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〈特集　マルクス派最適成長論の展開と周辺問題〉

Growth-Path Calibration and Panel Estimation of the Marxian Optimal Growth Model

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Abstract

This paper attempts to develop the Marxian optimal growth model in the field of computational economics and econometrics. The Marxian optimal growth model, established by Yamashita and Ohnishi (2002), is based on the labor theory of value and considers labor as the ultimate production factor. Using the mathematical method of the neoclassical optimal growth theory, Yamashita and Ohnishi (2002) illustrated the optimal capital-labor ratio of the communist society. In this paper, we calculated the stable path to that optimal capital-labor ratio in a stochastic process, using computational programming. Further, with panel macro data of Japan, we conducted empirical studies for the basic model. We used instrumental variable estimation, three-stage least squares, and panel analysis. The statistically significant estimation results provide evidence for the reality of the basic version of the Marxian optimal growth model in the real economy.

Keywords: Growth-path, the Marxian optimal growth model, panel data of Japan

Ⅰ. Introduction

The Marxian optimal growth model was first developed by Yamashita and Ohnishi (2002), who aimed to reinterpret Marxian economics within the framework of neoclassical optimal growth theory. It formulates a roundabout production system. Labor is considered the only ultimate production factor; however, in a capitalist society, the roundabout production system could be more efficient than inputting labor directly. We initially use one part of social labor to produce capital goods, and subsequently use these capital goods and other part of social labor to produce the final good. The final steady state is obtained theoretically, using dynamic optimization, with parameters.

Yamashita and Ohnishi (2002) constructed the model and reinterpreted Marxian theories. Several studies that followed paid attention to the extensions of the basic model, such as the two-class model, decentralization model, etc. In this study, we turned back to the basic model and calculated the path to the final steady state. Since the growing paths of dynamic models cannot be

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solved analytically using the standard mathematical techniques, we choose computer programming of MATLAB (matrix lab oratory) for calculation and simulation. Note that some dynamic models, which usually have a steady state theoretically, do not have closed-form solutions if we substitute actual values. We examine the Marxian optimal growth model, attempting to solve the model with actual values and derive the growth path to the steady state. The model is successfully solved in the program, and a steady growth path has been obtained, which demonstrates the reality of the theoretical results.

Further, we conduct empirical studies for the basic model, using macro panel data from Japan. The regional data set covers the period from 1975 to 2005, with the 47 regions of Japan. We estimate fixed-effect models, using three-stage least squares (3SLS). Unlike other macro econometric models of Japan, our model is based on the Marxian optimal growth theories. Statistically significant estimates are obtained.

Finally, we solve the econometric model, using deterministic simulation and stochastic simulation. The results indicate a good performance of the model.

Ⅱ. The Basic Theoretical Model

The basic Marxian optimal growth model considers two production sectors: the consumption goods sector and the production goods sector. Assume that social labor, which is the only ultimate production factor, is L. It is divided into two parts: xL and $(1-x)L$, where $0 \leq x \leq 1$. xL of labor is directly put into the final consumption goods sector; $(1-x)L$ of labor is first used to produce production goods ΔK, and then those production goods are invested into the consumption good sector as the input of capital stock K. The basic version of the Marxian optimal growth model is as follows.

The consumption goods sector:

$$Y = AK^{\alpha}(xL)^{1-\alpha}.$$

The production goods sector:

$$\Delta K = B(1-x)L.$$

The model is solved in the same way as dynamic programming. In stationary equilibrium, capital accumulation stops, and the total social labor is used to produce final consumption goods. That is,

$$\Delta K = 0, \text{ and } x = 1.$$

The long-term equilibrium of capital, K^*, is given by

$$K^* = \frac{\alpha L}{(1-\alpha)\rho}.$$

Further, the optimal capital-labor ratio is obtained as follows:

$$\left(\frac{K}{L}\right)^* = \frac{\alpha}{(1-\alpha)\rho}.$$

Ⅲ. The Stochastic Growing Path of the Marxian Optimal Growth Model

The growing paths of the basic models cannot be solved analytically using the standard mathematical techniques, because the unknown is not simply a vector, but rather an entire function. Computer methods are usually used to solve those problems. In this paper, we select the MATLAB program in order to compute the stochastic path of the Marxian optimal growth model.

