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WEIGHTED RICCI CURVATURE ESTIMATES FOR HILBERT AND FUNK GEOMETRIES

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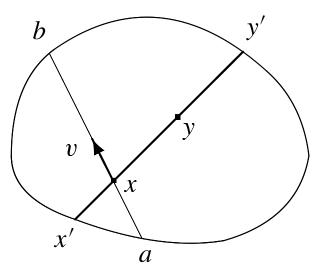
We consider Hilbert and Funk geometries on a strongly convex domain in Euclidean space. We show that, with respect to the Lebesgue measure on the domain, the Hilbert and Funk metrics have bounded and constant negative weighted Ricci curvature, respectively. As a corollary, these metric measure spaces satisfy the curvature-dimension condition in the sense of Lott, Sturm and Villani.

1. Introduction

Hilbert [1895] introduced the distance function $d_{\mathcal{H}}$ on a bounded convex domain $D \subset \mathbb{R}^n$, related to his fourth problem. Given distinct points $x, y \in D$, denoting by $x' = x + s(y - x)$ and $y' = x + t(y - x)$ the intersections of the boundary ∂D and the line passing through x and y with $s < 0 < t$ (see figure), Hilbert's distance $d_{\mathcal{H}}$ is given by

$$d_{\mathcal{H}}(x, y) = \frac{1}{2} \log \frac{|x' - y| \cdot |x - y'|}{|x' - x| \cdot |y - y'|},$$

where $|\cdot|$ stands for the Euclidean norm. This is indeed a distance function on D, and satisfies the interesting property that line segments between any points are minimizing. In the particular case where D is the unit ball, $(D, d_{\mathcal{H}})$ coincides with the Klein model of hyperbolic space. The structure of $(D, d_{\mathcal{H}})$ has been investigated



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from geometric and dynamical aspects; see, for example, [Egloff 1997; Benoist 2003; Colbois and Verovic 2004]. For instance, $(D, d_{\mathcal{H}})$ is known to be Gromov hyperbolic under mild smoothness and convexity assumptions on D.

Funk [1929] introduced a nonsymmetrization of $d_{\mathcal{H}}$, namely

$$d_{\mathcal{F}}(x, y) = \log \frac{|x - y'|}{|y - y'|}.$$

Note that $d_{\mathcal{F}}(x, y) \neq d_{\mathcal{F}}(y, x)$, while the triangle inequality

$$d_{\mathcal{F}}(x, z) \leq d_{\mathcal{F}}(x, y) + d_{\mathcal{F}}(y, z)$$

still holds. Clearly we have $2d_{\mathcal{H}}(x, y) = d_{\mathcal{F}}(x, y) + d_{\mathcal{F}}(y, x)$, and line segments are minimizing also with respect to Funk's distance.

If ∂D is smooth and D is strongly convex (in other words, ∂D is positively curved; see Definition 2.1), then $d_{\mathcal{H}}$ and $d_{\mathcal{F}}$ are realized by the smooth Finsler structures

$$\begin{aligned} F_{\mathcal{H}}(x, v) &= \frac{|v|}{2}\left(\frac{1}{|x-a|} + \frac{1}{|x-b|}\right), \\ F_{\mathcal{F}}(x, v) &= \frac{|v|}{|x-b|} \quad \text{for } v \in T_x D = \mathbb{R}^n, \end{aligned} \tag{1-1}$$

respectively (cf. [Shen 2001a, §2.3]), where $a = x + sv$ and $b = x + tv$ denote the intersections of ∂D and the line passing through x in the direction v with $s < 0 < t$ (see figure on page 185). Note that $2F_{\mathcal{H}}(x, v) = F_{\mathcal{F}}(x, v) + F_{\mathcal{F}}(x, -v)$. A remarkable feature of these metrics is that they have the constant negative flag curvatures -1 and $-\frac{1}{4}$, respectively; see [Okada 1983, Theorem 1; Shen 2001a, Theorem 12.2.11], provided that $n \geq 2$ as a matter of course. The flag curvature is a generalization of the sectional curvature in Riemannian geometry, so it is natural that $(D, d_{\mathcal{H}})$ and $(D, d_{\mathcal{F}})$ enjoy properties of negatively curved spaces.

Recently, the theory of the *weighted Ricci curvature* (see Definition 2.2) for Finsler manifolds equipped with arbitrary measures has been developed in connection with optimal transport theory. It turned out that the weighted Ricci curvature is a natural quantity and quite useful in the study of geometry and analysis on Finsler manifolds; see [Ohta 2009a; 2012; Ohta and Sturm 2009; 2011]. The aim of this article is to show that the weighted Ricci curvature for Hilbert and Funk geometries admits uniform bounds with respect to the Lebesgue measure m_L restricted on D.

