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| タイトル   | 空間の状態を決定する3次元多面体の従来方法をもとに、新しい数値的表現を導入した3次元空
|             | 一般多面体の状態をもとに、新しい数値的表現を導入した3次元空            |
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1. Introduction

In [TV], Turaev and Viro constructed a state sum invariant of 3-manifolds based on their triangulations, by using the $6j$-symbols of representations of the quantum group $U_q(sl_2)$. Further, Ocneanu [Oc] generalized the construction to the case of other types of $6j$-symbols, say, the $6j$-symbols of subfactors. When the $6j$-symbols can be obtained from representations of a quantum group, it is known (see [T]) that the state sum invariant is equal to the square of the absolute value of the Reshetikhin-Turaev invariant, and the calculation of the state sum invariant is reduced to the calculation of the Reshetikhin-Turaev invariant. However, in the case of the $6j$-symbols of the $E_6$ subfactor, such Reshetikhin-Turaev invariant can not be defined, and it is necessary to calculate the state sum invariant directly. For some calculations of the $E_6$ state sum invariant, see [SW, W], where they construct the $E_6$ state sum invariant directly from concrete values of the $6j$-symbols of the $E_6$ subfactor given in [I].

The $E_6$ subfactor is the subfactor $\mathcal{N} \subset \mathcal{M}$ whose principal graph is of the form of the right figure, where circled vertices are $\mathcal{N}$-$\mathcal{N}$ bimodules and the other vertices are $\mathcal{N}$-$\mathcal{M}$ bimodules. The $6j$-symbols are coefficients of a transformation between bases of the intertwiner spaces

\begin{equation}
\text{Hom}\left(V_i, (V_i \otimes V_j) \otimes V_k\right) \quad \text{and} \quad \text{Hom}\left(V_i, V_i \otimes (V_j \otimes V_k)\right),
\end{equation}

though it is difficult in general to calculate their concrete values directly from the subfactor, since these bimodules are infinite dimensional.

As a graphical approach to subfactors, Jones [J] introduced planar algebras, which are a kind of algebras given graphically in the plane. As the Kuperberg program says, it is a problem to (i) give a presentation by generators and relations for each planar algebra, and (ii) show basic properties of the planar algebra based on such a presentation. For the $E_6$ planar algebras, Bigelow [B] has done (i), and has partially done (ii) by using the existence of the subfactor planar algebra, though idempotents corresponding to $\mathcal{N}$-$\mathcal{N}$ bimodules are not given in the $E_6$ planar algebra in [B].

In this article, we introduce the $E_6$ linear skein, motivated by Bigelow’s generators and relations of the $E_6$ planar algebra. We define the $E_6$ linear skein $\mathcal{S}(\mathbb{R}^2)$ of $\mathbb{R}^2$ to be the vector space spanned by certain 6-valent graphs (which we call planar diagrams) subject to certain relations (Definition 2.1). We show that $\mathcal{S}(\mathbb{R}^2)$ is 1-dimensional (Proposition 2.2), which means the key point that we can calculate the value of any planar diagram by graphical calculation using the defining relations of the linear skein recursively.

Further, we introduce idempotents (gray boxes in Section 3) in our linear skein, corresponding to the irreducible $\mathcal{N}$-$\mathcal{N}$ bimodules $V_0, V_2, V_4$ in the above figure. Moreover, we consider the linear skein $H(i_1, i_2, \cdots, i_n)$ spanned by planar diagrams on a disk bounded
by the gray boxes over \(i_1\) strands, \(i_2\) strands, \(\cdots\), \(i_n\) strands, corresponding to the intertwiner space \(\text{Hom}(V_0, V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n})\). We show that a basis of this space is given by colored trivalent trees (Proposition 3.3). In particular, when \(n = 4\), we can describe the \(6j\)-symbols in terms of colored planar trivalent graphs as coefficients of a transformation between bases of certain forms.

By using these \(6j\)-symbols, we give a construction of our state sum invariant in terms of colored planar trivalent graphs (Definition 4.1). We show that our state sum invariant is equal to the \(E_6\) state sum invariant (Proposition 5.1), since our \(6j\)-symbols can be transformed into the \(6j\)-symbols of the \(E_6\) subfactor given in [I].

