WEGNER ESTIMATES FOR GENERALIZED ALLOY TYPE POTENTIALS

JYUNICHI TAKAHARA

Introduction

Let V^ω be a random field on $\mathbf{R^d}$ defined on a probability space $(\Omega, \mathbf{P})$ and H^ω_Λ be the restriction of the Schrödinger operator $-\Delta + V^\omega$ to a cube Λ by a boundary condition.

Then, the estimate

$$\mathbf{P}\{\mathrm{dist}(\sigma(H^\omega_\Lambda), E_0) \leq \eta\} \leq C|\Lambda|^A \eta^B,$$

with $A \geq 1$ and $B \leq 1$ is called a Wegner estimate, where $E_0 \in \mathbf{R}$, $\eta > 0$, $\sigma(H^\omega_\Lambda)$ is the spectrum of H^ω_Λ and $|\Lambda|$ is the volume of a cube Λ. From the fact that $\mathbf{P}\{\mathrm{dist}(\sigma(H^\omega_\Lambda), E_0) \leq \eta\}$ is dominated by $\mathbf{E}[\sharp\{\sigma(H^\omega_\Lambda) \cap [E_0 - \eta, E_0 + \eta]\}]$ and $\mathbf{E}[\sharp\{\sigma(H^\omega_\Lambda) \cap [E_0 - \eta, E_0 + \eta]\}]/|\Lambda|$ converges to the density of states (DS) as $|\Lambda| \to \infty$, we expect that $A = B = 1$ are the best exponents.

Wegner firstly obtained this estimate for the Anderson model [15]. After that the estimate with general exponents A and B is applied to the proof of the Anderson localization [3, 4, 13].

There had been many prior results on a Wegner estimate for multidimensional and continuous Schrödinger operators with Anderson-type random potentials $V^\omega(x) = \sum_{a \in \mathbf{Z^d}} f^\omega_a u(x-a)$ consisting of single site potentials around fixed positions on the lattice $\mathbf{Z^d}$ [12, 1, 8, 13]. Among them, Combes, Hislop and Nakamura obtained a bound with $A = 1$ and $B < 1$ which is arbitrarily close to 1 for the Schrödinger operators with the Anderson-type positive potentials by the method of the spectral shift function [2]. Kirsch and Veselić used this method to prove a bound with $A = 1$ and B arbitrarily close to 1 for the negative potentials $V^\omega(x) = \sum_{a \in S} f^\omega_a u(x - a)$ called generalized alloy type potentials where the positions S of the impurities were fixed randomly on $\mathbf{R}^d$ [9]. With the method of [9], the author proved a bound with

A and B arbitrarily close to 1 at negative energies for Schrödinger operators with negative potentials $V^\omega(x) = \sum_{a\in\Xi^{\omega_2}} f_a^{\omega_1} u(x-a)$ where Ξ^{ω_2} is a Poisson point process independent of real random variables $f_a^{\omega_1}$ in [14]. We recall the work in the first part of this paper.

These results on a Wegner estimate were obtained by the property that the eigenvalues monotonously depended on the coupling constants. Therefore, this method needs the property that a single-site potential u has the definite sign. We next drop this condition.

In 2002, Hislop and Klopp obtained a bound with $A = 1$ and B arbitrarily close to 1 for the nonsign definite Anderson-type potentials using Klopp's vector field method [11] and the spectral shift function method [7]. Applying their method and [9], we obtain a Wegner estimate for the Schödinger operator with the random potential $V^\omega(x) = \sum_{a\in\Xi^{\omega_2}} f_a^{\omega_1} u(x-a)$ in the second part of this paper, where u takes both positive and negative values. In our estimates, the exponents A and B are only arbitrarily close to 1. However this is the best result up to now.

In 2007, Germinet, Hislop and Klein proved the localization for a Schrödinger operator with the random potential $V^\omega(x) = \sum_{a\in\Xi^\omega} u(x-a)$ defined by the Poisson point process Ξ^ω, which is more difficult problem than ours. The inequality corresponding to the Wegner estimate for their proof of the localization is restricted to $\eta \sim |\Lambda|^{-|\Lambda|^\rho}$ with $\rho > 0$. Therefore their exponent A depends on $|\Lambda|$ [5, 6].

References

[1] J.M. Combes and P.D.Hislop: *Localization for some continuous random Hamiltonian in d-dimensions,* J.Funct.Anal. **124**(1994), 149-180.

[2] J.M. Combes, P.D. Hislop and S. Nakamura: *The L^p-theory of the spectral shift function,the Wegner estimate,and the integrated density of states for some random operators,* Comm.Math.Phys. **218**(2001), 113-130.

[3] R. Carmona and J. Lacroix: Spectral theory of random Schrödinger operators, Birkhäuser, Boston, 1990.

[4] A. Figotin and L.A. Pastur: Spectra of random and almost-periodic operators, Splinger-Verlag, Berlin, 1992.

[5] F. Germinet, P.D. Hislop and A. Klein: *Localization for Schrödinger operators with Poisson random potential,* J.Eur.Math.Soc. **9**(2007), no.3, 577-607.

[6] F. Germinet, P.D. Hislop and A. Klein: *Localization at low energies for attractive Poisson random Schrödinger operators,* Probability and mathematical physics, CRM Proc. Lecture Notes, **42**, 153-165, Amer.Math.Soc.,Providedence,RI,2007.

[7] P.D. Hislop and F. Klopp: *The Integrated density of states for some random operators with nonsign definite potentials,* J.Func.Anal. **195**(2002), 12-47.

[8] W. Kirsch: *Wegner estimates and localization for alloy-type potentials,* Math.Z. **221**(1996), 507-512.

[9] W. Kirsch and I. Veselić: *Wegner estimate for sparse and other generalized alloy type potentials,* Proc.Indian Acad.Sci **112**(2002), 131–146.

[10] A. Klein, A. Koines and M. Seifert: *Generalized eigenfunctions for waves in inhomogeneous media,* J.Funct.Anal **190**(2002), 255–291.

[11] F. Klopp: *Localization for some continuous random Schrödinger operators,* Comm.Math.Phys. **167**(1995), 553-569.

[12] S. Kotani and B. Simon: *Localization in general one dimensional systems,* Comm.Math.Phys. **112**(1987), 103-120.

[13] P. Stollmann: Caught by disorder, Bound states in random media, Birkhäuser, Boston, 2001.

[14] J.Takahara: *Wegner estimate for a generalized alloy type potential,* J.Math.Kyoto.Univ. **49**(2009) 255–265.

[15] F.Wegner: *Bounds on the density of states in disordered systems,* Z.Phys.B, **44**(1981), 9-15.

Contents

Part 1

Wegner estimate for a generalized alloy type potential

Abstract

W.Kirsch and I.Veselić proved a generalized Wegner estimate for Schrödinger operators with generalized alloy type potentials at negative energies for each fixed position of impurities. In this paper, a similar estimate is proven treating also the position of impurities as random variables.

1.1. Introduction

In this paper, we will give a Wegner estimate and state the properties of the exponential localization of eigenfunctions for a Schrödinger operator,

$$H^{\omega} := -\Delta + V^{\omega}(x) \text{ with } V^{\omega}(x) := -\sum_{i\in\mathbf{N}} f_i^{\omega_1} u(x-\xi_i^{\omega_2}), \tag{1.1.1}$$

where u is a nonnegative continuous function with a compact support, $\{f_i^{\omega_1}, i \in \mathbf{N},\ \omega_1 \in \Omega_1\}$ are independently and identically distributed random variables obeying the uniform distribution on the interval $[0,\ 1]$, and $\{\xi_i^{\omega_2},\ i \in \mathbf{N}, \omega_2 \in \Omega_2\}$ is a Poisson point process independent of $\{f_i^{\omega_1}\}$ with the Lebesgue measure as its intensity. We write $\omega = (\omega_1, \omega_2)$. For any $a \in \mathbf{R}^d$ and $L > 0$, we set $\Lambda_L(a) = \{x \in \mathbf{R}^d : |x_i - a_i| < L/2 \text{ for } 1 \le i \le d\}$ and $\Lambda_L := \Lambda_L(0)$. For simplicity we assume $\operatorname{supp} u \subset \Lambda_1$.

The investigations of the localization of eigenfunctions of the Schrödinger operators H^{ω} were begun by P.W.Anderson [1]. It has been discussed mainly about potential energies called alloy type potentials as $\sum_{i\in\mathbf{Z}^d} f_i^{\omega_1} u(x-i)$. Recently, Kirsch and Veselić proved a general form of the Wegner estimate used to prove the localization for the potential energies called generalized alloy type potentials [8]. Since positions of impurities in the lattice are considered as random variables, these potential energies are regarded as a liquid crystal type. In [8], they proved the Wegner estimate for each fixed position for impurities. In this paper, we will prove the Wegner estimate treating also the position of impurities as random variables for a typical example of the generalized alloy type potential energy defined in (1.1.1). Based on this estimate we will next use the variable energy multiscale analysis [6] to obtain the results on the Anderson localization as the strong Hilbert-Schmidt dynamical localization.