Theorem 1.1 (Funk case). *Let $D \subset \mathbb{R}^n$ with $n \geq 2$ be a strongly convex domain such that ∂D is smooth. Then $(D, F_{\mathcal{F}}, m_L)$ has constant negative weighted Ricci curvature: specifically, for any unit vector $v \in TD$,*

$$\mathrm{Ric}_\infty(v) = -\frac{n-1}{4}, \qquad \mathrm{Ric}_N(v) = -\frac{n-1}{4} - \frac{(n+1)^2}{4(N-n)} \quad \textit{for } N \in (n, \infty).$$

Theorem 1.2 (Hilbert case). *Let $D \subset \mathbb{R}^n$ with $n \geq 2$ be a strongly convex domain such that ∂D is smooth. Then the weighted Ricci curvature of $(D, F_{\mathcal{H}}, m_L)$ is bounded*; *specifically*, *for any unit vector $v \in TD$*,

$$\mathrm{Ric}_\infty(v) \in \big(-(n-1), 2\big], \quad \mathrm{Ric}_N(v) \in \left(-(n-1) - \frac{(n+1)^2}{N-n},\ 2\right] \quad \textit{for } N \in (n, \infty).$$

We stress that our estimates are independent of the choice of the domain D. There are several applications (Corollaries 5.1, 5.2) via the theory of the weighted Ricci curvature.

The article is organized as follows. After preliminaries for Finsler geometry and the weighted Ricci curvature, we prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. We finally discuss applications and remarks in Section 5.

2. Preliminaries

We very briefly review the necessary notions in Finsler geometry; we refer to [Bao et al. 2000; Shen 2001a; 2001b] for further reading. Let M be a connected, n-dimensional $\mathscr{C}^\infty$-manifold without boundary such that $n \geq 2$. Given a local coordinate $(x^i)_{i=1}^n$ on an open set $\Omega \subset M$, we always use the coordinate $(x^i, v^j)_{i,j=1}^n$ of $T\Omega$ such that

$$v = \sum_{j=1}^n v^j \frac{\partial}{\partial x^j}\Big|_x \in T_x M \quad \text{for } x \in \Omega.$$

Definition 2.1 (Finsler structures). A nonnegative function $F : TM \to [0, \infty)$ is called a $\mathscr{C}^\infty$-*Finsler structure* of M if the following three conditions hold.

(1) (*Regularity*) F is $\mathscr{C}^\infty$ on $TM \setminus 0$, where 0 stands for the zero section.

(2) (*Positive* 1-*homogeneity*) It holds $F(cv) = cF(v)$ for all $v \in TM$ and $c > 0$.

(3) (*Strong convexity*) The $n \times n$ matrix

$$\big(g_{ij}(v)\big)_{i,j=1}^n := \left(\frac{1}{2}\frac{\partial^2(F^2)}{\partial v^i \partial v^j}(v)\right)_{i,j=1}^n \tag{2-1}$$

is positive definite for all $v \in TM \setminus 0$.

For $x, y \in M$, we can define the *distance* from x to y in a natural way by

$$d(x, y) := \inf_\eta \int_0^1 F\big(\dot\eta(t)\big)\, dt,$$

where the infimum is taken over all $\mathscr{C}^1$-curves $\eta : [0, 1] \to M$ with $\eta(0) = x$ and $\eta(1) = y$. This distance can be *nonsymmetric* (namely $d(y, x) \neq d(x, y)$), since F is only positively homogeneous. A $\mathscr{C}^\infty$-curve η on M is called a *geodesic* if it is locally minimizing and has a constant speed (i.e., $F(\dot\eta)$ is constant).

Given $v \in T_xM$, if there is a geodesic $\eta : [0, 1] \to M$ with $\dot\eta(0) = v$, then we define the *exponential map* by $\exp_x(v) := \eta(1)$. We say that (M, F) is *forward complete* if the exponential map is defined on whole TM. If the *reverse* Finsler manifold $(M, \overleftarrow{F})$ with $\overleftarrow{F}(v) := F(-v)$ is forward complete, then (M, F) is said to be *backward complete*. We remark that $(D, F_{\mathcal{H}})$ is both forward and backward complete (they are indeed equivalent since $\overleftarrow{F_{\mathcal{H}}} = F_{\mathcal{H}}$), while $(D, F_{\mathcal{F}})$ is only forward complete.

For each $v \in T_xM \setminus 0$, the positive definite matrix $(g_{ij}(v))_{i,j=1}^n$ in (2-1) induces the Riemannian structure g_v of T_xM as

$$g_v\biggl(\sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\Big|_x, \sum_{j=1}^n b_j \frac{\partial}{\partial x^j}\Big|_x\biggr) := \sum_{i,j=1}^n a_i b_j g_{ij}(v). \tag{2-2}$$

Note that $g_{cv} = g_v$ for $c > 0$. This inner product is regarded as the best Riemannian approximation of $F|_{T_xM}$ in the direction v, in the sense that the unit sphere of g_v is tangent to that of $F|_{T_xM}$ at $v/F(v)$ up to the second order. In particular, we have $g_v(v, v) = F(v)^2$.

