ON HIGHER FITTING IDEALS OF IWASAWA MODULES OF IDEAL CLASS GROUPS OVER IMAGINARY QUADRATIC FIELDS AND EULER SYSTEMS OF ELLIPTIC UNITS

TATSUYA OHSHITA

Abstract. Kurihara described all higher Fitting ideals of the minus part of Iwasawa modules of ideal class groups over totally real fields by using Stickelberger elements and Euler systems of "Gauss sums". In this paper, we obtain some partial results for elliptic units which are analogues of his result. By using Kolyvagin derivative classes of Euler systems of elliptic units, we construct some ideals $\mathfrak{C}^{\mathrm{ell}}_{i,\chi}$ of Iwasawa algebras, and prove that they give "upper bounds" of higher Fitting ideals of one and two-variable Iwasawa modules of ideal class groups over imaginary quadratic fields.

1. Introduction

Let K be an imaginary quadratic field. We fix an algebraic closure $\overline{\mathbb{Q}} = \overline{K}$ of K. In this paper, an algebraic number field is a finite extension of $\mathbb{Q}$ in this fixed algebraic closure $\overline{\mathbb{Q}}$. For each algebraic number field F, we denote the ring of integers of F by $\mathcal{O}_F$. If F_2/F_1 is a finite extension of fields, we write $F_1 \subseteq_f F_2$.

We fix an abelian extension K_0 of K, and put $\Delta := \mathrm{Gal}(K_0/K)$. Let p be a prime number which does not divide $\#(\mathcal{O}_{K_0}^\times)_{\mathrm{tors}}\#\Delta$. We consider an abelian extension K_∞/K which contains K_0. We assume that $\Gamma := \mathrm{Gal}(K_\infty/K_0)$ is isomorphic to $\mathbb{Z}_p$ or $\mathbb{Z}_p^2$ as a topological group. We put $\mathcal{G} := \mathrm{Gal}(K_\infty/K) = \Delta \times \Gamma$. We define $\Lambda := \mathbb{Z}_p[[\mathcal{G}]]$.

Put $\widehat{\Delta} := \mathrm{Hom}(\Delta, \overline{\mathbb{Q}}_p^\times)$. For any character $\chi \in \widehat{\Delta}$, we denote by $\mathcal{O}_\chi$ the $\mathbb{Z}_p[\Delta]$-algebra, which is a $\mathbb{Z}_p$-algebra isomorphic to $\mathbb{Z}_p[\mathrm{Im}\,\chi]$ with action of Δ via χ. The Λ-algebra Λ_χ is defined by $\mathcal{O}_\chi[[\Gamma]]$. Note that for any $\chi \in \widehat{\Delta}$, the algebra Λ_χ is flat over Λ since we assume that p does not divide $\#\Delta$. The ring Λ is decomposed into $\Lambda = \prod_{\chi \in \widehat{\Delta}} \Lambda_\chi$ as Λ-algebra. For any Λ-module M, we put $M_\chi := M \otimes_\Lambda \Lambda_\chi$.

Let X be a projective limit of the systems

$$\left\{ N_{F'/F} \colon A'_F \longrightarrow A_F \;\middle|\; K_0 \subseteq_f F' \subseteq_f F' \subset K_\infty \right\},$$

where A_F is the p-Sylow subgroup of the ideal class group of F and $N_{F'/F}$ is the norm map. Note that X is a finitely generated torsion Λ-module. Let X_{fin} be the largest pseudo-null Λ-submodule of X, and $X' := X/X_{\mathrm{fin}}$. In our paper, we study the higher Fitting ideals $\{\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi)\}_{i \in \mathbb{Z}_{\geq 0}}$ for each $\chi \in \widehat{\Delta}$ by using the Euler systems of elliptic units. In §4, we will define ideals $\mathfrak{C}^{\mathrm{ell}}_{i,\chi}$ of Λ_χ for all $i \in \mathbb{Z}_{\geq 0}$, which are analogues of Kurihara's higher Stickelberger ideals in [Ku] for elliptic units, and we will prove

2010 *Mathematics Subject Classification.* Primary 11R23, Secondary 11R29.

that they give "upper bounds" (admitting some "error factors") of $\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi)$ (cf. Theorem 1.1 and Theorem 5.1).

To state our main theorem, we define some ideals, which appear in "error factors" of our main theorem. For each place v of K, we denote the decomposition group of v in Δ (resp. $\mathcal{G}$) by $D_{\Delta,v}$ (resp. D_v). For any subgroup $\mathcal{H}$ of $\mathcal{G}$, let $\mathcal{I}(\mathcal{H})$ be the ideal in Λ generated by $\{\gamma - 1 \mid \gamma \in \mathcal{H}\}$. Let T be the set of places of K above p which ramify in K_∞/K. We define $\mathcal{I}_T := \prod_{\mathfrak{p}\in T}\mathcal{I}(D_{\mathfrak{p}})$.

Let n be a positive integer. For each ring R, we denote the group of all n-th roots of unity by $\boldsymbol{\mu}_n(R)$. For simplicity, we write $\mu_n := \boldsymbol{\mu}_n(\overline{K})$ and $\mu_{p^\infty} := \bigcup_{m\geq 1}\boldsymbol{\mu}_{p^m}(\overline{K})$. We put $\mathcal{I}_\mu = \mathrm{ann}_\Lambda(\boldsymbol{\mu}_{p^\infty}(K_\infty))$.

The following is a rough form of the main theorem of our paper. (For precise version, see Theorem 5.1.)

Theorem 1.1. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. If K_0 contains μ_p, we assume $\chi \neq \omega$ and $\chi \neq \chi^{-1}\omega$, where*

$$\omega\colon \Delta \longrightarrow \mathrm{Gal}(K(\mu_p)/K) \longrightarrow \mathbb{Z}_p^\times$$

is the Teichmüller character. Assume one of the following:

- *p splits completely in $K/\mathbb{Q}$;*
- *p does not split (i.e. p ramifies or inerts) in $K/\mathbb{Q}$, and for the element $\mathfrak{p} \in T$, the character χ is non-trivial on $D_{\Delta,\mathfrak{p}}$. (Note that in this case, T is a singleton.)*

Then, the following holds:

(1) *If the character χ is non-trivial on $D_{\Delta,\mathfrak{p}}$ for any $\mathfrak{p} \in T$, then we have*

$$\mathfrak{C}^{\mathrm{ell}}_{0,\chi} \subseteq \mathrm{Fitt}_{\Lambda_\chi,0}(X'_\chi).$$

(2) *For any $i \in \mathbb{Z}_{\geq 0}$, there exists a height-two ideal $J_{i,\chi}$ of Λ_χ satisfying*

$$J_{i,\chi}\mathcal{I}^3_{T,\chi}\,\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}^{\mathrm{ell}}_{i,\chi}.$$

Moreover, if $\Gamma \simeq \mathbb{Z}_p$ and $\mathcal{I}_{T,\chi} = \mathcal{I}_{\mu,\chi} = \Lambda_\chi$, we have

$$\mathrm{ann}_{\Lambda_\chi}(X_{\mathrm{fin}})\,\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}^{\mathrm{ell}}_{i,\chi}$$

for any $i \in \mathbb{Z}_{\geq 0}$.

Remark 1.2. Here, we remark briefly on the structure of X_χ in the case of $\chi = 1$. For the two-variable cases, the generalized Greenberg conjecture predicts that the Iwasawa module X of the $\mathbb{Z}_p^2$-extension of any imaginary quadratic field $K_0 = K$ is pseudo-null. (For details, see [Gr] Conjecture 3.5.) In [Mi], Minardi proved the generalized Greenberg conjecture for imaginary quadratic fields when p does not divide the class number of K. So, the assertion of our main theorem holds trivially in this case.

For the one-variable cases, the following results are known.

(1) Assume that p does not split in $K/\mathbb{Q}$. Then, we have $X = X' = 0$ for any $\mathbb{Z}_p$-extensions K_∞ of $K_0 = K$ (a special case of Iwasawa's result in [Iw]).
(2) Assume that p splits in $K/\mathbb{Q}$ and the class number of K is prime to p. Then for all but finitely many $\mathbb{Z}_p$-extensions K_∞ of $K_0 = K$, the Λ-module $X' = X'_1$ associated to K_∞/K is free of rank 1 as a $\mathbb{Z}_p$-module ([Oz] Theorem 1).

Assume p splits in $K/\mathbb{Q}$, and p does not divide the class number of K. Then, the result (2) by Ozaki and the Iwasawa main conjecture imply that we have

$$\mathrm{Fitt}_{\Lambda,i}(X') = \begin{cases} \mathrm{char}_\Lambda(\mathcal{E}_\infty/\mathcal{C}_\infty) & \text{if } i = 0 \\ \Lambda & \text{if } i > 0 \end{cases}$$

for all but finitely many $\mathbb{Z}_p$-extensions K_∞ of $K_0 = K$, where $\mathcal{E}_\infty$ (resp. $\mathcal{C}_\infty$) is the Λ-module of global units (resp. the Λ-module of elliptic units) defined in §2.1 (resp. §2.2) of this paper.

In this paper, we prove Theorem 1.1 by using Kurihara's Euler system argument in [Ku] for elliptic units. Kurihara's methods are not "usual" Euler system arguments which appear in the proof of Iwasawa main conjectures in [Ru1] or [Ru3]. Note that "usual" Euler system arguments work well for Iwasawa modules with a diagonal relation matrix, but Kurihara's arguments work for Iwasawa modules with a square relation matrix. (Recall that when we prove the Iwasawa main conjecture for X, instead of X, we study an Iwasawa module with a diagonal relation matrix which is pseudo-isomorphic to X.) Though we also treat non-cyclotomic extensions in our paper, our Euler system arguments work completely parallel to those of [Ku] and [Oh], which treat only cyclotomic $\mathbb{Z}_p$-extensions.

Remark 1.3. In one-variable case, we can give some bounds of "error factors" $J_{i,\chi}\mathcal{I}^3_{T,\chi}$ of Theorem 1.1 (cf. Theorem 5.1, which is the precise form of our main theorem for one-variable case). Kurihara's Euler system arguments work well only for Iwasawa modules whose relation matrices can be written by square matrices. Under the assumption $\Gamma \simeq \mathbb{Z}_p$, the relation of an Iwasawa module M is written by a square matrix if (and only if) M has no non-trivial submodule whose order is finite (cf. Lemma 2.11). So, in one-variable case, we can apply Kurihara's argument directly to X', and we obtain some bounds of "error factors" $J_{i,\chi}\mathcal{I}^3_{T,\chi}$ when we observe Kurihara's Euler system argument carefully. In two-variable case, we cannot bound "error factors" since we have no "canonical" modification of X to Iwasawa modules with square relation matrix. (In two-variable case, X' may not have a relation matrix written by square matrix.) Indeed, as we will see later in §6, our result for two-variable case follows from the standard Euler system argument for the proof of Iwasawa main conjecture without using Kurihara's methods, and it is not so new or strong.

In particular, when $\Gamma \simeq \mathbb{Z}_p$ and X_χ has no non-trivial pseudo-null submodule, then our theorem give upper bounds of higher Fitting ideals directly.

Corollary 1.4. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. If K_0 contains μ_p, we assume $\chi \neq \omega$ and $\chi \neq \chi^{-1}\omega$. Assume one of the following:*

- *p splits completely in $K/\mathbb{Q}$;*

- *p does not split in $K/\mathbb{Q}$, and for the element $\mathfrak{p} \in T$, the character χ is non-trivial on $D_{\Delta,\mathfrak{p}}$.*

Further, we assume $\Gamma \simeq \mathbb{Z}_p$, $\mathcal{I}_{T,\chi} = \mathcal{I}_{\mu,\chi} = \Lambda_\chi$, and $X_{\mathrm{fin},\chi} = 0$. Then, we have the following:

(1) *If the character χ is non-trivial on $D_{\Delta,\mathfrak{p}}$ for any $\mathfrak{p} \in T$, then we have*
$$\mathrm{Fitt}_{\Lambda_\chi,0}(X_\chi) = \mathfrak{C}^{\mathrm{ell}}_{0,\chi}.$$
(2) $\mathrm{Fitt}_{\Lambda_\chi,i}(X_\chi) \subseteq \mathfrak{C}^{\mathrm{ell}}_{i,\chi}$ *for any $i \in \mathbb{Z}_{\geq 0}$.*

Remark 1.5. Here, we give an example satisfying $X_\chi = X'_\chi$. (Note that we usually have $X_\chi \neq X'_\chi$ for many cases.) When X is the Iwasawa module associated to the anti-cyclotomic $\mathbb{Z}_p$-extension K_∞ of a imaginary quadratic field $K_0 = K$, Theorem 4.2 in [Fu] gives a sufficient condition which make X a cyclic Λ-module satisfying $X = X'$. For instance, let K_∞ be the anti-cyclotomic $\mathbb{Z}_3$-extension of $K = \mathbb{Q}(\sqrt{-461})$, and γ a topological generator of $\Gamma = \mathrm{Gal}(K_\infty/K)$, then we have
$$X = X' \simeq \Lambda/(\gamma^3 - 1)\Lambda.$$
This implies
$$\mathrm{Fitt}_{\Lambda,i}(X) = \mathrm{Fitt}_{\Lambda,i}(X') = \begin{cases} (\gamma^3 - 1)\Lambda & \text{if } i = 0 \\ \Lambda & \text{if } i > 0. \end{cases}$$
For details of this example, see the examples below Theorem 2 and Theorem 4.2 in [Fu]. Note that we cannot apply Corollary 1.4 in this case since the corollary requires that χ is a non-trivial character.

Remark 1.6. Recall the fact that the Iwasawa main conjecture for Iwasawa modules X of ideal class groups implies the Iwasawa main conjecture for Iwasawa modules of Selmer groups of elliptic curves over $\mathbb{Q}$ with complex multiplication. (See §12 in [Ru1].) But this fact follows from *multiplicativity* of characteristic ideals for exact sequences of Iwasawa modules. Since higher Fitting ideals do not have multiplicativity, our main theorem does not imply any bounds of higher Fitting ideals of Iwasawa modules of Selmer groups of elliptic curves with complex multiplication.

Notation. In this paper, we use the following notation.

Let L/K be a finite Galois extension of algebraic number fields. Let λ be a prime ideal of K, and λ' a prime ideal of L above λ. We denote the completion of K at λ by K_λ, and the completion of L at λ' by $L_{\lambda'}$. If λ is unramified in L/K, the arithmetic Frobenius at λ' is denoted by $(\lambda', L/K) \in \mathrm{Gal}(L/K)$.

We fix a family of embeddings $\{\mathfrak{l}_{\overline{K}} \colon \overline{K} \hookrightarrow \overline{K}_\mathfrak{l}\}_{\mathfrak{l}:\mathrm{prime}}$ satisfying a technical condition (A) as follows.

(A) *For any subfield $L \subset \overline{K}$ which is a finite Galois extension of K and any element $\sigma \in \mathrm{Gal}(L/K)$, there exist infinitely many prime ideals $\mathfrak{l}$ of $\mathcal{O}_K$ such that $\mathfrak{l}$ is unramified in L/K and $(\mathfrak{l}_L, L/K) = \sigma$, where $\mathfrak{l}_L$ is the prime ideal corresponding to the embedding $\mathfrak{l}_{\overline{K}}|_L$.*

The existence of a family satisfying the condition (A) is easily proved by using the Chebotarev density theorem.

Let $\mathfrak{l}$ be a prime ideal $\mathfrak{l}$ of $\mathcal{O}_K$. For an algebraic number field L, let $\mathfrak{l}_L$ be the prime ideal of $\mathcal{O}_L$ corresponding to the embedding $\mathfrak{l}_{\overline{K}}|_L$. Then, if $L_2 \supseteq L_1$ is an extension of algebraic number fields containing K, we have $\mathfrak{l}_{L_2} \mid \mathfrak{l}_{L_1}$.

For an abelian group M and a positive integer n, we write M/n in place of M/nM for simplicity. In particular, for the multiplicative group $K^\times$ of a field K, we write $K^\times/p^N$ in place of $K^\times/(K^\times)^{p^N}$.

Let F be a finite extension field of K_0 contained in K_∞. We put $\Gamma_F := \mathrm{Gal}(K_\infty/F)$. For a Λ-module M, we denote the Γ_F-invariants (resp. Γ_F-coinvariants) of M by M^{Γ_F} (resp. M_{Γ_F} or M_F).

Let R be a commutative ring. For an R-module M, we define $\mathrm{ann}_R(M)$ to be the annihilator of M. Namely,

$$\mathrm{ann}_R(M) := \{a \in R \mid am = 0 \text{ for any } m \in M\}.$$

The maximal torsion submodule of M is denoted by M_{tors}.

Acknowledgement

The author would like to thank Professors Kazuya Kato, Masato Kurihara, Masataka Chida and Tetsushi Ito for their helpful advices. The author is also grateful to the reviewer's valuable comments that improved the manuscript. This work was supported by Grant-in-Aid for JSPS Fellows (22-2753) from Japan Society for the Promotion of Science.

2. Preliminaries

In this section, we review some preliminary results. We use the same notation as in §1. This section consists of three subsections. In the first subsection, we recall some Iwasawa theoretical results on unit groups and ideal class groups. In the second section, we recall the definition and some properties of elliptic units. In the last subsection, we recall the notion of higher Fitting ideals.

2.1. In this subsection, we recall some preliminary results on Iwasawa theory which is used in our paper. For each finite extension field F of K_0 contained in K_∞, we put

$$\mathcal{E}_F := \mathcal{O}_F^\times \otimes \mathbb{Z}_p,$$

and we define a Λ-module $\mathcal{E}_\infty$ to be the projective limit of the system

$$\left\{ N_{F'/F} \colon \mathcal{E}_{F'} \longrightarrow \mathcal{E}_F \;\middle|\; K_0 \subseteq_f F' \subseteq_f F' \subset K_\infty \right\}$$

where $N_{F'/F}$ are the norm maps.

Let F be a number field satisfying $K_0 \subseteq_f F \subset K_\infty$. Recall that we put $\Gamma_F := \mathrm{Gal}(K_\infty/F)$, and for a Λ-module M, we denote the Γ_F-coinvariants of M by M_F. We consider the natural homomorphisms

$$\pi_{\mathcal{E},F} \colon (\mathcal{E}_\infty)_F \longrightarrow \mathcal{E}_F,$$
$$\pi_{A,F} \colon (X_\infty)_F \longrightarrow A_F.$$

We define the ideals $\mathcal{I}_\mathcal{E}$, $\mathcal{J}_\mathcal{E}$, $\mathcal{I}_A$ and $\mathcal{J}_A$ of Λ by

$$\mathcal{I}_\mathcal{E} := \bigcap_F \mathrm{ann}_\Lambda(\mathrm{Ker}\, \pi_{\mathcal{E},F}), \ \ \mathcal{J}_\mathcal{E} := \bigcap_F \mathrm{ann}_\Lambda(\mathrm{Coker}\, \pi_{\mathcal{E},F}),$$
$$\mathcal{I}_A := \bigcap_F \mathrm{ann}_\Lambda(\mathrm{Ker}\, \pi_{A,F}), \ \ \mathcal{J}_A := \bigcap_F \mathrm{ann}_\Lambda(\mathrm{Coker}\, \pi_{A,F}),$$

where F runs all intermediate fields of K_∞/K satisfying $K_0 \subseteq_f F$.

Recall that we denote the set of places of K above p which ramify in K_∞/K by T, and we define $\mathcal{I}_T := \prod_{\mathfrak{p}\in T} \mathcal{I}(D_\mathfrak{p})$, where we denote the decomposition group of $\mathfrak{p}$ in $\mathcal{G}$ by D_v, and let $\mathcal{I}(D_v)$ be the ideal in Λ generated by $\{\gamma - 1 \mid \gamma \in D_v\}$.

Proposition 2.1. (1) *There exist a height two ideal $\mathcal{A}$ satisfying*

$$\mathcal{I}_T\mathcal{A} \subseteq \mathcal{I}_\mathcal{E} \ \ \textit{and} \ \ \mathcal{I}_T^2\mathcal{A} \subseteq \mathcal{J}_\mathcal{E}.$$

Further, if we assume that $\Gamma \simeq \mathbb{Z}_p$, and let $\chi \in \Delta$ be a character satisfying $\mathcal{I}_{T,\chi} = \Lambda_\chi$, then we have

$$\mathcal{I}_{\mathcal{E},\chi} = \Lambda_\chi \ \ \textit{and} \ \ \mathrm{ann}_{\Lambda_\chi}(X_{\mathrm{fin},\chi}) \subseteq \mathcal{J}_{\mathcal{E},\chi}.$$

([Ru1] *Theorem 7.6.*)

(2) *We have*

$$\mathcal{I}_T\mathcal{I}_0 \subseteq \mathcal{I}_A \ \ \textit{and} \ \ \mathcal{I}(\mathcal{G}) \subseteq \mathcal{J}_A,$$

where $\mathcal{I}(\mathcal{G})$ is the augmentation ideal, which is an ideal of Λ generated by $\{\gamma - 1 \mid \gamma \in \mathcal{G}\}$, and

$$\mathcal{I}_0 := \begin{cases} \Lambda & \textit{if } \Gamma \simeq \mathbb{Z}_p, \\ \mathcal{I}(\mathcal{G}) & \textit{if } \Gamma \simeq \mathbb{Z}_p^2. \end{cases}$$

In particular, if we assume that $\Gamma \simeq \mathbb{Z}_p$, then the natural homomorphism

$$\pi_{A,F} \colon (X_\infty)_F \longrightarrow A_F$$

is an isomorphism for any number field F satisfying $K_0 \subseteq_f F \subset K_\infty$, and any character $\chi \in \Delta$ satisfying $\mathcal{I}_{T,\chi} = \Lambda_\chi$. So, if $\Gamma \simeq \mathbb{Z}_p$, and if $\mathcal{I}_{T,\chi} = \Lambda_\chi$, then we have

$$\mathcal{I}_{A,\chi} = \mathcal{J}_{A,\chi} = \Lambda_\chi.$$

([Ru1] *Theorem 5.4.*)

2.2. Here, we briefly recall the definition and some properties of elliptic units. We fix an embedding $\infty_{\overline{K}}\colon \overline{K} \longrightarrow \mathbb{C}$, and regard $\overline{K}$ as a subfield of $\mathbb{C}$ by $\infty_{\overline{K}}$. Let F be an intermediate field of $\mathbb{C}/K$, and E an elliptic curve over F with complex multiplication by $\mathcal{O}_K$. In this paper, we always identify $\mathcal{O}_K$ with $\mathrm{End}(E)$ by unique isomorphism $\mathcal{O}_K \xrightarrow{\simeq} \mathrm{End}(E)$ such that the composite map

$$\mathcal{O}_K \longrightarrow \mathrm{End}(E) \longrightarrow \mathrm{End}_F(\mathrm{Lie}(E)) = F$$

coincides with the inclusion map. For each ideal $\mathfrak{a}$ of $\mathcal{O}_K$, we denote the $\mathfrak{a}$-torsion subgroup scheme of E by ${}_{\mathfrak{a}}E$.