The main theorem of this paper is the following:

Theorem 1.1.1. *For any $L > 0$, let H_L^ω be the restriction of the operator H^ω to $L^2(\Lambda_L)$ under the Dirichlet boundary condition and P_L^ω be its spectral projection. Then we have the following: for any $0 < \beta, \varepsilon < 1$ and $\delta > 0$, there exists $C_{\beta,\varepsilon,\delta} > 0$ such that*

$$\mathbf{E}[\mathrm{Tr}\, P_L^\omega((E-\eta, E+\eta))] \le C_{\beta,\varepsilon,\delta} L^{(1+\beta)d} \eta^\varepsilon,$$

for any $L > 1$, $E < 0$ and $\eta > 0$ satisfying $E + 2\eta \le -\delta$.

Remark 1. We can prove Theorem 1.1.1 under weaker assumptions on $\{f_i^{\omega_1}, i \in \mathbf{N}, \omega_1 \in \Omega_1\}$. For example, it is enough that the conditional probability of each $f_i^{\omega_1}$ with respect to other random variables has a bounded density as in Assumption 1 (iv) in [8]. However for the proof of localization by the multiscale analysis, we need extra assumptions on the correlations of $\{f_i^{\omega_1}, i \in \mathbf{N}, \omega_1 \in \Omega_1\}$. For example, it is enough that they are independently and identically distributed. These extensions are straightforward. Therefore we choose our assumption for the sake of simplicity.

The organization of this paper is as follows. In Section 2 we give the basic properties of the Schrödinger operators. In Section 3 we prove the main theorem. In Section 4 we modify Germinet and

Klein's theory on the multiscale analysis. Finally, in Section 5 we state the results of Germinet and Klein's theory on the strong dynamical localization.

1.2. The basic properties of the Schrödinger operators H^ω

In this section, we will prove the essential self-adjointness of the Schrödinger operators H^ω based on Kirsch-Veselić [8].

Lemma 1.2.1. *For any $j \in \mathbf{Z}^d$, let $\mathcal{L}_\omega(j) = \sharp\{i \in \mathbf{N} | \xi_i^\omega \in \Lambda(j)\}$, where $\Lambda(j) = \Lambda_1(j)$. Then, for almost all ω, there exists a finite constant $C(\omega)$ such that*

$$\mathcal{L}_\omega(j) \leq \|j\|_\infty^2 + C(\omega) \ for\ any\ j \in \mathbf{Z}^d, \tag{1.2.1}$$

where $\|j\|_\infty := \sup_{1 \leq i \leq d} \ |j_i| \ for\ j = (j_i)_{i=1}^d$.

Proof. We require only showing that

$$\mathbf{P}\{\text{for infinitely many } j \in \mathbf{Z}^d, \mathcal{L}_\omega(j) > \|j\|_\infty^2\} = 0. \tag{1.2.2}$$

In fact, when (1.2.2) holds, there exists some $\Omega' \subset \Omega$ such that $\mathbf{P}(\Omega')$ $= 1$, and every $\omega \in \Omega'$ has a finite set $\Gamma(\omega) \subset \mathbf{Z}^d$ satisfying

$$\mathcal{L}_\omega(j) \leq \|j\|_\infty^2, \text{ for all } j \in \mathbf{Z}^d \setminus \Gamma(\omega).$$

Then, (1.2.1) holds with $C(\omega) = \sharp\{i \in \mathbf{N} | \xi_i^\omega \in \bigcup_{j \in \Gamma(\omega)} \Lambda(j)\}$.

On the other hand, by Chebyshev's inequality, we have

$$\sum_{j \in \mathbf{Z}^d} e^{-\|j\|_\infty^2} \mathbf{E}(e^{\mathcal{L}_\omega(j)}) \geq \sum_{j \in \mathbf{Z}^d} e^{-\|j\|_\infty^2} \cdot e^{\|j\|_\infty^2} \cdot \mathbf{P}[\omega | \mathcal{L}_\omega(j) > \|j\|_\infty^2]$$

$$= \sum_{j \in \mathbf{Z}^d} \mathbf{P}[\omega | \mathcal{L}_\omega(j) > \|j\|_\infty^2].$$

Since $\mathbf{E}(e^{\mathcal{L}_\omega(j)}) = e^{(e-1)|\Lambda(j)|} = e^{e-1}$, the left hand side is dominated by $\sum_{j \in \mathbf{Z}^d} \exp(-\|j\|_\infty^2 + e - 1)$, which is finite.

Therefore, we have $\sum_{j\in\mathbf{Z}^d}\mathbf{P}[\omega|\mathcal{L}_\omega(j) > \|j\|_\infty^2] < \infty$, from which we have (1.2.2).

□

Proposition 1.2.1. *For almost all ω, the Schrödinger operator H^ω is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

Proof. By Lemma 1.2.1, for almost all ω, we have finite constants $C_1,\ C_2(\omega)$ such that

$$V_\omega \geq -C_1\|x\|_\infty^2 - C_2(\omega)$$

(cf. [8]). Consequently, by Faris-Lavine theorem [10], H^ω is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.

□

In the rest of the paper we denote the unique self-adjoint extension by the same symbol H^ω .

By Proposition 1.2.1 and Proposition V.3.1. in [3], the measurability of the self-adjoint operator H^ω in ω is obtained. Moreover $\{H^\omega\}_{\omega\in\Omega}$ is an ergodic family of self-adjoint operators. Therefore, by Theorem (5.34) in [5], the spectrum $\sigma(H^\omega)$ satisfies that $\sigma(H^\omega) = \mathbf{R}$ for almost all ω.

1.3. The proof of the main theorem

In this section, we will prove the Wegner estimate, Theorem 1.1.1, using the method in [8] and the theory of the spectral shift function.

By the method in [8], we have

$$\begin{aligned} &\mathbf{E}^{\omega_1}[\mathrm{Tr}P_L^\omega((E-\eta,E+\eta))] \qquad (1.3.1)\\ &\leq \mathbf{E}^{\omega_1}\{\sum_{j\in\Lambda_L^+}\frac{1}{\delta}\int_{-3\eta/2}^{3\eta/2} dt\,\mathrm{Tr}[\rho(H_0^{L,j}-E+t)-\rho(H_1^{L,j}-E+t)]\}, \end{aligned}$$

In (1.3.1), $\mathbf{E}^{\omega_1}$ is the expectation with respect to the randomness of ω_1, $\Lambda_L^+ = \{k \in \mathbf{N} | \Lambda_L \cap \mathrm{supp}\ u_k(\cdot - \xi_k^{\omega_2}) \neq \emptyset\}$, and ρ is a smooth monotone increasing function satisfying $0 \leq \rho \leq 1$, $\rho(t) = 0$ for $t \leq -\eta/2$, $\rho(t) = 1$ for $t \geq \eta/2$ and ρ'/η^{-1} is bounded. Moreover $H_0^{L,j}$ and $H_1^{L,j}$ are the operators obteined by replacing $f_j^{\omega_1}$ by 0 and 1, respectively, in the definition of H_L^ω.

Now we use the following proposition on spectral shift functions([2] Theorem 2.1, [14] Chapter 8 §3 Theorem 3 and Theorem 6) :

Proposition 1.3.1. *Let A_1 and A_0 be self-adjoint operators such that $A_1 - A_0 \in \mathcal{I}_{1/p}$ for $p > 1$, where $\mathcal{I}_{1/p}$ is the family of compact operators of the super trace class, which we define as follows: we say that $A \in \mathcal{I}_{1/p}$ if for some $p > 1$, $|||A|||_{1/p} := \big(\sum_j \mu_j(A)^{1/p}\big)^p < \infty$, where $\mu_j(A)$ denotes the j-th singular value of A.*

Then, there exists some $\pi(\cdot; A_1, A_0) \in L^p(\mathbf{R})$ such that for $\phi \in C^\infty(\Gamma)$ where $\Gamma \subset \mathbf{R}$: a compact interval which contains $\sigma(A_0)$ and $\sigma(A_1)$,

$$\mathrm{Tr}[\phi(A_1) - \phi(A_0)] = \int_\Gamma \pi(\lambda; A_1, A_0)\phi'(\lambda)d\lambda,$$

and

$$\|\pi\|_p \leq |||A_1 - A_0|||_{1/p}^{1/p}.$$

We set $A_1 = (H_1^{L,j} + M_{\omega_2})^{-\ell}$ and $A_0 = (H_0^{L,j} + M_{\omega_2})^{-\ell}$ where $M_{\omega_2} = 2\sup_{x\in\Lambda_L} \sum_{i\in\mathbf{N}} u(x - \xi_i^{\omega_2}) + 1$. Then, by Proposition 5.1 in [2], $A_1 - A_0 \in \mathcal{I}_{1/p}$ for any $p > 1$ and $2\mathbf{N} + 1 \ni \ell > dp/2 + 2$. Moreover, for any $J \in C_o^\infty(\mathbf{R}^d)$, $|||J \times A_k^{1/\ell}|||_{\ell/p} \leq |||J \times (-\Delta + 1)^{-1}|||_{\ell/p} < \infty$ (cf. [11] Theorem 2.13 and Theorem 4.1) and the operator norms of $A_k^{1/\ell}, (\partial/\partial x^i)A_k^{1/\ell}, A_k^{1/\ell}(\partial/\partial x^i)$ and $(\partial/\partial x^i)A_k^{1/\ell}(\partial/\partial x^j)$ are bounded by $1 (k = 0$ or 1 , $1 \leq i, j \leq d)$.

Therefore $|||A_1 - A_0|||_{1/p} \le C$, where C is independent of ω_2 and L.

