqcup J \sqcup i}$$

for $j \leq i \in [m]$, where \hat{I} is the maximum of I and the base point of the two points set $\hat{I} \sqcup i$ is \hat{I} as in the proof of Proposition 6.2. Notice that $\mathcal{S}^m(i, j)$ is a wedge summand of $\mathcal{W}_{[j, m]}^m$ and that the restriction of $\lambda_{[j', m], [j, m]}^m$ to $\mathcal{S}^m(i, j)$ factors through $\mathcal{S}^m(i, j') \subset \mathcal{W}_{[j', m]}^m$ for $j \geq j'$. Let $\delta^m(i, j)$ be the composite

$$X^{[i-1]} \times CX_i \times X^{[i+1, m]} \xrightarrow{\text{proj}} X^{[i-1]} \times \Sigma(X_i \wedge X^{[i+1, m]}) \xrightarrow{\delta^m(i)} \mathcal{S}^m(i, i) \rightarrow \mathcal{S}^m(i, j),$$

where the last arrow is the restriction of $\lambda_{[j, m], [i, m]}^m$. For $i \in [k+1, m-1]$, put $\tau_K^m(i) : \mathcal{S}^m(i, k+1) \rightarrow \mathcal{W}_K^m$ to be the restriction of $\lambda_{K, [k+1, m]}^m$. By the definition of $\theta_K^m(i)$ and the construction of the homotopy equivalence of Lemma 3.2, one sees that

$$\theta_K^m(i) = \tau_K^m(i) \circ \delta^m(i, k+1)$$

for $i \in [k+1, m-1]$. Let us now state the relation of $\theta_K^m(i)$ and the pushout (6.4).

Proposition 6.5. *If K is connected and shifted, for $i \in [k+1, m-1]$, the composite*

$$X^{[i-1]} \times CX_i \times X^{[i+1, m-1]} \times X_m \xrightarrow{1 \times \kappa_i \times 1 \times 1} X^{[i-1]} \times CX_i \times X^{[i+1, m-1]} \times X_m \\ \xrightarrow{\theta_{\text{rest}_K(m)}^{m-1}(i) \times 1} \mathcal{W}_{\text{rest}_K(m)}^{m-1} \times X_m \xrightarrow{\mu} \overline{\mathcal{W}}_K^m \xrightarrow{\bar{\varphi}_K^m} \mathcal{W}_K^m$$

coincides with $\theta_K^m(i)$, where $\kappa_i : CX_i \rightarrow CX_i$ pinches the upper half of CX_i and μ is the bottom arrow of (6.1).

Proof. By the construction of the homotopy equivalence of Lemma 3.2, if $k+1 = m-1$, then the proposition holds for $\theta_K^m(k+1)$. Suppose $k+1 < m-1$ and take $k+1 \leq i < m$. For $j = 1, 2, 3$, let $W_K(j)$ be as in the proof of Proposition 6.1. By Proposition 5.6 and 5.7, $\tau_K^m(i)$ factors through $X^{[k]} \times \bigvee_I \Sigma \overline{K}_I \wedge \widehat{X}^I \subset \mathcal{W}_K^m$, where I runs over all non-empty subsets of $[k+1, m]$ such that K_I is discrete. Then it follows from Proposition 5.9 and (6.6) that there is a map $f : \mathcal{S}^{m-1}(i, k+1) \rightarrow X^{[k]} \times W_K(0)$ such that the composite

$$(6.11) \quad \mathcal{S}^m(i, k+1) \xrightarrow{\text{proj}} \mathcal{S}^{m-1}(i, k+1) \xrightarrow{f} X^{[k]} \times W_K(0) \xrightarrow{\text{incl}} \mathcal{W}_K^m$$

is equal to $\tau_K^m(i)$. Moreover, the map f also satisfies that the composite

$$\mathcal{S}^{m-1}(i, k+1) \xrightarrow{f} X^{[k]} \times W_K(0) \xrightarrow{\text{incl}} \mathcal{W}_{\text{rest}_K(m)}^{m-1}$$

coincides with $\tau_{\text{rest}_K(m)}^{m-1}(i)$. By the construction of $\bar{\varphi}_K^m$ and of the homotopy equivalence of Lemma 3.2, the restriction of $\bar{\varphi}_K^m \circ \mu$ to $X^{[k]} \times W_K(0) \times X_m$ is given as the composite

$$X^{[k]} \times W_K(0) \times X_m \xrightarrow{\text{proj}} X^{[k]} \times W_K(0) \xrightarrow{\text{incl}} \mathcal{W}_K^m.$$