Proposition 2.2. *Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ which is prime to 6. Then, there exists a unique element ${}_{\mathfrak{a}}\theta_E$ of $\mathcal{O}(E \setminus {}_{\mathfrak{a}}E)^\times$ satisfying the following conditions (i), (ii).*

(i) *The divisor of ${}_{\mathfrak{a}}\theta_E$ is $N(\mathfrak{a}) \cdot (0) - {}_{\mathfrak{a}}E$.*
(ii) *For any integer b prime to $\mathfrak{a}$, we have*

$$N_{[b]}({}_{\mathfrak{a}}\theta_E|_{E\setminus {}_{b\mathfrak{a}}E}) = {}_{\mathfrak{a}}\theta_E,$$

where $N_{[b]}\colon \mathcal{O}(E \setminus_{b\mathfrak{a}} E)^\times \longrightarrow \mathcal{O}(E \setminus {}_{\mathfrak{a}}E)^\times$ is the norm map associated to the multiplication map

$$[b]\colon E \setminus {}_{b\mathfrak{a}}E \longrightarrow E \setminus {}_{\mathfrak{a}}E.$$

We use the notion of "CM-pair" in [Ka] §15. Let F be an intermediate field of $\mathbb{C}/K$, and $\mathfrak{f}$ be an ideal of $\mathcal{O}_F$ which makes the natural homomorphism $\mathcal{O}_K^\times \longrightarrow (\mathcal{O}_K/\mathfrak{f})^\times$ be injective. (For instance, if $\mathfrak{f}$ is a proper ideal of $\mathcal{O}_K$ prime to 6, then this injectivity holds.) We call a pair (E, α) a CM-pair of modulus $\mathfrak{f}$ over F if E is an elliptic curve over F with complex multiplication by $\mathcal{O}_K$, and α is a torsion point of $E(F)$ satisfying $\mathrm{ann}_{\mathcal{O}_K}(\alpha) = \mathfrak{f}$. A CM-pair (E, α) over F is isomorphic to a CM-pair (E', α') if and only if there exists an isomorphism $\iota\colon E \xrightarrow{\simeq} E'$ satisfying $\iota(\alpha) = \iota(\alpha')$. Note that since we assume the natural homomorphism $\mathcal{O}_K^\times \longrightarrow (\mathcal{O}_K/\mathfrak{f})^\times$ is injective, if a CM-pair (E, α) over F is isomorphic to a CM-pair (E', α'), then there exists only one isomorphism from (E, α) to (E', α').

Let $\mathfrak{n}$ be a non-zero ideal of $\mathcal{O}_K$. Then, we denote the ray class field of K of the modulus $\mathfrak{n}$ by $K(\mathfrak{n})$. In particular, $K(\mathcal{O}_K)$ is the Hilbert class field H_K of K. The following facts are well-known.

- There exists a CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})$ which is isomorphic to $(\mathbb{C}/\mathfrak{f}, 1 \bmod \mathfrak{f})$ over $\mathbb{C}$. This CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})$ is unique up to unique isomorphism. We call this CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})$ the canonical CM-pair over $K(\mathfrak{f})$, and denote it by $(E^{\mathfrak{f}}_{\mathrm{can}}, \alpha^{\mathfrak{f}}_{\mathrm{can}})$.
- Let F be an intermediate field of $\mathbb{C}/K$, and (E, α) a CM-pair of modulus $\mathfrak{f}$ over F. Then, there exists a unique embedding $\iota\colon K(\mathfrak{f}) \longrightarrow F$ such that the base change $(\iota^* E^{\mathfrak{f}}_{\mathrm{can}}, \iota^*\alpha^{\mathfrak{f}}_{\mathrm{can}})$ of the canonical CM-pair is isomorphic to (E, α).

Definition 2.3. Let $\mathfrak{a}$ and $\mathfrak{f}$ be ideals of $\mathcal{O}_K$ satisfying the following condition(I).

(I) *The ideal $\mathfrak{a}$ is prime to $6\mathfrak{f}$, and the ideal $\mathfrak{f}$ makes the natural homomorphism $\mathcal{O}_K^\times \longrightarrow (\mathcal{O}_K/\mathfrak{f})^\times$ injective.*

Then, we define

$$ {}_{\mathfrak{a}}z_{\mathfrak{f}} := {}_{\mathfrak{a}}\theta_{E^{\mathfrak{f}}_{\mathrm{can}}}(\alpha^{\mathfrak{f}}_{\mathrm{can}}) \in K(\mathfrak{f})^{\times}. $$

The following properties of ${}_{\mathfrak{a}}z_{\mathfrak{f}}$'s are well-know.

Proposition 2.4 ([dS] Chapter II, Proposition 2.5, norm compatibility)**.** *Let* $\mathfrak{a}$ *and* $\mathfrak{f}$ *be ideals of* $\mathcal{O}_K$ *satisfying the condition* (I).

(1) *If* $\mathfrak{f}$ *is a power of one prime ideal of* $\mathcal{O}_K$*, we have* ${}_{\mathfrak{a}}z_{\mathfrak{f}} \in \mathcal{O}_{K(\mathfrak{f})}[1/\mathfrak{f}]^{\times}$*. Otherwise, we have* ${}_{\mathfrak{a}}z_{\mathfrak{f}} \in \mathcal{O}_{K(\mathfrak{f})}^{\times}$*.*

(2) *Let* $\mathfrak{l}$ *be a prime ideal of* $\mathcal{O}_K$ *not dividing* $\mathfrak{a}$*. Then, we have*

$$ N_{K(\mathfrak{f}\mathfrak{l})/K(\mathfrak{f})}({}_{\mathfrak{a}}z_{\mathfrak{f}\mathfrak{l}}) = \begin{cases} {}_{\mathfrak{a}}z_{\mathfrak{f}}^{1-\mathrm{Fr}_{\mathfrak{l}}^{-1}} & \text{if } \mathfrak{l} \text{ is prime to } \mathfrak{f} \\ {}_{\mathfrak{a}}z_{\mathfrak{f}} & \text{if } \mathfrak{l} \text{ divides } \mathfrak{f}, \end{cases} $$

where $\mathrm{Fr}_{\mathfrak{l}} \in \mathrm{Gal}(K(\mathfrak{f})/K)$ *is the arithmetic Frobenius element at* $\mathfrak{l}$*.*

Here, we define elliptic units.

Definition 2.5. Let F be a finite abelian extension field of K which contains H_K. We denote the conductor of F/K by $\mathrm{Cond}(F)$.

(1) Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_K$ prime to $\mathfrak{a}$. We define

$$ {}_{\mathfrak{a}}z_{\mathfrak{f}}(F,\mathfrak{n}) := N_{F(\mathfrak{f})/F}({}_{\mathfrak{a}}z_{\mathfrak{f}\mathfrak{n}}). $$

For simplicity, we put ${}_{\mathfrak{a}}z_{\mathfrak{f}}(F) := {}_{\mathfrak{a}}z_{\mathfrak{f}}(F, \mathcal{O}_K)$.

(2) We denote by D_F the $\mathbb{Z}[\mathrm{Gal}(F/K)]$-submodule of $F^{\times}$ generated by

$$ \left\{ {}_{\mathfrak{a}}z_{\mathfrak{f}}(F) \;\middle|\; \begin{array}{l} \mathfrak{a} \text{ and } \mathfrak{f} \text{ are ideals of } \mathcal{O}_K \text{ satisfying} \\ \text{the condition (I) and } \mathfrak{f} \mid \mathrm{Cond}(F) \end{array} \right\} \cup (\mathcal{O}_F^{\times})_{\mathrm{tors}}. $$

We denote the intersection $D_F \cap \mathcal{O}_F^{\times}$ by C_F, and we call C_F the group of elliptic units of F.

(3) We denote by $\mathcal{C}_F$ the $\mathbb{Z}_p[\mathrm{Gal}(F/K)]$-submodule of $\mathcal{E}_F$ generated by the image of C_F, and we define a Λ-module $\mathcal{C}_{\infty}$ to be the projective limit of the system

$$ \left\{ N_{F'/F} \colon \mathcal{C}_{F'} \longrightarrow \mathcal{C}_F \;\middle|\; K_0 \subseteq_f F \subseteq_f F' \subset K_{\infty} \right\} $$

where $N_{F'/F}$ are the norm maps.

Here, we recall the statement of the Iwasawa main conjecture proved in [Ru1] and [Ru2] briefly. Let $\chi \in \Delta$ be an arbitrary character. It is well-known fact that $\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi}$ is a torsion Λ_{χ}-module. Assume one of the following:

- p splits completely in $K/\mathbb{Q}$;
- p does not split in $K/\mathbb{Q}$, and for the element $\mathfrak{p} \in T$, the character χ is non-trivial on $D_{\mathfrak{p}}$.

Then, we have

$$ \mathrm{char}_{\Lambda_{\chi}}(X_{\infty,\chi}) = \mathrm{char}_{\Lambda_{\chi}}(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi}). $$

(See [Ru1] Theorem 4.1 and [Ru2] Theorem 2.)

Here, we recall some results on the Λ-modules $\mathcal{E}_\infty$ and $\mathcal{C}_\infty$.

Proposition 2.6 ([Ru1] Proposition 7.7 and Corollary 7.8)**.** *Recall we put* $\mathcal{I}_\mu = \mathrm{ann}_\Lambda(\boldsymbol{\mu}_{p^\infty}(K_\infty))$*.*

(1) *Let* $(\mathcal{E}_\infty)_{\mathrm{tors}}$ *(resp.* $(\mathcal{C}_\infty)_{\mathrm{tors}}$*) be the maximal torsion* Λ*-submodule of* $\mathcal{E}_\infty$ *(resp.* $\mathcal{C}_\infty$*). Then, we have*

$$(\mathcal{E}_\infty)_{\mathrm{tors}} = (\mathcal{C}_\infty)_{\mathrm{tors}} = \begin{cases} \varprojlim \mu_{p^n} & \text{if } K_\infty = K_0(\mu_{p^\infty}) \\ 0 & \text{otherwise.} \end{cases}$$

(2) *We have*

$$\mathcal{C}_\infty \simeq \begin{cases} \mathcal{I}_\mu & \text{if } \Gamma \simeq \mathbb{Z}_p,\ K_\infty \neq K_0(\mu_{p^\infty}) \\ \Lambda \oplus \varprojlim \mu_{p^n} & \text{if } K_\infty = K_0(\mu_{p^\infty}) \\ \mathcal{I}(\mathcal{G})\mathcal{I}_\mu & \text{if } \Gamma \simeq \mathbb{Z}_p^2. \end{cases}$$

In this paper, we fix a generator $\theta_\chi \in \Lambda_\chi$ of the ideal $\mathrm{char}_{\Lambda_\chi}(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi})$. For each homomorphism $\varphi\colon \mathcal{E}_{\infty,\chi} \longrightarrow \Lambda_\chi$ of Λ_χ-modules, we write

$$\mathcal{I}(\mathcal{C}_{\infty,\chi};\varphi) := \theta_\chi^{-1}\varphi(\mathcal{C}_{\infty,\chi}).$$

Note that it follows from Proposition 2.6 (1) that $\mathcal{I}(\mathcal{C}_{\infty,\chi};\varphi)$ is an integral ideal of Λ_χ.

Definition 2.7. We define $\mathcal{I}_{\mathcal{C},\chi}$ to be the ideal of Λ_χ generated by $\bigcup_\varphi \mathcal{I}(\mathcal{C}_{\infty,\chi};\varphi)$, where φ runs through all homomorphism $\varphi\colon \mathcal{E}_{\infty,\chi} \longrightarrow \Lambda_\chi$ of Λ_χ-modules.

Note that $\mathcal{I}_{\mathcal{C},\chi}$ is an ideal of Λ_χ of height at least two. The following corollary follows from Proposition 2.6.

Corollary 2.8. *Assume* $\Gamma \simeq \mathbb{Z}_p$*. Let* $\chi \in \Delta$ *be a character satisfying* $\mathcal{I}_{\mu,\chi} = \Lambda_\chi$*. Then, there exists a* Λ_χ*-homomorphism* $\varphi\colon \mathcal{E}_{\infty,\chi} \longrightarrow \Lambda_\chi$ *satisfying*

$$\varphi(\mathcal{C}_{\infty,\chi}) = \mathrm{char}_{\Lambda_\chi}(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi}).$$

In particular, $\mathcal{I}_{\mathcal{C},\chi} = \Lambda_\chi$ *if* $\mathcal{I}_{\mu,\chi} = \Lambda_\chi$*.*

2.3. Here, we recall the notion of higher Fitting ideals.

Definition 2.9 (Higher Fitting ideals, see [No] §3.1)**.** Let R be a commutative ring, and M be a finitely presented R-module. Let

$$R^m \xrightarrow{f} R^n \longrightarrow M \longrightarrow 0$$

be an exact sequence of R-modules. For each $i \geq 0$, we define *the i-th Fitting ideal* $\mathrm{Fitt}_{R,i}(M)$ to be the ideal of R generated by all $(n-i)\times(n-i)$ minors of the matrix corresponding to f. Note that when $0 \leq i < n$ and $m < n-i$ (resp. $i \geq n$), we define $\mathrm{Fitt}_{R,i}(M) := 0$ (resp. $\mathrm{Fitt}_{R,i}(M) := R$). Definition of these ideals depends only on M, and does not depend on the choice of the above exact sequence. We have the ascending filtration

$$\mathrm{Fitt}_{R,0}(M) \subseteq \mathrm{Fitt}_{R,1}(M) \subseteq \cdots \subseteq \mathrm{Fitt}_{R,n}(M) = \mathrm{Fitt}_{R,n+1}(M) = \cdots = R.$$

We denote the smallest number of generators of an R-module M by $\mathfrak{n}_R(M)$. If $\mathrm{Fitt}_{R,n}(M) \neq R$, then $\mathfrak{n}_R(M) \geq n+1$. Note that when R is a local ring or a PID, we have $\mathfrak{n}_R(M) = i+1$ if and only if $\mathrm{Fitt}_{R,i}(M) \neq R$ and $\mathrm{Fitt}_{R,i+1}(M) = R$.

Example 2.10. Let $\mathcal{O}$ be the valuation ring of some finite extension field of $\mathbb{Q}_p$. Suppose R is a ring isomorphic to $\mathcal{O}[[T]]$ or $\mathcal{O}[[S,T]]$, and M is a finitely generated torsion R-module. (For example, $R = \Lambda_\chi$ for some $\chi \in \widehat{\Delta}$.) Assume

$$M \sim \bigoplus_{i=1}^{n} R/f_i R$$

and f_i divides f_{i+1} for $1 \leq i \leq n-1$. Then, for each i with $i \geq 0$, there exists an ideal I_i of height at least two in R such that

$$\mathrm{Fitt}_{R,i}(M) = \begin{cases} (\prod_{k=1}^{n-i} f_k) I_i & \text{if } i < n \\ I_i & \text{if } i \geq n \end{cases}$$

(cf. [Ku] Lemma 9.2). This implies that the family $\{\mathrm{Fitt}_{R,i}(M)\}_{i\geq 0}$ of Fitting ideals of M determines the pseudo-isomorphism class of M. Note that for two pseudo-isomorphic R-modules which have no non-trivial pseudo-null submodules, their higher Fitting ideals may be different. For example, we consider the following. Let $f, g \in R$ be distinguished polynomial which are prime to each other, and put $M_1 := R/(fg)$ and $M_2 := R/(f) \oplus R/(g)$. Then, R-modules M_1 and M_2 have no non-trivial pseudo-null submodules, and they are pseudo-isomorphic, but their first Fitting ideals are different: $\mathrm{Fitt}_{R,1}(M_1) = R$ and $\mathrm{Fitt}_{R,1}(M_2) = (f, g) \neq R$. Note that higher Fitting ideals do not determine the isomorphism classes of R-modules. See [Ku] Remark 9.4.

We need the following lemma in the proof of Theorem 1.1.

Lemma 2.11 (for example, see [Ku] Theorem 9.1)**.** *Let $\mathcal{O}$ be the valuation ring of some finite extension field of $\mathbb{Q}_p$, $R := \mathcal{O}[[T]]$ and M a finitely generated torsion R-module. Suppose M contains no non-trivial pseudo-null R-submodule. Then, there exists an exact sequence*

$$0 \longrightarrow R^n \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

for some integer $n > 0$, and we have

$$\mathrm{Fitt}_{R,0}(M) = \mathrm{char}_R(M).$$

3. Euler systems of elliptic units and Kurihara's element

In this section, we set up some notions related to Euler systems of elliptic units, and prove some preliminary propositions to prove our main theorem. This section contains four subsections. In the first section, we recall the notion of Kolyvagin derivative classes. In the second subsection, we define two homomorphisms which play key roles in Euler system arguments. In the third subsection, we define elements $x_{\mathfrak{n},\mathfrak{q}}(\eta, a) \in (F^\times/p^N)_\chi$, which are analogues of Kurihara's elements defined in [Ku] §7 for elliptic units. We define them by using the Kolyvagin derivative classes of the Euler system of elliptic units. In the final subsection, we prove an important proposition for induction arguments in the proof of our main result.

3.1. Here, we recall the definition of the Kolyvagin derivative classes $\kappa_{\mathfrak{f},\mathfrak{a}}(F, N; \mathfrak{n})$ of the Euler system of elliptic units (cf. for example, [Ru1]).

We denote the ideal class group of K by Cl_K, and we fix a decomposition

$$\mathrm{Cl}_K = \bigoplus_{i=1}^{k} \mathbb{Z}\bar{\mathfrak{a}}_i$$

of Cl_K into a direct sum of cyclic subgroups, where $\bar{\mathfrak{a}}_i$ is the ideal class of a prime ideal $\mathfrak{a}_i$ of $\mathcal{O}_K$ for each i. We denote the order of $\bar{\mathfrak{a}}_i$ in Cl_K by n_i, and fix a generator a_i of the principal ideal $\mathfrak{a}_i^{n_i}$.

Let F be a finite extension field of K_0. For an integer $N \geq 1$, let $\mathcal{S}_N^{\mathrm{prime}}(F)$ be the set of all prime ideals $\mathfrak{l}$ of $\mathcal{O}_K$ satisfying the following conditions:

(1) $\mathfrak{l}$ does not divide $\#\mathcal{O}_K^\times$;
(2) $\mathfrak{l}$ splits completely in $F(\mu_{p^N}, a_1^{1/p^N}, \ldots, a_k^{1/p^N})/K$.

We denote the set of all square-free integral ideals $\mathfrak{n}$ of $\mathcal{O}_K$ such that all prime divisors of $\mathfrak{n}$ belong to $\mathcal{S}_N^{\mathrm{prime}}(F)$ by $\mathcal{S}_N(F)$. For simplicity, we put $\mathcal{S}_N^{\mathrm{prime}} := \mathcal{S}_N^{\mathrm{prime}}(K_0)$ and $\mathcal{S}_N := \mathcal{S}_N(K_0)$. Recall the following lemma in [Ru2].

Lemma 3.1 ([Ru2] Lemma 3)**.** *Let N be a positive integer. For any $\mathfrak{l} \in \mathcal{S}_N^{\mathrm{prime}}$, there exists a cyclic extension $K_0(\mathfrak{l}; N)$ of F of degree p^N contained in the composite field $K_0 \cdot K(\mathfrak{l})$, which is totally ramified at all primes above $\mathfrak{l}$, and unramified at all primes not dividing $\mathfrak{l}$.*

Definition 3.2. Let F be a finite extension field of K_0 contained in K_∞, and N a positive integer. Let $\mathfrak{n} \in \mathcal{S}_N(F)$ be any element, and assume $\mathfrak{n}$ is decomposed as $\mathfrak{n} = \prod_{i=1}^{r} \mathfrak{l}_i$, where $\mathfrak{l}_1, \ldots, \mathfrak{l}_r$ are distinct prime ideals of $\mathcal{O}_K$. For each $\mathfrak{l}_i$, let $K_0(\mathfrak{l}_i) = K_0(\mathfrak{l}_i; N)$ be as in Lemma 3.1.

- We denote the composite field $F \cdot K_0(\mathfrak{l}_1; N) \cdots K_0(\mathfrak{l}_r; N)$ by $F(\mathfrak{n}; N)$, or by $F(\mathfrak{n})$ for simplicity. In particular, we put $F(\mathcal{O}_K) := F$.
- We put
$$\mathcal{H}_{\mathfrak{n}} = \mathcal{H}_{\mathfrak{n},N} := \mathrm{Gal}\left(K_0(\mathfrak{n})/K_0\right).$$
Note that $K_0(\mathfrak{l}_1), \ldots, K_0(\mathfrak{l}_r)$ and F are linearly disjointed over K_0, we have the natural isomorphism
$$\mathrm{Gal}\left(F(\mathfrak{n})/F\right) \simeq \mathcal{H}_{\mathfrak{n}} \simeq \mathcal{H}_{\mathfrak{l}_1} \times \cdots \times \mathcal{H}_{\mathfrak{l}_r},$$
and we identify them by this natural isomorphism.

As in §2.2, we regard $\overline{K}$ as a subfield of $\mathbb{C}$ by the fixed embedding $\infty_{\overline{K}} \colon \overline{K} \longrightarrow \mathbb{C}$. We put $\zeta_n := e^{2\pi i/n} \in \overline{K}$ for any positive integer n. Let F be a finite extension field of K_0 contained in K_∞, and $\mathfrak{l} \in \mathcal{S}_N^{\mathrm{prime}}(F)$. Recall $\mathcal{H}_{\mathfrak{l}} = \mathcal{H}_{\mathfrak{l},N}$ is a cyclic group of order p^N. We take a generator $\sigma_{\mathfrak{l}}$ of $\mathcal{H}_{\mathfrak{l}}$ as follows:

Note that since the prime ideal $\mathfrak{l}$ splits completely in $K_0(\mu_{p^N})/K$, we have $(K_0)_{\mathfrak{l}_{K_0}} = K_{\mathfrak{l}}$ and $\zeta_{p^N} \in K_{\mathfrak{l}}$. We put $L := K_0(\mathfrak{l})_{\mathfrak{l}_{K_0(\mathfrak{l})}}$. We identify $\mathrm{Gal}(L/K_{\mathfrak{l}})$

with $\mathcal{H}_{\mathfrak{l}}$ by the isomorphism induced by the embedding

$$\iota_{K_0(\mathfrak{l})} \colon K_0(\mathfrak{l}) \hookrightarrow L$$

fixed in §1 Notation. Let π be a uniformizer of $\mathcal{O}_L$. We fix a generator $\sigma_{\mathfrak{l}}$ of $\mathcal{H}_{\mathfrak{l}}$ such that

$$\pi^{\sigma_{\mathfrak{l}}-1} \equiv \zeta_{p^N} \pmod{\mathfrak{m}_L},$$

where $\mathfrak{m}_L$ is the maximal ideal of $\mathcal{O}_L$. Note that the definition of $\sigma_{\mathfrak{l}}$ does not depend on the choice of π.