We consider an infinite horizon, discrete time, and continuous state dynamic decision model, using the collocation methods. Assume an i. i. d. lognormal $(0, \sigma^2)$ shock on production of K and a discount factor δ. For simplicity, we set $A=3$, $B=2$, $L=5$, $\delta=0.9$, $\sigma=0.1$, and $\alpha=0.2$.

The amount of capital invested in the consumption goods sector, which is produced in the production goods sector, is a controlled continuous-valued Markov process that is given as follows :

$$K_{t+1}=\delta K_t+\varepsilon_{t+1}\Delta K_t$$

The state variable is capital K, and $K\in(0, \infty)$.

The action variable is the ratio of labor used to produce consumption goods, x (in the theoretical model it is s), and $x\in(0.1)$.

The Bellman equation is given by

$$V(k)=\max\{AK^{\alpha}(xL)^{1-\alpha}+\delta E_{\varepsilon}(K+\varepsilon B(1-x)L)\}.$$

See appendices for details of the program.

The results are plotted in Figs. 1 and 2.

Capital K rises at a declining rate, converging asymptotically to the steady-state value. It indicates that the growth of capital is quick at the beginning of a capitalist society, then slows down gradually, and reaches a steady state in the end, as proved theoretically by the basic model. In the same way, the labor rate for the final consumption goods sector, x, begins from a level of 0.2, grows at a declining rate, and finally converges to the steady-state value of 1. This also confirms the theoretical model, which predicted that the labor used for capital goods production would become zero at the end of the capitalist society. As shown in Fig. 2, once x reaches the value of 1, all the labor is used to produce the final consumption goods, and capital accumulation stops.

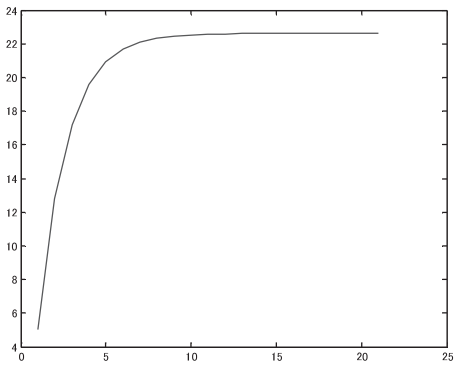

Fig. 1 Path of K

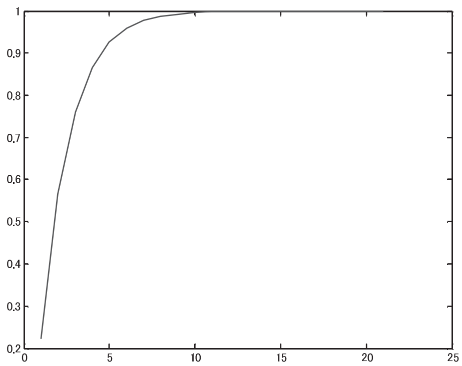

Fig. 2 Path of x

Ⅳ. Estimation of the Marxian Optimal Model

We estimate the econometric Marxian optimal model of Japan in this section. In the real economy, the return to scale could be constant, increasing, or decreasing. Therefore, we loosen some assumptions of the basic model and rewrite it as follows.

For the consumption goods sector:

$$Y = AK^{\alpha}(xL)^{\beta}$$

For the production goods sector:

$$\Delta K = B[(1-x)L]^{\phi}$$

We rewrite the estimation equations again as follows:

For the consumption goods sector:

$$\ln Y_{it} = \ln A + \alpha \ln K_{it} + \beta \ln(xL_{it}) + \varepsilon^{C}_{it},$$

and for the production goods sector:

$$\ln \Delta K_{it} = \ln B + \phi \ln(1-x)L_{it}) + \varepsilon^{P}_{it},$$

where i represents the cross sections of 47 regions, t denotes the time series, and ε^{C}_{it} and ε^{P}_{it} denote the error terms of consumption goods equation and production goods equation, respectively.