Then we get a commutative diagram

$$\begin{array}{ccccc} & & \xrightarrow{\tau_{\text{rest}_K(m)}^{m-1}(i) \times 1} & & \\ & & \searrow & & \\ & \mathcal{S}^{m-1}(i, k+1) \times X_m & \xrightarrow{f \times 1} & X^{[k]} \times W_K(0) \times X_m & \xrightarrow{\text{incl}} \mathcal{W}_{\text{rest}_K(m)}^{m-1} \times X_m \\ & \downarrow \text{proj} & & \downarrow \text{proj} & \downarrow \bar{\varphi}_K^m \circ \mu \\ \mathcal{S}^m(i, k+1) & \xrightarrow{\text{proj}} \mathcal{S}^{m-1}(i, k+1) & \xrightarrow{f} & X^{[k]} \times W_K(0) & \xrightarrow{\text{incl}} \mathcal{W}_K^m \\ & & \searrow \tau_K^m(i) & & \end{array}$$

On the other hand, by the definition of $\delta^m(i, k+1)$, there is also a commutative diagram

$$\begin{array}{ccc} X^{[i-1]} \times CX_i \times X^{[i+1, m-1]} \times X_m & \xrightarrow{\delta^{m-1}(i, k+1) \times 1} & \mathcal{S}^{m-1}(i, k+1) \times X_m \\ \uparrow 1 \times \kappa_i \times 1 \times 1 & & \downarrow \text{proj} \\ X^{[i-1]} \times CX_i \times X^{[i+1, m-1]} \times X_m & \xrightarrow{\delta^m(i, k+1)} \mathcal{S}^m(i, k+1) & \xrightarrow{\text{proj}} \mathcal{S}^{m-1}(i, k+1). \end{array}$$

Therefore, by combining the above two commutative diagrams, the proof is completed. \square

7. PROOF OF THEOREM 1.7

The aim of this section is to prove:

Theorem 7.1. *If K, L are shifted, there is a homotopy equivalence*

$$\epsilon_K^m : \mathcal{Z}_K^m \xrightarrow{\simeq} \mathcal{W}_K^m$$

satisfying $\epsilon_K^m \circ \rho_{K,L}^m = \lambda_{K,L}^m \circ \epsilon_L^m$.

Let us first prove Theorem 1.7 and Corollary 1.8 by assuming Theorem 7.1 holds.

Proof of Theorem 1.7. As is noted above, if $k = 0$,

$$\mathcal{Z}_K^m = \mathcal{Z}_K(C\underline{X}, \underline{X}) \quad \text{and} \quad \mathcal{W}_K^m \simeq \bigvee_{\emptyset \neq I \subset [m]} |K_I| * \widehat{K}^I.$$

Therefore the proof is completed by Theorem 7.1. \square

Proof of Corollary 1.8. Buchstaber and Panov proved in [BP] that $\mathcal{Z}_K(\underline{D}^2, \underline{S}^1)$ is a deformation retract of $\mathcal{Z}_K(\underline{\mathbb{R}}^2, \underline{\mathbb{R}}^2 - 0)$. Their proof can be easily generalized to arbitrary $r \geq 1$, implying

$$\mathcal{Z}_K(\underline{\mathbb{R}}^r, \underline{\mathbb{R}}^r - 0) \simeq \mathcal{Z}_K(\underline{D}^r, \underline{S}^{r-1}).$$

Therefore the proof of Corollary 1.8 is completed by Proposition 5.7 and Theorem 1.7. \square

We now give a proof of Theorem 7.1.

Proof of Theorem 7.1. By induction on m , we construct the homotopy equivalence ϵ_K^m satisfying $\epsilon_K^m \circ \rho_{K,L}^m = \lambda_{K,L}^m \circ \epsilon_L^m$ together with the equality $\theta_K^m(i) = \epsilon_K^m \circ \hat{\rho}_{K,i}^m$ for $i \in [k+1, m]$, where $\hat{\rho}_{K,i}^m : X^{[i-1]} \times CX_i \times X^{[i+1, m]} \rightarrow \mathcal{Z}_K^m$ denotes the inclusion. We abbreviate ϵ_K^m by ϵ^m when K is clear in the context.