Let $\mathfrak{n} \in \mathcal{S}_N(F)$. We define the element $D_{\mathfrak{n}}$ of the group ring $\mathbb{Z}[\mathcal{H}_{\mathfrak{n}}]$ as follows.

Definition 3.3. Let $\mathfrak{n} = \prod_{i=1}^r \mathfrak{l}_i \in \mathcal{S}_N(F)$ such that $\mathfrak{l}_i \in \mathcal{S}_N^{\mathrm{prime}}(F)$ for $i = 1, \dots, r$. We define

$$D_{\mathfrak{l}_i} := \sum_{k=1}^{p^N-1} k\sigma_{\mathfrak{l}_i}^k \in \mathbb{Z}[\mathcal{H}_{\mathfrak{l}_i}] \subseteq \mathbb{Z}[\mathcal{H}_{\mathfrak{n}}]$$

for $i = 1, \dots, r$, and

$$D_{\mathfrak{n}} := \prod_{i=1}^r D_{\mathfrak{l}_i} \in \mathbb{Z}[\mathcal{H}_{\mathfrak{n}}].$$

The following lemma is well-known.

Lemma 3.4. *Let $\mathfrak{a}$ and $\mathfrak{f}$ be ideals of $\mathcal{O}_K$ satisfying the condition* (I) *in Definition 2.3. Let $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{S}_N(F)$. Assume $\mathfrak{l} \in \mathcal{S}_N^{\mathrm{prime}}\big(F(\mathfrak{n}_1)\big)$ for each prime divisor $\mathfrak{l}$ of $\mathfrak{n}_2$. We put $\mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2$. Then, the image of ${}_{\mathfrak{a}}z_{\mathfrak{f}}(F,\mathfrak{n})^{D_{\mathfrak{n}_2}}$ in $F(\mathfrak{n})^\times/p^N$ is fixed by $\mathcal{H}_{\mathfrak{n}_2} = \mathrm{Gal}\big(F(\mathfrak{n})/F(\mathfrak{n}_1)\big)$.*

The Kolyvagin derivative class

$$\kappa_{\mathfrak{f},\mathfrak{a}}^{\mathfrak{n}_1}(F, N; \mathfrak{n}) \in F(\mathfrak{n}_1)^\times/p^N$$

is an element of $F(\mathfrak{n}_1)^\times/p^N$ such that its image in $F(\mathfrak{n})^\times/p^N$ by the natural homomorphism

$$\iota \colon F(\mathfrak{n}_1)^\times/p^N \longrightarrow \big(F(\mathfrak{n})^\times/p^N\big)^{\mathcal{H}_{\mathfrak{n}_2}}$$

coincides with the class of ${}_{\mathfrak{a}}z_{\mathfrak{f}}(F,\mathfrak{n})^{D_{\mathfrak{n}_2}}$. Note that the natural homomorphism ι is not injective or surjective in general, so the inverse image $\iota^{-1}({}_{\mathfrak{a}}z_{\mathfrak{f}}(F,\mathfrak{n})^{D_{\mathfrak{n}_2}})$ may not be a singleton. In order to construct Kolyvagin derivative classes, we recall the notion of universal Euler systems. Let $F, N, \mathfrak{f}, \mathfrak{a}$ and $\mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2$ be as in Lemma 3.4. Let $\mathcal{Y}_{F(\mathfrak{n}_1)}(\mathfrak{n}_2)$ be the free $\mathbb{Z}[\mathcal{H}_{\mathfrak{n}_2}]$-module whose basis is symbols

$$\big\{ y(\mathfrak{d}) \bigm| \mathfrak{d} \text{ is an ideal of } \mathcal{O}_K \text{ dividing } \mathfrak{n}_2 \big\}.$$

We write the group law multiplicatively. Let $\mathcal{Z}_{F(\mathfrak{n}_1)}(\mathfrak{n}_2)$ be the $\mathbb{Z}[\mathcal{H}_{\mathfrak{n}_2}]$-submodule of $\mathcal{Y}_{F(\mathfrak{n}_1)}(\mathfrak{n}_2)$ generated by

$$\big\{ y(\mathfrak{d})^{\sigma-1} \bigm| \mathfrak{d} \text{ is an ideal of } \mathcal{O}_K \text{ dividing } \mathfrak{n}_2, \text{ and } \sigma \in \mathrm{Gal}\big(F(\mathfrak{n})/F(\mathfrak{n}_1\mathfrak{d})\big) \big\}$$
$$\cup \left\{ y(\mathfrak{d})^{N_{\mathfrak{l}}} y(\mathfrak{d}/\mathfrak{l})^{\mathrm{Fr}_{\mathfrak{l}}^{-1}-1} \;\middle|\; \begin{array}{l} \mathfrak{d} \text{ is an ideal of } \mathcal{O}_K \text{ dividing } \mathfrak{n}_2, \\ \text{and } \mathfrak{l} \text{ is a prime ideal of } \mathcal{O}_K \text{ dividing } \mathfrak{d} \end{array} \right\},$$

where $N_{\mathfrak{l}} := \sum_{\sigma \in \mathcal{H}_{\mathfrak{l}}} \sigma \in \mathbb{Z}[\mathcal{H}_{\mathfrak{l}}]$, and $\mathrm{Fr}_{\mathfrak{l}}$ is the arithmetic Frobenius at $\mathfrak{l}$ in $\mathcal{H}_{\mathfrak{n}_2/\mathfrak{l}}$. (Note that we regard $\mathcal{H}_{\mathfrak{n}_2/\mathfrak{l}}$ as a subgroup of $\mathcal{H}_{\mathfrak{n}_2}$.) Then, we define the module $\mathcal{X}_{F(\mathfrak{n}_1)}(\mathfrak{n}_2)$ of "universal Euler systems at $F(\mathfrak{n}_1)$" by

$$\mathcal{X}_{F(\mathfrak{n}_1)}(\mathfrak{n}_2) := \mathcal{Y}_{F(\mathfrak{n}_1)}(\mathfrak{n}_2)/\mathcal{Z}_{F(\mathfrak{n}_1)}(\mathfrak{n}_2).$$

In order to define Kolyvagin derivatives, we use the following lemma.

Lemma 3.5 ([Ru1] Lemma 2.1). (i) *The $\mathbb{Z}[\mathcal{H}_{\mathfrak{n}_2}]$-module $\mathcal{X}_{F(\mathfrak{n}_1)}(\mathfrak{n}_2)$ is torsion free.*
(ii) *For any ideal $\mathfrak{d}$ of $\mathcal{O}_K$ dividing n and any $\sigma \in \mathcal{H}_{\mathfrak{n}_2}$, we have*

$$y(\mathfrak{d})^{D_{\mathfrak{d}}(\sigma-1)} \in \mathcal{X}_{F(\mathfrak{n}_1)}(\mathfrak{n}_2)^{p^N}.$$

By Lemma 2.4, we define a homomorphism

$$\delta \colon \mathcal{X}_{F(\mathfrak{n}_1)}(\mathfrak{n}_2) \longrightarrow F(\mathfrak{n})^\times$$

of $\mathbb{Z}[\mathcal{H}_{\mathfrak{n}_2}]$-modules by $\delta(y(\mathfrak{d})) := {}_{\mathfrak{a}}z_{\mathfrak{f}}(F(\mathfrak{n}_1), \mathfrak{n}_1\mathfrak{d})$ for each ideal $\mathfrak{d}$ of $\mathcal{O}$ dividing $\mathfrak{n}_2$. Then, by Lemma 3.5, we (uniquely) define a 1-cocycle $c \colon \mathcal{H}_{\mathfrak{n}_2} \longrightarrow F(\mathfrak{n})^\times$ by

$$c(\sigma) := \delta\big((y(\mathfrak{d})^{D_{\mathfrak{d}}(\sigma-1)})^{1/p^N}\big).$$

By Hilbert's Theorem 90, there exists an element $\beta \in F(\mathfrak{n})^\times$ such that $\beta^{\sigma-1} = c(\sigma)$ for any $\sigma \in \mathcal{H}_{\mathfrak{n}_2}$.

Now, we define the Kolyvagin derivative class $\kappa_{\mathfrak{f},\mathfrak{a}}^{\mathfrak{n}_1}(F, N; \mathfrak{n})$.

Definition 3.6. Let $F, N, \mathfrak{f}, \mathfrak{a}, \mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2$ be as in Lemma 3.4. We define

$$\kappa_{\mathfrak{f},\mathfrak{a}}^{\mathfrak{n}_1}(F, N; \mathfrak{n}) := {}_{\mathfrak{a}}z_{\mathfrak{f}}(F, \mathfrak{n})^{D_{\mathfrak{n}_2}}/\beta^{p^N} \in F(\mathfrak{n}_1)^\times/p^N.$$

Note that the definition of $\kappa_{\mathfrak{f},\mathfrak{a}}^{\mathfrak{n}_1}(F, N; \mathfrak{n})$ is independent of the choice of β. When $\mathfrak{n}_1 = \mathcal{O}_K$, the element $\kappa_{\mathfrak{f},\mathfrak{a}}^{\mathcal{O}_K}(F, N; \mathfrak{n})$ is denoted by $\kappa_{\mathfrak{f},\mathfrak{a}}(F, N; \mathfrak{n})$.

3.2. Let F be a finite extension field of K_0 contained in K_∞. We put $R_{F,N} := \mathbb{Z}/p^N[\mathrm{Gal}(F/K)]$ and $R_{F,N,\chi} := R_{F,N} \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\chi$ for any character $\chi \in \widehat{\Delta}$. Let $\mathfrak{n}$ be an element of $\mathcal{S}_N(F)$. Here, for each $\mathfrak{l} \in \mathcal{S}_N^{\mathrm{prime}}(F(\mathfrak{n}))$, we define of two homomorphisms

$$[\cdot]_{F,N,\chi}^{\mathfrak{l}} \colon (F^\times/p^N)_\chi \longrightarrow R_{F,N,\chi} \qquad \text{(cf. Definition 3.7)}$$

and

$$\bar{\phi}_{F(\mathfrak{n}),N,\chi}^{\mathfrak{l}} \colon (F(\mathfrak{n})^\times/p^N)_\chi \longrightarrow R_{F,N,\chi}[\mathcal{H}_{\mathfrak{n}}] \qquad \text{(cf. Definition 3.8)},$$

which play important roles in Euler system arguments.

First, we define $[\cdot]_{m,N,\chi}^{\mathfrak{l}}$. Let F be an algebraic number field. We define

$$\mathcal{I}_F := \mathrm{Div}\big(\mathrm{Spec}(\mathcal{O}_F)\big)$$

to be the divisor group, and we write its group law additively. We define the homomorphism $(\cdot)_F \colon F^\times \longrightarrow \mathcal{I}_F$ by

$$(x)_F = \sum_\lambda \mathrm{ord}_\lambda(x)\lambda,$$

where λ runs through all prime ideals of $\mathcal{O}_F$, and $\mathrm{ord}_\lambda\colon F^\times \twoheadrightarrow \mathbb{Z}$ is the normalized valuation of λ. For any prime ideal $\mathfrak{l}$ of $\mathcal{O}_K$, we define $\mathcal{I}_F^{\mathfrak{l}}$ to be the subgroup of $\mathcal{I}_F$ generated by all prime divisors above $\mathfrak{l}$. Then, we define $(\cdot)_F^{\mathfrak{l}}\colon F^\times \longrightarrow \mathcal{I}_F^{\mathfrak{l}}$ by

$$(x)_F^{\mathfrak{l}} = \sum_{\lambda|\mathfrak{l}} \mathrm{ord}_\lambda(x)\lambda.$$

Recall that we fix a family of embeddings $\{\mathfrak{l}_{\overline{K}}\colon \overline{K} \hookrightarrow \overline{K}_{\mathfrak{l}}\}_{\mathfrak{l}:\mathrm{prime}}$ satisfying the condition (A) (cf. §1 Notation). For each prime number $\mathfrak{l}$ and algebraic number field F, we denote the ideal of $\mathcal{O}_F$ corresponding to the embedding $\mathfrak{l}_{\overline{K}}|_K$ by $\mathfrak{l}_F$. Note that $\mathfrak{l}$ splits completely in F/K. Then, $\mathcal{I}_F^{\mathfrak{l}}$ is a free $\mathbb{Z}[\mathrm{Gal}(F/K)]$-module generated by $\mathfrak{l}_F$, and we identify $\mathcal{I}_F^{\mathfrak{l}}$ with $\mathbb{Z}[\mathrm{Gal}(F/K)]$ by the isomorphism $\iota\colon \mathbb{Z}[\mathrm{Gal}(F/K)] \xrightarrow{\simeq} \mathcal{I}_F^{\mathfrak{l}}$ defined by $x \longmapsto x \cdot \mathfrak{l}_F$ for $x \in \mathbb{Z}[\mathrm{Gal}(F/K)]$. We also denote the composite map $F^\times \longrightarrow \mathcal{I}_F^{\mathfrak{l}} \xrightarrow{\iota^{-1}} \mathbb{Z}[\mathrm{Gal}(F/K)]$ by $(\cdot)_F^{\mathfrak{l}}$.

Definition 3.7. We define the $R_{F,N,\chi}$-homomorphism

$$[\cdot]_{F,N,\chi}\colon (F^\times/p^N)_\chi \longrightarrow (\mathcal{I}_F/p^N)_\chi$$

to be the homomorphism induced by $(\cdot)_F^{\mathfrak{l}}\colon F^\times \longrightarrow \mathcal{I}_F$. For each $\mathfrak{l} \in \mathcal{S}_N^{\mathrm{prime}}(F)$, we define the $R_{F,N,\chi}$-homomorphism

$$[\cdot]_{F,N,\chi}^{\mathfrak{l}}\colon (F^\times/p^N)_\chi \longrightarrow R_{F,N,\chi}$$

to be the homomorphism induced by $(\cdot)_F^{\mathfrak{l}}\colon F^\times \longrightarrow \mathbb{Z}[\mathrm{Gal}(F/K)]$.

Second, we will define $\bar{\phi}_{F(\mathfrak{n}),N,\chi}^{\mathfrak{l}}$. Let $\mathfrak{l} \in \mathcal{S}_N(F(\mathfrak{n}))$. Note $\mathfrak{l}$ splits completely in $F(\mathfrak{n})/K$, so we have $F(\mathfrak{n})_\lambda = K_{\mathfrak{l}}$ for any prime ideal λ of F above $\mathfrak{l}$. The groups $\bigoplus_{\lambda|\mathfrak{l}} F(\mathfrak{n})_\lambda^\times$ and $\bigoplus_{\lambda|\mathfrak{l}} \mathcal{H}_{\mathfrak{l}}$ are regarded as $\mathbb{Z}[\mathrm{Gal}(F(\mathfrak{n})/K)]$-modules by the identification

$$\bigoplus_{\lambda|\mathfrak{l}} F(\mathfrak{n})_\lambda^\times = \mathcal{I}_{F(\mathfrak{n})}^{\mathfrak{l}} \otimes_{\mathbb{Z}} K_{\mathfrak{l}}^\times \quad \text{and} \quad \bigoplus_{\lambda|\mathfrak{l}} \mathcal{H}_{\mathfrak{l}} = \mathcal{I}_{F(\mathfrak{n})}^{\mathfrak{l}} \otimes \mathcal{H}_{\mathfrak{l}},$$

respectively. (Here, we regard $K_{\mathfrak{l}}^\times$ as a $\mathbb{Z}[\mathrm{Gal}(F(\mathfrak{n})/K)]$-modules on which the group $\mathrm{Gal}(F(\mathfrak{n})/K)$ acts trivially.) We denote by

$$\phi_{K_{\mathfrak{l}}}\colon K_{\mathfrak{l}}^\times \longrightarrow \mathrm{Gal}\left(K(\mathfrak{l})_{\mathfrak{l}_{K(\mathfrak{l})}}/K_{\mathfrak{l}}\right) = \mathrm{Gal}\left(K(\mathfrak{l})/K\right) = \mathcal{H}_{\mathfrak{l}}$$

the homomorphism induced by the reciprocity map

$$\phi_{K_{\mathfrak{l}}}^{\mathrm{rec}}\colon K_{\mathfrak{l}}^\times \longrightarrow \mathrm{Gal}\left(\overline{K_{\mathfrak{l}}}/K_{\mathfrak{l}}\right)$$

of local class field theory. (Let π be a uniformizer of $K_{\mathfrak{l}}$ and $k(\mathfrak{l}) := \mathcal{O}/\mathfrak{l}$. Then $\phi_{K_{\mathfrak{l}}}^{\mathrm{rec}}(\pi)$ induces the $\mathrm{N}(\mathfrak{l})$-power map on $\overline{k(\mathfrak{l})}$.) The homomorphism

$$\phi_{F(\mathfrak{n})}^{\mathfrak{l}}\colon F(\mathfrak{n})^\times \longrightarrow \mathbb{Z}[\mathrm{Gal}(F(\mathfrak{n})/K)] \otimes \mathcal{H}_{\mathfrak{l}}$$

is defined to be the composite of the three homomorphisms of $\mathbb{Z}[\mathrm{Gal}(F(\mathfrak{n})/K)]$-modules:

$$\mathrm{diag}\colon F(\mathfrak{n})^\times \longrightarrow \bigoplus_{\lambda|\mathfrak{l}} F(\mathfrak{n})^\times,$$

$$\oplus\phi_{K_\mathfrak{l}}\colon \bigoplus_{\lambda|\mathfrak{l}} F(\mathfrak{n})^\times_\lambda \longrightarrow \bigoplus_{\lambda|\mathfrak{l}} \mathcal{H}_\mathfrak{l},$$

$$\iota_{\mathcal{H}}^{-1}\colon \bigoplus_{\lambda|\mathfrak{l}} \mathcal{H}_\mathfrak{l} \xrightarrow{\simeq} \mathbb{Z}[\mathrm{Gal}(F(\mathfrak{n})/K)] \otimes \mathcal{H}_\mathfrak{l},$$

which are defined as follows:

(1) the first homomorphism diag is the diagonal inclusion;
(2) the second homomorphism $\oplus\phi_{K_\mathfrak{l}}$ is the direct sum of the reciprocity maps;
(3) the third isomorphism ι_H^{-1} is the inverse of the isomorphism

$$\iota_{\mathcal{H}}\colon \mathbb{Z}[\mathrm{Gal}(F(\mathfrak{n})/K)] \otimes \mathcal{H}_\mathfrak{l} \xrightarrow{\simeq} \bigoplus_{\lambda|\mathfrak{l}} \mathcal{H}_\mathfrak{l} = \mathcal{I}^{\mathfrak{l}}_{F(\mathfrak{n})} \otimes \mathcal{H}_\mathfrak{l},$$

which is induced by the isomorphism

$$\iota\colon \mathbb{Z}[\mathrm{Gal}(F(\mathfrak{n})/K)] \xrightarrow{\simeq} \mathcal{I}^{\mathfrak{l}}_{F(\mathfrak{n})}$$

given by $x \longmapsto x \cdot \mathfrak{l}_{F(\mathfrak{n})}$.

Definition 3.8. Let $\mathfrak{l} \in \mathcal{S}_N(F(\mathfrak{n}))$. We define

$$\phi^{\mathfrak{l}}_{F(\mathfrak{n}),N,\chi}\colon (F(\mathfrak{n})^\times/p^N)_\chi \longrightarrow \mathbb{Z}/p^N[\mathrm{Gal}(F(\mathfrak{n})/K)]_\chi \otimes \mathcal{H}_\mathfrak{l}$$

to be the homomorphism of $R_{F,N,\chi}[\mathcal{H}_\mathfrak{n}]$-modules induced by $\phi^{\mathfrak{l}}_{F(\mathfrak{n})}$. The choice of a generator $\sigma_\mathfrak{l}$ of $\mathcal{H}_\mathfrak{l}$ induces the $R_{F,N,\chi}[\mathcal{H}_\mathfrak{n}]$-homomorphism

$$\bar{\phi}^{\mathfrak{l}}_{F(\mathfrak{n}),N,\chi}\colon (F(\mathfrak{n})^\times/p^N)_\chi \longrightarrow \mathbb{Z}[\mathrm{Gal}(F(\mathfrak{n})/K)]_\chi = R_{F,N,\chi}[\mathcal{H}_\mathfrak{n}].$$

The following formulas on Kolyvagin derivative classes are well-known. (For example, see [Ru1] Proposition 2.4 for the proof. Note that our $\bar{\phi}^{\mathfrak{l}}_{F,N,\chi}$ is the map $\varphi_\mathfrak{l}$ in [Ru1].)

Proposition 3.9. *Let $\mathfrak{a}$ and $\mathfrak{f}$ be ideals of $\mathcal{O}_K$ satisfying the condition* (I) *in Definition 2.3. Let $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{S}_N(F)$. Assume $\mathfrak{l} \in \mathcal{S}_N^{\mathrm{prime}}\big(F(\mathfrak{n}_1)\big)$ for each prime divisor $\mathfrak{l}$ of $\mathfrak{n}_2$. We put $\mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2$.*

(1) *If λ is a finite place of K not dividing $\mathfrak{n}_2$, the λ-component of $[\kappa^{\mathfrak{n}_1}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})_\chi]_{F,N,\chi}$ is* 0. *In particular, if $\mathfrak{q} \in \mathcal{S}_N^{\mathrm{prime}}(F)$ is a prime ideal of $\mathcal{O}_K$ not dividing $\mathfrak{n}_2$, we have*

$$[\kappa_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})_\chi]^{\mathfrak{q}}_{F,N,\chi} = 0.$$

(2) *Let $\mathfrak{l}$ be a prime ideal of $\mathcal{O}_K$ dividing $\mathfrak{n}$. Then,*

$$[\kappa_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})_\chi]^{\mathfrak{l}}_{F,N,\chi} = \bar{\phi}^{\mathfrak{l}}_{F,N,\chi}(\kappa_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n}/\mathfrak{l})_\chi).$$

To prove our main theorem, we need not only Proposition 3.9 but another relations of Kolyvagin derivative classes (cf. Proposition 3.11). As in [Ku] §6.2, we need the notion *well-ordered*.

Definition 3.10. Let $\mathfrak{n} \in \mathcal{S}_N(F)$. We call $\mathfrak{n}$ *well-ordered* if and only if $\mathfrak{n}$ has a factorization $\mathfrak{n} = \prod_{i=1}^{r} \mathfrak{l}_i$ with $\mathfrak{l}_i \in \mathcal{S}_N^{\mathrm{prime}}(F)$ for each i such that $\mathfrak{l}_{i+1}$ splits in $F(\prod_{j=1}^{i} \mathfrak{l}_j)/K$ for $i = 1, \ldots, r-1$.