1. Data

We use the regional panel data of Japan, X_{it}, for the estimation. The cross sections i represent the 47 regions of Japan. The time series t covers the years from 1975 to 2005 (some of the data sets are for shorter periods). The data of labor inputs for the consumption goods sector, xL_{it}, include the number of workers for the agricultural industry, provisions industry, textile industry, etc. (a total of 28 industries). The data of labor inputs for the production goods sector, $(1-x)L_{it}$, include the number of workers for the machinery and appliances industry, iron and steel industry, nonferrous metal industry, construction industry, mining industry, etc. (a total of 13 industries). For the consumption goods data Y_{it}, we choose the total individual and government consumption (including stock). We use the data of total capital stock in the private sector as the surrogate variable of capital stock, K_{it}, because direct data of total capital stock are usually not available in official statistics. In the production goods sector, the data of production goods, ΔK_{it} (which is $\dot{K}$ in the basic model), comprise the gross capital formation data. Most of the data cover every year from 1975 to 2005. Note that since the worker number data at the industrial level are from the census that is held every five years, they are available only for the following years: 1975, 1980, 1985, 1990, 1995, 2000, and 2005.

The data list is as follows.

Table I. Data List

	Y_{it}	K_{it}	xL_{it}	$(1-x)L_{it}$	ΔK_{it}
Mean	5273294.	11491714	1028566.	232350.0	1113909.
Median	2980738.	6558643.	662039.0	144509.0	631841.0
Maximum	35464144	1.37E+08	5527044.	909434.0	14028066
Minimum	762053.0	829411.0	266585.0	41093.00	93849.00
Std. Dev.	6071976.	15506498	957343.3	214063.5	1571302.
Skewness	2.740709	4.217872	2.391728	1.698653	4.591477
Kurtosis	11.05954	27.25967	9.405875	4.951403	31.00599
Jarque-Bera	898.5620	6239.598	604.5453	145.1822	8216.097
Probability	0.000000	0.000000	0.000000	0.000000	0.000000
Sum	1.20E+09	2.61E+09	2.33E+08	52743443	2.53E+08
Sum Sq. Dev.	8.33E+15	5.43E+16	2.07E+14	1.04E+13	5.58E+14
Observations	227	227	227	227	227

2. Estimation Results

(1) Single-Equation Estimation

In this part of the paper, we estimate the equations of the Marxian optimal model separately. Our panel data with long cross sections and short time series avoid autocorrelation and serial correlation of the disturbances across periods of nonstationary time-series data, and therefore, the production functions could be estimated directly using least squared estimation. Further, we introduce instruments for a more general estimation of the instrumental variable (IV) method. The exogeneity and relevance of instruments have been examined.

We choose panel two-stage least squares with fixed effects for our single-equation estimation. The results of ordinary-least squares estimation (OLS) and generalized method of moments (GMM) are also listed for comparison.

a. Estimation of the Consumption Goods Sector

After data adjustment, the time series cover 4 years, and cross sections include the 47 regions of Japan. We choose $\ln K_{it}(-1)$ and $\ln(xL_{it})(-5)$ for instruments, which are uncorrelated with disturbance, but correlated with the independent variables, $\ln K_{it}$ and $\ln(xL_{it})$. The instrumental variable estimation (panel two-stage least squares) leads to the following results.

$$\ln Y_{it}=0.34 \ln K_{it}+0.31 \ln(xL_{it})+5.47+\theta_i+\gamma_t+\varepsilon_{it}^C$$

$$(2.44)^{***} \quad (1.96)^{**} \quad (2.25)^{***}$$

Adjusted R-squared: 0.99

Note that ***, **, and * denote statistical significant values at the 1%, 5%, and 10% levels, respectively; θ_i denotes the fixed effects of cross sections of region i; γ_t denotes the fixed effects

Table II. Comparison of Estimation Results for the Consumption Goods Sector

Variables	OLS	TSLS	GMM
$\ln K_{it}$	0.26^{***}	0.34^{***}	0.34^{***}
	(2.42)	(2.44)	(2.44)
$\ln xL_{it}$	0.53^{***}	0.31^{***}	0.31^{***}
	(4.14)	(1.96)	(1.96)

Notes: T-statistics are shown in parentheses; ***, **, and * denote statistically significant values at the 1%, 5%, and 10% levels, respectively.

of the time series in year t; and ε^{C}_{it} denotes the error term of region i in year t.