The *Ricci curvature* (as the trace of the *flag curvature*) for a Finsler manifold is defined by using the Chern connection. Instead of giving the precise definition in coordinates, we explain a useful interpretation due to Shen [2001b, §6.2; 1997, Lemma 2.4]. Given a unit vector $v \in T_xM \cap F^{-1}(1)$, we extend it to a nonvanishing $\mathscr{C}^\infty$-vector field V on a neighborhood of x in such a way that every integral curve of V is geodesic, and consider the Riemannian structure g_V induced from (2-2). Then the Ricci curvature $\mathrm{Ric}(v)$ of v with respect to F coincides with the Ricci curvature of v with respect to g_V (in particular, it is independent of the choice of V).

Let us fix a positive $\mathscr{C}^\infty$-measure m on M. Inspired by the above interpretation of the Finsler Ricci curvature and the theory of weighted Riemannian manifolds, the weighted Ricci curvature for the triple (M, F, m) was introduced in [Ohta 2009a] as follows.

Definition 2.2 (weighted Ricci curvature). Given a unit vector $v \in T_xM \cap F^{-1}(1)$, let $\eta : (-\varepsilon, \varepsilon) \to M$ be the geodesic such that $\dot\eta(0) = v$. We decompose m along η using the Riemannian volume measure $\mathrm{vol}_{\dot\eta}$ of $g_{\dot\eta}$ as $m = e^{-\Psi}\,\mathrm{vol}_{\dot\eta}$, where $\Psi : (-\varepsilon, \varepsilon) \to \mathbb{R}$. Then we define the *weighted Ricci curvature* involving a parameter $N \in [n, \infty]$ by

(1) $\mathrm{Ric}_n(v) := \begin{cases} \mathrm{Ric}(v) + \Psi''(0) & \text{if } \Psi'(0) = 0, \\ -\infty & \text{if } \Psi'(0) \neq 0, \end{cases}$

(2) $\mathrm{Ric}_N(v) := \mathrm{Ric}(v) + \Psi''(0) - \dfrac{\Psi'(0)^2}{N-n} \quad \text{for } N \in (n, \infty),$

(3) $\mathrm{Ric}_\infty(v) := \mathrm{Ric}(v) + \Psi''(0).$

We also set $\mathrm{Ric}_N(cv) := c^2\,\mathrm{Ric}_N(v)$ for $c \geq 0$.

We will say that $\mathrm{Ric}_N \geq K$ holds for some $K \in \mathbb{R}$ if $\mathrm{Ric}_N(v) \geq KF(v)^2$ for all $v \in TM$. Observe that $\mathrm{Ric}_N(v) \leq \mathrm{Ric}_{N'}(v)$ for $N < N'$, and that for the scaled space $M' = (M, F, am)$ with $a > 0$ we have $\mathrm{Ric}_N^{M'}(v) = \mathrm{Ric}_N^{M}(v)$. It was shown in [Ohta 2009a, Theorem 1.2] that $\mathrm{Ric}_N \geq K$ is equivalent to Lott, Sturm and Villani's *curvature-dimension condition* $\mathsf{CD}(K, N)$. (Roughly speaking, the curvature-dimension condition is a convexity condition of an entropy functional on the space of probability measures; we refer to [Sturm 2006a; 2006b; Lott and Villani 2007; 2009; Villani 2009, Part III] for details and further theories.) This equivalence extends the corresponding result on (weighted) Riemannian manifolds, and has many analytic and geometric applications; see [Ohta 2009a].

3. The Funk case

We turn to the proof of Theorem 1.1. For brevity, we denote the Funk metric simply by F, and we consider the standard coordinate of $D \subset \mathbb{R}^n$. The following lemma enables us to translate all the vertical derivatives $(\partial/\partial v^i)$ into horizontal derivatives $(\partial/\partial x^i)$.