Proposition 3.11. *Let $\mathfrak{a}$ and $\mathfrak{f}$ be ideals of $\mathcal{O}_K$ satisfying the condition* (I) *in Definition 2.3. Let $\mathfrak{n} \in \mathcal{S}_N(F)$ be prime to $\mathfrak{af}$. If $\mathfrak{n}$ is well-ordered, then*

$$\bar{\phi}^{\mathfrak{l}}_{F,N,\chi}(\kappa_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})_\chi) = 0$$

for each prime ideal $\mathfrak{l}$ of $\mathcal{O}_K$ dividing $\mathfrak{n}$.

Proof. In the theory of Kolyvagin systems, this proposition is proved in more general situation. (For example, see [MR] Theorem A.4 for the case of Euler systems over $\mathbb{Q}$.) But in our case, we can give a more elementary proof by using the similar method to [Ku] Lemma 6.3.

We may assume $\mathfrak{n} \neq \mathcal{O}_K$. Since $\mathfrak{n}$ is well-ordered, we put $\mathfrak{n} = \prod_{i=1}^{r} \mathfrak{l}_i$, where $\mathfrak{l}_i$'s are elements of $\mathcal{S}_N^{\mathrm{prime}}(F)$ satisfying $\mathfrak{l}_{i+1}$ splits in $F(\prod_{j=1}^{i} \mathfrak{l}_j)/K$ for $i = 1, \ldots, r-1$. Assume $\mathfrak{l} = \mathfrak{l}_i$, and put $\mathfrak{n}_1 := \prod_{j=1}^{i-1} \mathfrak{l}_j$. (If $\mathfrak{l} = \mathfrak{l}_1$, then we put $\mathfrak{l}_1 = \mathcal{O}_K$.) Note that the image of $\kappa_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})$ in $F(\mathfrak{n}_1)^\times/p^N$ coincides with $\kappa^{\mathfrak{n}_1}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})^{D_{\mathfrak{n}_1}}$. Since the diagram

$$\begin{array}{ccc}
\big(F(\mathfrak{n}_1) \otimes_K K_{\mathfrak{l}}\big)^\times/p^N = \mathcal{I}^{\mathfrak{l}}_{F(\mathfrak{n}_1)} \otimes_{\mathbb{Z}} (K_{\mathfrak{l}}^\times/p^N) & \xrightarrow{\oplus_{\lambda'}\phi_{K_{\mathfrak{l}}}} & \mathcal{I}^{\mathfrak{l}}_{F(\mathfrak{n}_1)} \otimes_{\mathbb{Z}} (\mathcal{H}_{\mathfrak{l}}/p^N) \\
\uparrow & & \uparrow \\
(F \otimes_K K_{\mathfrak{l}})^\times/p^N = \mathcal{I}^{\mathfrak{l}}_F \otimes_{\mathbb{Z}} (K_{\mathfrak{l}}^\times/p^N) & \xrightarrow{\oplus_{\lambda}\phi_{K_{\mathfrak{l}}}} & \mathcal{I}^{\mathfrak{l}}_F \otimes_{\mathbb{Z}} (\mathcal{H}_{\mathfrak{l}}/p^N),
\end{array}$$

commutes, in order to prove our proposition, it is sufficient to show that

$$\bar{\phi}^{\mathfrak{l}}_{F(\mathfrak{n}_1),N,\chi}\big(\kappa^{\mathfrak{n}_1}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})_\chi\big) = 0. \tag{1}$$

Let λ be a place of $F(\mathfrak{n}_1)$ above $\mathfrak{l}$, and λ' a place of $F(\mathfrak{n}_1\mathfrak{l})$ above λ. We fix a uniformizer π' of $F(\mathfrak{n}_1\mathfrak{l})_{\lambda'}$, and put $\pi = N_{F(\mathfrak{n}_1\mathfrak{l})_{\lambda'}/F(\mathfrak{n}_1)_{\lambda'}}(\pi')$. We denote the residue field of $F(\mathfrak{n}_1)$ by $k(\lambda)$, and fix a generator α of a cyclic group $k^\times/p^N$. Then, we have a decomposition $F(\mathfrak{n}_1)^\times_\lambda/p^N = \langle\bar{\pi}\rangle \times \langle\alpha\rangle$, where $\bar{\pi}$ is the image of π in $F(\mathfrak{n}_1)^\times_\lambda/p^N$. Let

$$\phi_{F(\mathfrak{n})_\lambda} \colon F(\mathfrak{n})^\times_\lambda/p^N \longrightarrow \mathcal{H}_{\mathfrak{l}}/p^N$$

be the local reciprocity map. To prove (1), it is sufficient to prove that the image of $\kappa^{\mathfrak{n}_1}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})$ is contained in $\mathrm{Ker}(\phi_{F(\mathfrak{n})_\lambda})$ for all λ above $\mathfrak{l}$. By local class field theory, we have $\mathrm{Ker}(\phi_{F(\mathfrak{n})_\lambda}) = \langle\bar{\pi}\rangle$ since the image of norm map

$$N_{F(\mathfrak{n}_1\mathfrak{l})_{\lambda'}/F(\mathfrak{n}_1)_{\lambda'}} \colon F(\mathfrak{n}_1\mathfrak{l})^\times_{\lambda'}/p^N \longrightarrow F(\mathfrak{n}_1)^\times_{\lambda'}/p^N$$

coincides with $\langle\bar{\pi}\rangle$. Note that we can check easily that the kernel of the natural homomorphism

$$\iota_\lambda \colon F(\mathfrak{n}_1)^\times_\lambda/p^N = K_{\mathfrak{l}}^\times/p^N \longrightarrow F(\mathfrak{n}_1\mathfrak{l})^\times_{\lambda'}/p^N$$

is also $\langle\bar{\pi}\rangle$. So, it is sufficient to prove that the image of $\iota_\lambda(\kappa^{\mathfrak{n}_1}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})) = 1$. The image of $\kappa^{\mathfrak{n}_1}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})$ in $F(\mathfrak{n}_1\mathfrak{l})^\times_{\lambda'}/p^N$ coincides with $\kappa^{\mathfrak{n}_1\mathfrak{l}}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})^{D_{\mathfrak{l}}}$, so let us prove $\kappa^{\mathfrak{n}_1\mathfrak{l}}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})^{D_{\mathfrak{l}}} = 1$ in $F(\mathfrak{n}_1\mathfrak{l})^\times_{\lambda'}/p^N$. By proposition 3.9 (1), we have

$$\kappa^{\mathfrak{n}_1\mathfrak{l}}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n}) \in (\mathcal{O}_{F(n_1\mathfrak{l})} \otimes \mathcal{O}_{K,\mathfrak{l}})^\times,$$

where $\mathcal{O}_{K,\mathfrak{l}}$ is localization of $\mathcal{O}_K$ at $\mathfrak{l}$. The group $\mathcal{H}_\mathfrak{l} = \mathrm{Gal}\left(F(\mathfrak{n}_1\mathfrak{l})/F(\mathfrak{n}_1)\right)$ acts trivially on

$$\left(\mathcal{O}_{F(\mathfrak{n}_1\mathfrak{l})}/\mathfrak{l}_{K(\mathfrak{l})}\mathcal{O}_{F(\mathfrak{n}_1\mathfrak{l})}\right)^\times = \mathcal{I}^{\mathfrak{l}}_{F(\mathfrak{n}_1)} \otimes_\mathbb{Z} \left(\mathcal{O}_K/\mathfrak{l}\right)^\times$$

since all prime ideals above $\mathfrak{l}$ ramifies completely in $F(\mathfrak{n}_1\mathfrak{l})/F(\mathfrak{n}_1)$. Therefore, we have

$$\begin{aligned}
\kappa^{\mathfrak{n}_1}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n}) &= \kappa^{\mathfrak{n}_1\mathfrak{l}}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})^{D_\mathfrak{l}} \\
&= \kappa^{\mathfrak{n}_1\mathfrak{l}}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})^{\sum_{k=1}^{p^N-1} k\sigma_\mathfrak{l}^k} \\
&= \kappa^{\mathfrak{n}_1\mathfrak{l}}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})^{p^N(p^N-1)/2} \\
&= 1
\end{aligned}$$

in $F(\mathfrak{n}_1\mathfrak{l})/F(\mathfrak{n}_1)$. This implies the image of $\kappa^{\mathfrak{n}_1}_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})$ belongs to $\mathrm{Ker}(\iota_\lambda) = \mathrm{Ker}(\phi_{F(\mathfrak{n})_\lambda})$ for all places λ of $F(\mathfrak{n}_1)$ above $\mathfrak{l}$, and completes the proof. $\square$

3.3. In this subsection, we will define some elements $x_{\mathfrak{n},\mathfrak{q}}(\eta,a) \in (F^\times/p^N)_\chi$, which are analogues of Kurihara's elements defined in [Ku] §7 for elliptic units. Elements $x_{\mathfrak{n},\mathfrak{q}}(\eta,a)$ become a key of the proof of our main theorem for the one-variable cases.

Let F be a finite extension field of K_0 contained in K_∞, $\mathrm{Cond}(F)$ the conductor ideal of F/K, and N a positive integer. We consider an elliptic unit $\eta \in C_F$. Let a be a map

$$(\mathcal{I}_K)^2 = \{\text{non-zero ideals of } \mathcal{O}_K\}^2 \longrightarrow R_{F,N,\chi};\ (\mathfrak{f},\mathfrak{a}) \longmapsto a_{\mathfrak{f},\mathfrak{a}}$$

satisfying the following condition (R):

(R) *We have $a_{\mathfrak{f},\mathfrak{a}} = 0$ for all but finitely many $(\mathfrak{a},\mathfrak{f})$, and there exists an element $\zeta \in (\mathcal{O}_F^\times)_{\mathrm{tors}}$ satisfying*

$$\eta = \zeta \prod_{(\mathfrak{f},\mathfrak{a})\in(\mathcal{I}_K)^2} {}_\mathfrak{a}z_\mathfrak{f}(F)^{a_{\mathfrak{f},\mathfrak{a}}}.$$

Further, if $a_{\mathfrak{f},\mathfrak{a}} \neq 0$, then the pair $(\mathfrak{f},\mathfrak{a})$ satisfies $\mathfrak{f} \mid \mathrm{Cond}(F)$ and the condition (I) *in Definition 2.3.*

By the definition of elliptic units, there exists such a map a. We define the ideal $\mathfrak{a}(\eta;a)$ of $\mathcal{O}_K$ by the product of the all ideals $\mathfrak{a}$ satisfying $a_{\mathfrak{f},\mathfrak{a}} \neq 0$ for some $\mathfrak{f}$. We put

$$\kappa(\eta,a;\mathfrak{n}) := \prod_{\mathfrak{f},\mathfrak{a}} \kappa_{\mathfrak{f},\mathfrak{a}}(F,N;\mathfrak{n})_\chi^{a_{\mathfrak{f},\mathfrak{a}}} \in F^\times/p^N.$$

Note that for any character $\chi \in \widehat{\Delta}$ satisfying $\chi \neq \omega$, we have

$$\eta_\chi = \kappa(\eta,a;\mathcal{O}_K)_\chi \in (F^\times/p^N)_\chi.$$

Definition 3.12. Let $\mathfrak{q}\mathfrak{n} = \mathfrak{q}\prod_{i=1}^r \mathfrak{l}_i \in \mathcal{S}_N$, where $\mathfrak{q},\mathfrak{l}_1,\ldots,\mathfrak{l}_r$ are distinct prime ideals of $\mathcal{O}_K$ prime to $\mathfrak{a}(\eta;a)$. For any ideal $\mathfrak{d}$ of $\mathcal{O}_K$ dividing $\mathfrak{n}$, we define the element $\tilde{\kappa}_{\{\mathfrak{d},\mathfrak{q}\}}(\eta) \in (F^\times/p^N) \otimes (\bigotimes_{\mathfrak{l}\mid\mathfrak{d}} \mathcal{H}_\mathfrak{l})$ by

$$\tilde{\kappa}_{\{\mathfrak{d},\mathfrak{q}\}}(\eta,a) := \kappa(\eta,a;\mathfrak{q}\mathfrak{d}) \otimes \Big(\bigotimes_{\mathfrak{l}\mid\mathfrak{d}} \sigma_\mathfrak{l}\Big).$$

Fix a character $\chi \in \widehat{\Delta}$. Let $\mathfrak{q}\mathfrak{n} \in \mathcal{S}_N$ be an ideal of $\mathcal{O}_K$ satisfying $(\mathfrak{n}, \mathfrak{a}(\eta; a)) = 1$ and assume $\mathfrak{q}\mathfrak{n}$ is *well-ordered*. Assume that for each prime number $\mathfrak{l}$ dividing $\mathfrak{n}$, an element $w_{\mathfrak{l}} \in R_{F,N,\chi} \otimes \mathcal{H}_{\mathfrak{l}}$ is given. Then, we have an element $\bar{w}_{\mathfrak{l}} \in R_{F,N,\chi}$ such that $w_{\mathfrak{l}} = \bar{w}_{\mathfrak{l}} \otimes \sigma_{\mathfrak{l}}$. Note that we will take $\{w_{\mathfrak{l}}\}_{\mathfrak{l}|\mathfrak{n}}$ explicitly later, but here, we take arbitrary one. For any ideal $\mathfrak{d}$ of $\mathcal{O}_K$ dividing $\mathfrak{n}$, we define

$$w_{\mathfrak{d}} := \bigotimes_{\mathfrak{l}|\mathfrak{d}} w_{\mathfrak{l}} \in R_{F,N,\chi} \otimes \Big(\bigotimes_{\mathfrak{l}|\mathfrak{d}} \mathcal{H}_{\mathfrak{l}} \Big).$$

We also define the element $\bar{w}_{\mathfrak{d}} \in R_{m,N,\chi}$ by $w_{\mathfrak{d}} = \bar{w}_{\mathfrak{d}} \otimes \big(\bigotimes_{\mathfrak{l}|\mathfrak{d}} \sigma_{\mathfrak{l}} \big)$.

Definition 3.13. We write the group law of $(F^\times/p^N)_\chi \otimes \big(\bigotimes_{\mathfrak{l}|\mathfrak{d}} \mathcal{H}_{\mathfrak{l}} \big)$ multiplicatively. We define the element $\tilde{x}_{\mathfrak{n},\mathfrak{q}}(\eta)$ by

$$\tilde{x}_{\mathfrak{n},\mathfrak{q}}(\eta, a) := \prod_{\mathfrak{d}|\mathfrak{n}} w_{\mathfrak{d}} \otimes \tilde{\kappa}_{\{\mathfrak{n}/\mathfrak{d},\mathfrak{q}\}}(\eta, a)_\chi \in (F^\times/p^N)_\chi \otimes \Big(\bigotimes_{\mathfrak{l}|\mathfrak{d}} \mathcal{H}_{\mathfrak{l}} \Big).$$

Note that we naturally identify the $R_{F,N,\chi}$-module $(F^\times/p^N)_\chi \otimes \big(\bigotimes_{\mathfrak{l}|\mathfrak{d}} \mathcal{H}_{\mathfrak{l}} \big)$ with

$$R_{F,N,\chi} \otimes \Big(\bigotimes_{\mathfrak{l}|\mathfrak{d}} \mathcal{H}_{\mathfrak{l}} \Big) \otimes_{R_{F,N,\chi}} (F^\times/p^N)_\chi.$$

The element $x_{\mathfrak{n},\mathfrak{q}}(\eta, a) \in (F^\times/p^N)_\chi$ is defined by $\tilde{x}_{\mathfrak{n},\mathfrak{q}}(\eta, a) = x_{\mathfrak{n},\mathfrak{q}}(\eta, a) \otimes \big(\bigotimes_{\mathfrak{l}|\mathfrak{n}} \sigma_{\mathfrak{l}} \big)$.

The following formulas follows from Proposition 3.9 straightforward.

Proposition 3.14 (cf. [Ku] Proposition 5.2)**.** *Let $\eta \in F$ be an elliptic unit as above, and $\mathfrak{n}\mathfrak{q} \in \mathcal{S}_N(F)$. Fix a map $a\colon (\mathcal{I}_K)^2 \longrightarrow R_{F,N,\chi}$ satisfying the condition* (R) *for η. We assume that $\mathfrak{n}\mathfrak{q}$ is well-ordered.*

(1) *If λ is a prime ideal of K not dividing $\mathfrak{n}$, the λ-component of $[x_{\mathfrak{n},\mathfrak{q}}]_{F,N,\chi}$ is 0. In particular, if $\mathfrak{s}$ is a prime ideal of $\mathcal{O}_K$ not dividing $\mathfrak{n}\mathfrak{q}$, we have*
$$[x_{\mathfrak{n},\mathfrak{q}}(\eta, a)]^{\mathfrak{s}}_{F,N,\chi} = 0.$$
(2) *Let $\mathfrak{l}$ be a prime ideal of $\mathcal{O}_K$ dividing $\mathfrak{n}$. Then, we have*
$$[x_{\mathfrak{n},\mathfrak{q}}(\eta, a)]^{\mathfrak{l}}_{F,N,\chi} = \bar{\phi}^{\mathfrak{l}}_{F,N,\chi}(x_{\mathfrak{n}/\mathfrak{l},\mathfrak{q}}(\eta, a)).$$
(3) *Let $\mathfrak{l}$ be a prime ideal of $\mathcal{O}_K$ not dividing $\mathfrak{n}$. Then, we have*
$$\bar{\phi}^{\mathfrak{l}}_{F,N,\chi}(x_{\mathfrak{n}/\mathfrak{l},\mathfrak{q}}(\eta, a)) = \bar{w}_{\mathfrak{l}} \bar{\phi}^{\mathfrak{l}}_{F,N,\chi}(x_{\mathfrak{n}/\mathfrak{l},\mathfrak{q}}(\eta, a)).$$

3.4. Recall that we fix a family of embeddings $\{\mathfrak{l}_{\overline{K}}\colon \overline{K} \hookrightarrow \overline{K}_{\mathfrak{l}}\}_{\mathfrak{l}:\mathrm{prime}}$ satisfying the condition (A) for families of embeddings as follows.

(A) *For any subfield $L \subset \overline{K}$ which is a finite Galois extension field of K and any element $\sigma \in \mathrm{Gal}(L/K)$, there exist infinitely many prime numbers $\mathfrak{l}$ such that $\mathfrak{l}$ is unramified in L/K and $(\mathfrak{l}_L, L/K) = \sigma$, where $\mathfrak{l}_L$ is the prime ideal of L corresponding to the embedding $\mathfrak{l}_{\overline{L}}|_L$.*

Note that the existence of such a family of embeddings follows from the Chebotarev density theorem. Here, we prove the following proposition, which plays key roles in induction arguments in the proof of our main theorem.

Proposition 3.15. *Let F be an intermediate field of K_∞/K satisfying $K \subseteq_f F$, and $\chi \in \widehat{\Delta}$ a non-trivial character. If K_0 contains μ_p, we assume $\chi \neq \omega$ and $\chi \neq \chi^{-1}\omega$. Let $\mathfrak{q}$ be a non-zero prime ideal of $\mathcal{O}_K$, and $\mathfrak{n} \in \mathcal{S}_N(F)$ an ideal of $\mathcal{O}_K$ prime to $\mathfrak{q}$. Assume $\mathfrak{n}$ has a factorization $\mathfrak{n} = \prod_{i=1}^r \mathfrak{l}_i$ into the product of prime ideals. Suppose the following are given:*

- *a finite $R_{F,N,\chi}$-submodule W of $(F^\times/p^N)_\chi$;*
- *an $R_{F,N,\chi}$-homomorphism $\lambda\colon W \longrightarrow R_{F,N,\chi}$.*

Then, there exist infinitely many $\mathfrak{q}' \in \mathcal{S}_N(F(\mathfrak{n}))$ which have the following properties:

(1) *the class of $\mathfrak{q}'_F$ in $A_{F,\chi}$ coincides with that of $\mathfrak{q}_F$;*
(2) *there exists an element $z \in (F^\times \otimes \mathbb{Z}_p)_\chi$ such that*
$$(z)_{F,\chi} = (\mathfrak{q}'_F - \mathfrak{q}_F)_\chi \in \left(\mathcal{I}_F \otimes \mathbb{Z}_p\right)_\chi,$$
and
$$\phi^{\mathfrak{l}_i}_{F,N,\chi}(z) = 0$$
for each $i = 1, \dots, r$;
(3) *the group W is contained in the kernel of $[\cdot]^{\mathfrak{q}'}_{F,N,\chi}$, and*
$$\lambda(x) = \bar{\phi}^{\mathfrak{q}'}_{F,N,\chi}(x)$$
for any $x \in W$.

Proof. Let F be an intermediate field of K_∞/K satisfying $K \subseteq_f F$. For a finite place v of F, we denote the valuation ring of the completion F_v of F at v by $\mathcal{O}_{F_v}$, and put
$$\mathcal{O}^1_{F_v} := \{x \mid x \equiv 1 \mod \mathfrak{m}_v\},$$
where $\mathfrak{m}_v$ is the maximal ideal of $\mathcal{O}_{F_v}$. We denote the residue field of F at v by $k(v)$.

In the first step of the proof, by using global class field theory, we construct a finite Galois extension L_1 and an element $\sigma \in \mathrm{Gal}(L_1/F)$, which are related to the condition (1) and (2) in the assertion of Proposition 3.15. Let $F\{\mathfrak{n}\}$ be the maximal abelian p-extension of F unramified outside $\mathfrak{n}$. Note that $F\{\mathfrak{n}\}$ is Galois over K. By global class field theory, we have the $\mathrm{Gal}(F/K)$-equivariant isomorphism
$$\frac{(\prod_{v|\mathfrak{n}} F_v^\times/\mathcal{O}^1_{F_v}) \times (\bigoplus_{u\nmid\mathfrak{n}} F_u^\times/\mathcal{O}^\times_{F_u})}{\text{the image of } F^\times} \otimes \mathbb{Z}_p \xrightarrow{\simeq} \mathrm{Gal}\left(F\{\mathfrak{n}\}/F\right),$$
where u runs all finite places outside $\mathfrak{n}$. We naturally regard $\mathrm{Gal}(F\{\mathfrak{n}\}/F)_\chi$ as a quotient group of $\mathrm{Gal}(F\{\mathfrak{n}\}/F)$. Let $F\{\mathfrak{n}\}_\chi$ be the intermediate field of $F\{\mathfrak{n}\}/F$ satisfying $\mathrm{Gal}\left(F\{\mathfrak{n}\}_\chi/F\right) = \mathrm{Gal}(F\{\mathfrak{n}\}/F)_\chi$.