The signs of all the coefficients are consistent with the theoretical model and all of them are statistically significant. For comparison and confirmation, the results of other estimation methods are listed in the following table.

The results of OLS, two-stage least squares TSLS, and GMM are consistent with the theoretical model, and they all indicate diminishing returns to scale.

b. Estimation of the Production Goods Sector

We also use panel two-stage least squares for estimation; the time series are five years, and the cross sections are the 47 regions.

$$\ln \Delta K_{it}=2.22\ln((1-x)L_{it})-12.4+v_i+\varepsilon^{P}_{it}$$
$$(4.20)^{***} \qquad (-1.95)^{***}$$

Adjusted R-squared: 0.99

In this equation, ***, **, and * denote statistical significance at the 1%, 5%, and 10% levels, respectively, and v_i represent the fixed effects of cross sections.

The result also indicate a statistically significant positive coefficient for $\ln(1-x)L_{it}$, which is also consistent with the basic model.[1]

For comparison, the estimation results of OLS, two-stage least square (TSLS), and GMM are shown in the following table.

Table III. Comparison of Estimation Results for the Production Goods Sector

Variables	OLS	TSLS	GMM
$\ln(1-xL_{it})$	1.03^{***}	2.22^{***}	3.33^{***}
	(32.1)	(4.20)	(26.5)

Notes: The t-statistics are shown in parentheses; ***, **, and * denote statistically significant values at the 1%, 5%, and 10% levels, respectively.

In this section, we estimated the single-equation econometric model for the Marxian optimal model. Note that the two sectors, the consumption goods sector and the production goods sector, combine a roundabout production system. It is more common to estimate the multiple-equations model using system estimation, such as seemingly unrelated regressions (SUR), 3SLS, GMM of system estimation, and full information maximum likelihood (FIML).

1) For identification, it would be better to have at least one more independent variable. However, the formulation of the original theoretical model would change if we introduce other variables.

Table IV. System Estimation Results

System estimation (T-statistic in parentheses)
The consumption goods sector: $\ln Y_{it}=0.46 \ln K_{it}+0.30 \ln (xL_{it})+3.43+0.49dum_1+\cdots+0.16dum_{46}+\varepsilon_{it}^{CS}$ (20.6)*** (2.20)*** (2.23)***   Instrument list: $K_{it}(-1)$, $xL_{it}(-5)$ Adjusted R-squared: 0.99
The production goods sector: $\ln \Delta K_{it}=2.22 \ln ((1-x)L_{it}-11.18-2.27dum_1+\cdots-0.35dum_{46}+\varepsilon_{it}^{ps}$ (4.87)*** (−2.18)***   Instrument list: $(1-x)L_{it}(-5)$ Adjusted R-squared: 0.95

Notes: (1) $dum_1 \cdots dum_{46}$ are dummy variables of regions.
(2) ***, **, and * denote statistical significant values at the 1%, 5%, and 10% levels, respectively.

(2) System Estimation (Multiple-Equations Estimation)

In this part of the paper, we estimate the Marxian optimal growth model as a system by taking the interaction of all the variables into account.

The estimation system is

$$\ln Y_{it}=\ln A+\alpha \ln K_{it}+\beta \ln (xL_{it})+\varepsilon_{it}^{CS},$$
$$\ln \Delta K_{it}=\ln B+\phi \ln ((1-x)L_{it})+\varepsilon_{it}^{PS}.$$

ε_{it}^{CS} and ε_{it}^{PS} are the error terms of consumption goods equation and production goods equation in the system, respectively.