Lemma 3.1 [Okada 1983, Proposition 1; Shen 2001a, Lemma 2.3.1]. *For any $v \in TD \setminus 0$ and $i = 1, 2, \dots, n$, we have*

$$\frac{\partial F}{\partial x^i}(v) = F(v)\frac{\partial F}{\partial v^i}(v).$$

Proof of Theorem 1.1. On $TD \setminus 0$,

$$\frac{1}{2}\frac{\partial^2(F^2)}{\partial v^i \partial v^j} = \frac{\partial}{\partial v^i}\left(\frac{\partial F}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\left(\frac{1}{F}\frac{\partial F}{\partial x^i}\right) = \frac{1}{F}\frac{\partial^2 F}{\partial x^i \partial x^j} - \frac{1}{F^2}\frac{\partial F}{\partial x^i}\frac{\partial F}{\partial x^j}. \tag{3-1}$$

Now, we fix a unit vector $v \in T_x D \cap F^{-1}(1)$ and choose a coordinate such that x is the origin, $v = \partial/\partial x^n$ and $g_{in}(v) = 0$ for all $i = 1, 2, \dots, n-1$. Such a coordinate exchange multiplies the Lebesgue measure merely by a positive constant, so the weighted Ricci curvature does not change. Put $V := \partial/\partial x^n$ on D and recall that the all integral curves of V are minimizing (and hence reparametrizations of geodesics). Therefore it suffices to calculate the weighted Ricci curvature of (D, g_V, m_L).

We can represent $\partial D \cap \{x \in \mathbb{R}^n \mid x^n > 0\}$ as the graph of a $\mathscr{C}^\infty$-function $h : U \to (0, \infty)$ for a sufficiently small neighborhood $U \subset \mathbb{R}^{n-1}$ of 0, namely

$$\partial D \cap \{(z, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid z \in U,\ t > 0\} = \{(z, h(z)) \mid z \in U\}. \tag{3-2}$$

Then (1-1) yields

$$F(V(z, t)) = \frac{1}{h(z) - t} \quad \text{for } (z, t) \in D \subset \mathbb{R}^{n-1} \times \mathbb{R}.$$

Putting $\partial_i := \partial/\partial x^i$ for simplicity, we deduce from (3-1) that

$$\begin{aligned} g_{ij}(V) &= (h-t)\partial_i\partial_j\Big(\frac{1}{h-t}\Big) - (h-t)^2\partial_i\Big(\frac{1}{h-t}\Big)\partial_j\Big(\frac{1}{h-t}\Big) \\ &= (h-t)\bigg(-\frac{\partial_i\partial_j(h-t)}{(h-t)^2} + \frac{2\partial_i(h-t)\partial_j(h-t)}{(h-t)^3}\bigg) - \frac{\partial_i(h-t)\partial_j(h-t)}{(h-t)^2} \\ &= -\frac{\partial_i\partial_j(h-t)}{h-t} + \frac{\partial_i(h-t)\partial_j(h-t)}{(h-t)^2}, \end{aligned}$$

where the evaluations at $(z,t)\in D$ were omitted. We remark that, for $i, j \neq n$,

$$g_{ij}(V) = -\frac{\partial_i\partial_j h}{h-t} + \frac{\partial_i h\partial_j h}{(h-t)^2}, \quad g_{in}(V) = -\frac{\partial_i h}{(h-t)^2}, \quad g_{nn}(V) = \frac{1}{(h-t)^2}.$$

Hence, when differentiating $g_{ij}(V(z,t))$ by t, we need to take only the denominators into account. Thus we find

$$\begin{aligned} \frac{\partial[g_{ij}(V)]}{\partial t} &= -\frac{\partial_i\partial_j(h-t)}{(h-t)^2} + \frac{2\partial_i(h-t)\partial_j(h-t)}{(h-t)^3} \\ &= \frac{1}{h-t}\bigg(g_{ij}(V) + \frac{\partial_i(h-t)\partial_j(h-t)}{(h-t)^2}\bigg). \end{aligned}$$

Decomposing $m_{\boldsymbol{L}}$ as

$$m_{\boldsymbol{L}} = e^{-\Psi}\sqrt{\det(g_{ij}(V))}\,dx^1dx^2\cdots dx^n$$

along the curve $\eta(t) = (0,t)\in D$, we observe

$$\Psi(t) = \tfrac{1}{2}\log\det(g_{ij}(t)), \qquad \Psi'(t) = \tfrac{1}{2}\operatorname{trace}\big[(g^{ij}(t))\cdot(g'_{ij}(t))\big],$$

where we abbreviated as $g_{ij}(t) := g_{ij}(V(0,t))$ and $(g^{ij}(t))$ stands for the inverse matrix of $(g_{ij}(t))$. Dividing $\Psi'(t)$ by the speed $F(\dot\eta(t)) = F(V(0,t)) = (h(0)-t)^{-1}$, we obtain

$$\big(h(0)-t\big)\Psi'(t) = \tfrac{1}{2}\operatorname{trace}\bigg[(g^{ij}(t))\cdot\bigg(g_{ij}(t) + \frac{\partial_i(h(0)-t)\partial_j(h(0)-t)}{(h(0)-t)^2}\bigg)\bigg] \equiv \frac{n+1}{2},$$

where the second equality follows from the fact that $g_{in}(t) = -\partial_i h(0)/(h(0)-t)^2 = 0$ for $i\neq n$, guaranteed by $g_{in}(v)=0$. As (D,F) has constant flag curvature $-\frac{1}{4}$, we therefore conclude that

$$\mathrm{Ric}_\infty(v) = -\frac{n-1}{4}, \qquad \mathrm{Ric}_N(v) = -\frac{n-1}{4} - \frac{(n+1)^2}{4(N-n)}. \qquad \square$$

4. The Hilbert case

We next consider the Hilbert case, where the calculation is similar but more involved. Now F will denote the Hilbert metric of D.