Recall that we fix a decomposition
$$\mathrm{Cl}_K = \bigoplus_{i=1}^k \mathbb{Z}\bar{\mathfrak{a}}_i$$

of Cl_K into a direct sum of cyclic subgroups. We denote the order of $\bar{\mathfrak{a}}_i$ in Cl_K by n_i, and fix a generator a_i of the principal ideal $\mathfrak{a}_i^{n_i}$. We put

$$F' := F(\mu_{p^N}, a_1^{1/p^N}, \dots, a_k^{1/p^N}).$$

Let $L_1 := F\{\mathfrak{n}\}_\chi \cdot F' \cdot F(\mathfrak{n})$ be the composite field.

We put $\Delta' := \mathrm{Gal}\big(K_0(\mu_p)/K\big)$. By the natural surjection $\Delta' \twoheadrightarrow \Delta$, we regard χ as a character of Δ'. Note that the subgroup Δ' of $\mathrm{Gal}(F/K)$ acts on $\mathrm{Gal}\big(F\{\mathfrak{n}\}_\chi/F\big)$ (resp. $\mathrm{Gal}\big(F(\mu_{p^N}) \cdot F(\mathfrak{n})/F\big)$ and $\mathrm{Gal}\big(F'/F(\mu_{p^N})\big)$) via χ (resp. trivial character and ω). Since we assume that χ is non-trivial and $\chi \neq \omega$, we have

$$F\{\mathfrak{n}\}_\chi \cap F' \cdot F(\mathfrak{n}) = F.$$

Then, we take the element $\sigma \in \mathrm{Gal}\big(L_1/F' \cdot F(\mathfrak{n})\big)$ such that

$$\sigma|_{F\{\mathfrak{n}\}_\chi} = (q_{F\{\mathfrak{n}\}_\chi}, F\{\mathfrak{n}\}_\chi/F).$$

In the second step, by using Kummer theory, we construct a finite Galois extension L_2/F' and an element $\lambda' \in \mathrm{Gal}(L_2/F')$, which are related to the condition (3) in the assertion of Proposition 3.15. We define a projection $\mathrm{pr}\colon R_{F,N} \longrightarrow \mathbb{Z}/p^N\mathbb{Z}$ by

$$\sum_{g \in \mathrm{Gal}(F/K)} a_g g \longmapsto a_1,$$

where $a_g \in \mathbb{Z}/p^N\mathbb{Z}$ for all $g \in \mathrm{Gal}(F/K)$, and $1 \in \mathrm{Gal}(F/K)$ is the identity element. We define $\lambda' \in \mathrm{Hom}(W, \mu_{p^N})$ by

$$x \longmapsto (\zeta_{p^N})^{\mathrm{pr} \circ \lambda(x)}$$

for all $x \in W$. (Recall ζ_{p^N} is a primitive p^N-th root of unity defined in §3.1.) We use the following well-known lemma.

Lemma 3.16. *Let $P\colon \mathrm{Hom}_{R_{F,N,\chi}}(W, R_{F,N,\chi}) \longrightarrow \mathrm{Hom}(W, \mathbb{Z}/p^N\mathbb{Z})$ be the map given by $f \longmapsto \mathrm{pr} \circ f$. Then, P is bijective.*

Indeed, the inverse of P is given by

$$h \longmapsto \left(x \longmapsto \sum_{g \in \mathrm{Gal}(F/K)} h(g^{-1}x) g \right) \in \mathrm{Hom}_{R_{F,N,\chi}}(W, R_{F,N,\chi}),$$

for $h \in \mathrm{Hom}(W, \mathbb{Z}/p^N\mathbb{Z})$. The group Δ' acts on W via χ, so we have

$$\mathrm{Hom}_{R_{F,N}}(W, R_{F,N}) = \mathrm{Hom}_{R_{F,N,\chi}}(W, R_{F,N,\chi}).$$

Note that Δ' acts on $H^1(F(\mu_{p^N})/F, \mu_{p^N})$ and $H^0\big(F'/F(\mu_{p^N}), H^1(F', \mu_{p^N})\big)$ via the trivial character. Since we assume that χ is non-trivial, we have

$$H^1\big(F'/F, \mu_{p^N}\big)_\chi = 0.$$

So, the natural homomorphism

$$W \subset (F^\times/p^N)_\chi \longrightarrow (F'^\times/p^N)_\chi$$

is injective. Then, we regard W as a subgroup of $(F'^{\times}/p^N)_{\chi}$. Let L_2 be the extension field of F' generated by all p^N-th roots of elements of $F^{\times}$ whose image in $F^{\times}/p^N$ is contained in W. We consider the Kummer pairing

$$\mathrm{Gal}(L_2/F') \times W \longrightarrow \mu_{p^N}.$$

This pairing induces a $\mathrm{Gal}(F(\mu_{p^N})/K)$-equivariant isomorphism

$$\mathrm{Hom}(W, \mu_{p^N}) \simeq \mathrm{Gal}(L_2/F').$$

(Note that L_2 is Galois over K since W is stable by the action of $\mathrm{Gal}(F'/K)$.) We regard λ' as an element of $\mathrm{Gal}(L_2/F')$ by this isomorphism.

In the final step, we complete the proof. By the isomorphism $\mathrm{Hom}(W, \mu_{p^N}) \simeq \mathrm{Gal}(L_2/F')$, the group Δ' acts on $\mathrm{Gal}\left(L_2/F'\right)$ via $\chi^{-1}\omega$. Comparing the action of Δ, we obtain

$$L_1 \cap L_2 = F'.$$

We put the composite field $\tilde{L} := L_1 L_2$. By the condition (A), there exists infinitely many prime numbers $\mathfrak{q}'$ such that

$$\begin{cases} (\mathfrak{q}'_{L_1}, L_1/K) = \sigma \in \mathrm{Gal}(L_1/F') \\ (\mathfrak{q}'_{L_2}, L_2/K) = \lambda'^{-1} \in \mathrm{Gal}(L_1/F'). \end{cases}$$

Let us prove that each of such $\mathfrak{q}'$ unramified in $\tilde{L}/K$ satisfies conditions (1)-(3) of Proposition 3.15.

First, we show $\mathfrak{q}'$ satisfies conditions (1) and (2). Let $\alpha = (\alpha_v)_v \in \mathbb{A}_F^{\times}$ be an idele whose $\mathfrak{q}'_F$-component is a prime element of $F_{\mathfrak{q}'_F}$, and other components are 1. Let $\beta = (\beta_v)_v \in \mathbb{A}_F^{\times}$ be an element whose $\mathfrak{q}_F$-component is a uniformizer of $F_{\mathfrak{q}_F}$, and other components are 1. By definition, ideles α and β have the same image in

$$\left(\frac{(\prod_{v|\mathfrak{n}} F_v^{\times}/\mathcal{O}_{F_v}^1) \times (\bigoplus_{u\nmid\mathfrak{n}} F_u^{\times}/\mathcal{O}_{F_u}^{\times})}{\text{the image of } F^{\times}} \otimes \mathbb{Z}_p \right)_{\chi} \simeq \mathrm{Gal}\left(F\{\mathfrak{n}\}_{\chi}/F\right).$$

This implies there exist $z \in (F^{\times} \otimes \mathbb{Z}_p)_{\chi}$ such that

$$\alpha = z\beta \ \text{ in } \left(((\prod_{v|n} F_v^{\times}/\mathcal{O}_{F_v}^1) \times (\bigoplus_{u\nmid n} F_u^{\times}/\mathcal{O}_{F_u}^{\times})) \otimes \mathbb{Z}_p \right)_{\chi}.$$

Hence, we have $(z)_{F_{\chi}} = (\mathfrak{q}'_F - \mathfrak{q}_F)_{\chi}$, and $\phi_{F,N,\chi}^{\mathfrak{l}_i}(z) = 0$ for any $i = 1, \ldots, r$. The prime ideal $\mathfrak{q}'$ of $\mathcal{O}_K$ satisfies conditions (1) and (2).

Next, we shall prove $\mathfrak{q}'$ satisfies condition (3). Since $\mathfrak{q}'$ is unramified in $\tilde{L}/K$, the group W is contained in the kernel of $[\cdot]_{F,N,\chi}^{\mathfrak{q}'}$. Since $(\mathfrak{q}'_{L_2}, L_2/K) = \lambda'^{-1}$, for any $x \in W$, we have

$$(\zeta_{p^N})^{\mathrm{pro}\lambda(x)} = \lambda'(x) = (x^{1/p^N})^{1-\mathrm{Fr}_{\mathfrak{q}'}},$$

where $\mathrm{Fr}_{\mathfrak{q}'} \in \mathrm{Gal}(\tilde{L}/K)$ is the arithmetic Frobenius at $\mathfrak{q}'$, and $x^{1/p^N} \in L_2$ is a p^N-th root of x. Then, we obtain

$$(\zeta_{p^N})^{\mathrm{pro}\lambda(x)} \equiv x^{(1-N(\mathfrak{q}'))/p^N} \pmod{\overline{\mathfrak{q}'_{L_2}}}.$$

Let π be a uniformizer of $M := F(\mathfrak{q}')_{\mathfrak{q}'_{F(\mathfrak{q}')}}$. By the definition of $\sigma_{\mathfrak{q}'}$, we have

$$\pi^{\sigma_{\mathfrak{q}'}-1} \equiv \zeta_{p^N} \pmod{\mathfrak{m}_M},$$

where $\mathfrak{m}_M$ is the maximal ideal of M. Recall that W is contained in the kernel of $[\cdot]^{\mathfrak{q}'}_{F,N,\chi}$. By [Se] Chapter XIV Proposition 6, we have

$$(\zeta_{p^N})^{\mathrm{pro}\bar{\phi}^{\mathfrak{q}'}_{F,N,\chi}(x)} \equiv \pi^{\phi(x)-1} \equiv x^{(1-N(\mathfrak{q}'))/p^N} \pmod{\mathfrak{m}_M}$$

for all $x \in W$, where we put

$$\phi(x) := \sigma_{\mathfrak{q}'}^{\mathrm{pro}\bar{\phi}^{\mathfrak{q}'}_{F,N,\chi}(x)}.$$

Hence, we obtain

$$(\zeta_{p^N})^{\mathrm{pro}\lambda(x)} = (\zeta_{p^N})^{\mathrm{pro}\bar{\phi}^{\mathfrak{q}'}_{F,N,\chi}(x)}$$

for all $x \in W$. By Lemma 3.16, we have $\lambda = \bar{\phi}^{\mathfrak{q}'}_{F,N,\chi}|_W$. Therefore $\mathfrak{q}'$ satisfies condition (3) of Proposition 3.15, and the proof is complete. $\square$

4. Analogue of Kurihara's ideals for elliptic units

Let $\chi \in \widehat{\Delta}$ be an arbitrary character. In this section, we define ideals $\mathfrak{C}^{\mathrm{ell}}_{i,\chi}$ of Λ_χ for each $i \in \mathbb{Z}_{\geq 0}$ by using elliptic units, and prove Theorem 1.1 for $i = 0$.

4.1. Let F be a finite extension field of K_0 contained in K_∞, and N a positive integer. Let $\mathfrak{n} \in \mathcal{S}_N(F)$ with a decomposition $\mathfrak{n} = \prod_{i=1}^r \mathfrak{l}_i$, where $\mathfrak{l}_i \in \mathcal{S}_N^{\mathrm{prime}}(F)$ for each i. We put $\epsilon(\mathfrak{n}) := r$. Namely, $\epsilon(\mathfrak{n})$ is the number of prime divisors of $\mathfrak{n}$. We denote by $\mathcal{S}_N^{\mathrm{w.o.}}(F)$ the set of all elements in $\mathfrak{n} \in \mathcal{S}_N(F)$ which are well-ordered. We define the $R_{F,N,\chi}$-submodule $\mathcal{W}_{F,N,\chi}(\mathfrak{n})$ of $(F^\times/p^N)_\chi$ to be the $R_{F,N,\chi}$-submodule generated by the image of

$$\left\{ \kappa(\eta, a; \mathfrak{n}) \;\middle|\; \eta \in C_F,\ (\mathfrak{n}, \mathfrak{a}(\eta; a)) = 1 \text{ for some } a \text{ satisfying (R) in §3.3} \right\} \cup (\mathcal{O}_F^\times)_{\mathrm{tors}}.$$

We put

$$\mathcal{HW}_{F,N,\chi}(\mathfrak{n}) := \mathrm{Hom}_{R_{F,N,\chi}}\big(\mathcal{W}_{F,N,\chi}(\mathfrak{n}), R_{F,N,\chi}\big).$$

Definition 4.1. We define $\mathfrak{C}^{\mathrm{ell}}_{i,F,N,\chi}$ to be the ideal of $R_{F,N,\chi}$ generated by the union of the images of all $f \in \mathcal{HW}_{F,N,\chi}(\mathfrak{n})$, where $\mathfrak{n}$ runs through all elements of $\mathcal{S}_N^{\mathrm{w.o.}}(F)$ satisfying $\epsilon(\mathfrak{n}) \leq r$.

Remark 4.2. Note that $R_{F,N,\chi}$ is injective as an $R_{F,N,\chi}$-module, since the $R_{F_1,N,\chi}$-module $\mathrm{Hom}_{\mathbb{Z}}(R_{F_i,N,\chi}, \mathbb{Q}/\mathbb{Z})$ is injective and free of rank 1. In particular, for any $\mathfrak{n} \in \mathcal{S}_N^{\mathrm{w.o.}}(F)$, the restriction map

$$\mathrm{Hom}_{R_{F,N,\chi}}\big((F_2^\times/p^N)_\chi, R_{F,N,\chi}\big) \longrightarrow \mathcal{HW}_{F,N,\chi}(\mathfrak{n})$$

is surjective. This implies that the ideal $\mathfrak{C}^{\mathrm{ell}}_{F,N,\chi}$ coincides with the ideal of $R_{F,N,\chi}$ generated by

$$\bigcup_{\mathfrak{n}} \bigcup_{f} f(\mathcal{W}_{F,N,\chi}(\mathfrak{n})),$$

where $\mathfrak{n}$ runs through all elements of $\mathcal{S}_N^{\text{w.o.}}(F)$ satisfying $\epsilon(\mathfrak{n}) \leq r$, and f runs through all elements of $\mathrm{Hom}_{R_{F,N,\chi}}\big((F_2^\times/p^N)_\chi, R_{F,N,\chi}\big)$.

In order to define the ideal $\mathfrak{C}_{i,\chi}^{\text{ell}}$ of Λ_χ, we need the following lemma.

Lemma 4.3. *Let N_1, N_2 be integers satisfying $N_1 \leq N_2$, and $F_1 \subseteq F_2$ finite extension fields of K_0 contained in K_∞. Then, the image of $\mathfrak{C}_{i,F_2,N_2,\chi}^{\text{ell}}$ by the natural projection $R_{F_2,N_2,\chi} \longrightarrow R_{F_1,N_1,\chi}$ is contained in $\mathfrak{C}_{i,F_1,N_1,\chi}^{\text{ell}}$.*

***Proof*.** It is sufficient to show our lemma in the following two cases: (1) $F_2 = F_1$; (2) $N_1 = N_2$. The first case is clear, so let us prove our lemma in the second case. Assume $N_1 = N_2 = N$. We put the natural surjection $\mathrm{pr}\colon R_{F_2,N,\chi} \longrightarrow R_{F_1,N,\chi}$. By Lemma 2.4, we have

$$N_{F_2/F_1}(\mathcal{W}_{F_2,N,\chi}(\mathfrak{n})) \subseteq \mathcal{W}_{F_1,N,\chi}(\mathfrak{n})$$

for any $\mathfrak{n} \in \mathcal{S}_N(F_2)$. So, it is sufficient to show the following claim:

Claim 4.4. *For any homomorphism*

$$f_2 \in \mathrm{Hom}_{R_{F_2,N,\chi}}\big((F_2^\times/p^N)_\chi, R_{F_2,N,\chi}\big),$$

there exists a homomorphism

$$f_1 \in \mathrm{Hom}_{R_{F_1,N}}\big((F_1^\times/p^N)_\chi, R_{F_1,N,\chi}\big)$$

which makes the diagram

$$\begin{tikzcd} (F_2^\times/p^n)_\chi \arrow[r, "f_2"] \arrow[d, "N_{F_1,F_2}"'] & R_{F_2,N,\chi} \arrow[d, "\mathrm{pr}"] \\ (F_1^\times/p^n)_\chi \arrow[r, dashed, "f_1"] & R_{F_1,N,\chi} \end{tikzcd}$$

commute.

For each elements $\sigma \in \mathrm{Gal}(F_1/K)$, we fix a lift $\bar{\sigma} \in \mathrm{Gal}(F_2/K)$ of σ. We have

$$(R_{F_2,N})^{\mathrm{Gal}(F_2/F_1)} = \left\{ \sum_{\sigma \in \mathrm{Gal}(F_1/K)} a_\sigma \bar{\sigma}\mathbf{n} \;\middle|\; a_\sigma \in \mathbb{Z}/p^N \right\},$$

where $\mathbf{n}$ is an element of $R_{F_2,N}$ defined by

$$\mathbf{n} := \sum_{\tau \in \mathrm{Gal}(F_2/F_1)} \tau.$$

We define the isomorphism $\varphi\colon (R_{F_2,N,\chi})^{\mathrm{Gal}(F_2/F_1)} \xrightarrow{\simeq} R_{F_1,N,\chi}$ of $R_{F_1,N,\chi}$-modules by

$$\sum_{\sigma \in \mathrm{Gal}(F_1/K)} a_\sigma \bar{\sigma}\mathbf{n} \longmapsto \sum_{\sigma \in \mathrm{Gal}(F_1/K)} a_\sigma \sigma.$$

Let $\iota\colon (F_1^\times/p^N)_\chi \longrightarrow (F_2^\times/p^N)_\chi$ be the natural homomorphism. We have

$$\mathrm{pr} \circ f_2 = \varphi \circ f_2 \circ \iota \circ N_{F_2/F_1}.$$

Since $R_{F_1,N,\chi}$ is an injective $R_{F_1,N,\chi}$-module, there exist a homomorphism

$$f_1\colon (F_1^\times/p^N)_\chi \longrightarrow R_{F_1,N,\chi}$$

satisfying

$$f_1|_{N_{F_2/F_1}(F_1^\times/p^N)} = \varphi \circ f_2 \circ \iota.$$

By the definition of f_1, we obtain the commutative diagram

$$\begin{tikzcd}
(F_2^\times/p^n)_\chi \arrow[r, "f_2"] \arrow[d, "N_{F_1,F_2}"'] & R_{F_2,N,\chi} \arrow[d, "\mathrm{pr}"] \\
(F_1^\times/p^n)_\chi \arrow[r, dashed, "f_1"] & R_{F_1,N,\chi}
\end{tikzcd}$$

as desired. This completes the proof of the claim, and our lemma follows from the claim immediately. □

Now, we can define the ideals $\mathfrak{C}^{\mathrm{ell}}_{i,\chi}$ of Λ_χ, which are analogues of Kurihara's higher Stickelberger ideals $\Theta^{(\delta),\chi}_{i,K_\infty}$ in [Ku] for elliptic units.

Definition 4.5. We define *the i-th elliptic ideal* $\mathfrak{C}^{\mathrm{ell}}_{i,\chi}$ to be the ideal of Λ_χ by $\mathfrak{C}^{\mathrm{ell}}_{i,\chi} := \varprojlim \mathfrak{C}^{\mathrm{ell}}_{i,F,N,\chi}$, where the projective limit is taken with respect to the system of the natural homomorphisms $\mathfrak{C}_{i,F_2,N_2,\chi} \longrightarrow \mathfrak{C}^{\mathrm{ell}}_{i,F_1,N_1,\chi}$ for integers N_1, N_2 satisfying $N_2 \geq N_1$ and intermediate fields F_1, F_2 of K_∞/K satisfying $K_0 \subseteq_f F_1 \subseteq_f F_2 \subseteq K_\infty$.

4.2. Recall that the Iwasawa main conjecture says

$$\mathrm{char}_{\Lambda_\chi}(X_{\infty,\chi}) = \mathrm{char}_{\Lambda_\chi}(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi}).$$

In order to obtain the "$i = 0$"-part of our main theorem (Theorem 1.1), we compare $\mathfrak{C}^{\mathrm{ell}}_{0,\chi}$ with $\mathrm{char}_{\Lambda_\chi}(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi})$.

Proposition 4.6. *Let $\chi \in \Delta$ be an arbitrary character. Then, we have the following:*

(1) $\mathcal{I}_{\mathcal{E},\chi}\mathcal{J}_{\mathcal{E},\chi}\mathcal{I}_{\mathcal{C},\chi}\,\mathrm{char}_{\Lambda_\chi}\big(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi}\big) \subseteq \mathfrak{C}^{\mathrm{ell}}_{0,\chi}$;
(2) *If the character χ is non-trivial on $D_{\Delta,\mathfrak{p}}$ for any $\mathfrak{p} \in T$, we have*

$$\mathfrak{C}^{\mathrm{ell}}_{0,\chi} \subseteq \mathrm{char}_{\Lambda_\chi}\big(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi}\big).$$

Proof. We fix a generator $\theta_\chi \in \Lambda_\chi$ with $\mathrm{char}_{\Lambda_\chi}\big(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi}\big)$. First, let us prove $\mathfrak{C}^{\mathrm{ell}}_{0,\chi}$ contains $\mathcal{I}_{\mathcal{E},\chi}\mathcal{J}_{\mathcal{E},\chi}\mathcal{I}_{\mathcal{C},\chi}\,\mathrm{char}_{\Lambda_\chi}\big(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi}\big)$. It is sufficient to show that

$$\bar{\theta}_\chi\bar{\mathcal{I}}_{\mathcal{E},\chi}\bar{\mathcal{J}}_{\mathcal{E},\chi}\bar{\mathcal{I}}_{\mathcal{C},\chi} \subseteq \mathfrak{C}^{\mathrm{ell}}_{0,F,N,\chi}$$

for any intermediate field F of K_∞/K satisfying $K \subseteq_f F$ and for any positive integer N, where $\bar{\theta}_\chi$ (resp. $\bar{\mathcal{I}}_{\mathcal{E},\chi}$, $\bar{\mathcal{J}}_{\mathcal{E},\chi}$ and $\bar{\mathcal{I}}_{\mathcal{C},\chi}$) is the image of θ_χ (resp. $\mathcal{I}_{\mathcal{E},\chi}$, $\mathcal{J}_{\mathcal{E},\chi}$ and $\mathcal{I}_{\mathcal{C},\chi}$) in $R_{F,N,\chi}$.