For a more general estimation, we introduce instruments. Unbiasedness and consistency are secured even if there are correlations between K_{it}, xL_{it}, $(1-x)L_{it}$, and their disturbances. We choose 3SLS for the estimation. The time series covers the period from 1980 to 1995, including 186 observations. We use instruments of $K_{it}(-1)$, $xL_{it}(-5)$, and $(1-x)L_{it}(-5)$, which are the previous terms of available observations.

The results are reported in Table IV.

Table V. Comparison of system estimation results

Equations	Variables	SUR	GMM	3SLS	FIML
$\ln Y_{it}$	$\ln K_{it}$	0.47***	0.46***	0.46***	0.50***
		(24.6)	(18.8)	(20.6)	(16.2)
	$\ln (xL_{it})$	0.50***	0.30*	0.30***	0.27*
		(4.29)	(1.62)	(2.20)	(1.61)
$\ln \Delta K_{it}$	$(1-x)L_{it}$	2.95***	3.33***	2.22***	3.26***
		(10.18)	(5.18)	(4.87)	(5.83)

Notes: The values in parentheses are t-statistics for SUR, GMM, and 3SLS, and z-statistics for FIML.

Other estimation methods are also operated. The results are reported in the following table. Similar results are obtained using the other estimation methods. Robustness of the econometric model is confirmed.

3. The Econometric Marxian Optimal Growth of Japan

We use the estimation results of 3SLS from the previous part of the paper to introduce the econometric Marxian optimal growth of Japan as follows

The consumption goods sector:

$$Y_{it}=30.9(xL)_{it}^{0.30}K_{it}^{0.46}.$$

The production goods sector:

$$\Delta K_{it}=1.39\times10^{-5}[(1-x)L]_{it}^{2.22}.$$

V. Simulations

We examine the performance of the econometric Marxian optimal growth of Japan in this section. The model is solved using both deterministic simulation and stochastic simulation.

1. Deterministic Simulation

Fig. 3 reports the deterministic solutions, $\ln \Delta Y_{it}^{Deter}$, for the consumption goods sector. Fig. 4 shows the actual values of $\ln Y_{it}$ for comparison. The distribution of the solutions is rather close to the actual values, with a mean of 15.061 and standard deviation of 0.84, the actual values of which are 15.069 and 0.85, respectively. Further, we compare every solution for all the observations by calculating the ratio of Y_{it}/Y_{it}^{Deter}. The results are shown in Fig. 5. All the ratios range from 0.8 to 1.2, and are mostly concentrated around 1.

For solution of the production goods sector, as shown in Figs. 6 and 7, the solutions, $\ln \Delta K_{it}^{Deter}$, are also distributed close to the actual values, with a mean of 14.271 and standard deviation of 0.742, the actual values of which are 14.228 and 0.788, respectively. Fig. 8 reports the ratios of