Proof of Theorem 1.2. Given a unit vector $v \in T_x D \cap F^{-1}(1)$, similarly to the previous section, we choose a coordinate such that x is the origin, $v = \partial/\partial x^n$ and that $g_{in}(v) = 0$ for all $i = 1, 2, \dots, n-1$. Put $V := \partial/\partial x^n$ again. In addition to $h : U \to (0, \infty)$ as in (3-2), we introduce the function $b : U \to (-\infty, 0)$ such that

$$\partial D \cap \{(z, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid z \in U,\ t < 0\} = \{(z, b(z)) \mid z \in U\}.$$

Using the Funk metric F_+ of D and its reverse $F_-(v) := F_+(-v)$, and recalling (1-1), we can write $F(V)$ as

$$F(V(z,t)) = \frac{F_+(V(z,t)) + F_-(V(z,t))}{2} = \frac{1}{2}\left(\frac{1}{h(z)-t} + \frac{1}{t-b(z)}\right).$$

It follows from Lemma 3.1 and $F_-(v) = F_+(-v)$ that

$$\frac{\partial F_-}{\partial x^i} = -F_- \frac{\partial F_-}{\partial v^i}.$$

This yields

$$\begin{aligned}
2\frac{\partial^2(F^2)}{\partial v^i \partial v^j} &= \frac{1}{2}\frac{\partial^2}{\partial v^i \partial v^j}(F_+^2 + 2F_+F_- + F_-^2) \\
&= \frac{1}{2}\frac{\partial^2(F_+^2)}{\partial v^i \partial v^j} + \frac{1}{2}\frac{\partial^2(F_-^2)}{\partial v^i \partial v^j} - \frac{\partial_i F_+}{F_+}\frac{\partial_j F_-}{F_-} - \frac{\partial_j F_+}{F_+}\frac{\partial_i F_-}{F_-} \\
&\quad + \left(\frac{\partial_i \partial_j F_+}{F_+^2} - \frac{2\partial_i F_+ \partial_j F_+}{F_+^3}\right)F_- + \left(\frac{\partial_i \partial_j F_-}{F_-^2} - \frac{2\partial_i F_- \partial_j F_-}{F_-^3}\right)F_+.
\end{aligned}$$

By (3-1) we have, omitting the evaluations at $(z, t) \in D$,

$$\begin{aligned}
4g_{ij}(V) &= -\frac{\partial_i\partial_j(h-t)}{h-t} + \frac{\partial_i(h-t)\partial_j(h-t)}{(h-t)^2} - \frac{\partial_i\partial_j(t-b)}{t-b} + \frac{\partial_i(t-b)\partial_j(t-b)}{(t-b)^2} \\
&\quad - \left(\frac{\partial_i(h-t)}{h-t}\frac{\partial_j(t-b)}{t-b} + \frac{\partial_j(h-t)}{h-t}\frac{\partial_i(t-b)}{t-b}\right) \\
&\qquad - \frac{\partial_i\partial_j(h-t)}{t-b} - \frac{\partial_i\partial_j(t-b)}{h-t} \\
&= -(\partial_i\partial_j(h-t) + \partial_i\partial_j(t-b))\left(\frac{1}{h-t} + \frac{1}{t-b}\right) \\
&\quad + \left(\frac{\partial_i(h-t)}{h-t} - \frac{\partial_i(t-b)}{t-b}\right)\left(\frac{\partial_j(h-t)}{h-t} - \frac{\partial_j(t-b)}{t-b}\right).
\end{aligned}$$

Note that the assumption $g_{in}(v) = 0$ implies

$$\frac{\partial_i h(0)}{h(0)} - \frac{\partial_i b(0)}{b(0)} = 0 \quad \text{for } i = 1, 2, \dots, n-1. \tag{4-1}$$

We also observe for later convenience that, for $i, j \neq n$,

$$4g_{ij}(v) = -(\partial_i\partial_j h(0) - \partial_i\partial_j b(0))\left(\frac{1}{h(0)} - \frac{1}{b(0)}\right), \quad 4g_{nn}(v) = \left(\frac{1}{h(0)} - \frac{1}{b(0)}\right)^2.$$