Then noting $\sigma(A_1)$ and $\sigma(A_0) \subset [0,1]$ we have

$$\begin{aligned}
&\mathrm{Tr}[\rho(H_0^{L,j} - E + t) - \rho(H_1^{L,j} - E + t)] \\
&= \mathrm{Tr}[\mu(A_0) - \mu(A_1)] \\
&= \int_{[0,1]} \frac{\partial\mu(\lambda)}{\partial\lambda}\pi(\lambda)d\lambda \\
&\le C\,\Big(\int_{[0,1]} |\frac{\partial\mu}{\partial\lambda}(\lambda)|^{p'}d\lambda\Big)^{1/p'},
\end{aligned} \tag{1.3.2}$$

where p' is $1/p + 1/p' = 1$, $\mu(\lambda) = \rho((1/\lambda)^{1/\ell} - M_{\omega_2} - E + t)$ and π is the spectral shift function for A_1 and A_0. By changing the variable as $\lambda = \gamma^{-\ell}$ we can show that the right hand side of (1.3.2) is dominated by

$$\sup_{-\eta/2 \le \gamma - M_{\omega_2} - E + t \le \eta/2} |\gamma|^{(1+\ell)(p'-1)/p'}[\int_{\mathbf{R}} |\rho'(\gamma)|^{p'}d\gamma]^{1/p'}. \tag{1.3.3}$$

We may assume that $|E| \le M_{\omega_2}$. In fact if

$$-\sup_{x\in\Lambda_L} |V^\omega(x)| > E + \eta, \tag{1.3.4}$$

it follows $[E-\eta, E+\eta] \subset \sigma(H_L^\omega)^c$. Since $2\sup_{x\in\Lambda_L} |V^\omega(x)| \le M_{\omega_2}$ and $E+\eta < E/2$, a sufficient condition for (1.3.4) is $-M_{\omega_2} > E$. Thus $[E-\eta, E+\eta] \cap \sigma(H_L^\omega) \ne \emptyset$ implies $|E| \le M_{\omega_2}$. We here note that the restriction $|E| \le M_{\omega_2}$ does not affect the estimate (1.3.1), since M_{ω_2} is independent of ω_1. Therefore, the first factor of (1.3.3) is bounded by $M_{\omega_2}^{(1+\ell)(1-1/p')}$.

By using also

$$\int_{\mathbf{R}} |\rho'(\gamma)|^{p'}d\gamma \le \int_{\mathbf{R}} |\rho'(\gamma)|d\gamma \sup(\rho')^{(p'-1)},$$

we can show that the second factor of the right hand side of (1.3.3) is dominated by $\eta^{(1/p'-1)}$. Therefore, the second factor of (2.2.7) is dominated by

$$\eta^{(1/p'-1)} M_{\omega_2}^{(1+\ell)(1-1/p')}. \tag{1.3.5}$$

Consequently, we obtain

$$\mathbf{E}\{\mathrm{Tr} P_L^{\omega}((E-\eta, E+\eta))\} \le \mathbf{E}\{\sum_{j\in\Lambda_L^+} \eta^{1/p'} C_1 (M_{\omega_2})^N\},$$

where $N = (\ell+1)(1-1/p')$. Since

$$M_{\omega_2} \le C_2 \sum_{k\in\mathbf{N}} \chi_{\Lambda_1}(x - \xi_k^{\omega_2}) + 1,$$

this is dominated by

$$\begin{aligned}
&\mathbf{E}\{\sum_{j\in\Lambda_L^+} \eta^{1/p'} (\sup_{x\in\Lambda_L} 2 \sum_{k\in\mathbf{N}} \chi_{\Lambda_1}(x-\xi_k^{\omega_2}) + 1)^N\} \\
&\le C_3 \eta^{1/p'} \mathbf{E}\{(\sharp\Lambda_L^+)(\sup_{x\in\Lambda_L} \sharp\{i\in\mathbf{N} | \xi_i^{\omega_2} \in \Lambda_1(x)\} + 1)^N\}.
\end{aligned}$$

By the Hölder inequality, this is dominated by

$$\eta^{1/p'} \mathbf{E}[(\sharp\Lambda_L^+)^u]^{\frac{1}{u}} \times \mathbf{E}[(\sup_{x\in\Lambda_L} \sharp\{i\in\mathbf{N}|\xi_i^{\omega_2}\in\Lambda_1(x)\} + 1)^{vN}]^{\frac{1}{v}},$$

for any u and $v > 1$ with $1/u + 1/v = 1$.

The first factor is dominated by

$$\mathbf{E}[(\sharp\{i\in\mathbf{N}|\xi_i^{\omega_2}\in\Lambda_{L+1}\})^u]^{\frac{1}{u}} \le C_4 L^d,$$

and the second factor is dominated by

$$\begin{aligned}
&\mathbf{E}[\sum_{a\in\Lambda_L\cap\mathbf{Z}^d} (\sup_{x\in\Lambda_1(a)} \sharp\{i\in\mathbf{N}|\xi_i^{\omega_2}\in\Lambda_1(x)\} + 1)^{vN}]^{\frac{1}{v}} \\
&\le \mathbf{E}[\sum_{a\in\Lambda_L\cap\mathbf{Z}^d} (\sharp\{i\in\mathbf{N}|\xi_i^{\omega_2}\in\Lambda_2(a)\} + 1)^{vN}]^{\frac{1}{v}} \\
&\le C_5 L^{d/v}.
\end{aligned}$$

By all these, we obtain the theorem.

1.4. Multiscale analysis

In [6] Germinet and Klein gave the theory of the bootstrap multiscale analysis in an abstract setting under several conditions. In this section we show our model satisfies these conditions in a weakened form. We use the following definitions in [4].

Definition 1.4.1. *Given $\theta > 0$, $E \in \mathbf{R}$, $x \in \mathbf{Z}^d$, and $L \in 6\mathbf{N}$, we say that the box $\Lambda_L(x)$ is (θ, E)-suitable for ω if $E \notin \sigma(H^\omega_{L,x})$ and*

$$\| \Gamma_{L,x}(H^\omega_{L,x} - E)^{-1}\chi_{L/3,x} \| \leq \frac{1}{L^\theta},$$

where $H^\omega_{L,x}$ is the restriction of the operator H^ω to $L^2(\Lambda_L(x))$ under the Dirichlet boundary condition and, $\Gamma_{L,x}$ and $\chi_{L,x}$ are characteristic functions of $\overline{\Lambda}_{L-1}(x) \setminus \Lambda_{L-3}(x)$ and $\Lambda_L(x)$ respectively.

Definition 1.4.2. *Given $m > 0$, $E \in \mathbf{R}$, $x \in \mathbf{Z}^d$, and $L \in 6\mathbf{N}$, we say that the box $\Lambda_L(x)$ is (m, E)-regular for ω if $E \notin \sigma(H^\omega_{L,x})$ and*

$$\| \Gamma_{L,x}(H^\omega_{L,x} - E)^{-1}\chi_{L/3,x} \| \leq \exp(-\frac{mL}{2}).$$

Based on the paper by Fischer, Leschke and Müller [4], we will give the initial length scale estimate under our setting. By the Combes-Thomas estimate (Lemma A.1 in [4]) we have

$$\begin{aligned} (1.4.1)\quad & \| \Gamma_L(H^\omega_L - E)^{-1}\chi_{L/3} \| \\ & \leq \frac{\sqrt{\{(L-1)^d - (L-3)^d\}(L/3)^d}}{2^{(d+1)/4}(\pi\delta)^{(d-1)/2}}(V^\omega_0 - E)^{(d-3)/4} \\ & \qquad \times \big(1 + \frac{d^2}{8\delta\sqrt{2(V^\omega_0 - E)}}\big)\exp(-\delta\sqrt{2(V^\omega_0 - E)}) \end{aligned}$$

for all $E < V^\omega_0 := \operatorname{ess\,inf}_{x\in\Lambda_L} V^\omega(x)$, where $\delta := (2L-9)/6$, $\Gamma_L := \Gamma_{L,0}$, $\chi_L := \chi_{L,0}$, and $H^\omega_L := H^\omega_{L,0}$. To control V^ω_0, we use the following:

Proposition 1.4.1. *For all $\eta > 0$, there exist finite positive constants C_1 and C_2 such that*

$$\mathbf{P}[\sup_{x\in\Lambda_L} |V^\omega(x)| > \eta] \le C_1 L^d \exp(-C_2 \eta \log \eta), \tag{1.4.2}$$

for any $L \in \mathbf{N}$.