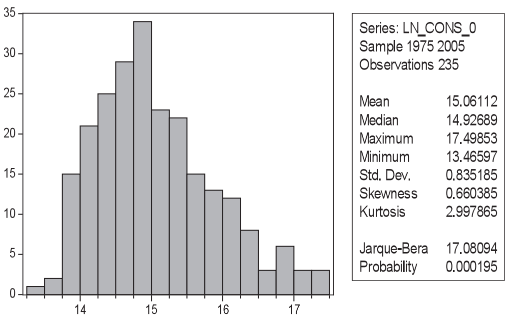

Fig. 3 Results of $\ln Y_{it}^{Deter}$

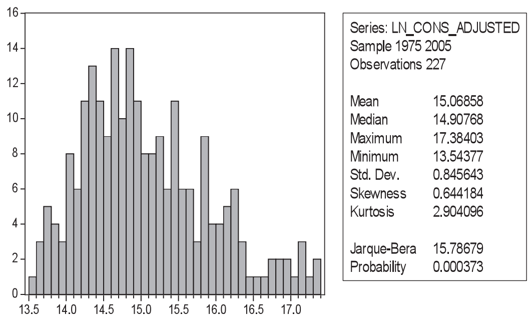

Fig. 4 Actual values of $\ln Y_{it}$

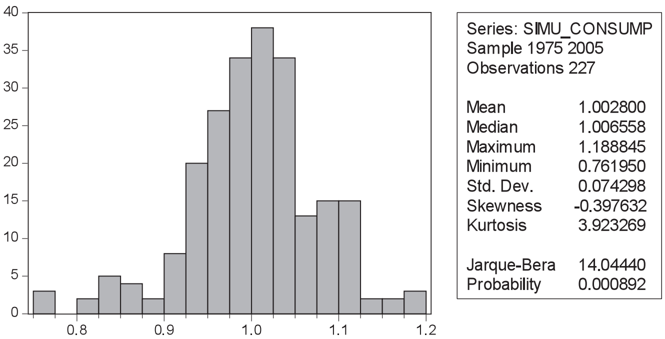

Fig. 5 Y_{it}/Y_{it}^{Deter}

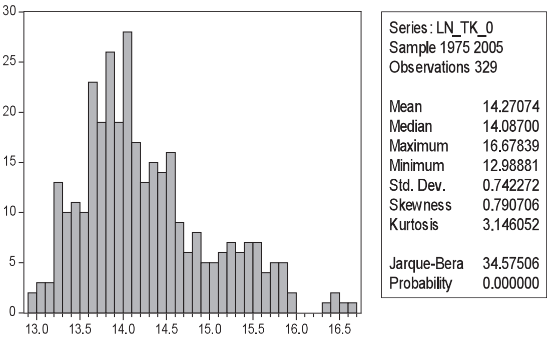

Fig. 6 Results of $\ln \Delta K_{it}^{Deter}$

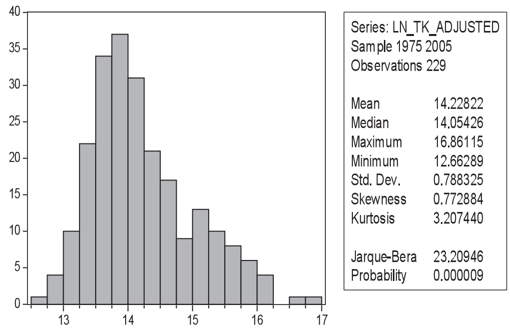

Fig. 7 Results of $\ln \Delta K_{it}$

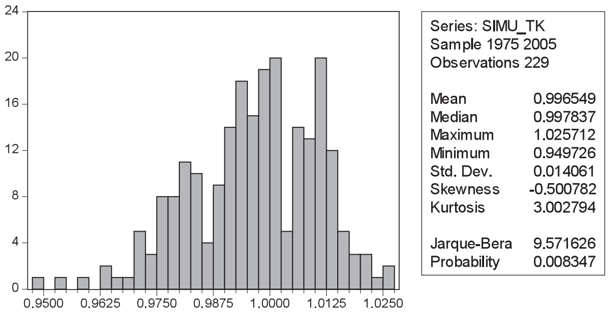

Fig. 8 Results of $\ln \Delta K_{it}/\ln K_{it}^{Deter}$

$\ln \Delta K_{it}^{Deter}$ to $\ln \Delta K_{it}$, which also concentrate around the value of 1. Accordingly, it is indicated that the model performs well by deterministic solutions.

2. Stochastic Simulations

There is uncertainty regarding error terms and coefficients in the estimation. In this part, we consider uncertainty and conduct stochastic simulations in order to examine our model.

With the help of a computer, in stochastic simulation I, we repeatedly solve our model 1,000 times for different draws of the stochastic components of error terms; in stochastic simulation II,

Table VI. Stochastic Simulation Results

	Actual	Stochastic Simulation I		Stochastic Simulation II	
		$Mean^{1000}$	Std.^{1000}	$Mean^{1000}$	Std.^{1000}
$\ln Y_{it}$	15.069 (0.846)	15.064 (0.833)	0.075 (0.0017)	15.065 (0.834)	0.082 (0.0021)
$\ln \Delta K_{it}$	14.228 (0.788)	14.275 (0.766)	0.198 (0.0045)	14.275 (0.767)	0.218 (0.0068)

the solution includes both uncertainty of error terms and coefficients, which is also repeated 1,000 times.