Fix a homomorphism $\varphi\colon \mathcal{E}_{\infty,\chi} \longrightarrow \Lambda_\chi$ with pseudo-null cokernel, and let

$$\delta_{\mathcal{C}} \in \mathcal{I}(\mathcal{C}_{\infty,\chi};\varphi) \subseteq \mathcal{I}_{\mathcal{C},\chi}$$

be an arbitrary element. Note that by the definition of $\mathcal{I}(\mathcal{C}_{\infty,\chi};\varphi)$, we have $\delta_{\mathcal{C}}\theta_\chi \in \varphi(\mathcal{C}_{\infty,\chi})$. We fix elements $\delta_{\mathcal{I}} \in \mathcal{I}_{\mathcal{E}}$ and $\delta_{\mathcal{J}} \in \mathcal{J}_{\mathcal{E}}$. Let F be an intermediate field of

K_∞/K satisfying $K \subseteq_f F$ and N a positive integer. Then, there exists a homomorphism $\psi\colon (\mathcal{E}_F/p^N)_\chi \longrightarrow R_{F,N,\chi}$ which makes the diagram

$$\begin{tikzcd}
(\mathcal{C}_{\infty,\chi})_F/p^N \arrow[r] \arrow[d] & (\mathcal{E}_{\infty,\chi})_F/p^N \arrow[rr, "\delta_{\mathcal{I}}\delta_{\mathcal{J}}\bar{\varphi}_{F,N}"] \arrow[d] & & R_{F,N,\chi} \\
\mathcal{W}_{F,N,\chi}(\mathcal{O}_K) \arrow[r, hook] & (\mathcal{E}_F/p^N)_\chi \arrow[urr, dashed, "\psi"'] & &
\end{tikzcd}$$

commute, where $\bar{\varphi}_{F,N}\colon (\mathcal{E}_{\infty,\chi})_F/p^N \longrightarrow R_{F,N,\chi}$ is a homomorphism of $R_{F,N,\chi}$-modules induced by φ. This implies

$$\delta_{\mathcal{I}}\delta_{\mathcal{J}}\delta_{\mathcal{C}}\theta_\chi \in \delta_{\mathcal{I}}\delta_{\mathcal{J}}\bar{\varphi}_{F,N}\big((\mathcal{C}_{\infty,\chi})_F/p^N\big) = \psi\big(\mathcal{W}_{F,N,\chi}(\mathcal{O}_K)\big) \subseteq \mathfrak{C}^{\mathrm{ell}}_{i,F,N,\chi}.$$

Vary a homomorphism φ, elements $\delta_{\mathcal{I}} \in \mathcal{I}_{\mathcal{E}}$ and $\delta_{\mathcal{J}} \in \mathcal{J}_{\mathcal{E}}$, and we have

$$\bar{\theta}_\chi \bar{\mathcal{I}}_{\mathcal{E},\chi}\bar{\mathcal{J}}_{\mathcal{E},\chi}\bar{\mathcal{I}}_{\mathcal{C},\chi} \subseteq \mathfrak{C}^{\mathrm{ell}}_{0,F,N,\chi}.$$

Therefore, taking projective limit, we obtain

$$\mathfrak{C}^{\mathrm{ell}}_{0,F,N,\chi} \supseteq \mathcal{I}_{\mathcal{E},\chi}\mathcal{J}_{\mathcal{E},\chi}\mathcal{I}_{\mathcal{C},\chi}\,\mathrm{char}_{\Lambda_\chi}\left(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi}\right).$$

Next, Let us prove that $\mathfrak{C}^{\mathrm{ell}}_{0,\chi}$ is contained in $\mathrm{char}_{\Lambda_\chi}(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi})$. Here, we assume that the character χ is non-trivial on $D_{\Delta,\mathfrak{p}}$ for any $\mathfrak{p} \in T$. Under this assumption, Lemma 2.4 implies that the natural homomorhism $\mathcal{C}_{\infty,\chi} \longrightarrow \mathcal{C}_{F,\chi}$ is surjective. (Recall that $\mathcal{C}_F$ is the $\mathbb{Z}_p[F/K]$-submodule of $\mathcal{E}_F$ generated by the image of the group C_F of elliptic units.)

Fix a homomorphism $\varphi\colon \mathcal{E}_{\infty,\chi} \longrightarrow \Lambda_\chi$ of pseudo-null cokernel. Let $\delta \in \operatorname{Ker}\varphi$ and $\delta' \in \operatorname{Coker}\varphi$. Let F be an intermediate field of K_∞/K satisfying $K \subseteq_f F$ and N a positive integer. Let $\bar{\varphi}_{F,N}\colon (\mathcal{E}_{\infty,\chi})_F/p^N \longrightarrow R_{F,N,\chi}$ be a homomorphism induced by φ. Let $f\colon \mathcal{W}_{F,N,\chi}(\mathcal{O}_K) \longrightarrow R_{F,N,\chi}$ be an arbitrary $R_{F,N,\chi}$- homomorphism. Since $R_{F,N,\chi}$ is an injective $R_{F,N,\chi}$-module, there exists a homomorphism $\tilde{f}\colon (\mathcal{E}_F/p^N)_\chi \longrightarrow R_{F,N,\chi}$ whose restriction to $\mathcal{W}_{F,N,\chi}(\mathcal{O}_K)$ coincides with f. Then, we have an element $a \in R_{F,N,\chi}$ which makes the following diagram

$$\begin{tikzcd}
\mathcal{C}_{\infty,\chi} \arrow[r] \arrow[d] & (\mathcal{E}_{\infty,\chi})_F/p^N \arrow[r, "\delta\delta'\bar{\varphi}_{F,N}"] \arrow[d] & R_{F,N,\chi} \arrow[d, dotted, "\times a"] \\
\left(C_F/p^N\right)_\chi \arrow[r] \arrow[dr] & \left(\mathcal{E}_F/p^N\right)_\chi \arrow[r, dashed, "\tilde{f}"] & R_{F,N,\chi} \\
& \mathcal{W}_{F,N,\chi}(\mathcal{O}_K) \arrow[u, hook] \arrow[ur, "f"'] &
\end{tikzcd}$$

commute, where $\times a$ is the homomorphism multiplying a. This diagram implies that

$$f(\mathcal{W}_{F,N,\chi}(\mathcal{O}_K)) = \delta\delta' a\bar{\varphi}_{F,N}(\left(C_F/p^N\right)_\chi) \subseteq a\delta\delta'\mathcal{I}_{\mathcal{C}}\bar{\theta}_\chi R_{F,N,\chi} \subseteq \bar{\theta}_\chi R_{F,N,\chi}$$

Then, we have $\mathfrak{C}^{\mathrm{ell}}_{0,F,N,\chi} \subseteq \bar{\theta}_\chi R_{F,N,\chi}$. Taking projective limit, we obtain

$$\mathfrak{C}_{0,\chi} \subseteq \theta_\chi\Lambda_\chi = \mathrm{char}_{\Lambda_\chi}(\mathcal{E}_{\infty,\chi}/C_{\infty,\chi}).$$

This completes the proof. $\square$

Theorem 1.1 for $i = 0$ follows from Proposition 4.6 and the Iwasawa main conjecture.

Corollary 4.7 (Theorem 1.1 for $i = 0$, precise form). *Let $\chi \in \widehat{\Delta}$ be any character. Assume one of the following:*

- *p splits completely in $K/\mathbb{Q}$;*
- *p does not split in $K/\mathbb{Q}$, and for the element $\mathfrak{p} \in T$, the character χ is non-trivial on $D_{\Delta,\mathfrak{p}}$.*

Then, the following holds:

(1) $\mathcal{I}_{\mathcal{E},\chi}\mathcal{J}_{\mathcal{E},\chi}\mathcal{I}_{\mathcal{C},\chi}\,\mathrm{Fitt}_{\Lambda_\chi,0}(X_\chi) \subseteq \mathfrak{C}^{\mathrm{ell}}_{0,\chi}$;
(2) *If the character χ is non-trivial on $D_{\Delta,\mathfrak{p}}$ for any $\mathfrak{p} \in T$, we have*
$$\mathfrak{C}^{\mathrm{ell}}_{0,\chi} \subseteq \mathrm{Fitt}_{\Lambda_\chi,0}(X_\chi).$$

Remark 4.8. By Proposition 2.1, there exists a height-two ideal $J_{0,\chi}$ of Λ_χ satisfying
$$\mathcal{I}_{\mathcal{E},\chi}\mathcal{J}_{\mathcal{E},\chi}\mathcal{I}_{\mathcal{C},\chi} = J_{0,\chi}\mathcal{I}^3_{T,\chi}.$$
So, Corollary 4.7 implies Theorem 1.1 for $i = 0$.

5. Proof of the main theorem for one-variable case

Here, we prove our main theorem for $\Gamma \simeq \mathbb{Z}_p$. First, we recall the notation and state the precise assertion of our main theorem. In this section, we assume $\Gamma := \mathrm{Gal}(K_\infty/K_0) \simeq \mathbb{Z}_p$. The Λ-module X is defined by the projective limit $X := \varprojlim A_F$ with respect to norm maps, where F runs through all finite extension field of K contained in K_∞, and A_F is the p-Sylow subgroup of the ideal class group of F. The Λ-module X' is defined by $X' := X/X_{\mathrm{fin}}$, where X_{fin} is the maximal pseudo-null Λ-submodule of X.

We denote the ideal of Λ_χ generated by i-th power of elements of $\mathcal{I}_{\mathcal{A}}$ (resp. $\mathcal{J}_{\mathcal{A}}$) by $\mathcal{I}_{\mathcal{A},i}$ (resp. $\mathcal{J}_{\mathcal{A},i}$) for each $i \in \mathbb{Z}_{i\geq 0}$. The precise assertion of our main theorem for one-variable case is as follows.

Theorem 5.1. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. If K_0 contains μ_p, we assume $\chi \neq \omega$ and $\chi \neq \chi^{-1}\omega$. Assume one of the following:*

- *p splits completely in $K/\mathbb{Q}$;*
- *p does not split in $K/\mathbb{Q}$, and for the element $\mathfrak{p} \in T$, the character χ is non-trivial on $D_{\mathfrak{p}}$.*

Further, we assume that $\Gamma \simeq \mathbb{Z}_p$. Then, the following holds:

(1) *If the character χ is non-trivial on $D_{\Delta,\mathfrak{p}}$ for any $\mathfrak{p} \in T$, we have*
$$\mathfrak{C}^{\mathrm{ell}}_{0,\chi} \subseteq \mathrm{Fitt}_{\Lambda_\chi,0}(X'_\chi).$$
(2) $\mathcal{I}_{\mathcal{E},\chi}\mathcal{J}_{\mathcal{E},\chi}\mathcal{I}_{\mathcal{C},\chi}\mathcal{I}_{\mathcal{A},i}\mathcal{J}_{\mathcal{A},i}\,\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}^{\mathrm{ell}}_{i,\chi}$ *for any $i \in \mathbb{Z}_{\geq 0}$.*

We have already proved Theorem 1.1 for $i = 0$ in the last section. Here, we prove the second assertion for $i \geq 1$.

5.1. We spend this subsection on the setting of notations. Fix a non-trivial character $\chi \in \widehat{\Delta}$. we assume that $\chi \neq \chi^{-1}\omega$ and $\chi \neq \omega$ if K_0 contains μ_p. Since X'_χ has no non-trivial pseudo-null submodules, we have an exact sequence

$$0 \longrightarrow \Lambda_\chi^h \xrightarrow{f} \Lambda_\chi^h \xrightarrow{g} X'_\chi \longrightarrow 0, \tag{2}$$

by Lemma 2.11. Let M be the matrix corresponding to f with respect to the standard basis $(\mathbf{e}_i)_{i=1}^h$ of Λ_χ^h. Let $\{m_1, \ldots, m_h\}$ and $\{n_1, \ldots, n_h\}$ be permutations of $\{1, \ldots, h\}$. For any integer i satisfying $1 \leq i \leq h-1$, consider the matrix M_i which is obtained from M by eliminating the n_j-th rows ($j = 1, \ldots, i$) and the m_k-th columns ($k = 1, \ldots, i$). If $\det M_i = 0$, it is trivial that $\det M_i \in \mathfrak{C}_{i,\chi}^{\mathrm{ell}}$. So we assume that $\det M_i \neq 0$. If necessary, we permute $\{m_1, \ldots, m_i\}$, and assume $\det M_r \neq 0$ for all integers r satisfying $0 \leq r \leq i$.

We fix a finite extension field F of K_0 contained in K_∞ and we put the group $\Gamma_F := \mathrm{Gal}(K_\infty/F)$ and the integer $N_F := \max\{\#A_F, \#(X'_\chi)_F\}$. (Recall we denote the Γ_F-coinvariants of a Λ-module M by M_F.) We fix positive integer $N > N_F$, and we put, for simplicity, $R := \mathbb{Z}_p[\mathrm{Gal}(F/K)]_\chi$ and $R_N := R_{F,N,\chi} = \mathbb{Z}/p^N[\mathrm{Gal}(F/K)]_\chi$. Let $A_{F,\mathrm{fin},\chi}$ be the image of $X_{\mathrm{fin},\chi}$ in $A_{F,\chi}$ by the natural homomorphism.

Let $\varepsilon_{\mathcal{I}} \in \mathcal{I}_{\mathcal{A},\chi}$ and $\varepsilon_{\mathcal{J}} \in \mathcal{J}_{\mathcal{A},\chi}$ be any non-zero elements. Then, we can consider a homomorphism

$$\iota_{\varepsilon_{\mathcal{I}},\varepsilon_{\mathcal{J}}} \colon A_{F,\chi}/A_{F,\mathrm{fin},\chi} \longrightarrow (X'_\chi)_F; [\mathfrak{a}]_\chi \longmapsto \varepsilon_{\mathcal{I}} b$$

of R-modules, where $b \in (X'_\chi)_F$ is an element whose image by the natural homomorphism $(X'_\chi)_F \longrightarrow A_{F,\chi}/A_{F,\mathrm{fin},\chi}$ is $\varepsilon_{\mathcal{J}}[\mathfrak{a}]_\chi$. Note the cokernel of $\iota_{\varepsilon_{\mathcal{I}},\varepsilon_{\mathcal{J}}}$ is annihilated by $\mathcal{I}_{\mathcal{A},\chi}\mathcal{J}_{\mathcal{A},\chi}$. From the exact sequence (2), we obtain the exact sequence

$$0 \longrightarrow R^h \xrightarrow{\bar{f}} R^h \xrightarrow{\bar{g}} (X'_\chi)_F \longrightarrow 0,$$

by taking the Γ_F-coinvariants. Note the injectivity of the homomorphism $\bar{f}$ follows from the finiteness of $(X'_\chi)_F$. This injectivity become a key of our argument.

The image of $\mathbf{e}_r$ in R^h is denoted by $\mathbf{e}_i^{(F)}$. We define $\mathbf{c}_1 := g(\mathbf{e}_1), \ldots, \mathbf{c}_h := g(\mathbf{e}_h)$, and $\mathbf{c}_r^{(F)}$ to be the image of $\mathbf{c}_r$ in $(X'_\chi)_F$, namely $\mathbf{c}_r^{(F)} := \bar{g}(\mathbf{e}_r^{(F)})$. We take sufficiently large F, and we may assume $\mathbf{c}_r^{(F)} \neq \mathbf{c}_s^{(F)}$ if $r \neq s$. We fix a lift $\tilde{\mathbf{c}}_r^{(F)} \in A_{F,\chi}$ of $\mathbf{c}_r^{(F)}$, and define

$$P_r := \left\{ \mathfrak{l} \in \mathcal{S}_N^{\mathrm{prime}}(F) \;\middle|\; \iota_{\varepsilon_{\mathcal{I}},\varepsilon_{\mathcal{J}}}([\mathfrak{l}_F]_\chi) = \tilde{\mathbf{c}}_r^{(F)} \right\},$$

where $[\mathfrak{l}_F]_\chi$ is the class of $\mathfrak{l}_F$ in $A_{F,\chi}$. We define

$$P := \bigcup_{r=1}^{i} P_r,$$

and P_F to be the set of all the prime ideals of F above P. Let J be the subgroup of $\mathcal{I}_F$ generated by P_F, and the R-submodule $\mathcal{F}$ of $(F^\times \otimes \mathbb{Z}_p)_\chi$ the inverse image of $(J \otimes \mathbb{Z}_p)_\chi$ by the homomorphism $(\cdot)_F \colon (F^\times \otimes \mathbb{Z}_p)_\chi \longrightarrow (\mathcal{I}_F \otimes \mathbb{Z}_p)_\chi$. We define a surjective homomorphism

$$\alpha \colon (J \otimes \mathbb{Z}_p)_\chi \longrightarrow R^h$$

by $\mathfrak{l}_F \mapsto \mathbf{e}_r$ for each $\mathfrak{l} \in P_r$ and r with $1 \leq r \leq h$. We define

$$\alpha_r := \mathrm{pr}_r \circ \alpha \colon (J \otimes \mathbb{Z}_p)_\chi \xrightarrow{\alpha} R^h \xrightarrow{\mathrm{pr}_r} R$$

to be the composite of α and the r-th projection pr_r.

We define the homomorphism $\beta \colon \mathcal{F} \longrightarrow R^h$ to make the following diagram

$$\begin{tikzcd} & \mathcal{F} \arrow[r, "(\cdot)_{F,\chi}"] \arrow[d, "\beta"] & (J \otimes \mathbb{Z}_p)_\chi \arrow[r, two heads, "\text{can.}"] \arrow[d, "\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}}\alpha"] & A_{F,\chi}/A_{F,\mathrm{fin},\chi} \arrow[d, "\iota_{\varepsilon_{\mathcal{I}},\varepsilon_{\mathcal{J}}}"] & \\ 0 \arrow[r] & R^h \arrow[r, "\bar{f}"] & R^h \arrow[r, "\bar{g}"] & \mathrm{ann}_{\Lambda_\chi}(X'_\chi) \arrow[r] & 0 \end{tikzcd} \tag{3}$$

commute, where can. is induced by the canonical homomorphism

$$J \longrightarrow A'_{F,\chi} = A_{F,\chi}/A_{F,\mathrm{fin},\chi}. \tag{4}$$

Note that since the second row of the diagram is exact, β is well-defined. We define

$$\beta_r := \mathrm{pr}_r \circ \beta \colon \mathcal{F} \xrightarrow{\beta} R^h \xrightarrow{\mathrm{pr}_r} R$$

to be the composite of β and the r-th projection pr_r.

We consider the diagram (3) by taking $(- \otimes \mathbb{Z}/p^N\mathbb{Z})$. We use the following lemmas.

Lemma 5.2. *The canonical homomorphism*

$$\mathcal{F}/p^N \longrightarrow (F^\times/p^N)_\chi$$

is injective.

Proof. Let x be an element in the kernel of the homomorphism $\mathcal{F}/p^N \longrightarrow (F^\times/p^N)_\chi$ and $\tilde{x}$ a lift of x in $\mathcal{F}$. Then, there exists $y \in (F^\times \otimes \mathbb{Z}_p)_\chi$ such that $\tilde{x} = y^{p^N}$. Since $(\tilde{x})_{F,\chi} \in (J \otimes \mathbb{Z}_p)_\chi$ and $(\mathcal{I}_F \otimes \mathbb{Z}_p)/(J \otimes \mathbb{Z}_p)$ is torsion free $\mathbb{Z}_p$-module, we have $(y)_{F,\chi} \in (J \otimes \mathbb{Z}_p)_\chi$. Hence, $y \in \mathcal{F}$, and we obtain $x = 1$. □

The R_N-module $\mathcal{F}/p^N$ is regarded as a submodule of $(F^\times/p^N)_\chi$ by Lemma 5.2.

We regard $(F^\times/p^N)_\chi$ as a Λ_χ-module. For an element $x \in (F^\times/p^N)_\chi$ and $\delta \in \Lambda_\chi$, we write x^δ for the scalar multiple of x by δ.

Lemma 5.3. *Let* $[\cdot]_{F,N,\chi}$ *be the homomorphism*

$$(F^\times/p^N)_\chi \longrightarrow (\mathcal{I}_F/p^N)_\chi$$

induced by $(\cdot)_F \colon F^\times \longrightarrow \mathcal{I}_F$. *Let* x *be an element of* $(F^\times/p^N)_\chi$ *satisfying* $[x]_{F,N,\chi} \in (J/p^N)_\chi$. *Then,* x^δ *is contained in* $\mathcal{F}/p^N \subset (F^\times/p^N)_\chi$ *for any* $\delta \in \mathrm{ann}_{\Lambda_\chi}(X_{\mathrm{fin},\chi})$.

Proof. We consider the natural exact sequence:

$$0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{I}_K \longrightarrow A_F \longrightarrow 0,$$

where $\mathcal{P}$ is defined by $\mathcal{P} = F^\times/\mathcal{O}_F^\times$. By the snake lemma for the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P} \arrow[r] \arrow[d, "\times p^N"] & \mathcal{I}_F \arrow[r] \arrow[d, "\times p^N"] & A_F \arrow[r] \arrow[d, "\times p^N"] & 0 \\
0 \arrow[r] & \mathcal{P} \arrow[r] & \mathcal{I}_F \arrow[r] & A_F \arrow[r] & 0,
\end{tikzcd}$$

we obtain the exact sequence

$$0 \longrightarrow A_F \longrightarrow \mathcal{P}/p^N \xrightarrow{[\cdot]_N} \mathcal{I}_F/p^N \longrightarrow A_F \longrightarrow 0.$$

(Recall that we assume $p^N > \# A_F$.) Let B_F be the image of J in A_F, and $\mathcal{P}_0 = \mathcal{F}/\mathcal{O}_F^\times$. Then, we have the exact sequence

$$0 \longrightarrow \mathcal{P}_0 \longrightarrow J \longrightarrow B_F \longrightarrow 0,$$

and by a similar argument as above, we obtain the exact sequence

$$0 \longrightarrow B_F \longrightarrow \mathcal{P}_0/p^N \xrightarrow{[\cdot]_{N,0}} J/p^N \longrightarrow B_F \longrightarrow 0.$$

Now, we obtain two commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & B_F \arrow[r] \arrow[d, hook, "f_1"] & \mathcal{P}_0/p^N \arrow[r, "{[\cdot]_{N,0}}"] \arrow[d, hook, "f_2"] & \mathrm{Im}[\cdot]_{N,0} \arrow[r] \arrow[d, hook, "f_3"] & 0 \\
0 \arrow[r] & A_F \arrow[r] & \mathcal{P}/p^N \arrow[r, "{[\cdot]_N}"] & \mathrm{Im}[\cdot]_N \arrow[r] & 0
\end{tikzcd} \tag{5}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}[\cdot]_{N,0} \arrow[r] \arrow[d, hook, "f_3"] & J/p^N \arrow[r] \arrow[d, hook, "f_4"] & B_F \arrow[r] \arrow[d, hook, "f_1"] & 0 \\
0 \arrow[r] & \mathrm{Im}[\cdot]_N \arrow[r] & \mathcal{I}_F/p^N \arrow[r] & A_F \arrow[r] & 0
\end{tikzcd} \tag{6}$$

whose all rows are exact, and all vertical arrows are injective.