In this manuscript, the scalar field for every vector space is the complex field \(\mathbb{C}\), and we put \([n] = (e^{n\pi \sqrt{-1}/12} - e^{-n\pi \sqrt{-1}/12})/(e^{\pi \sqrt{-1}/12} - e^{-\pi \sqrt{-1}/12}), \omega = e^{4\pi \sqrt{-1}/3}\). A thick strand attached with an integer \(n\) means \(n\) parallel strands.

2. The \(E_6\) linear skein

We define a planar diagram to be a 6-valent graph (possibly containing closed curves) embedded in \(\mathbb{R}^2\) such that each vertex is depicted by a disk whose boundary has a base point, as shown in the picture on the right.

**Definition 2.1.** We define the \(E_6\) linear skein of \(\mathbb{R}^2\), denoted by \(S(\mathbb{R}^2)\), to be the vector space spanned by planar diagrams subject to the following relations,

\[
\begin{align*}
(2.1) & \quad D \cup (\text{a closed curve}) = [2] D \quad \text{for any planar diagram } D, \\
(2.2) & \quad (\text{A planar diagram containing a cap}) = 0, \\
(2.3) & \quad \begin{array}{c}
\text{\includegraphics[scale=0.5]{diagram1.png}}
\end{array} = \omega \begin{array}{c}
\text{\includegraphics[scale=0.5]{diagram2.png}}
\end{array}, \\
(2.4) & \quad \begin{array}{c}
\text{\includegraphics[scale=0.5]{diagram3.png}}
\end{array} = [4] \begin{array}{c}
\text{\includegraphics[scale=0.5]{diagram4.png}}
\end{array} + [3][4] \begin{array}{c}
\text{\includegraphics[scale=0.5]{diagram5.png}}
\end{array}.
\end{align*}
\]

Here, in each of (2.3) and (2.4), pictures in the formula mean planar diagrams, which are identical except for a disk, where they differ as shown in the pictures. The white boxes are the Jones-Wenzl idempotents.

**Proposition 2.2.** There exists an isomorphism \(\langle \rangle : S(\mathbb{R}^2) \to \mathbb{C}\) which takes the empty diagram \(\emptyset\) to 1. Especially, \(S(\mathbb{R}^2)\) is 1-dimensional.

**Sketch of proof.** We first show that any planar diagram \(D\) is equal to a scalar multiple of the empty diagram \(\emptyset\) in the \(E_6\) linear skein, by applying the relations (2.1)–(2.4) repeatedly. From this, we see that \(S(\mathbb{R}^2)\) is at most 1-dimensional.

We next show that the scalar in the above argument is uniquely determined. From this, we can define the bracket \(\langle \rangle\) by \(D = \langle D \rangle \emptyset\), and we see that \(S(\mathbb{R}^2)\) is just 1-dimensional. \(\square\)

In the similar way as a planar diagram in \(\mathbb{R}^2\), we define a planar diagram in \((D^2, 2m)\) as shown in the picture on the right, where \((D^2, 2m)\) means a unit disk with \(2m\) distinct points on its boundary. We define the \(E_6\) linear skein of \((D^2, 2m)\), denoted by \(S(D^2, 2m)\), to be the vector space spanned by planar diagrams in \((D^2, 2m)\) subject to the relations (2.1)–(2.4). In the rest of the article, we omit to draw the disk \(D^2\) of a planar diagram.
3. Colored planar trivalent graphs

We define gray boxes \( \in S(D^2, 2n) \) for \( n = 0, 2, 4 \) by

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{image1} \\
\end{array} \\
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{image2} \\
\end{array} \\
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{image3} \\
\end{array} \\
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{image4} \\
\end{array} \\
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{image5} \\
\end{array} \\
\end{align*}
\]

A planar trivalent graph is a trivalent graph embedded in \( \mathbb{R}^2 \). We consider two kinds of vertices; one is depicted by \( \bullet \), and the other is depicted by a disk \( \odot \) whose boundary has a base point. A coloring of a planar trivalent graph \( \Gamma \) is a map from the set of edges of \( \Gamma \) to \( \{0, 2, 4\} \) and a map from the set of vertices of \( \Gamma \) to \( \{\bullet, \odot\} \). A coloring of a planar trivalent graph \( \Gamma \) is said to be admissible if the neighborhood of each vertex of \( \Gamma \) is colored as shown in either of the following pictures.