By the same reasoning as the Funk case, the numerators can be neglected when one differentiates $g_{ij}(V)$ with respect to t. Thus we find

$$\begin{aligned}4\frac{\partial g_{ij}(V)}{\partial t} &= -(\partial_i\partial_j(h-t) + \partial_i\partial_j(t-b))\left(\frac{1}{(h-t)^2} - \frac{1}{(t-b)^2}\right)\\ &\quad + \left(\frac{\partial_i(h-t)}{(h-t)^2} + \frac{\partial_i(t-b)}{(t-b)^2}\right)\left(\frac{\partial_j(h-t)}{h-t} - \frac{\partial_j(t-b)}{t-b}\right)\\ &\quad + \left(\frac{\partial_i(h-t)}{h-t} - \frac{\partial_i(t-b)}{t-b}\right)\left(\frac{\partial_j(h-t)}{(h-t)^2} + \frac{\partial_j(t-b)}{(t-b)^2}\right).\end{aligned}$$

We further calculate

$$\begin{aligned}4\frac{\partial^2[g_{ij}(V)]}{\partial t^2} &= -(\partial_i\partial_j(h-t) + \partial_i\partial_j(t-b))\left(\frac{2}{(h-t)^3} + \frac{2}{(t-b)^3}\right)\\ &\quad + \left(\frac{2\partial_i(h-t)}{(h-t)^3} - \frac{2\partial_i(t-b)}{(t-b)^3}\right)\left(\frac{\partial_j(h-t)}{h-t} - \frac{\partial_j(t-b)}{t-b}\right)\\ &\quad + \left(\frac{\partial_i(h-t)}{h-t} - \frac{\partial_i(t-b)}{t-b}\right)\left(\frac{2\partial_j(h-t)}{(h-t)^3} - \frac{2\partial_j(t-b)}{(t-b)^3}\right)\\ &\quad + 2\left(\frac{\partial_i(h-t)}{(h-t)^2} + \frac{\partial_i(t-b)}{(t-b)^2}\right)\left(\frac{\partial_j(h-t)}{(h-t)^2} + \frac{\partial_j(t-b)}{(t-b)^2}\right).\end{aligned}$$

We abbreviate as $g_{ij}(t) := g_{ij}(V(0,t))$ and deduce from (4-1) that, for $i, j \neq n$,

$$\begin{aligned}4g'_{ij}(0) &= 4g_{ij}(0)\left(\frac{1}{h(0)} + \frac{1}{b(0)}\right),\\ 4g'_{in}(0) &= -\left(\frac{\partial_i h(0)}{h(0)^2} - \frac{\partial_i b(0)}{b(0)^2}\right)\left(\frac{1}{h(0)} - \frac{1}{b(0)}\right),\\ 4g'_{nn}(0) &= 8g_{nn}(0)\left(\frac{1}{h(0)} + \frac{1}{b(0)}\right).\end{aligned}$$

We also obtain, for $i, j \neq n$,

$$\begin{aligned}4g''_{ij}(0) &= 8g_{ij}(0)\left(\frac{1}{h(0)^2} + \frac{1}{h(0)b(0)} + \frac{1}{b(0)^2}\right)\\ &\quad + 2\left(\frac{\partial_i h(0)}{h(0)^2} - \frac{\partial_i b(0)}{b(0)^2}\right)\left(\frac{\partial_j h(0)}{h(0)^2} - \frac{\partial_j b(0)}{b(0)^2}\right),\\ 4g''_{nn}(0) &= 8g_{nn}(0)\left(2\left(\frac{1}{h(0)^2} + \frac{1}{h(0)b(0)} + \frac{1}{b(0)^2}\right) + \left(\frac{1}{h(0)} + \frac{1}{b(0)}\right)^2\right).\end{aligned}$$

Put $\Psi(t) = 2^{-1}\log(\det(g_{ij}(t)))$ and observe

$$\begin{aligned}\Psi'(t) &= \tfrac{1}{2}\operatorname{trace}\big[(g^{ij}(t))\cdot(g'_{ij}(t))\big],\\ \Psi''(t) &= \tfrac{1}{2}\operatorname{trace}\big[(g^{ij}(t))\cdot(g''_{ij}(t)) - \big((g^{ij}(t))\cdot(g'_{ij}(t))\big)^2\big].\end{aligned}$$

Comparing $g_{ij}(0)$ and $g'_{ij}(0)$, we have

$$\Psi'(0) = \tfrac{1}{2}\left((n-1)\left(\frac{1}{h(0)}+\frac{1}{b(0)}\right)+2\left(\frac{1}{h(0)}+\frac{1}{b(0)}\right)\right) = \frac{n+1}{2}\left(\frac{1}{h(0)}+\frac{1}{b(0)}\right).$$