Let x be an element of $\mathcal{P}/p^N$ satisfying $[x]_N \in \operatorname{Im} f_4 = J/p^N$, and δ an arbitrary element of $\operatorname{ann}_{\Lambda_\chi}(X_{\mathrm{fin},\chi})$. Let us show that x^δ belongs to $\operatorname{Im} f_2 = \mathcal{P}_0/p^N$. By the snake lemma for the diagram (6), we have an exact sequence

$$0 = \operatorname{Ker} f_1 \longrightarrow \operatorname{Coker} f_3 \longrightarrow \operatorname{Coker} f_4,$$

so we obtain $[x]_N \in \operatorname{Im} f_3$. The exact sequence

$$\operatorname{Coker} f_1 \longrightarrow \operatorname{Coker} f_2 \longrightarrow \operatorname{Coker} f_3 \longrightarrow 0$$

follows from the diagram (5). Then, we obtain $x^\delta \in \operatorname{Im} f_2 = \mathcal{P}_0/p^N$ since the surjection (4) implies that δ annihilates $\operatorname{Coker} f_1$. $\square$

The following corollary follows as a byproduct of the proof of Lemma 5.3.

Corollary 5.4. *The kernel of the homomorphism*

$$[\cdot]_{F,\chi}\colon \mathcal{F}/p^N \longrightarrow J/p^N$$

is finite.

Let $\mathfrak{n}$ be an element of $\mathcal{S}_N(F)$ whose prime divisors are in P. We define $P_F^{\mathfrak{n}}$ to be the set of all elements of P dividing $\mathfrak{n}$. We define $J_{\mathfrak{n}}$ to be the subgroup of J generated by $P_F^{\mathfrak{n}}$, and the submodule $\mathcal{F}_{\mathfrak{n},N}$ of $\mathcal{F}/p^N$ the inverse image of $J_{\mathfrak{n}}$ by the restriction of $[\cdot]_{F,N,\chi}$ to $\mathcal{F}/p^N$. Note that $\mathcal{F}_{\mathfrak{n},N}$ is a *finite* R_N-submodule of $(F^\times/p^N)_\chi$ by Corollary 5.4. We have obtained the following commutative diagram

$$\begin{tikzcd} \mathcal{F}_{\mathfrak{n},N} \arrow[r, "{[\cdot]_{F,\chi}}"] \arrow[d, "\beta"] & (J_{\mathfrak{n}}/p^N)_\chi \arrow[d, "\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}}\alpha"] \\ R_N^h \arrow[r, "\bar{f}"] & R_N^h\ . \end{tikzcd}$$

5.2. First, we take a prime ideal $\mathfrak{q}$ of $\mathcal{O}_K$ by the following way. For each integer r with $1 \leq r \leq h$, we fix a prime number $\mathfrak{q}_r \in P_{n_r}$. We put $\mathfrak{Q} := \prod_{r=1}^h \mathfrak{q}_r \in \mathcal{S}_N(F)$. We fix a homomorphism $\varphi\colon \mathcal{E}_{\infty,\chi} \longrightarrow \Lambda_\chi$ with pseudo-null cokernel. By the Iwasawa main conjecture, we have

$$\varphi(\mathcal{C}_{\infty,\chi}) = (\det M_0) \cdot \mathcal{I}(\mathcal{C}_{\infty,\chi};\varphi).$$

Then, we fix elements $\delta_{\mathcal{C}} \in \mathcal{I}(\mathcal{C}_{\infty,\chi};\varphi)$ and $\boldsymbol{\eta} := (\eta_{F'})_{F'} \in \mathcal{C}_\infty$ satisfying $\varphi(\boldsymbol{\eta}_\chi) = \delta_{\mathcal{C}} \det M_0$. Let a be a map

$$(\mathcal{I}_K)^2 \longrightarrow R_{F,N,\chi};\ (\mathfrak{f},\mathfrak{a}) \longmapsto a_{\mathfrak{f},\mathfrak{a}}$$

satisfying the condition (R) in §3.3 for the elliptic unit $\eta := \eta_F \in C_F$. We assume $\chi \neq \omega$, so we have

$$\eta_\chi = \kappa(\eta, a; \mathcal{O}_K)_\chi = \prod_{(\mathfrak{f},\mathfrak{a})\in(\mathcal{I}_K)^2} {}_{\mathfrak{a}}z_{\mathfrak{f}}(F)_\chi^{a_{\mathfrak{f},\mathfrak{a}}} \in (F^\times/p^N)_\chi.$$

We fix non-zero elements $\delta_{\mathcal{I}} \in \mathcal{I}_{\mathcal{E}}$ and $\delta_{\mathcal{J}} \in \mathcal{J}_{\mathcal{E}}$. Then, as in the proof of Proposition 4.6, there exists a homomorphism $\psi\colon (\mathcal{E}_F/p^N)\chi \longrightarrow R_N$ which makes the diagram

$$\begin{tikzcd} (\mathcal{C}_{\infty,\chi})_F/p^N \arrow[r] \arrow[d] & (\mathcal{E}_{\infty,\chi})_F/p^N \arrow[r, "\delta_{\mathcal{I}}\delta_{\mathcal{J}}\bar{\varphi}_{F,N}"] \arrow[d] & R_N \\ \mathcal{W}_{F,N,\chi}(\mathcal{O}_K) \arrow[r, hook] & (\mathcal{E}_F/p^N)_\chi \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

commute, where $\bar{\varphi}_{F,N}\colon (\mathcal{E}_{\infty,\chi})_F/p_\chi^N \longrightarrow R_N$ is a homomorphism of R_N-modules induced by φ. By Proposition 3.15, we can take a prime ideal $\mathfrak{q} \in \mathcal{S}_N^{\mathrm{prime}}(F)$ prime to $\mathfrak{a}(\eta; a)$ satisfying the following two conditions:

(q1) the class of $\mathfrak{q}_F$ in $A_{F,\chi}$ coincides with the class of $\mathfrak{q}_{1F}$;
(q2) For all $x \in (\mathcal{E}_F/p^N)_\chi$, we have

$$\bar{\phi}^{\mathfrak{q}}(x) = \psi(x).$$

(Note that the natural homomorphism $(\mathcal{E}_F/p^N)_\chi \longrightarrow (F^\times/p^N)_\chi$ is injective, and we regard $(\mathcal{E}_F/p^N)_\chi$ as an R_N-submodule of $(F^\times/p^N)_\chi$ by this homomorphism.) In particular, we have

$$\begin{aligned}\bar{\phi}^q(\eta_\chi) = \psi(\eta_\chi) &= \delta_{\mathcal{I}}\delta_{\mathcal{J}}\bar{\varphi}_{F,N}(\boldsymbol{\eta}_\chi) \\ &= \delta_{\mathcal{I}}\delta_{\mathcal{J}}\delta_{\mathcal{C}} \det M_0.\end{aligned}$$

Next, we shall take $\mathfrak{n}$ and $\{w_\mathfrak{l}\}_{\mathfrak{l}|\mathfrak{n}}$. We fix an element $\delta_{\mathrm{fin}} \in \mathrm{ann}_{\Lambda_\chi}(X_{\mathrm{fin}})$. First, we consider the homomorphism

$$\beta_{m_1} : \mathcal{F}_{\mathfrak{Q}\mathfrak{q},N} \longrightarrow R_N.$$

Applying Proposition 3.15, we can take $\mathfrak{l}_2 \in \mathcal{S}_N^{\mathrm{prime}}(F(\mathfrak{Q}\mathfrak{q}))$ prime to $\mathfrak{a}(\eta_F; a)$ such that $\mathfrak{l}_2 \in P_{n_2}$, $\mathfrak{l} \neq \mathfrak{q}_2$, and

$$\bar{\phi}^{\mathfrak{l}_2}(x) = \beta_{m_1}(x)$$

for all $x \in \mathcal{F}_{\mathfrak{Q}\mathfrak{q},N}$. We put $\mathfrak{n}_1 := \mathcal{O}_K$.

In the case $i = 1$, we put $\mathfrak{n} := \mathfrak{n}_1 = \mathcal{O}_K$, and

$$x_{\mathfrak{n},\mathfrak{q}} = x_{\mathcal{O}_K,\mathfrak{q}} := \kappa(\eta, a).$$

It follows from Proposition 3.14 (1) and Lemma 5.3 that $x_{\mathcal{O}_K,\mathfrak{q}}^{\delta_{\mathrm{fin}}}$ is an element of $\mathcal{F}_{\mathfrak{Q}\mathfrak{q},N}$.

Suppose $i \geq 2$. To take $\mathfrak{n}$ and $\{w_\mathfrak{l}\}_{\mathfrak{l}|\mathfrak{n}}$, we choose prime ideals $\mathfrak{l}_r$ for each r with $2 \leq r \leq i+1$ by induction on r as follows. Let r be an integer satisfying $2 < r \leq i+1$, and suppose that we have chosen distinct prime ideals $\mathfrak{l}_s \in \mathcal{S}_N^{\mathrm{prime}}(F(\mathfrak{Q}\mathfrak{q}\mathfrak{n}_{s-1}))$ for each s with $2 \leq s \leq r-1$. We put $\mathfrak{n}_{r-1} := \prod_{s=2}^{r-1} \mathfrak{l}_s$. We consider the homomorphism $\beta_{m_1} : \mathcal{F}_{\mathfrak{Q}\mathfrak{q}\mathfrak{n}_{r-1},N} \longrightarrow R_N$. Applying Proposition 3.15, we can take $\mathfrak{l}_r \in \mathcal{S}'_N(F(\mathfrak{Q}\mathfrak{q}\mathfrak{n}_{r-1}))$ prime to $\mathfrak{a}(\eta; a)$ satisfying the following conditions:

(x1) $\mathfrak{l}_r \in P_{n_r}$, and $\mathfrak{l}_r \neq \mathfrak{q}_r$;
(x2) there exists $b_r \in (F^\times \otimes \mathbb{Z}_p)_\chi$ such that $(b_r)_{F,\chi} = (\mathfrak{l}_{r,F} - \mathfrak{q}_{r,F})_\chi$ and $\bar{\phi}^{\mathfrak{l}_s}(b_r) = 0$ for any s with $2 \leq s < r$;
(x3) $\bar{\phi}^{\mathfrak{l}_r}(x) = \beta_{m_{r-1}}(x)$ for any $x \in \mathcal{F}_{\mathfrak{Q}\mathfrak{q}\mathfrak{n}_{r-1},N}$.

Thus, we have taken $\mathfrak{l}_2, \ldots, \mathfrak{l}_{i+1}$, and we put $\mathfrak{n} := \mathfrak{n}_i = \prod_{r=2}^{i} \mathfrak{l}_r \in \mathcal{S}_N(F)$. Note that the ideal $\mathfrak{n}$ of $\mathcal{O}_K$ satisfies $(\mathfrak{n}, \mathfrak{a}(\eta; a)) = 1$. For each r with $2 \leq r \leq i$, we put $w_{\mathfrak{l}_r} := -\phi^{\mathfrak{l}_r}(b_r) \in R_N \otimes \mathcal{H}_{\mathfrak{l}_r}$, and we obtain

$$x_{\mathfrak{n},\mathfrak{q}} := x_{\mathfrak{n},\mathfrak{q}}(\eta, a) \in (F^\times/p^N)_\chi.$$

(See Definition §3.13.) It follows from Proposition 3.14 (1) and Lemma 5.3 that $x_{\mathfrak{n},\mathfrak{q}}^{\delta_{\mathrm{fin}}}$ is an element of $\mathcal{F}_{\mathfrak{Q}\mathfrak{n},N}$. Note that $\mathfrak{q}\mathfrak{n}$ is *well-ordered.*

Lemma 5.5 (cf. [Ku] Lemma 10.2)**.** *Suppose $i \geq 2$. Then,*

(1) $\beta_{m_{r-1}}(x_{\mathfrak{n},\mathfrak{q}}^{\delta_{\mathrm{fin}}}) = 0$ *for all r with $2 \leq r \leq i$;*
(2) $\alpha_j([x_{\mathfrak{n},\mathfrak{q}}]_{F,\chi}) = 0$ *for any $j \neq n_1, \ldots, n_i$.*

Proof. The assertion (2) of this lemma follows straightforward from Proposition 3.14 (1). Let us prove the assertion (1). We have $\alpha([b_r]_{F,\chi}) = 0$ for any r satisfying $2 \le r \le i$ since $(b_r)_{F,\chi} = (\mathfrak{l}_{r,F} - \mathfrak{q}_{r,F})_\chi$. By definition of β, we have $\beta(b_r) = 0$. We put

$$y_r = x_{\mathfrak{n},\mathfrak{q}} \prod_{s=r}^{i} b_s^{\bar{\phi}^{\mathfrak{l}_s}(x_{\mathfrak{n}/\mathfrak{l}_s,\mathfrak{q}})},$$

then we have $\beta(x_{\mathfrak{n},\mathfrak{q}}^{\delta_{\mathrm{fin}}}) = \beta(y_r^{\delta_{\mathrm{fin}}})$. So, let us show $\beta_{m_{r-1}}(y_r^{\delta}) = 0$ for any r satisfying $2 \le r \le i$. Note that by Proposition 3.14 (2), we have $[y_r]_{F,N,\chi} \in J_{\mathfrak{Q}\mathfrak{n}_{r-1}}$. Then, we have $y_r^{\delta_{\mathrm{fin}}} \in \mathcal{F}_{\mathfrak{Q}\mathfrak{n}_{r-1},N}$. Therefore, we obtain

$$\delta_{\mathrm{fin}}\bar{\phi}^{\mathfrak{l}_r}(y_r) = \beta_{m_{r-1}}(y_r^{\delta_{\mathrm{fin}}})$$

by the condition (x3). Since $\bar{\phi}^{\mathfrak{l}_r}(b_s) = 0$ for all integers s satisfying $r+1 \le s \le i$ by the condition (x2), we have

$$\bar{\phi}^{\mathfrak{l}_r}(y_r) = \bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n},\mathfrak{q}} b_r^{\bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n}/\mathfrak{l}_r,\mathfrak{q}})}).$$

By Proposition 3.14 (3), we have

$$\begin{aligned}\bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n},\mathfrak{q}} b_r^{\bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n}/\mathfrak{l}_r,\mathfrak{q}})}) &= \bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n},\mathfrak{q}}) + \bar{\phi}^{\mathfrak{l}_r}(b_r^{\bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n}/\mathfrak{l}_r,\mathfrak{q}})}) \\ &= w_{\mathfrak{l}_r}\bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n}/\mathfrak{l}_r,\mathfrak{q}}) + \bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n}/\mathfrak{l}_r,\mathfrak{q}})\bar{\phi}^{\mathfrak{l}_r}(b_r) \\ &= -\bar{\phi}^{\mathfrak{l}_r}(b_r)\bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n}/\mathfrak{l}_r,\mathfrak{q}}) + \bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n}/\mathfrak{l}_r,\mathfrak{q}})\bar{\phi}^{\mathfrak{l}_r}(b_r) \\ &= 0.\end{aligned}$$

Hence, we obtain $\beta_{m_{r-1}}(x_{\mathfrak{n},\mathfrak{q}}^{\delta_{\mathrm{fin}}}) = \delta_{\mathrm{fin}}\bar{\phi}^{\mathfrak{l}_r}(y_r) = 0$, and this completes the proof. □

As in the fourth step of the proof of Theorem 2.1 of [Ku] in §10.2, we obtain the following proposition from Lemma 5.5.

Proposition 5.6. *We have the following equalities on elements of $R_{N,\chi}$:*

(1) $\delta_{\mathrm{fin}}(\det M) \cdot \bar{\phi}^{\mathfrak{l}_2}(x_{\mathcal{O}_K,q}) = \pm\delta_{\mathrm{fin}}\delta_{\mathcal{I}}\delta_{\mathcal{J}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}} \cdot (\det M_1) \cdot \bar{\varphi}_{F,N}(\boldsymbol{\eta})$*;*
(2) *For any integer r satisfying $2 \le r \le i$, we have*

$$\delta_{\mathrm{fin}}(\det M_{r-1}) \cdot \bar{\phi}^{\mathfrak{l}_{r+1}}(x_{\mathfrak{n}_r,\mathfrak{q}}) = \pm\delta_{\mathrm{fin}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}} \cdot (\det M_r) \cdot \bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n}_{r-1},\mathfrak{q}}).$$

The signs $\pm$ in (1) *and* (2) *do not depend on F.*

Proof. For simplicity, we put

$$\mathbf{x}^{(r)} := \beta(x_{\mathfrak{n}_r,q}^{\delta_{\mathrm{fin}}}) \in R_N^h \quad \text{and} \quad \mathbf{y}^{(r)} := \varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}}\alpha(x_{\mathfrak{n}_r,\mathfrak{q}}^{\delta_{\mathrm{fin}}}) \in R_N^h,$$

for each integer r satisfying $1 \le r \le i$, and regard them as column vectors. Then, we have $\mathbf{y}^{(r)} = M\mathbf{x}^{(r)}$ in R_N^h.

First, we prove the assertion (1) of this proposition. Note that $x_{1,\mathfrak{q}}^{\delta_{\mathrm{fin}}}$ is an element of $\mathcal{F}_{\mathfrak{q},N}$. By Proposition 3.9 (2) and condition (q2), we have

$$\begin{aligned}\mathbf{y}^{(1)} &= \delta_{\mathrm{fin}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}} \cdot [\kappa_{\mathcal{O}_K,\mathfrak{q}}(\eta)_\chi]^{\mathfrak{q}}_{F,N,\chi}\mathbf{e}_{n_1}^{(F)} \\ &= \delta_{\mathrm{fin}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}} \cdot \bar{\phi}^{\mathfrak{q}}(\eta_\chi)\mathbf{e}_{n_1}^{(F)} \\ &= \delta_{\mathrm{fin}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}} \cdot \delta_{\mathcal{I}}\delta_{\mathcal{J}} \cdot \bar{\varphi}_{F,N}(\eta_\chi)\mathbf{e}_{n_1}^{(F)}\end{aligned}$$

Let $\widetilde{M}$ be the matrix of cofactors of M. Multiplying the both sides of $\mathbf{y}^{(1)} = M\mathbf{x}^{(1)}$ by $\widetilde{M}$, and comparing the m_1-st components, we obtain

$$(-1)^{n_1+m_1}\delta_{\mathrm{fin}}\delta_{\mathcal{I}}\delta_{\mathcal{J}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}}(\det M_1) \cdot \bar{\varphi}_{F,N}(\eta_{F,\chi}) = (\det M)\beta_{m_1}(x_{\mathcal{O}_K,\mathfrak{q}}^{\delta_{\mathrm{fin}}}).$$

By condition (x3) for $\mathfrak{l}_2$, we have $\beta_{m_1}(x_{\mathcal{O}_K,\mathfrak{q}}^{\delta_{\mathrm{fin}}}) = \delta_{\mathfrak{mrm}}\bar{\phi}^{\mathfrak{l}_2}(x_{\mathcal{O}_K,\mathfrak{q}})$. Then, the assertion (1) follows.

Next, we assume $i \geq 2$, and let us show the second assertion. This can be proved similarly to the proof of assertion (1). It is sufficient to prove the assertion when $r = i$. We write $\mathbf{x} = \mathbf{x}^{(i)}$ and $\mathbf{y} = \mathbf{y}^{(i)}$. Let $\mathbf{x}' \in R_N^{h-i+1}$ be the vector obtained from $\mathbf{x}$ by eliminating the m_j-th rows for $j = 1, \dots, i-1$, and $\mathbf{y}'$ the vector obtained from $\mathbf{y}$ by eliminating the n_k-th rows for $k = 1, \dots, i-1$. Since the m_r-th rows of $\mathbf{x}$ are 0 for all r with $1 \leq r \leq i-1$ by Lemma 5.5 (1), we have $\mathbf{y}' = M_{i-1}\mathbf{x}'$. We assume the m_i'-th component of $\mathbf{x}'$ corresponds to the m_i-th component of $\mathbf{x}$, and the n_i'-th component of $\mathbf{y}'$ corresponds to the n_i-th component of $\mathbf{y}$. By Lemma 5.5 (2) and Proposition 3.14 (2), we have

$$\mathbf{y}' = \delta_{\mathrm{fin}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}} \cdot \bar{\phi}^{\mathfrak{l}_i}(x_{\mathfrak{n}_{i-1},\mathfrak{q}})\mathbf{e}'^{(F)}_{n_i'},$$

where $(\mathbf{e}'^{(F)}_i)_{i=1}^{h-i+1}$ denotes the standard basis of R_N^{h-i+1}. Let $\widetilde{M}_{i-1}$ be the matrix of cofactors of M_{i-1}. Multiplying the both sides of $\mathbf{y}' = M_{i-1}\mathbf{x}'$ by $\widetilde{M}_{i-1}$, and comparing the m_i'-th components, we obtain

$$(-1)^{n_i'+m_i'}(\det M_i)\delta_{\mathrm{fin}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}} \cdot \bar{\phi}^{\mathfrak{l}_i}(x_{\mathfrak{n}_{i-1},\mathfrak{q}}) = (\det M_{i-1}) \cdot \beta_{m_i}(x_{\mathfrak{n},\mathfrak{q}}^{\delta_{\mathrm{fin}}}).$$

By condition (x3) for $\mathfrak{l}_{i+1}$, and since $x_{\mathfrak{n},\mathfrak{q}}^{\delta_{\mathrm{fin}}}$ is an element of $\mathcal{F}_{\mathfrak{Q}\mathfrak{q}\mathfrak{n},N}$, we have

$$\beta_{m_i}(x_{\mathfrak{n},\mathfrak{q}}^{\delta_{\mathrm{fin}}}) = \delta_{\mathrm{fin}}\bar{\phi}^{\mathfrak{l}_{i+1}}(x_{\mathfrak{n},\mathfrak{q}}).$$

This completes the proof. □

5.3. Now let us prove the main theorem.

Proof of Theorem 5.1 . Here, we vary F and N. So, the element

$$\bar{\phi}^{\mathfrak{l}_{r+1}}(x_{\mathfrak{n}_r,\mathfrak{q}}) \in R_N = R_{F,N,\chi} = (\mathbb{Z}/p^N)[\mathrm{Gal}(F/K)]_\chi$$

defined in §5.2 is denoted by $\bar{\phi}^{\mathfrak{l}_{r+1}}(x_{\mathfrak{n}_r,\mathfrak{q}})_{F,N}$.

Let D be a set of pairs (F, N) of an intermediate field F of K_∞/K_0 finite over K_0 and a positive integer N satisfying the following property:

(D) *For any intermediate field F of K_∞/K_0 satisfying $K_0 \subseteq_f F$, there exists a positive integer N_F such that $(F, N) \in D$ for any integer N satisfying $N \geq N_F$.*

Let b be an element of Λ_χ and $b_{F,N}$ the image of b for any intermediate field F of K_∞/K satisfying $K \subseteq_f F$ and any positive integer N. We say a sequence $(a_{F,N})_{(F,N)\in D}$ converge to $b = (b_{F,N}) \in \Lambda$ if and only if there exists a subset D' of D satisfying the condition (D) such that $a_{F,N} = b_{F,N}$ for any $(F, N) \in D'$. If a sequence $(a_{F,N})_{(F,N)\in D}$ converge to b, we write $\lim(a_{F,N}) := b$.