Proof. Noting that $0 \le f_i^{\omega_1} \le 1$ and $\operatorname{supp} u \subset \Lambda_1(0)$,
we have

$$\mathbf{P}[\sup_{x\in\Lambda_L} |V^\omega(x)| > \eta] \le \mathbf{P}[\sup_{x\in\Lambda_L} \sharp\{i \in \mathbf{N} : \xi_i^{\omega_2} \in \Lambda_1(x)\} > \eta/||u||_\infty]$$
$$\le \sum_{a\in\Lambda_L\cap\mathbf{Z}^d} \mathbf{P}[\sup_{x\in\Lambda_1(a)} \sharp\{i \in \mathbf{N} : \xi_i^{\omega_2} \in \Lambda_1(x)\} > \eta/||u||_\infty]$$
$$\le L^d \mathbf{P}[\sup_{x\in\Lambda_1} \sharp\{i \in \mathbf{N} : \xi_i^{\omega_2} \in \Lambda_1(x)\} > \eta/||u||_\infty]$$
$$\le L^d \mathbf{P}[\mathcal{L}'_{\omega_2} > \eta/||u||_\infty],$$

where $\mathcal{L}'_{\omega_2} := \sharp\{i \in \mathbf{N} : \xi_i^{\omega_2} \in \Lambda_2\}$.
Since $\mathcal{L}'_{\omega_2}$ obeys the Poisson distribution, for $N \in \mathbf{N}$

$$\mathbf{P}[\mathcal{L}'_{\omega_2} > N] = \sum_{n=N}^{\infty} e^{-2^d} \frac{2^{dn}}{n!} = \frac{\gamma(N, 2^d)}{\Gamma(N)},$$

where γ is the incomplete gamma function and Γ is the gamma function. By estimating the integral representation of the gamma functions, we obtain

$$\mathbf{P}[\mathcal{L}'_{\omega_2} > N] \le C_1 \exp(-C_2 N \log N)$$

Using this formula, we obtain (1.4.2). □

By (1.4.1) and Proposition 1.4.1, we can prove the initial length scale estimate as follows:

Proposition 1.4.2 (Initial length scale estimate)**.** *For all $L_0 \in 6\mathbf{N}$ and $0 < \theta$, there exists $E_0 < 0$ such that*
$\mathbf{P}\{\omega : \Lambda_{L_0}$ *is* (θ, E)*-suitable* $\} > 1 - 841^{-d}$ *for all* $E \le E_0$.

Proof. On the event $V_0^\omega > E$, we set $V_0^\omega - E = \Delta E$ where $V_0^\omega := \inf_{x\in\Lambda_{L_0}} V^\omega$. Then, by (1.4.1), we have

$$\| \Gamma_{L_0}(H^\omega_{L_0} - E)^{-1}\chi_{L_0/3} \| \leq C_1 \times L_0^{d/2}(\Delta E)^{(d-3)/4} \times \big(1 + \frac{3d^2}{2L_0\sqrt{2\Delta E}}\big)\exp(-C_2L_0\sqrt{2\Delta E}).$$

For this to be dominated by $L_0^{-\theta}$ for all $L_0 \in 6\mathbf{N}$, it should hold that

$$L_0^{(\theta+d/2)}(\Delta E)^{(d-3)/4}\times\big(1+\frac{3d^2}{2L_0\sqrt{2\Delta E}}\big)\exp(-C_2L_0\sqrt{2\Delta E}) \leq 1/C_1.$$

If we take $C_3 > 0$ sufficiently large, then this inequality holds whennever $\Delta E \geq C_3$. Consequently, we have only to take E_0 so that

$$\mathbf{P}[V_0^\omega \geq E_0 + C_3] \geq 1 - 841^{-d}.$$

This is possible by Proposition 1.4.1.

□

The condition on the average number of eigenvalues is satisfied in the following form:

Proposition 1.4.3 (Number of eigenvalues)**.** *For any compact interval I, there exists a finite constant C_I such that*

$$\mathbf{E}[\mathrm{Tr}[P_L^\omega(I)]\,] \leq C_I L^{2d} \qquad \textit{for all } L \in 2\mathbf{N}. \tag{1.4.3}$$

Proof. We dominate the spectral projection by the heat semigroup $\exp(-tH_L^\omega)$ generated by H_L^ω:

$$\mathrm{Tr}[P_L^\omega(I)] \leq e^b\,\mathrm{Tr}[\exp(-H_L^\omega)],$$

where $b = \sup I$. By Mercer's theorem, we have

$$\mathrm{Tr}[\exp(-H_L^\omega)] = \int_{\Lambda_L} \exp(-H_L^\omega)(x,x)dx,$$

where $\exp(-H_L^\omega)(x,y),\ x,y \in \Lambda_L$, is the integral kernel of $\exp(-H_L^\omega)$. By the Feynman-Kac formula [3], we have

$$\exp(-H_L^\omega)(x,y) \leq \frac{1}{4\pi}\exp(-\inf_{x\in\Lambda_L} V_\omega(x)).$$

By Proposition 1.4.1, there is a finite positive constant C' such that

$$\mathbf{E}[\exp(-\inf_{x\in\Lambda} V_\omega(x))] \leq C'L^d.$$

Consequently, we obtain (1.4.3). □

The random fields $V^\omega(x)|_{\Lambda_L(y)}$ and $V^\omega(x)|_{\Lambda_{L'}(y')}$ are independent if $d(\Lambda_L(y),\ \Lambda_{L'}(y'))>1$. This means that the condition on the independence at distance is satisfied in our setting(cf. [13] p59 (IAD)). The Simon-Lieb inequality in our setting is as follows: for all compact interval I, there exists $\gamma_I \in (0,\infty)$ such that for all $L, \ell', \ell'' \in 2\mathbf{N}$, $y, y' \in \mathbf{Z}^d$ which satisfy $\Lambda_{\ell''}(y) \sqsubset \Lambda_{\ell'}(y') \sqsubset \Lambda_L$, and $E \in I - \sigma(H_L^\omega) - \sigma(H_{\ell',y'}^\omega)$,

$$\| \Gamma_L(H_L^\omega - E)^{-1}\chi_{\ell'',y} \| \leq \gamma_I(1 + \sup_{x\in\Lambda_{\ell'}(y')} |V^\omega(x)|)$$
$$\times \| \Gamma_{\ell',y'}(H_{\ell',y'}^\omega - E)^{-1}\chi_{\ell'',y} \| \times \| \Gamma_L(H_L^\omega - E)^{-1}\chi_{\ell',y'} \|,$$

where $\Lambda_{\ell'}(y') \sqsubset \Lambda_L(x)$ denotes $\Lambda_{\ell'}(y') \subset \Lambda_{L-3}(x)$. In the inequality in Germinet and Klein theory, the term $\sup|V^\omega(x)|$ does not appear. However, this term $\sup|V^\omega(x)|$ is controlled in our setting using Propposituion 1.4.1(cf. [13]). Now the conditions in Germinet and Klein theory [6] are satisfied in a weakened form. For this situation, their theory is extended as follows(cf. [13]):

Proposition 1.4.4 (Bootstrap multiscale analysis [6]). *For any $\delta > 0$ and $\theta > 2d/\varepsilon$, there exists $\tilde{L} \in 6\mathbf{N}$ satisfying the* *following: if* $\mathbf{P}[\omega : \Lambda_L$ *is* $(\theta, E_0) -$ *suitable*$] > 1 - 1/841^d$ *holds for some* $\tilde{L} \le L \in 6\mathbf{N}$ *and* $E_0 \le -\delta$*, then there exists* $\delta_0 > 0$ *such that for any* $0 < \zeta < 1$ *and* $1 < \alpha < \zeta^{-1}$*, there exist* $L_0 \in 6\mathbf{N}$ *and* $m_\zeta > 0$ *satisfying*

$$[\mathbf{P}[R(m_\zeta, L_k, I(E_0, \delta_0), x, y)] \ge 1 - \exp(-L_k^\zeta)]$$

for any $k \in \mathbf{Z}^+$ *and* $x, y \in \mathbf{Z}^d$ *with* $||x - y||_\infty > L_k + 1$*, where* $L_{k+1} = [L_k^\alpha]_{6\mathbf{N}} := \max\{N \in 6\mathbf{N} : N \le L_k^\alpha\}$ *and* $I(E_0, \delta_0) = [E_0 - \delta_0, E_0 + \delta_0] \cap (-\infty, -\delta]$*, for an interval* I*, we set* $R(m,\ L,\ I,\ x,\ y) := \{\omega :$ *for all* $E \in I$*,* $\Lambda_L(x)$ *or* $\Lambda_L(y)$ *is* (m, E)*-regular*$\}$. (*see* [13]).

Proof. This theorem is proven by extending the four theorems in Section 5 in [6]. In the proof of Theorem 5.1, as in [13], we require s satisfies

$$(p + 2d)/\varepsilon < s \text{ and } s < \theta.$$

Moreover, in the definition of the event $\mathcal{F}_{L,\ell}$ in [6], we add the condition on $\sup_{\Lambda_L} |V^\omega|$ as follows:

$$\begin{aligned}\mathcal{F}_{L,\ell} = \{\omega :&\text{there exist } n\ (\theta,\ E_0)\text{-non suitable boxes}\\ &\{\Lambda_\ell(y_i)\}_{i=1}^n, \text{ where } n \ge S+1, \text{ in } C_{L,\ell} \text{ such that}\\ &\text{dist}(\Lambda_\ell(y_i), \Lambda_\ell(y_j)) > 1\ for\ i \ne j\}\\ &\cup \{\omega : \text{dist}(\sigma(H^\omega_{\ell',x}), E_0) \le t_L \text{ for some } x \in \Xi'_{L,\ell}\\ &\text{and } \ell' = (7k + 2/3)\ell\ (1 \le k \le S)\}\\ &\cup \{\omega : \text{dist}(\sigma(H^\omega_{L,x}), E_0) \le t_L\}\\ &\cup \{\omega : \sup_{\Lambda_L} |V^\omega| \ge \log L\},\end{aligned}$$

where

$$\Xi_{L,\ell} := \Lambda_L \cap (\ell/3)\mathbf{Z}^d \subset \mathbf{Z}^d,$$
$$C_{L,\ell} := \{\Lambda_\ell(y) : \ y \in \Xi_{L,\ell}, \Lambda_\ell(y) \sqsubset \Lambda_L\},$$
$$\Xi'_{L,\ell} := \Lambda_L \cap (\ell/6)\mathbf{Z}^d \subset \mathbf{Z}^d.$$

The rest of the proof is same as in [6, 13].