Similarly,

$$\begin{aligned}&\tfrac{1}{2}\operatorname{trace}\big[(g^{ij}(0))\cdot(g''_{ij}(0))\big]\\ &\quad= (n-1)\left(\frac{1}{h(0)^2}+\frac{1}{h(0)b(0)}+\frac{1}{b(0)^2}\right)\\ &\qquad\quad+\frac{1}{4}\sum_{i,j=1}^{n-1} g^{ij}(0)\left(\frac{\partial_i h(0)}{h(0)^2}-\frac{\partial_i b(0)}{b(0)^2}\right)\left(\frac{\partial_j h(0)}{h(0)^2}-\frac{\partial_j b(0)}{b(0)^2}\right)\\ &\qquad\qquad\qquad+2\left(\frac{1}{h(0)^2}+\frac{1}{h(0)b(0)}+\frac{1}{b(0)^2}\right)+\left(\frac{1}{h(0)}+\frac{1}{b(0)}\right)^2\\ &\quad= (n+1)\left(\frac{1}{h(0)^2}+\frac{1}{h(0)b(0)}+\frac{1}{b(0)^2}\right)+\left(\frac{1}{h(0)}+\frac{1}{b(0)}\right)^2\\ &\qquad\qquad\qquad+\frac{1}{4}\sum_{i,j=1}^{n-1} g^{ij}(0)\left(\frac{\partial_i h(0)}{h(0)^2}-\frac{\partial_i b(0)}{b(0)^2}\right)\left(\frac{\partial_j h(0)}{h(0)^2}-\frac{\partial_j b(0)}{b(0)^2}\right).\end{aligned}$$

Combining this with

$$\begin{aligned}&\operatorname{trace}\big[((g^{ij}(0))\cdot(g'_{ij}(0)))^2\big]\\ &\quad= (n-1)\left(\frac{1}{h(0)}+\frac{1}{b(0)}\right)^2+4\left(\frac{1}{h(0)}+\frac{1}{b(0)}\right)^2\\ &\qquad+\frac{g^{nn}(0)}{8}\sum_{i,j=1}^{n-1} g^{ij}(0)\left(\frac{\partial_i h(0)}{h(0)^2}-\frac{\partial_i b(0)}{b(0)^2}\right)\left(\frac{\partial_j h(0)}{h(0)^2}-\frac{\partial_j b(0)}{b(0)^2}\right)\left(\frac{1}{h(0)}-\frac{1}{b(0)}\right)^2\\ &\quad=(n+3)\left(\frac{1}{h(0)}+\frac{1}{b(0)}\right)^2+\frac{1}{2}\sum_{i,j=1}^{n-1} g^{ij}(0)\left(\frac{\partial_i h(0)}{h(0)^2}-\frac{\partial_i b(0)}{b(0)^2}\right)\left(\frac{\partial_j h(0)}{h(0)^2}-\frac{\partial_j b(0)}{b(0)^2}\right),\end{aligned}$$

we obtain

$$\Psi''(0) = (n+1)\left(\frac{1}{h(0)^2} + \frac{1}{h(0)b(0)} + \frac{1}{b(0)^2}\right) - \frac{n+1}{2}\left(\frac{1}{h(0)} + \frac{1}{b(0)}\right)^2$$
$$= \frac{n+1}{2}\left(\frac{1}{h(0)^2} + \frac{1}{b(0)^2}\right).$$

Therefore we have, as $F(v) = (h(0)^{-1} - b(0)^{-1})/2 = 1$,

$$\frac{d}{dt}\left[\frac{\Psi'(t)}{F(V(0,t))}\right]_{t=0} = \Psi''(0) - \frac{\Psi'(0)}{2}\left(\frac{1}{h(0)^2} - \frac{1}{b(0)^2}\right) = -\frac{n+1}{h(0)b(0)}.$$

Since

$$0 < -\frac{1}{h(0)b(0)} \le \frac{1}{4}\left(\frac{1}{h(0)} - \frac{1}{b(0)}\right)^2 = 1,$$

this yields $\mathrm{Ric}_\infty(v) \in (-(n-1), 2]$. Moreover,

$$\Psi'(0)^2 = \frac{(n+1)^2}{4}\left(\frac{1}{h(0)} + \frac{1}{b(0)}\right)^2 = (n+1)^2\left(1 + \frac{1}{h(0)b(0)}\right) \in [0, (n+1)^2)$$

shows that

$$\mathrm{Ric}_N(v) \in \left(-(n-1) - \frac{(n+1)^2}{N-n}, 2\right]. \qquad \square$$

5. Applications and remarks

As mentioned in Section 2, $\mathrm{Ric}_N \ge K$ is equivalent to the curvature-dimension condition $\mathsf{CD}(K, N)$. Spaces satisfying $\mathsf{CD}(K, N)$ enjoy a number of properties similar to Riemannian manifolds of $\mathrm{Ric} \ge K$ and $\dim \le N$. Since $\mathsf{CD}(K, N)$ (between compactly supported measures) is preserved under the *pointed measured Gromov–Hausdorff convergence* of locally compact, complete metric measure spaces [Villani 2009, Theorem 29.25], we can deal with merely bounded, convex domains D.