The results are reported in Table VI. $Meay^{1000}$ and Std.^{1000} are the mean and standard deviation of 1,000 results for every observation, respectively. Owing to space limitations, the results of all 200 observations cannot be listed. We report the mean of the 200 observations in the table, with standard deviations in parentheses.

Comparing with the first column of the actual values, the results also indicate a good performance of our model.

Ⅵ. Conclusions

Our study developed the Marxian optimal growth model in the field of computational economics and econometrics.

We obtained the steady path of the Marxian optimal growth model. Both capital accumulation and labor ratio converge asymptotically to the steady-state value, which has theoretically been proven in previous studies. Based on the materialistic interpretation of history, the steady state is actually the end of the capitalist society (Ohnishi, 2006); thus, capital accumulation and labor reallocation could describe the entire history of capitalism, which began with primitive accumulation — initially, increasing rapidly, and subsequently, slowing down gradually — and ended with the disappearance of the capitalist society.

Using macro panel data of Japan, we conducted empirical studies for the basic model. Using strict estimations based on the instrumental variable method and multiple equations, we obtained statistically significant results. Further, the econometric model also performed rather well using deterministic simulations and stochastic simulations. These empirical studies indicated that the Marxian optimal growth model not only developed the Marxian theory, but also applied it to the real economy. The econometric macro model of Japan, which is based on the Marxian theory, is also developed in our study.

Owing to time constraints, we have not been able to use the econometric Marxian optimal growth model for prediction. We leave this as a potential subject for future study.

APPENDIX

In order to obtain the path of capital growth and labor ratio changes, we wrote the MATLAB program. The program, using CompEcon Toolbox routines, is as follows.

First, our model was as follows

```
function [out1, out2, out3] = growth1 (flag, k, x, e, alpha, a, b, l)
switch flag
    case 'b';
        out1 = zeros (size (k));
        out2 = k;
    case'f';
        out1 = a.*k.^alpha.*x.^(1-alpha).*l.^(1-alpha);
        out2 = (1-alpha).*a.*k.^alpha.*x.^(-alpha).*l.^(1-alpha);
        out3 = (1-alpha).*(-alpha).*a.*k.^alpha.*x.^(-alpha-1).*l.^(1-alpha);
    case'g';
        out1 = k + e.*b.*(1-x).*l;
        out2 = -e.*b.*l.*ones (size (k));
        out3 = zeros (size (k));
end
```

We ran our program as follows:

The model parameters are

```
alpha = .2,
a = 3,
b = 2,
l = 5,
delta = .9,
sigma = 0.1.
```

We used a 10-function Chebychev polynomial basis on the interval [5,100].

```
n = 10,
smin = 5,
smax = 100,
fspace = fundefn ('cheb', n, smin, smax),
snodes = funnode (fspace),
```

wherein fundefn is a structured variable that contains the information that is required to define the approximation basis, and snodes is the n × 1 vector of standard collocation nodes associated with the basis.

At last, we solve the model as follows:

```
estar = exp (sigma^2/2);
xstar = 1 ;
```

```
nshocks = 3 ;
[e, w] = qnwlogn (nshocks, 0, sigma^2);
model. func = 'growth1',
model. discount = delta,
model. e = e,
model. w = w,
model. params = {alpha a b l},

kstar = (alpha.*l.*b)/(((1 - alpha).*delta)),
[vlg, xlg] = lqapprox (model, snodes, kstar, xstar, estar),
[c, k, v, x, resid] = dpsolve (model, fspace, snodes, vlg, xlg),

nyrs = 20,
npath = 2000,
sinit = 5*ones (npath, 1),
[kpath, xpath] = dpsimul (model, sinit, nyrs, k, x),

plot (mean (xpath)),
plot (mean (kpath)).
```

This is the MATLAB program that was written by us in order to calculate the paths.

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