□

For the application to the Anderson localization, we need also conditions on the generalized eigenfunctions. These are also satisfied in a form which is enough for our purpose(cf. [7], [13]).

1.5. Dynamical localization

In this section, we will state the results on the Anderson localization obtained by the direct application of Germinet and Klein theory [6] on the basis of the results of Section 1.4.:

Proposition 1.5.1 (Decay of kernel)**.** *We take a compact interval I such that $\sup I \le E_0$, where E_0 is the negative number given in Proposition 1.4.2.*

Then for all $0 < \zeta < 1$ there exists some C_ζ, such that for all $x,y \in \mathbf{Z}^d$,

$$\mathbf{E}[\sup_{f \in G} |||\chi_{1,x} f(H^\omega)\mathbf{P}^\omega(I)\chi_{1,y}|||_2^2] \le C_\zeta(\exp(-||x-y||_\infty^\zeta),$$

where G is the set of all Borel measurable functions such that $\| f \|_\infty \le 1$ and $\mathbf{P}^\omega(I)$ is the restriction of the projection operator of H^ω to the energy region I.

From this, we obtain the following:

Corollary 1.5.1.

(1) (Strong Hilbert-Schmidt dynamical localization) *We take a compact interval I as in the last proposition. Then we*

have

$$\mathbf{E}[\sup_t ||||x|^q e^{-itH^\omega}\mathbf{P}^\omega(I)\chi_0|||_2^2] < \infty$$

for any q>0.

(2) (Semi Uniformly Localized Eigenfunction) *We take a compact interval I as in the last proposition. For any $\epsilon > 0$ there exists m_ϵ and for a.e. ω there are constants $C_{\epsilon,\omega}$, $\tilde{C}_\omega \in (0,\infty)$ and $\{x_{n,\omega}\}_{n\in\mathbf{N}} \subset \mathbf{Z}^d$, such that, if we let $\{\phi_{n,\omega}\}_{n\in\mathbf{N}}$ be the normalized eigenfunctions of H_ω with energy $E_{n,\omega}$ in I, we have*

$$\| \chi_{1,x}\phi_{n,\omega} \|_2 \leq C_{\epsilon,\omega} \exp\{m_\epsilon(\log ||x_{n,\omega}||_\infty)^{1+\epsilon}\} \times \exp\{-m_\epsilon||x - x_{n,\omega}||_\infty\}$$

and

$$||x_{n,\omega}||_\infty \geq \tilde{C}_\omega n^{1/(4\nu)}$$

for any $n \in \mathbf{N}$, $x \in \mathbf{Z}^d$ and $\nu > d/4$.

Acknowledgments. The author would like to express his gratitude to Professor N.Ueki for many helpful advices.

References

[1] P.W. Anderson, *Absence of diffusion in certain random lattice,* Phys.Rev. **109**(1958), 1492–1505.
[2] J.M. Combes, P.D. Hislop and S. Nakamura, *The L^p-theory of the spectral shift function,the Wegner estimate,and the integrated density of states for some randam operators,* Commun.Math.Phys. **218**(2001), 113–130.
[3] R. Carmona and J. Lacroix, *Spectral theory of random Schrödinger operators,* Birkhäuser, Boston, 1990.
[4] W. Fischer, H. Leschke and P. Müller, *Spectral localization by Gaussian random potentials in multi-dimensional continuous space,* J.Stat.Phys. **101**(2000), 935-985.
[5] A. Figotin and L.A. Pastur, *Spectra of random and almost-periodic operators,* Splinger-Verlag, Berlin, 1992.
[6] F. Germinet and A. Klein, *Bootstrap multiscale analysis and localization in random media,* Commun.Math.phys. **222**(2001), 415–448.
[7] A. Klein, A. Koines and M. Seifert, *Generalized eigenfunctions for waves in inhomogenious media,* J.Funct.Anal **190**(2002), 255–291.
[8] W. Kirsch and I. Veselić, *Wegner estimate for sparse and other generalized alloy type potentials,* Proc.Indian Acad.Sci **112**(2002), 131–146.
[9] S. Nakamura, *A remark on the Dirichlet-Neumann decoupling and the integrated density of states,* J.Func.Anal. **179**(2001), 136–152.

[10] M. Reed and B. Simon, *Methods of modern mathematical physics*(II) Academic Press, Inc. New York, 1975.
[11] B. Simon *Trace ideals and their applications* American Mathematical Society, 2005
[12] P. Stollmann, *Caught by disorder,* Birkhäuser, Boston, 2001.
[13] N. Ueki, *Wegner estimates and localization for Gaussian random potentials,* Publ.RIMS.Kyoto Univ. **40**(2004), 29–90.
[14] D. R. Yafaev, *Mathematical scattering theory,* American Mathematical Society, 1992.

Part 2

Wegner estimate for a nonsign definite generalized alloy type potential

Abstract

P.D.Hislop and F.Klopp proved a Wegner estimate for Schrödinger operators with nonsign definite potentials for each fixed position of impurities [12]. In this paper, a similar estimate is proven treating also the position of impurities as random variables.

2.1. Introduction

In this paper, we will give a Wegner estimate for a Schrödinger operator,

$$H^\omega := H_0 + V^\omega(x) \quad \text{with } V^\omega(x) := \sum_{i\in\mathbf{N}} f_i^{\omega_1} u(x-\xi_i^{\omega_2}) \text{ and } H_0 = -\Delta, \tag{2.1.1}$$

where u is a continuous function with a compact support and does not have a fixed sign, $\{f_i^{\omega_1}, i \in \mathbf{N},\ \omega_1 \in \Omega_1\}$ on some probability space $(\Omega_1, F_1, \mathbf{P}_1)$ are independently and identically distributed random variables with the probability density function $h_0 \in C_0^1$ such that $\mathbf{P}_1(f_i^{\omega_1} \in d\lambda_i) = h_0(\lambda_i)d\lambda_i$ satisfying $\operatorname{supp} h_0 = [m, m']$, and $\{\xi_i^{\omega_2},\ i \in \mathbf{N}, \omega_2 \in \Omega_2\}$ on some probability space $(\Omega_2, F_2, \mathbf{P}_2)$ is a Poisson point process independent of $\{f_i^{\omega_1}\}$ with the Lebesgue measure as its intensity. We write $\omega = (\omega_1, \omega_2)$. For any $y \in \mathbf{R}^d$ and $L >0$, we set $\Lambda_L(y) = \{x \in \mathbf{R}^d : |x_i - y_i| < L/2 \text{ for } 1 \le i \le d\}$ and $\Lambda_L := \Lambda_L(0)$. For simplicity we assume $\operatorname{supp} u \subset \Lambda_a$ and $\|u\|_\infty = 1$, where $a > 0$. As in [2, 22], we can prove the essential self-adjointness and the measurability in ω of the Schrödinger operator H^ω, and that the spectrum $\sigma(H^\omega) = \mathbf{R}$ for almost all ω. In the rest of this paper we denote the unique self-adjoint extension by the same symbol H^ω.

We consider the approximation of H^ω defined by the self-adjoint operator

$$H^\omega_{\Lambda_L} := H_0 + V^\omega_{\Lambda_L}, V^\omega_{\Lambda_L} := \sum_{i:\xi_i^{\omega_2} \in \Lambda_L} f_i^{\omega_1} u(x - \xi_i^{\omega_2})$$

on $L^2(\mathbf{R}^d)$ following Klopp [16] and Hislop and Klopp [12].

The main theorem of this paper is the following:

Theorem 2.1.1. *For any $\delta > 0$, $q > 1$ and arbitrarily small $\zeta > 0$, there exists a finite positive constant $C_{q,\delta,\zeta}$ such that*

$$\mathbf{P}\{\mathrm{dist}(\sigma(H^\omega_{\Lambda_L}), E_0) \le \eta\} \le C_{q,\delta,\zeta} L^{(1+\zeta)d} \eta^{1/q}, \tag{2.1.2}$$

for $L > 0$, $E_0 < 0$ and $\eta > 0$ satisfying $\eta < \delta/4$ and $E_0 + \eta < -\delta$.

Remark 1. We can prove Theorem2.1.1 for $H_0 = -\triangle + V$, where V is a periodic non-random potential. However in this paper, we will assume $V = 0$ for simplicity.

Remark 2. We can extend Theorem2.1.1 for more general distributions of impurities' positions $\{\xi_i^{\omega_2}\}$. (see Remark 5, 6 below)

Remark 3. The initial length scale estimate, which is an important estimate together with a Wegner estimate to prove the Anderson localization, holds in this case as in [22].

As one of the applications, we obtain the strong Hilbert-Schmidt dynamical localization which is deduced in the same way as [22] based on the main theorem.

Corollary 2.1.1. *There exists $E_0 < 0$ such that*

$$\mathbf{E}[\sup_t ||||x|^r e^{-itH^\omega} \mathbf{P}^\omega(I)\chi_0|||_2^2] < \infty$$

for any $r > 0$ *and any compact interval* I *satisfying* $\sup I \leq E_0$.