\[
(3.1)
\]

We define a colored planar trivalent graph to be a planar trivalent graph with an admissible coloring, for example, as shown in the figure on the right. We regard a colored planar trivalent graph as in the \( E_6 \) linear skein, by substituting \( \in S(D^2, 2n) \) into each of the edges colored by \( n \), and substituting the following diagrams into vertices, \( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{image6} \\
\end{array} \), where we put \( a = \frac{-i+j+k}{2} \), \( b = \frac{-i+j+k}{2} \), \( c = \frac{-i+j-k}{2} \). In the rest of the article, summations run all admissible colorings of the colored planar trivalent graph in the summand.

We put

\[
d_n = \langle (0, j, k) \rangle \quad \text{and} \quad \theta(i, j, k, \bullet) = \langle (0, j, k) \rangle \quad \text{and} \quad \theta(2, 2, 2, \odot) = \langle (0, j, k) \rangle \]

for a triple \( i, j, k \in \{0, 2, 4\} \) such that \( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{image7} \\
\end{array} \) is one of the pictures in (3.1) up to rotation. By direct calculation, we can verify that the values \( d_n \) and \( \theta(\cdot) \) are positive real numbers.

For \( i_1, \ldots, i_n \in \{0, 2, 4\} \) \( (n \geq 2) \), we define the vector space \( H(i_1, \ldots, i_n) \) to be the subspace of \( S(D^2, i_1 + \cdots + i_n) \) spanned by planar diagrams of the form \( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{image8} \\
\end{array} \)

with \( T \) being a planar diagram in \( (D^2, i_1 + \cdots + i_n) \).

The following two Lemmas are shown by checking some properties of the \( E_6 \) linear skein.\]

**Lemma 3.1.** For any \( i, j \in \{0, 2, 4\} \) and \( T \in S(D^2, i + j) \), \( H(i, j) \) is 1-dimensional if \( i = j \), and is equal to 0 if \( i \neq j \).

**Lemma 3.2.** For any \( i, j \in \{0, 2, 4\} \), \( \begin{array}{c}
\includegraphics[width=0.1\textwidth]{image9} \\
\end{array} = \sum_{k, A} \frac{d_k}{\theta(i, j, k, A)} \begin{array}{c}
\includegraphics[width=0.1\textwidth]{image10} \\
\end{array} \in S(D^2, 2(i + j)) \).

The following proposition is shown by induction on \( n \), using Lemmas 3.1 and 3.2.

**Proposition 3.3.** For \( i_1, \ldots, i_n \in \{0, 2, 4\} \) \( (n \geq 3) \), the vector space \( H(i_1, \ldots, i_n) \) has a basis

\[
\left\{ \begin{array}{c}
\includegraphics[width=0.1\textwidth]{image11} \\
\end{array} \right\}_{(j_1, \ldots, j_{n-1}, A_1, \ldots, A_{n-2})}
\]

The following Proposition is obtained by using Lemmas 3.1 and 3.2.
Proposition 3.4 (the defining relation of the 6j-symbols). For \( i, j, k, l, m \in \{0, 2, 4\} \) and \( A, B \in \{\bullet, \varnothing\} \) with \( \frac{i}{A} \frac{j}{l} \frac{m}{D} \frac{j}{k} \) being admissible,

\[
\sum_{n, C, D} d_n \frac{\theta(k, l, n, C) \theta(i, j, n, D)}{\theta(i, j, n, D)} = \sum_{n, C, D} d_n \frac{\theta(k, l, n, C) \theta(i, j, n, D)}{\theta(i, j, n, D)}.
\]

From this proposition, we see that the coefficient values of the above formula give the 6j-symbols of the \( E_6 \) linear skein.

4. A state sum invariant of 3 manifolds

We relate an oriented tetrahedron to a planar trivalent graph, as the following picture.

We define an admissible coloring of a tetrahedron as the dual of an admissible coloring of a planar trivalent graph. In the definition, we consider a marking such as \( \triangleleft \) at a vertex of a triangle of the face corresponding to the base point of \( \varnothing \). For the details, see [Ok].