Corollary 5.1. *Let $D \subset \mathbb{R}^n$ be a bounded convex domain with $n \ge 2$. Then the metric measure spaces $(D, d_{\mathcal{F}}, m_{\boldsymbol{L}})$ and $(D, d_{\mathcal{H}}, m_{\boldsymbol{L}})$ satisfy* $\mathsf{CD}(K, N)$ *for $N \in (n, \infty]$ with*

$$K = -\frac{n-1}{4} - \frac{(n+1)^2}{4(N-n)}, \qquad K = -(n-1) - \frac{(n+1)^2}{N-n},$$

respectively, where we read $K = -(n-1)/4$ and $K = -(n-1)$ when $N = \infty$. In particular, they satisfy

- *the Brunn–Minkowski inequality by* $\mathsf{CD}(K, N)$ *with $N \in (n, \infty]$,*
- *the Bishop–Gromov volume comparison by* $\mathsf{CD}(K, N)$ *with $N \in (n, \infty)$.*

See [Sturm 2006b, Proposition 2.1, Theorem 2.3] (and, for $N = \infty$, also [Villani 2009, Theorem 30.7; Ohta 2010, Theorem 6.1]) for the precise statements of the Brunn–Minkowski inequality and the Bishop–Gromov volume comparison. Beyond the general theory of the curvature-dimension condition, the weighted Ricci curvature bound implies the following.

Corollary 5.2. *Let $D \subset \mathbb{R}^n$ with $n \geq 2$ be a strongly convex domain such that ∂D is smooth. For K as in Corollary 5.1, $(D, F_{\mathcal{F}}, m_{\boldsymbol{L}})$ and $(D, F_{\mathcal{H}}, m_{\boldsymbol{L}})$ satisfy*

- *the Laplacian comparison for $N \in (n, \infty)$,*
- *the Bochner–Weitzenböck inequality for $N \in (n, \infty]$.*

See [Ohta and Sturm 2009, Theorem 5.2] for the Laplacian comparison, and [Ohta and Sturm 2011, Theorems 3.3, 3.6] for the Bochner–Weitzenböck formula (by the Bochner–Weitzenböck inequality we meant the inequality given by plugging the weighted Ricci curvature bound into the Bochner–Weitzenböck formula).

We conclude the article with remarks on possible improvements of the estimates in Theorems 1.1, 1.2. Our estimates on Ric_N with respect to $m_{\boldsymbol{L}}$ are independent of the shape of D. In particular, Theorem 1.2 provides the same (far from optimal) estimates even for the Klein model of the hyperbolic spaces. Thus there would be a better choice of a measure depending on the shape of D. Then, as an arbitrary measure is represented by $e^{-\psi} m_{\boldsymbol{L}}$, its weighted Ricci curvature is calculated by combining Theorems 1.1, 1.2 and the convexity of ψ. One may think of the squared distance function from some point as a candidate of ψ, however, in order to estimate its convexity along geodesics, we need to bound not only the flag curvature but also the *uniform convexity* as well as the *tangent curvature* (also called the **S**-*curvature*; see [Ohta 2009b, Theorem 5.1]). The uniform convexity is measured by the constant

$$\mathbf{C} = \sup_{x \in M} \sup_{v, w \in T_x M \setminus 0} \frac{F(w)}{g_v(w, w)^{1/2}},$$

and it is infinite for Funk metrics. As for Hilbert geometry, one could bound $\mathbf{C}$ by the convexity of ∂D (but this seems unclear; see [Egloff 1997, Remark 2.1]). The author has no idea about the tangent curvature, which measures how the tangent spaces are distorted as one moves in M.

There are several natural constructive measures m on D, and it is interesting to consider the corresponding weighted Ricci curvature $\mathrm{Ric}_N^m(V)$. Then, however, it seems not easy (at least more difficult than $m_{\boldsymbol{L}}$) to calculate $\mathrm{Ric}_N^m(V)$ because m should depend on the shape of whole ∂D, while g_V is induced only from the behavior of $F_{\mathcal{F}}$ or $F_{\mathcal{H}}$ near the direction V.

We also remark that, in Hilbert geometry (which is both forward and backward complete), Ric_N with $N < \infty$ cannot be nonnegative for any measure. Otherwise, g_V splits isometrically, which is a contradiction [Ohta 2012, Proposition 4.3]. Due to the same reasoning, Ric_∞ can be nonnegative only when $\sup \Psi = \infty$.

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