The estimate

$$\mathbf{P}\{\mathrm{dist}(\sigma(H^{\omega}_{\Lambda_L}), E_0) \leq \eta\} \leq CL^{Ad}\eta^{B}, \tag{2.1.3}$$

with $A \geq 1$ and $B \leq 1$ as (2.1.2) is called a Wegner estimate. From the fact that $\mathbf{P}\{\mathrm{dist}(\sigma(H^{\omega}_{\Lambda_L}), E_0) \leq \eta\}$ is dominated by $\mathbf{E}[\sharp\{\sigma(H^{\omega}_{\Lambda_L}) \cap [E_0 - \eta, E_0 + \eta]\}]$ and $\mathbf{E}[\sharp\{\sigma(H^{\omega}_{\Lambda_L}) \cap [E_0 - \eta, E_0 + \eta]\}]/L^d$ converges to the density of states (DS) as $L \to \infty$, we expect that $A = B = 1$ are the best exponents.

Wegner [24] firstly obtained this estimate for the Anderson model. After that the estimate (2.1.3) with general exponents A and B is applied to the proof of the Anderson localization [6, 8, 21].

There had been many prior results on a Wegner estimate for multidimensional and continuous Schrödinger operators with Anderson-type random potentials whose positions corresponding to $\xi_i^{\omega_2}$ in equation (2.1.1) were fixed on the lattice [17, 3, 13, 21]. Among them, Combes, Hislop and Nakamura obtained a bound as (2.1.3) with $A = 1$ and $B < 1$ which is arbitrarily close to 1 for the Schrödinger operators with the Anderson-type positive potentials by the method of the spectral shift function [5]. Kirsch and Veselić used this method to prove a bound as (2.1.3) with $A = 1$ and B arbitrarily close to 1 for the negative potentials called generalized alloy type potentials where the positions of the impurities were fixed randomly on $\mathbf{R}^d$ [14]. With the method of [14], the author proved a bound as (2.1.3) with A and B arbitrarily close to 1 at negative energies for Schrödinger operators with negative potentials V^{ω} under the conditions that $f_i^{\omega_1} \in [0, 1]$ and u is negative and, showed the results of the localization [22].

These results on a Wegner estimate were obtained by the property that the eigenvalues monotonously depended on the coupling constants. Therefore, this method needs the property that a single-site potential has the definite sign. We would like to drop this condition for this paper.

In 2002, Hislop and Klopp obtained a bound as (2.1.3) with $A = 1$ and B arbitrarily close to 1 for the nonsign definite Anderson-type potentials using Klopp's vector field method [16] and the spectral shift function method [12]. Applying their method and [14], we obtain Theorem 2.1.1, which is a bound as (2.1.3) with A and B arbitrarily close to 1. Theorem2.1.1 may not have the optimal exponents of L and η for the Wegner estimate. However, this is the best result that we are able to derive up to now. In 2007, Germinet, Hislop and Klein proved the localization for a sign definite Poisson potential model without $f_i^{\omega_1}$, which is more difficult problem than ours. The inequality corresponding to the Wegner estimate for their proof of the localization is restricted to $\eta \sim L^{-L^\rho}$ with $\rho > 0$, for that reason the exponent A depends on L [9, 10].

Remark 4. The best estimate of Wegner-type for continuous Schrödinger operators with Anderson-type random potentials was proved by Combes, Hislop and Klopp with $A = 1$ and $B \leq 1$ [4]. This estimate was obtained by the method of the spectral averaging mainly for Schrödinger operators with a nonnegative single-site potential. It seems to be very difficult to extend this result to a nonsign definite single-site potential with a randomly distributed position, Corollary 2.1.1 is obtained for the first time by treating the positions ξ of the single-site potentials as random variables. This is the different point from [14].

2.2. The proof of the main theorem

In this section, we will prove a Wegner estimate, Theorem 2.1.1, using the method in [12].

The resolvent $R_{\Lambda_L}(E_0,\omega) := (H^\omega_{\Lambda_L} - E_0)^{-1}$ is written as

$$R_{\Lambda_L}(E_0,\omega) = (H_0 - E_0)^{-1/2}(1+\Gamma_{\Lambda_L}(E_0,\omega))^{-1}(H_0-E_0)^{-1/2},$$

where $\Gamma_{\Lambda_L}(E,\omega)$ is a compact operator defined by $(H_0-E)^{-1/2}V^\omega_{\Lambda_L} \times (H_0 - E)^{-1/2}$. Then, using the inequality $\|R_{\Lambda_L}(E_0,\omega)\| \leq \delta^{-1}\|(1+\Gamma_{\Lambda_L}(E_0,\omega))^{-1}\|$, we have

$$\mathbf{P}\{\mathrm{dist}(\sigma(H_{\Lambda_L}), E_0) \leq \eta\} \leq \mathbf{P}\{\mathrm{dist}(\sigma(\Gamma_{\Lambda_L}(E_0,\omega)), -1) \leq \eta/\delta\}.$$

Consequently, we have only to show

$$\mathbf{P}\{\mathrm{dist}(\sigma(\Gamma_{\Lambda_L}(E_0,\omega)), -1) \leq \kappa\} \leq C_{q,\delta,\zeta} L^{(1+\zeta)d}\eta^{1/q},$$

where $\kappa := \eta/\delta$.

We now apply Chebyshev's inequality,

$$\begin{aligned} &\mathbf{P}\{\mathrm{dist}(\sigma(\Gamma_{\Lambda_L}(E_0,\omega)), -1) \leq \eta/\delta\} \\ &\qquad = \mathbf{P}[\mathrm{Tr}(P^\omega_{\Lambda_L}(I_\kappa)) \geq 1] \leq \mathbf{E}[\mathrm{Tr}(P^\omega_{\Lambda_L}(I_\kappa))], \end{aligned} \tag{2.2.1}$$

where $P^\omega_{\Lambda_L}$ denotes the spectral projection of $\Gamma_{\Lambda_L}(E_0,\omega)$ and $I_\kappa := [-1-\kappa, -1+\kappa]$.

We will at first estimate the expectation of the right hand side of (2.2.1) with respect to the randomness of ω_1. This estimate holds for any point processes. Then we will calculate its expectation with respect to the randomness of ω_2, only which Poisson process affects(see (2.2.9)).

We define $\mathbf{E}^{\omega_1}$ as the expectation with respect to the randomness of ω_1 and $\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)$ as $\Gamma_{\Lambda_L}(E_0, \omega)$ in which $f_i^{\omega_1}$ for any $i \in \{i \in \mathbf{N} : \xi_i^{\omega_2} \in \Lambda_L\}$ is replaced by λ_i for any $\lambda = (\lambda_i)_{i:\xi_i^{\omega_2} \in \Lambda_L} \in [m, m']^{\sharp\{i:\xi_i^{\omega_2} \in \Lambda_L\}}$. By the method in [12], we have

$$
\begin{aligned}
(2.2.2) \qquad & \mathbf{E}^{\omega_1}[\mathrm{Tr} P_{\Lambda_L}^{\omega}(I_\kappa)] \\
& \le \mathbf{E}^{\omega_1}\{\int_{-3\kappa/2}^{3\kappa/2} \frac{d}{dt} \mathrm{Tr}[\rho(\Gamma_{\Lambda_L}(E_0, \omega) + 1 - t)] dt\} \\
= \prod_{\ell} \int_m^{m'} & h_0(\lambda_\ell) d\lambda_\ell \{\int_{-3\kappa/2}^{3\kappa/2} \frac{d}{dt} \mathrm{Tr}[\rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) + 1 - t)] dt\},
\end{aligned}
$$

where ρ is a nonnegative, smooth, monotone decreasing function such that $\rho(x) = 1$ for $x < -\kappa/2$ and $\rho(x) = 0$ for $x \ge \kappa/2$.

Since $\mathrm{supp}\,\rho$ is included in $(-\infty, \kappa/2]$, $\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)$ of the right hand side of (2.2.2) is restricted to the spectral subspace where the operator is smaller than $(-1 + 2\kappa)$ which is negative. Therefore, noting that ρ' is negative,

$$
\begin{aligned}
(2.2.3) \quad & \frac{d}{dt} \mathrm{Tr}[\rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) + 1 - t)] \\
& = \sum_{j:E_j \in [-1-2\kappa, -1+2\kappa]} \frac{d}{dt} \rho(E_j + 1 - t) \\
& \le \sum_{j:E_j \in [-1-2\kappa, -1+2\kappa]} \frac{-E_j}{-1 + 2\kappa} \rho'(E_j + 1 - t) \\
& = \frac{-1}{-1 + 2\kappa} \mathrm{Tr}[\rho'(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) + 1 - t) \Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)],
\end{aligned}
$$

where $\{E_j\}_{j \in \mathbf{N}}$ are the eigenvalues of $\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)$.