For $m = 1$, K must be the one point set 1 or the empty set \emptyset . We put ϵ_\emptyset^1 to be the identity map of X_1 and ϵ_1^1 to be the constant map, where $\mathcal{W}_1^1 = *$. Then it follows that $\epsilon_1^1 \circ \rho_{1,\emptyset} = \lambda_{1,\emptyset} \circ \epsilon_\emptyset^1$. By definition, $\theta_1^1(1)$ is the constant map, implying $\theta_1^1(1) = \epsilon_1^1 \circ \hat{\rho}_{1,1}^1$.

Suppose the desired homotopy equivalence ϵ_A^b has been constructed for any simplicial complex A on the index set $[a, b]$ with $b < m$. Let us first construct the homotopy equivalence ϵ_K^m satisfying $\epsilon_K^m \circ \rho_{K,\emptyset}^m = \lambda_{K,\emptyset}^m \circ \epsilon_\emptyset^m$ by considering the following two cases: Let K_0 be the connected component of the vertex m . We first construct $\epsilon_{K_0}^m$ by Construction 1 and then next ϵ_K^m by Construction 2 and the induction on k .

Let $\kappa_K^m : \mathcal{Z}_K^m \rightarrow \mathcal{Z}_K^m$ be the restriction of $\kappa_1 \times \cdots \times \kappa_m : \prod_{i \in [m]} CX_i \rightarrow \prod_{i \in [m]} CX_i$ to \mathcal{Z}_K^m , where $\kappa_i : CX_i \rightarrow CX_i$ pinches the upper half of CX_i as in the previous section. Then one has

$$(7.1) \quad \kappa_K^m \circ \rho_{K,L}^m = \rho_{K,L}^m \circ \kappa_L^m.$$

Construction 1 : K is connected.

By the induction hypothesis, there is a commutative diagram

$$\begin{array}{ccccc}
\mathcal{Z}_{\text{rest}_K(m)}^{m-1} \times X_m & \xleftarrow{\rho^{m-1} \times 1} & \mathcal{Z}_{\text{link}_K(m)}^{m-1} \times X_m & \xrightarrow{1 \times \iota} & \mathcal{Z}_{\text{link}_K(m)}^{m-1} \times CX_m \\
\downarrow \epsilon^{m-1} \times 1 & & \downarrow \epsilon^{m-1} \times 1 & & \downarrow \epsilon^{m-1} \times 1 \\
\mathcal{W}_{\text{rest}_K(m)}^{m-1} \times X_m & \xleftarrow{\rho^{m-1} \times 1} & \mathcal{W}_{\text{link}_K(m)}^{m-1} \times X_m & \xrightarrow{1 \times \iota} & \mathcal{W}_{\text{link}_K(m)}^{m-1} \times CX_m.
\end{array}$$

By Proposition 4.1 and Lemma 3.3, this commutative diagram induces a homotopy equivalence $\bar{\epsilon}_K^m : \mathcal{Z}_K^m \rightarrow \bar{\mathcal{W}}_K^m$ between pushouts, and then we put

$$\epsilon_K^m = \bar{\varphi}_K^m \circ \bar{\epsilon}_K^m \circ \kappa_K^m$$

where $\bar{\varphi}_K^m$ is as in Proposition 6.1. In order to make the equality for $\theta_K^m(i)$ hold, the map κ_K^m is inserted. By construction, $\epsilon_K^m \circ \rho_{K,\emptyset}^m = \lambda_{K,\emptyset}^m \circ \epsilon_\emptyset^m$.

Construction 2 : $K \sqcup k$ from K .

By Proposition 4.1 and Lemma 3.3, the map $\hat{\epsilon}_{K \sqcup k}^m : \mathcal{Z}_{K \sqcup k}^m \rightarrow \widehat{\mathcal{W}}_{K \sqcup k}^m$ between pushouts induced from the following commutative diagram is a homotopy equivalence.

$$\begin{array}{ccccc}
\mathcal{Z}_K^m & \xleftarrow{\rho_{K,\emptyset}^m} & X^{[m]} & \xrightarrow{1 \times \iota \times 1} & X^{[k-1]} \times CX_k \times X^{[k+1,m]} \\
\downarrow \epsilon_K^m & & \parallel & & \parallel \\
\mathcal{W}_K^m & \xleftarrow{\lambda_{K,\emptyset}^m} & X^{[m]} & \xrightarrow{1 \times \iota \times 1} & X^{[k-1]} \times CX_k \times X^{[k+1,m]}
\end{array}$$