Let \( M \) be a closed oriented 3-manifold, and \( \mathcal{T} \) a triangulation of \( M \). A coloring of \( \mathcal{T} \) is a map from the set of faces of \( \mathcal{T} \) to \( \{\bullet, \varnothing\} \) and a map from the set of edges of \( \mathcal{T} \) to \( \{0, 2, 4\} \). A coloring of \( \mathcal{T} \) is said to be admissible if the coloring of each tetrahedron of \( \mathcal{T} \) is admissible. When a face is colored by \( \varnothing \), we consider a marking such as \( \triangleleft \) at a vertex of a triangle of the face. We define a weight \( |\cdot| \) of a colored oriented tetrahedron by

\[
|\cdot| = \sqrt{\theta(i, j, n, A \theta(j, k, l, B) \theta(l, m, n, C) \theta(i, k, m, D)},
\]

where the colored planar trivalent graph in the right-hand side is obtained from the tetrahedron as mentioned above. The position of the base points of the colored planar trivalent graph in the right-hand side is defined by the markings of the tetrahedron in the left-hand side. We denote by \( v \) the number of vertices of \( \mathcal{T} \). We put \( w = d_0^2 + d_2^2 + d_4^2 \).

Definition 4.1. We define the \( E_6 \) state sum of a closed oriented 3-manifold \( M \) with a triangulation \( \mathcal{T} \) by

\[
Z_{E_6}(M, \mathcal{T}) = w^{-v} \sum_{\lambda} \prod_{E} d_{\lambda(E)} \prod_{T} \langle \mathcal{T}, \lambda \rangle,
\]

where the sum of \( \lambda \) runs over all admissible colorings of \( \mathcal{T} \), the product of \( E \) runs over all edges of \( \mathcal{T} \), and the product of \( T \) runs all tetrahedra of \( \mathcal{T} \).
**Theorem 4.2.** $Z_{E_6}(M, T)$ is a topological invariant of a closed oriented 3-manifold $M$, independently of a choice of a triangulation $T$.

*Sketch of proof.* It is sufficient to show that $Z_{E_6}(M, T)$ is invariant under the Pachner moves. In order to show such an invariance, it is sufficient to show the orthogonal relation and the pentagon relation, which can be shown by using Proposition 3.4 repeatedly. For the details, see [Ok]. □

5. **Equality to the $E_6$ state sum invariant**

Bigelow [B] defined a planar algebra $\{S'(D^2, 2n)\}_{n=0,1,\ldots}$ (in his paper this is denoted by $P$) by giving generators and relations, and proved that its principal graph is the $E_6$ Dynkin diagram. $S'(D^2, 2n)$ is defined to be the vector space spanned by planar diagrams in $(D^2, 2n)$ subject to the relations (2.1)–(2.3),

$$\begin{align*}
\begin{tikzpicture}[scale=0.5]
\draw[thick] (0,0) -- (1,0);
\draw[thick] (0,0) -- (0,1);
\draw[thick] (1,0) -- (1,1);
\end{tikzpicture}
+ [2] [3]
\begin{tikzpicture}[scale=0.5]
\draw[thick] (0,0) -- (0,1);
\draw[thick] (1,0) -- (1,1);
\draw[thick] (0,0) -- (1,1);
\end{tikzpicture}
, \quad \text{and} \quad 
\begin{tikzpicture}[scale=0.5]
\draw[thick] (0,0) -- (1,0);
\draw[thick] (0,0) -- (0,1);
\draw[thick] (1,0) -- (1,1);
\end{tikzpicture}
= 0.
\end{align*}$$

By checking some properties of the $E_6$ linear skein, we can show that $S(D^2, 2n)$ is isomorphic to $S'(D^2, 2n)$ for any $n \geq 0$, which says our relations are equivalent to Bigelow’s relations.

We also show the following proposition.

**Proposition 5.1.** Our state sum invariant of a closed oriented 3-manifold $M$ defined in Section 4 is equal to the $E_6$ state sum invariant of $M$.

*Sketch of proof.* We transfer the colors of faces by an appreciate transformation, and we verify that the substitution of $S_3, S_4$ can be transformed into the substitution of $\bullet, \varnothing$ in the definition of the state sum invariant.

It is shown, see [Ok, Lemma B.1] (due to T. Ohtsuki), that our $6j$-symbols can be transformed into the $6j$-symbols of the $E_6$ subfactor by such a transformation as above. Hence, our state sum invariant is equal to the $E_6$ state sum invariant. □

**REFERENCES**


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