On the other hand, using the Hellmann-Feynman theorem [21] and the equation

$$
\sum_{i:\xi_i^{\omega_2} \in \Lambda_L} \lambda_i \frac{\partial}{\partial \lambda_i} \Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) = \Gamma_{\Lambda_L}(E_0, \lambda, \omega_2),
$$

we can show

$$
\begin{aligned}
&\sum_{i:\xi_i^{\omega_2}\in\Lambda_L} \lambda_i \frac{\partial}{\partial\lambda_i}\mathrm{Tr}[\rho(\Gamma_{\Lambda_L}(E_0,\lambda,\omega_2)+1-t)] \\
&= \sum_{j:E_j\in[-1-2\kappa,-1+2\kappa]} \{\rho'(E_j+1-t)\cdot \sum_{i:\xi_i^{\omega_2}\in\Lambda_L} \lambda_i \frac{\partial}{\partial\lambda_i}E_j\} \\
&= \mathrm{Tr}[\rho'(\Gamma_{\Lambda_L}(E_0,\lambda,\omega_2)+1-t)\Gamma_{\Lambda_L}(E_0,\lambda,\omega_2)].
\end{aligned}
\tag{2.2.4}
$$

By (2.2.3) and (2.2.4), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathrm{Tr}[\rho(\Gamma_{\Lambda_L}(E_0,\lambda,\omega_2)+1-t)] \\
&\le \frac{-1}{-1+2\kappa}\sum_{i:\xi_i^{\omega_2}\in\Lambda_L} \lambda_i \frac{\partial}{\partial\lambda_i}\mathrm{Tr}[\rho(\Gamma_{\Lambda_L}(E_0,\lambda,\omega_2)+1-t)].
\end{aligned}
$$

By this estimate and the integration by parts with respect to λ_i, the right hand side of (2.2.2) is less than or equal to

$$
\begin{aligned}
&\frac{-1}{-1+2\kappa}\sum_{i:\xi_i^{\omega_2}\in\Lambda_L}\int_{-3\kappa/2}^{3\kappa/2} dt\{\prod_{\ell}\int_m^{m'} h_0(\lambda_\ell)d\lambda_\ell\} \\
&\quad\times\{\lambda_i\frac{\partial}{\partial\lambda_i}\mathrm{Tr}[\rho(\Gamma_{\Lambda_L}(E_0,\lambda,\omega_2)+1-t)]\} \\
&\le 2\frac{((m'-m)\|\tilde{h_0}'\|_\infty)\vee\tilde{h}_0(m')}{(1-2\kappa)} \\
&\times\sum_{i:\xi_i^{\omega_2}\in\Lambda}\int_{-3\kappa/2}^{3\kappa/2} dt\prod_{\ell\neq i}\int_m^{m'} h_0(\lambda_\ell)d\lambda_\ell|\mathrm{Tr}\{D(i,E_0,m,\lambda_i^+)\}|,
\end{aligned}
\tag{2.2.5}
$$

where $\tilde{h}_0(\lambda)$ is the function $\lambda h_0(\lambda)$ and $D(i,E_0,m,\lambda_0)$ is the operator $\rho(\Gamma_{\Lambda_L}(E_0,\lambda,\omega_2)^{m,i}+1-t)-\rho(\Gamma_{\Lambda_L}(E_0,\lambda,\omega_2)^{\lambda_0,i}+1-t)$. We denote $\Gamma_{\Lambda_L}(E_0,\lambda,\omega_2)^{\varpi,i}$ for $\varpi\in[m,m']$ by the operator $\Gamma_{\Lambda_L}(E_0,\lambda,\omega_2)$ with the fixed coupling constant $\lambda_i=\varpi$ at the i-th site, and $\lambda_i^+\in[m,m']$ by the value of the coupling constant λ_i where the maximum of $|\mathrm{Tr}\{D(i,E_0,m,\lambda_0)\}|$ is attained.

Now we use the following proposition on spectral shift functions ([5] Theorem 2.1, [25] Chapter 8 §3 Theorem 3 and Theorem 6) :

Proposition 2.2.1. *Let A_1 and A_0 be self-adjoint operators such that $A_1 - A_0 \in \mathcal{I}_{1/p}$ for $p > 1$, where $\mathcal{I}_{1/p}$ is the family of compact operators of the super trace class, which we define as follows: we say that $A \in \mathcal{I}_{1/p}$ if for some $p > 1$, $|||A|||_{1/p} := \big(\sum_j \mu_j(A)^{1/p}\big)^p < \infty$, where $\mu_j(A)$ denotes the j-th singular value of A.*

Then, there exists some $\pi(\cdot; A_1, A_0) \in L^p(\mathbf{R})$ such that for $\phi \in C^\infty(\Gamma)$ where $\Gamma \subset \mathbf{R}$: a compact interval which contains $\sigma(A_0)$ and $\sigma(A_1)$,

$$\mathrm{Tr}[\phi(A_1) - \phi(A_0)] = \int_\Gamma \pi(\lambda; A_1, A_0)\phi'(\lambda)d\lambda,$$

and

$$\|\pi\|_p \le |||A_1 - A_0|||_{1/p}^{1/p}.$$

We fix $p > 1$ and set $A_1 = (\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{\lambda_i^+, i})^\ell$ and $A_0 = (\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{m,i})^\ell$ for any ℓ greater than $dp/2+1$, and $V_i = (\lambda_i^+ - m)R_0(E_0)^{1/2}u(x - \xi_i^{\omega_2})R_0(E_0)^{1/2}$, where $R_0(E_0) := (H_0 - E_0)^{-1}$. Then,

$$\begin{aligned}
V_{eff} &:= A_1 - A_0 \\
&= \sum_{j=0}^{\ell-1} (\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{\lambda_i^+, i})^{\ell-j-1} V_i (\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{m,i}))^j \\
&= (\lambda_i^+ - m) \sum_{j=0}^{\ell-1} [J_i^{\ell-j-1}(R_0(E_0)V_{\Lambda_L}^{\lambda_i^+, i})^{\ell-j-1} R_0(E_0)^{1/2}]^* \\
&\times u_i [J_i^j (R_0(E_0)V_{\Lambda_L}^{m,i})^j R_0(E_0)^{1/2}],
\end{aligned}$$

where $J \in C_0^\infty$ such that $J(x)u(x) = u(x)$. We denote $u_i := u(x - \xi_i^{\omega_2})$, $J_i := J(x - \xi_i^{\omega_2})$ and for $\varpi \in [m, m']$, $V_{\Lambda_L}^{\varpi,i}$ is the potential V_{Λ_L} with the fixed coupling constant $\lambda_i = \varpi$ at the i-th site.

According to Proposition 12 in [18], for any $\tau \in \operatorname{supp} h_0$ and $r \in \mathbf{N}$,

$$J_i^r (R_0(E_0)\tilde{V}_{\Lambda_L}^{\tau,i})^r R_0(E_0)^{1/2} = \sum_{\alpha=1}^{N} \{\prod_{\beta=1}^{r} J_i^{\alpha,\beta} R_0(E_0) B_i^{\alpha,\beta}\}, \tag{2.2.6}$$

where the bounded operators $J_i^{\alpha,\beta}$ are combinations of the derivatives of J_i, and the operators $B_i^{\alpha,\beta}$ are the polynomials of the bounded operators containing $V_{\Lambda_L}^{\tau,i}$.

Let $s > d/2$. Using the estimates of the norms in Theorem 4.1 of [20],

$$\|J_i^{\alpha,\beta} R_0(E_0)\|_{\mathcal{I}_s} \le \|J_i^{\alpha,\beta}\|_{L^s(\mathbf{R}^d)} \left\| \frac{1}{|x|^2 - E_0} \right\|_{L^s(\mathbf{R}^d)},$$

which is bounded by a constant independent of Λ_L. The norms of the operators $B_i^{\alpha,\beta}$ are estimated as follows:

$$\begin{aligned} \|B_i^{\alpha,\beta}\|_{\mathcal{B}(L^s(\mathbf{R}^d))} &\le C |\|V_\Lambda^{\tau,i}\|_\infty + 1|^\gamma \\ &\le C[\sup_{x \in \Lambda_{L+a}} \sharp(k \in \mathbf{N} : \xi_k^{\omega_2} \in \Lambda_{3a}(x)) + 1]^\gamma, \end{aligned}$$

for some γ and C independent of Λ_L. Therefore, we can obtain the estimate of the norm of V_{eff} as follows:

$$|||V_{eff}|||_{1/p}^{1/p} \le C(\ell - 1)[\sup_{x \in \Lambda_{L+a}} \sharp(k \in \mathbf{N} : \xi_k^{\omega_2} \in \Lambda_{3a}(x)) + 1]^{\gamma'},$$

for some γ'.

Noting $\operatorname{supp}\rho'(\cdot+1-t)$ is included in $\Sigma := [-1-3\kappa, 0)$, we have

$$
\begin{aligned}
(2.2.7)\qquad & |\mathrm{Tr} D(i, E_0, m, \lambda_i)| \\
&= |\int_\Sigma \frac{\partial\rho(\lambda'+1-t)}{\partial\lambda'}\pi(\lambda')d\lambda'| \\
&\leq C[\sup_{x\in\Lambda_{L+a}} \sharp(k\in\mathbf{N} : \xi_k^{\omega_2}\in\Lambda_{3a}(x))+1]^{\gamma'} \\
&\times \Big(\int_\Sigma |\frac{\partial\rho}{\partial\lambda}(\lambda+1-t)|^q d\lambda\Big)^{1/q},
\end{aligned}
$$

where q is the positive number such that $1/p+1/q=1$, and π is the spectral shift function for A_1 and A_0.

By using also

$$\int_{\mathbf{R}} |\rho'(\gamma)|^q d\gamma \leq \int_{\mathbf{R}} |\rho'(\gamma)| d\gamma \sup(\rho')^{(q-1)},$$

we can show that the second factor of the right hand side of (2.2.7) is dominated by $\kappa^{(1/q-1)}$.