Then we define

$$\epsilon_{K \sqcup k}^m = \hat{\varphi}_{K \sqcup k}^m \circ \hat{\epsilon}_{K \sqcup k}^m$$

which satisfies $\epsilon_{K \sqcup k}^m \circ \rho_{K \sqcup k,\emptyset}^m = \lambda_{K \sqcup k,\emptyset}^m \circ \epsilon_\emptyset^m$ by construction, where $\hat{\varphi}_{K \sqcup k}^m$ is as in Proposition 6.2

Let us next show the naturality of ϵ_K^m by considering the following three cases, in between the equality for $\theta_K^m(i)$ is also shown. These three cases are used as: Let K_0 and L_0 be the connected component of K and L , respectively. First, the naturality for the pair $K_0 \supset L_0$ is proved by Naturality 1. Next, the naturality for $K \supset L_0$ is proved by Naturality 2 and the induction on k , and then, finally, the naturality for $K \supset L$ is proved by Naturality 3 and the induction on ℓ . Therefore the proof of Theorem 7.1 is completed by proving the following.

Naturality 1 : $K \supset L$ where K, L are connected.

By the induction hypothesis, there is a commutative diagram

$$\begin{array}{ccccc}
 \mathcal{Z}_{\text{rest}_L(m)}^{m-1} \times X_m & \xleftarrow{\rho^{m-1} \times 1} & \mathcal{Z}_{\text{link}_L(m)}^{m-1} \times X_m & \xrightarrow{1 \times \iota} & \mathcal{Z}_{\text{link}_L(m)}^{m-1} \times CX_m \\
 \downarrow \epsilon^{m-1} \times 1 & & \downarrow \epsilon^{m-1} \times 1 & & \downarrow \epsilon^{m-1} \times 1 \\
 \mathcal{W}_{\text{rest}_L(m)}^{m-1} \times X_m & \xleftarrow{\lambda^{m-1} \times 1} & \mathcal{W}_{\text{link}_L(m)}^{m-1} \times X_m & \xrightarrow{1 \times \iota} & \mathcal{W}_{\text{link}_L(m)}^{m-1} \times CX_m \\
 \downarrow \rho^{m-1} \times 1 & & \downarrow \rho^{m-1} \times 1 & & \downarrow \rho^{m-1} \times 1 \\
 \mathcal{Z}_{\text{rest}_K(m)}^{m-1} \times X_m & \xleftarrow{\rho^{m-1} \times 1} & \mathcal{Z}_{\text{link}_K(m)}^{m-1} \times X_m & \xrightarrow{1 \times \iota} & \mathcal{Z}_{\text{link}_K(m)}^{m-1} \times CX_m \\
 \downarrow \epsilon^{m-1} \times 1 & & \downarrow \epsilon^{m-1} \times 1 & & \downarrow \epsilon^{m-1} \times 1 \\
 \mathcal{W}_{\text{rest}_K(m)}^{m-1} \times X_m & \xleftarrow{\lambda^{m-1} \times 1} & \mathcal{W}_{\text{link}_K(m)}^{m-1} \times X_m & \xrightarrow{1 \times \iota} & \mathcal{W}_{\text{link}_K(m)}^{m-1} \times CX_m.
 \end{array}$$

It is clear from Proposition 4.1 that the pushout of the back face is the map $\rho_{K,L}^m : \mathcal{Z}_L^m \rightarrow \mathcal{Z}_K^m$. By the definition of $\lambda_{K,L}^m$, the pushout of the front face is the map $\bar{\lambda}_{K,L}^m : \bar{\mathcal{W}}_L^m \rightarrow \bar{\mathcal{W}}_K^m$. By the definition of $\bar{\epsilon}_L^m$ and $\bar{\epsilon}_K^m$ in Construction 1, the pushouts of the top and the bottom faces are the maps $\bar{\epsilon}_L^m : \mathcal{Z}_L^m \rightarrow \bar{\mathcal{W}}_L^m$ and $\bar{\epsilon}_K^m : \mathcal{Z}_K^m \rightarrow \bar{\mathcal{W}}_K^m$, respectively. Then the above commutative diagram yields a commutative square

$$\begin{array}{ccc}
 \mathcal{Z}_L^m & \xrightarrow{\bar{\epsilon}_L^m} & \bar{\mathcal{W}}_L^m \\
 \downarrow \rho_{K,L}^m & & \downarrow \bar{\lambda}_{K,L}^m \\
 \mathcal{Z}_K^m & \xrightarrow{\bar{\epsilon}_K^m} & \bar{\mathcal{W}}_K^m.
 \end{array}$$