By induction on r, we shall prove that

$$\lim\left(\bar{\phi}^{\mathfrak{l}_{r+1}}(x_{\mathfrak{n}_r,\mathfrak{q}})_{F,N}\right)_{F,N} = \pm\delta_{\mathcal{I}}\delta_{\mathcal{J}}\delta_{\mathcal{C}}(\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}})^r \det M_r \in \Lambda_\chi.$$

First, we consider the equality

$$\delta_{\mathrm{fin}} \det M \cdot \bar{\phi}^{\mathfrak{l}_2}(x_{\mathcal{O}_K,q}) = \pm\delta_{\mathrm{fin}}\delta_{\mathcal{I}}\delta_{\mathcal{J}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}} \det M_1 \cdot \bar{\varphi}_{F,N}(\boldsymbol{\eta}_\chi).$$

Since the right hand side converges to

$$\pm\delta_{\mathrm{fin}}\delta_{\mathcal{I}}\delta_{\mathcal{J}}\delta_{\mathcal{C}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}} \det M_1 \cdot \det M$$

and $\delta_{\mathrm{fin}} \det M$ is non-zero element, we obtain

$$\lim\left(\bar{\phi}^{\mathfrak{l}_2}(x_{\mathcal{O}_K,\mathfrak{q}})_{F,N}\right)_{F,N} = \pm\delta_{\mathcal{I}}\delta_{\mathcal{J}}\delta_{\mathcal{C}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}} \det M_1.$$

(Note the sign $\pm$ does not depend on F, see Proposition 5.6).

Next, we assume

$$\lim\left(\bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n}_{r-1},\mathfrak{q}})_{F,N}\right)_{F,N} = \pm\delta_{\mathcal{I}}\delta_{\mathcal{J}}\delta_{\mathcal{C}}(\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}})^{r-1} \det M_{r-1}.$$

Then, the right hand side of

$$\delta_{\mathrm{fin}} \det M_{r-1} \cdot \bar{\phi}^{\mathfrak{l}_{r+1}}(x_{\mathfrak{n}_r,\mathfrak{q}}) = \pm\delta_{\mathrm{fin}}\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}} \det M_r \cdot \bar{\phi}^{\mathfrak{l}_r}(x_{\mathfrak{n}_{r-1},q})$$

converges to

$$\pm\delta_{\mathrm{fin}}\delta_{\mathcal{I}}\delta_{\mathcal{J}}\delta_{\mathcal{C}}(\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}})^r \det M_r \cdot \det M_{r-1}.$$

Since we take $\det M_{r-1} \neq 0$, we obtain

$$\lim\left(\bar{\phi}^{\mathfrak{l}_{r+1}}(x_{\mathfrak{n}_r,q})_{F,N}\right)_{F,N} = \pm\delta_{\mathcal{I}}\delta_{\mathcal{J}}\delta_{\mathcal{C}}(\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}})^r \det M_r.$$

By induction, in particular, we conclude $\left(\bar{\phi}^{\mathfrak{l}_{i+1}}(x_{\mathfrak{n},\mathfrak{q}})_{F,N}\right)$ converges to

$$\pm\delta_{\mathcal{I}}\delta_{\mathcal{J}}\delta_{\mathcal{C}}(\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}})^i \det M_i.$$

Since $(x_{\mathfrak{n},\mathfrak{q}})_{F,N}$ is contained in an R_N-submodule of $(F^\times/p^N)_\chi$ generated by

$$\bigcup_{\mathfrak{d}|\mathfrak{qn}} \mathcal{W}_{F,N,\chi}(\mathfrak{d})$$

with $\epsilon(\mathfrak{qn}) = i$, we have $\bar{\phi}^{\mathfrak{l}_{i+1}}(x_{\mathfrak{n},\mathfrak{q}})_{F,N} \in \mathfrak{C}_{i,F,N\chi}$ for any finite extension field F of K contained in K_∞ and any positive integer N. Hence we have

$$\delta_{\mathcal{I}}\delta_{\mathcal{J}}\delta_{\mathcal{C}}(\varepsilon_{\mathcal{I}}\varepsilon_{\mathcal{J}})^i \det M_i \in \mathfrak{C}_{i,\chi},$$

and this complete the proof of Theorem 5.1. □

6. Higher Fitting ideals for two-variable cases

Here, let us consider the two-variable case. We assume $\Gamma \simeq \mathbb{Z}_p^2$. Note that for any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ above p, the decomposition subgroup $D_{\mathfrak{p}}$ in $\mathcal{G}$ has finite index by the global class field theory. So, the height of the ideal $\mathcal{I}_{T,\chi}$ of Λ_χ is at least two if $\Gamma \simeq \mathbb{Z}_p^2$. In the two-variable cases, our main theorem is stated only in the following form, which is weaker than the results in the one-variable cases, Theorem 5.1.

Theorem 6.1. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. If K_0 contains μ_p, we assume $\chi \neq \omega$ and $\chi \neq \chi^{-1}\omega$. Assume $\Gamma \simeq \mathbb{Z}_p^2$ and one of the following:*

- *p splits completely in $K/\mathbb{Q}$;*
- *p does not split in $K/\mathbb{Q}$, and for the element $\mathfrak{p} \in T$, the character χ is non-trivial on $D_{\Delta,\mathfrak{p}}$.*

Then, the following holds:

(1) *If the character χ is non-trivial on $D_{\Delta,\mathfrak{p}}$ for any $\mathfrak{p} \in T$, we have*
$$\mathfrak{C}^{\mathrm{ell}}_{0,\chi} \subseteq \mathrm{Fitt}_{\Lambda_\chi,0}(X'_\chi).$$
(2) *For each $i \in \mathbb{Z}_{\geq 0}$, there exists a height-two ideal $J_{i,\chi}$ of Λ_χ satisfying*
$$J_{i,\chi}\,\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}^{\mathrm{ell}}_{i,\chi}.$$

Note that in the two-variable cases, we cannot give bounds for error factors $J_{i,\chi}\mathcal{I}^3_{T,\chi}$. Indeed, as we will see later, our result for the two-variable cases follow from the standard Euler system arguments for the proof of Iwasawa main conjecture, so it is not so new or strong.

Proof of Theorem 6.1. The first assertion is proved in Corollary 4.7, so it is sufficient to prove the second assertion. Note that X_χ is a finitely generated torsion Λ_χ-module, so we have a pseudo-isomorphism
$$\iota_X\colon \bigoplus_{i=1}^{r} \Lambda_\chi/f_i\Lambda_\chi \longrightarrow X_\chi,$$
where r is a positive integer, and f_i's are non-zero elements of Λ_χ satisfying $f_i \mid f_{i+1}$ for all i. By Example 2.10, it is sufficient to show that for any integer i satisfying $0 \leq i \leq r-1$, there exists a height-two ideal I_i of Λ_χ satisfying
$$(\prod_{j=1}^{r-i} f_j) \cdot I_i \subseteq \mathfrak{C}^{\mathrm{ell}}_{i,\chi}.$$

First, we set up the notation. Let $e_i \in \bigoplus_{i=1}^{r} \Lambda_\chi/f_i\Lambda_\chi$ be the element 1 in the i-th summand Λ_χ/f_i. For an intermediate field F of K_∞/K_0 which is finite over K_0 and for any $i \in \mathbb{Z}$ satisfying $0 \leq i \leq r$, we denote the image of e_i by the composite map
$$\bigoplus_{i=1}^{r} \Lambda_\chi/f_i\Lambda_\chi \xrightarrow{\iota_X} X_\chi \longrightarrow A_{F,\chi}$$

by $\mathfrak{c}_{i,F} \in A_{F,\chi}$.

Fix a homomorphism $\varphi \colon \mathcal{E}_{\infty,\chi} \longrightarrow \Lambda_\chi$ of Λ_χ-modules with pseudo-null cokernel. Let $\theta_\chi \in \Lambda_\chi$ be a generator of $\mathrm{char}_{\Lambda_\chi}(\mathcal{E}_{\infty,\chi}/\mathcal{C}_{\infty,\chi})$. We put

$$\mathcal{B} := \big(\mathcal{I}_{0,\chi} \cdot \mathrm{Coker}\, \iota_X\big) \cap \big(\mathcal{A}_\chi \cdot \mathcal{I}(\mathcal{C}_{\infty,\chi}; \varphi)\big),$$

where $\mathcal{I}_0$ and $\mathcal{A}$ are as in Proposition 2.1.

We denote the set of all continuous homomorphisms from Γ to the discrete group μ_{p^∞} by $\mathfrak{X}$. Note that any element $\rho \in \mathfrak{X}$ uniquely extends to a continuous ring homomorphism $\rho \colon \Lambda_\chi \longrightarrow \mathcal{O}_\chi[\mu_p^\infty]$. For any $f \in \Lambda_\chi$, we define a subset $\mathfrak{X}(f) \subseteq \mathfrak{X}$ and an ideal $\mathcal{I}(f)$ by

$$\begin{aligned} \mathfrak{X}(f) &= \{\rho \in \mathfrak{X} \mid \rho(f) = 0\}, \\ \mathcal{I}(f) &= \{g \in \Lambda_\chi \mid \rho(g) = 0 \text{ for any } \rho \in \mathfrak{X}(f)\}. \end{aligned}$$

If $I \subseteq \Lambda_\chi$ is a principal ideal generated by f, we define $\mathcal{I}(I) := \mathcal{I}(f)$. Note that if $\Gamma \simeq \mathbb{Z}_p^2$, then the ideal

$$\mathcal{I}(\mathrm{char}(X_\chi)) = \mathcal{I}(\theta_\chi)$$

is height-two ideal of Λ_χ. (See [Ru1] Proposition 7.11.) We define a height-two ideal $\mathcal{B}'$ of Λ_χ by

$$\mathcal{B}' := \mathcal{B} \cdot \Big(\mathcal{I}_{T,\chi} \cap \mathcal{I}(\theta_\chi)\Big),$$

and fix an element $\delta \in \mathcal{B}'$. Note that $\mathcal{B}' \subseteq \mathcal{I}(\mathcal{C}_{\infty,\chi}; \varphi)$, so there exists an element $\boldsymbol{\eta} = \{\eta_F\}_F \in \mathcal{C}_{\infty,\chi}$ satisfying $\varphi(\boldsymbol{\eta}_\chi) = \delta\theta_\chi$.

Let F be an intermediate field of K_∞/K_0 satisfying $K_0 \subseteq_f F$. We put $R_{F,\chi} := \mathbb{Z}_p[\mathrm{Gal}(F/K)]_\chi$. Let N be any positive integer satisfying

$$p^N \delta R_{F,\chi} \subseteq ([F : K_0]\# A_{F,\chi} \delta^{4r} \theta_\chi) R_{F,\chi}.$$

Let a be a map

$$(\mathcal{I}_K)^2 = \{\text{non-zero ideals of } \mathcal{O}_K\}^2 \longrightarrow R_{F,N,\chi}$$

satisfying the condition (R) in §3.3 for the elliptic unit $\eta := \eta_F \in C_F$.

By the argument in the proof of [Ru1] Theorem 8.3, let us construct a sequence $\{\mathfrak{l}_i\}_{i=1}^{r+1}$ of distinct prime ideals $\mathcal{O}_K$ prime to $\mathfrak{a}(\eta, a)$ satisfying the following properties:

- (S1) $\mathfrak{l}_i \in \mathcal{S}_N^{\mathrm{prime}}\big(F(\mathfrak{n}_{i-1})\big)$ for any i with $1 \leq i \leq r+1$, where we put $\mathfrak{n}_0 := \mathcal{O}_K$ and $\mathfrak{n}_i := \prod_{j=1}^i \mathfrak{l}_j$ for $i \geq 1$;
- (S2) the image of the ideal class of $\mathfrak{l}_i$ in $A_{F,\chi}$ coincides with $\mathfrak{c}_{i,F}$ for any i with $1 \leq i \leq r$;
- (S3) $\bar{\phi}^{\mathfrak{l}_1}_{F,N,\chi}(\kappa(\eta, a; \mathcal{O}_K)) = \delta^4\theta_\chi$;
- (S4) for any i with $2 \leq i \leq r+1$, we have

$$f_{r-i+2}\bar{\phi}^{\mathfrak{l}_i}_{F,N,\chi}\big((\kappa(\eta, a; \mathfrak{n}_{i-1})\big) = \bar{\phi}^{\mathfrak{l}_{i-1}}_{F,N,\chi}\big(\kappa(\eta, a; \mathfrak{n}_{i-2})\big).$$

First, we choose $\mathfrak{l}_1$. By Proposition 2.1, we have $\delta \in \mathcal{I}_{\mathcal{E}}$ and $\delta^2 \in \mathcal{J}_{\mathcal{E}}$. So, as in the proof of Proposition 4.6, there exists a homomorphism $\psi_\delta \colon (\mathcal{E}_F/p^N)\chi \longrightarrow R_N$ which makes the diagram

$$\begin{tikzcd}
(\mathcal{C}_{\infty,\chi})_F/p^N \arrow[r] \arrow[d] & (\mathcal{E}_{\infty,\chi})_F/p^N_\chi \arrow[r, "\delta^3\bar{\varphi}_{F,N,\chi}"] \arrow[d] & R_N \\
\mathcal{W}_{F,N,\chi}(\mathcal{O}_K) \arrow[r, hook] & (\mathcal{E}_F/p^N)_\chi \arrow[ur, dashed, "\psi_\delta"'] &
\end{tikzcd}$$

commute. By Proposition 3.15, we can take the prime ideal $\mathfrak{l}_1$ of $\mathcal{O}_K$ prime to $\mathfrak{a}(\eta_F, a)$ satisfying the following:

- the prime $\mathfrak{l}_1$ satisfies (S1) and (S2);
- $\bar{\phi}^{\mathfrak{l}_1}_{F,N,\chi}|_{(\mathcal{E}_F/p^N)_\chi} = \psi_\delta$.

Note that the second condition on $\mathfrak{l}_1$ implies the condition (S3).

Next we choose $\mathfrak{l}_i$ for $i \geq 2$ inductively on i. Let i be an integer satisfying $1 < i \leq r+1$, and suppose that we have chosen distinct prime ideals $\{\mathfrak{l}_j\}_{j=1}^{i-1}$ satisfying the condition (S1)-(S4). Now let us find a prime ideal $\mathfrak{l}_i$. Let W_{i-1} be the $R_{F,N,\chi}$-submodule of $(F^\times/p^N)_\chi$ generated by $\kappa(\eta, a; \mathfrak{n}_{i-1})_\chi$. By [Ru1] Lemma 8.2, there exists a homomorphism $\psi_{i-1} \colon W_{i-1} \longrightarrow R_{F,N,\chi}$ satisfying

$$f_{r-i+2}\psi_{i-1} = \delta^4[\cdot]^{\mathfrak{l}}_{F,N,\chi} \colon W_{i-1} \longrightarrow R_{F,N,\chi}.$$

(For details, see Lemma 8.2 and the arguments in the proof of Theorem 8.3 in [Ru1].) By Proposition 3.15, we can find the prime ideal $\mathfrak{l}_i$ of $\mathcal{O}_K$ prime to $\mathfrak{a}(\eta, a) \cdot n_{i-1}$ satisfying the following:

- the prime $\mathfrak{l}_i$ satisfies (S1);
- the prime $\mathfrak{l}_i$ satisfies (S2) if $i \leq r$;
- $\bar{\phi}^{\mathfrak{l}_1}_{F,N,\chi}|_{W_{i-1}} = \psi_{i-1}$.

By the second condition on $\mathfrak{l}_i$ and Proposition 3.9, we have

$$\begin{aligned}
f_{r-i+2}\bar{\phi}^{\mathfrak{l}_i}_{F,N,\chi}\big(\kappa(\eta, a; \mathfrak{n}_{i-1})\big) &= f_{r-i+2}\bar{\psi}_{i-1}\big(\kappa(\eta, a; \mathfrak{n}_{i-1})\big) \\
&= \big[\kappa(\eta, a; \mathfrak{n}_{i-1})\big]^{\mathfrak{l}_{i-1}}_{F,N,\chi} \\
&= \bar{\phi}^{\mathfrak{l}_{i-1}}_{F,N,\chi}\big(\kappa(\eta, a; \mathfrak{n}_{i-2})\big).
\end{aligned}$$

So, the sequence $\{\mathfrak{l}_j\}_{j=1}^{i-1}$ satisfies the condition (S4). By induction on i, we obtain the sequence $\{\mathfrak{l}_j\}_{j=1}^{r+1}$ satisfying the conditions (S1)-(S4).

Now, we shall vary F and N, and prove Theorem 6.1 by using the arguments in §5.3. For any intermediate field F of K_∞/K_0 satisfying $K_0 \subseteq_f F$, for any positive integer N and for any integer i satisfying $1 \leq i \leq r+1$, the element

$$\bar{\phi}^{\mathfrak{l}_i}\big(\kappa(\eta, a; \mathfrak{n}_{i-1})\big) \in R_{F,N,\chi}$$

is denoted by $\bar{\phi}^{\mathfrak{l}_i}(\kappa(\eta, a; \mathfrak{n}_{i-1}))_{F,N}$. By induction on i, we shall prove

$$\lim \left(\bar{\phi}^{\mathfrak{l}_i}(\kappa(\eta, a; \mathfrak{n}_{i-1}))_{F,N}\right)_{F,N} = \delta^4 \prod_{j=1}^{r-i+1} f_j \in \Lambda_\chi$$

in the sense of §5.3.

First, by the condition (S3), the sequence $\left(\bar{\phi}^{\mathfrak{l}_1}(\kappa(\eta, a; \mathfrak{n}_0))_{F,N}\right)_{F,N}$ converges to the element

$$\delta^4\theta_\chi = \delta^4 \prod_{j=1}^{r} f_j \in \Lambda_\chi.$$

Next, let i be an integer with $1 \le i \le r$, and assume

$$\lim \left(\bar{\phi}^{\mathfrak{l}_i}(\kappa(\eta, a; \mathfrak{n}_{i-1}))_{F,N}\right)_{F,N} = \delta^4 \prod_{j=1}^{r-i+1} f_j \in \Lambda_\chi.$$

Then, the condition (S4) implies

$$\lim \left(f_{r-i+1} \cdot \bar{\phi}^{\mathfrak{l}_{i+1}}(\kappa(\eta, a; \mathfrak{n}_{i}))_{F,N}\right)_{F,N} = \delta^4 \cdot \prod_{j=1}^{r-i+1} f_j.$$

Since $f_i \in \Lambda_\chi$ is non-zero element, we obtain

$$\lim \bar{\phi}^{\mathfrak{l}_{i+1}}(\kappa(\eta, a; \mathfrak{n}_{i}))_{F,N}\big)_{F,N} = \delta^4 \cdot \prod_{j=1}^{r-i} f_j.$$

Therefore, by induction on i, we conclude

$$\lim \left(\bar{\phi}^{\mathfrak{l}_i}(\kappa(\eta, a; \mathfrak{n}_{i-1}))_{F,N}\right)_{F,N} = \delta^4 \prod_{j=1}^{r-i+1} f_j \in \Lambda_\chi$$

for any i satisfying $1 \le i \le r$, and

$$\lim \left(\bar{\phi}^{\mathfrak{l}_{r+1}}(\kappa(\eta, a; \mathfrak{n}_{r}))_{F,N}\right)_{F,N} = \delta^4 \in \Lambda_\chi.$$

This implies

$$(\prod_{j=1}^{r-i} f_j) \cdot \mathcal{B}'' \subseteq \mathfrak{C}^{\mathrm{ell}}_{i,\chi}.$$

for any $i \in \mathbb{Z}_{\ge 0}$, where $\mathcal{B}''$ is the ideal of Λ_χ generated by $\{\delta^4 \mid \delta \in \mathcal{B}'\}$. Note that $\mathcal{B}''$ is a height-two ideal of Λ_χ, so this completes the proof. □

References

[dS] de Shalit, E., *Iwasawa Theory of Elliptic Curves with Complex Multiplication*, Perspect. Math. vol. 3 (1987).

[Fu] Fujii, S., *On a bound of λ and the vanishing of μ of $\mathbb{Z}_p$-extensions of an imaginary quadratic field*, J. Math. Soc. Japan **65** (2013), 277-298. MR 3034405. DOI 10.2969/jmsj/06510277.

[Gr] Greenberg, R., *Iwasawa theory—past and present*, Class field theory—its centenary and prospect, Adv. Stud. Pure Math., **30**, Math. Soc. Japan, Tokyo (2001), 335–385.

[Iw] Iwasawa, K., *A note on class numbers of algebraic number fields*, Abh. Math. Sem. Univ. Hamburg **20** (1956), 257–258.

[Ka] Kato, K., *p-adic Hodge theory and values of zeta functions of modular forms*, Astérisque **295** (2004), 117–290.

[Ku] Kurihara, M., *Refined Iwasawa theory and Kolyvagin systems of Gauss sum type*, Proceedings of the London Mathematical Society (3) **104** (2012), 728–769.

[Mi] Minardi, J., *Iwasawa modules for $\mathbb{Z}_p^d$-extensions of algebraic number fields*, Thesis, Washington University (1986).

[MR] Mazur, B., and Rubin, K., *Kolyvagin systems*, Memoirs of the AMS Vol **168**, Number **799** (2004).

[No] Northcott, D. G., *Finite free resolutions*, Cambridge Univ. press (1976).

[Oh] Ohshita, T., *The Euler system of cyclotomic units and higher Fitting ideals*, preprint.

[Oz] Ozaki, M., *Iwasawa invariants of $\mathbb{Z}_p$-extensions over an imaginary quadratic field*, Class field theory—its centenary and prospect, Adv. Stud. Pure Math., **30**, Math. Soc. Japan, Tokyo (2001), 387–399.

[Ru1] Rubin, K., *The "main conjectures" of Iwasawa theory for imaginary quadratic fields*, Invent. math. **103** (1991), 25–68.

[Ru2] Rubin, K., *More "Main Conjecture" for Imaginary Quadratic Fields*, in "Elliptic Curves and Related Topics" (edited by Kisilevsky, H. and Murty, R.), CRM proceedings Lecture Notes, Vol. 4 (1994), 23–28.

[Ru3] Rubin, K., *Euler systems*, Hermann Weyl lectures, Ann. of Math. Studies, vol. **147**, Princeton Univ. Press (2000)

[Se] Serre, J.-P., *Local Field*, Graduate Texts in mathematics **67**, Springer-Verlag (1979).

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

E-mail address: ohshita@math.kyoto-u.ac.jp