Consequently, according to (2.2.2), (2.2.5), (2.2.7) and above comments, we obtain

$$
\begin{aligned}
(2.2.8)\qquad & \mathbf{E}\{\mathrm{Tr}P^\omega_{\Lambda_L}(I_\kappa)\} \leq C_1\eta^{1/q} \\
&\times\mathbf{E}^{\omega_2}[\sharp(j\in\mathbf{N} : \xi_j^{\omega_2}\in\Lambda_L)\{\sup_{x\in\Lambda_{L+a}} \sharp(k\in\mathbf{N} : \xi_k^{\omega_2}\in\Lambda_{3a}(x))+1\}^{\gamma'}].
\end{aligned}
$$

For (2.2.8), using the Hölder inequality, we obtain

$$
\begin{aligned}
(2.2.9)\quad & \mathbf{E}\{\mathrm{Tr}P^\omega_{\Lambda_L}(I_\kappa)\} \\
&\leq C_1\eta^{1/q}\mathbf{E}^{\omega_2}[\sharp(j\in\mathbf{N} : \xi_j^{\omega_2}\in\Lambda_L)^{1+\theta}]^{1/(1+\theta)} \\
&\times \mathbf{E}^{\omega_2}[\{\sup_{x\in\Lambda_{L+a}} \sharp(k\in\mathbf{N} : \xi_k^{\omega_2}\in\Lambda_{3a}(x))+1\}^{(\theta+1)\gamma'/\theta}]^{\theta/(1+\theta)},
\end{aligned}
$$

for $\theta>0$.

Noting that the third factor of the right hand side of (2.2.9) is bounded by CL^d and the fourth factor is less than or equal to

$$[\sum_{e\in\Lambda_{L+a}\cap\mathbf{Z}}\mathbf{E}^{\omega_2}[\sup_{x\in\Lambda_1(e)}\sharp(k\in\mathbf{N}:\xi_k^{\omega_2}\in\Lambda_{3a}(x))+1]^{(\theta+1)\gamma'/\theta}]^{\theta/(1+\theta)}$$
$$\leq CL^{d\theta/(1+\theta)},$$

we obtain the theorem.

Remark 5. By (2.2.8), the main theorem holds for a model of which the impurity position $\xi_i(\omega_2)=a_i+y_i(\omega_2)$ with a uniformly bounded $y_i(\omega_2)$ as A=1 similarly to [12], where $\{a_i:i\in\mathbf{N}\}=\mathbf{Z}^d$.

Remark 6. Our proof of this paper needs only the Z^d stationary property and the finite moments' property for all orders for the number of impurities in the finite cube. Moreover, if we treat point processes with finite moments of some $n>d/2$, then our results hold for $A>1+\gamma'/n$. The condition $n>d/2$ is for the essential self-adjointness of our Schrödinger operators on $C_0^\infty(\mathbf{R}^d)$ [14].

Acknowledgments. The author would like to express his gratitude to Professor N.Ueki for many helpful advices. He would also like to thank the referee for valuable advices. Remark 5 and 6 are based on referee's advices.

References

[1] P.W. Anderson: *Absence of diffusion in certain random lattice,* Phys.Rev. **109**(1958), 1492–1505.

[2] K. Ando, A. Iwatsuka, M. Kaminaga and F. Nakano: *The Spectrum of Schrödinger operators with Poisson type random potential,* Ann.Henri Poincaré **7**(2006), 145-160.

[3] J.M. Combes and P.D.Hislop: *Localization for some continuous random Hamiltonian in d-dimensions,* J.Funct.Anal. **124**(1994), 149-180.

[4] J.M. Combes, P.D.Hislop and F.Klopp: *An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators,* Duke Math.J. **140**(2007), 469-498.

[5] J.M. Combes, P.D. Hislop and S. Nakamura: *The L^p-theory of the spectral shift function,the Wegner estimate,and the integrated density of states for some random operators,* Comm.Math.Phys. **218**(2001), 113-130.

[6] R. Carmona and J. Lacroix: Spectral theory of random Schrödinger operators, Birkhäuser, Boston, 1990.

[7] W. Fischer, H. Leschke and P. Müller: *Spectral localization by Gaussian random potentials in multi-dimensional continuous space,* J.Statist.Phys. **101**(2000), 935-985.

[8] A. Figotin and L.A. Pastur: Spectra of random and almost-periodic operators, Splinger-Verlag, Berlin, 1992.

[9] F. Germinet, P.D. Hislop and A. Klein: *Localization for Schrödinger operators with Poisson random potential,* J.Eur.Math.Soc. **9**(2007), no.3, 577-607.

[10] F. Germinet, P.D. Hislop and A. Klein: *Localization at low energies for attractive Poisson random Schrödinger operators,* Probability and mathematical physics, CRM Proc. Lecture Notes, **42**, 153-165, Amer.Math.Soc.,Providedence,RI,2007.

[11] F. Germinet and A. Klein: *Bootstrap multiscale analysis and localization in random media,* Comm.Math.phys. **222**(2001), 415–448.

[12] P.D. Hislop and F. Klopp: *The Integrated density of states for some random operators with nonsign definite potentials,* J.Func.Anal. **195**(2002), 12-47.

[13] W. Kirsch: *Wegner estimates and localization for alloy-type potentials,* Math.Z. **221**(1996), 507-512.

[14] W. Kirsch and I. Veselić: *Wegner estimate for sparse and other generalized alloy type potentials,* Proc.Indian Acad.Sci **112**(2002), 131–146.

[15] A. Klein, A. Koines and M. Seifert: *Generalized eigenfunctions for waves in inhomogeneous media,* J.Funct.Anal **190**(2002), 255–291.

[16] F. Klopp: *Localization for some continuous random Schrödinger operators,* Comm.Math.Phys. **167**(1995), 553-569.

[17] S. Kotani and B. Simon: *Localization in general one dimensional systems,* Comm.Math.Phys. **112**(1987), 103-120.

[18] S. Nakamura: *A remark on the Dirichlet-Neumann decoupling and the integrated density of states,* J.Func.Anal. **179**(2001), 136–152.

[19] M. Reed and B. Simon: Methods of modern mathematical physics(II), Academic Press, New York, 1975.

[20] B. Simon: Trace ideals and their applications, American Mathematical Society, 2005

[21] P. Stollmann: Caught by disorder, Bound states in random media, Birkhäuser, Boston, 2001.

[22] J.Takahara: *Wegner estimate for a generalized alloy type potential,* J.Math.Kyoto.Univ. **49**(2009) 255–265.

[23] N. Ueki: *Wegner estimates and localization for Gaussian random potentials,* Publ.RIMS.Kyoto Univ. **40**(2004), 29–90.

[24] F.Wegner: *Bounds on the density of states in disordered systems,* Z.Phys.B, **44**(1981), 9-15.

[25] D. R. Yafaev: Mathematical scattering theory, general theory, American Mathematical Society, Providence, RI, 1992.

List of symbols

f^ω	2	random coupling constant
u	2	single site potential
ξ^ω	6	random impurity site
$(\Omega, \mathbf{P})$, $(\Omega_i, \mathbf{P}_i)$	6	probability space
Λ	2	cube in $\mathbf{R^d}$
$\Lambda_L(a)$	6	$= \{x \in \mathbf{R}^d : \|x_i - a_i\| < L/2$ for $1 \le i \le d\}$ for any $a \in \mathbf{R}^d$ and $L > 0$
Λ_L	6	$= \Lambda_L(0)$
$\|\Lambda_L(a)\|$, $\|\Lambda\|$	2	volume of a cube
V^ω	2	random field
$V^\omega_{\Lambda_L}$	22	$= \sum_{i:\xi_i^{\omega_2} \in \Lambda_L} f_i^{\omega_1} u(x - \xi_i^{\omega_2})$
Δ	2	Laplacian of $\mathbf{R^d}$
H_0	21	$= -\Delta$
$\Gamma_{L,x}$	13	characteristic functions of $\overline{\Lambda}_{L-1}(x) \setminus \Lambda_{L-3}(x)$
Γ_L	13	$= \Gamma_{L,0}$
$\chi_{L,x}$	13	characteristic functions of $\Lambda_L(x)$
H^ω	6	random Schrödinger operator
$H^\omega_{L,x}$	13	the restriction of the operator H^ω to $L^2(\Lambda_L(x))$ under the Dirichlet boundary condition
H^ω_L	7	$=H^\omega_{L,0}$
$H^\omega_{\Lambda_L}$	22	$= H_0 + V^\omega_{\Lambda_L}$
$\Gamma_{\Lambda_L}(E, \omega)$	25	$= (H_0 - E)^{-1/2} V^\omega_{\Lambda_L} \times (H_0 - E)^{-1/2}$
P^ω_L	7	spectral projection of a self adjoint operator H^ω_L
$P^\omega_{\Lambda_L}$	25	spectral projection of a self adjoint operator $\Gamma_{\Lambda_L}(E, \omega)$

μ_j	10	j-th singular value of a compact operator
$\vert\vert\vert\cdot\vert\vert\vert_{1/p}$	10	super trace norm of compact operator for $p > 1$
$\mathcal{I}_{1/p}$	10	family of the super trace class which has a finite super trace norm for $p > 1$
$\vert\vert \cdot \vert\vert_p$	10	L_p norm
π	10	spectral shift function

論文の公表

1. Part 1

論文題目：Wegner estimate for a generalized alloy type potential

著者：Jyunichi Takahara

雑誌名：Journal of Mathematics of Kyoto University

Volume 49, Number 2, 2009, P255-265.

2 Part 2

.論文題目：Wegner estimate for a nonsign definite generalized alloy type potential

著者：Jyunichi Takahara

雑誌名：Osaka Journal of Mathematics に掲載決定。