Thus by the definition of ϵ_K^m , (7.1) and Proposition 6.1, we obtain

$$\begin{aligned}
 \epsilon_K^m \circ \rho_{K,L}^m &= \bar{\varphi}_K^m \circ \bar{\epsilon}_K^m \circ \kappa_K^m \circ \rho_{K,L}^m = \bar{\varphi}_K^m \circ \bar{\epsilon}_K^m \circ \rho_{K,L}^m \circ \kappa_L^m \\
 &= \bar{\varphi}_K^m \circ \bar{\lambda}_{K,L}^m \circ \bar{\epsilon}_L^m \circ \kappa_L^m = \lambda_{K,L}^m \circ \bar{\varphi}_L^m \circ \bar{\epsilon}_L^m \circ \kappa_L^m = \lambda_{K,L}^m \circ \epsilon_L^m.
 \end{aligned}$$

Naturality 2 : $K \sqcup k \supset K$.

By Proposition 4.1 and 6.2, there is a commutative diagram

$$\begin{array}{ccc}
 \mathcal{Z}_K^m & \xrightarrow{\rho_{K \sqcup k, K}^m} & \mathcal{Z}_{K \sqcup k}^m \\
 \downarrow \epsilon_K^m & & \downarrow \hat{\epsilon}_{K \sqcup k}^m \\
 \mathcal{W}_K^m & \xrightarrow{\nu} & \widehat{\mathcal{W}}_{K \sqcup k}^m
 \end{array}$$

where ν is as in Proposition 6.2. Then by the definition of $\epsilon_{K \sqcup k}^m$ and Proposition 6.2,

$$\epsilon_{K \sqcup k}^m \circ \rho_{K \sqcup k, K}^m = \hat{\varphi}_{K \sqcup k}^m \circ \hat{\epsilon}_{K \sqcup k}^m \circ \rho_{K \sqcup k, K}^m = \hat{\varphi}_{K \sqcup k}^m \circ \nu \circ \epsilon_K^m = \lambda_{K \sqcup k, K}^m \circ \epsilon_K^m.$$

Equality for $\theta_K^m(i)$.

Before proceeding to the last case, let us prove the equality $\theta_K^m(i) = \epsilon_K^m \circ \hat{\rho}_{K,i}^m$. Let K_0 be the connected component of the vertex m . If i is a vertex of K_0 and $i < m$, then by Proposition

6.3, the naturality of $\hat{\rho}_{K,i}^m$ and Naturality 2, it holds that

$$\epsilon_K^m \circ \hat{\rho}_{K,i}^m = \epsilon_K^m \circ \rho_{K,K_0}^m \circ \hat{\rho}_{K_0,i}^m = \lambda_{K,K_0}^m \circ \epsilon_{K_0}^m \circ \hat{\rho}_{K_0,i}^m.$$

It also holds that

$$\begin{aligned} \epsilon_{K_0}^m \circ \hat{\rho}_{K_0,i}^m &= \bar{\varphi}_{K_0}^m \circ \bar{\epsilon}_{K_0}^m \circ \kappa_K^m \circ \hat{\rho}_{K_0,i}^m && \text{definition of } \epsilon_{K_0}^m \\ &= \bar{\varphi}_{K_0}^m \circ \bar{\epsilon}_{K_0}^m \circ \hat{\rho}_{K_0,i}^m \circ (1 \times \kappa_i \times 1) && (7.1) \\ &= \bar{\varphi}_{K_0}^m \circ \bar{\epsilon}_{K_0}^m \circ \rho \circ (\hat{\rho}_{\text{rest}_{K_0}(m),i}^{m-1} \times 1) \circ (1 \times \kappa_i \times 1) && \text{naturality of } \hat{\rho}_{K,i}^m \\ &= \bar{\varphi}_{K_0}^m \circ \mu \circ (\epsilon_{\text{rest}_{K_0}(m)}^{m-1} \circ \hat{\rho}_{\text{rest}_{K_0}(m),i}^{m-1} \times 1) \circ (1 \times \kappa_i \times 1) && \text{definition of } \bar{\epsilon}_{K_0}^m \\ &= \bar{\varphi}_{K_0}^m \circ \mu \circ (\theta_{\text{rest}_{K_0}(m)}^{m-1}(i) \times 1) \circ (1 \times \kappa_i \times 1) && \text{induction hypothesis} \\ &= \theta_{K_0}^m(i) && \text{Proposition 6.5} \end{aligned}$$

where $\rho : \mathcal{Z}_{\text{rest}_{K_0}(m)}^{m-1} \times X_m \rightarrow \mathcal{Z}_{K_0}^m$ denotes the inclusion and μ is as in Proposition 6.5. Therefore

$$\theta_K^m(i) = \lambda_{K,K_0}^m \circ \theta_{K_0}^m(i) = \lambda_{K,K_0}^m \circ \epsilon_{K_0}^m \circ \hat{\rho}_{K_0,i}^m = \epsilon_K^m \circ \hat{\rho}_{K,i}^m$$

for $i \in [k+1, m-1]$. If $i = m$, the equality is obvious. If $i \in [k+1, m]$ but not in K_0 , the equality follows from Proposition 6.3 and Naturality 2.

Naturality 3 : $K \supset L \sqcup \ell$ for $k < \ell$.

By the equality for $\theta_K^m(i)$ proved above, there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{Z}_L^m & \xleftarrow{\rho_{L,\emptyset}^m} & X^{[m]} & \xrightarrow{1 \times \iota \times 1} & X^{[\ell-1]} \times CX_\ell \times X^{[\ell+1,m]} \\ \downarrow \epsilon_L^m & \searrow & \downarrow & \swarrow & \downarrow \\ & & \mathcal{W}_L^m & \xleftarrow{\lambda_{K,\emptyset}^m} & X^{[m]} & \xrightarrow{1 \times \iota \times 1} & X^{[\ell-1]} \times CX_\ell \times X^{[\ell+1,m]} \\ \downarrow \rho_{K,L}^m & & \downarrow \rho_{K,\emptyset}^m & & \downarrow \hat{\rho}_{K,i}^m & & \downarrow \\ \mathcal{Z}_K^m & \xleftarrow{\epsilon_K^m} & \mathcal{Z}_K^m & \xleftarrow{\lambda_{K,L}^m} & \mathcal{Z}_K^m & \xleftarrow{\lambda_{K,\emptyset}^m} & \mathcal{Z}_K^m \\ \downarrow \epsilon_K^m & \searrow & \downarrow \epsilon_K^m & \swarrow & \downarrow \epsilon_K^m & \swarrow & \downarrow \theta_K^m(i) \\ & & \mathcal{W}_K^m & \xleftarrow{\lambda_{K,L}^m} & \mathcal{W}_K^m & \xleftarrow{\lambda_{K,\emptyset}^m} & \mathcal{W}_K^m \end{array}$$

Since the map $\mathcal{Z}_{L \sqcup \ell}^m \rightarrow \mathcal{Z}_K^m$ induced from the back face is $\rho_{K,L \sqcup \ell}^m$ by Proposition 4.1, the above diagram implies

$$\epsilon_K^m \circ \rho_{K,L \sqcup \ell}^m = \nu_{K,L \sqcup \ell}^m \circ \hat{\epsilon}_{L \sqcup \ell}^m$$

and then by Corollary 6.4, the naturality is obtained as

$$\epsilon_K^m \circ \rho_{K,L \sqcup \ell}^m = \nu_{K,L \sqcup \ell}^m \circ \hat{\epsilon}_{L \sqcup \ell}^m = \lambda_{K,L \sqcup \ell}^m \circ \hat{\varphi}_{L \sqcup \ell}^m \circ \hat{\epsilon}_{L \sqcup \ell}^m = \lambda_{K,L \sqcup \ell}^m \circ \epsilon_{L \sqcup \ell}^m.$$

□

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DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES, OSAKA PREFECTURE UNIVERSITY, SAKAI,
599-8531, JAPAN

E-mail address: kiriye@mi.s.osakafu-u.ac.jp

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KYOTO, 606-8502, JAPAN

E-mail address: kishi@math.kyoto-u.ac.jp