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Bisubmodular Polyhedra, Simplicial Divisions, and Discrete Convexity

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Abstract

We consider a class of integer-valued discrete convex functions, called BS-convex functions, defined on integer lattices whose affinity domains are sets of integral points of integral bisubmodular polyhedra. We examine discrete structures of BS-convex functions and give a characterization of BS-convex functions in terms of their convex conjugate functions by means of (discordant) Freudenthal simplicial divisions of the dual space.

Keywords: BS-convex functions, bisubmodular polyhedra, the Freudenthal simplicial divisions, discrete convexity

2000 MSC: 90C27, 52B40, 52A41, 05C22

1. Introduction

Kazuo Murota [13] has developed the theory of discrete convex functions such as M- and M♮-convex functions and L- and L♮-convex functions (also see [10, Chapter VII]). The class of integer-valued such discrete convex functions defined on integer lattices is the most fundamental, where M♮-convex functions have generalized polymatroids [8, 12] as their affinity domains and L♮-convex functions have convex extensions with respect to the Freudenthal simplicial divisions.

Murota’s M- and M♮-convex functions and L- and L♮-convex functions arise in many discrete optimization problems that have efficient solution al-

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gorithms (see, e.g., [15, 18]). Nice combinatorial structures of such discrete convex functions come from a kind of local submodularity or matroidal structure, i.e., the greediness property of Jack Edmonds [7] or the structures of affinity domains, of the conjugate functions, composed by the Freudenthal simplices (the details will be discussed in Sections 3 and 4).

In this paper we consider a class of integer-valued discrete convex functions, called BS-convex functions, which are defined on integer lattices and whose affinity domains are sets of integral points of integral bisubmodular polyhedra. As shown in [1, 3, 4, 5, 10, 16], (integral) bisubmodular polyhedra have a signed greediness property and their conjugate functions have affinity domains composed of reflected Freudenthal simplices. The class of BS-convex functions has not yet been fully investigated in the paradigm of Murota’s discrete convex analysis.

We introduce, in Section 2, the concept of BS-convex function. In Section 3 we examine the combinatorial structures of BS-convex functions in detail, especially the half-integrality property of their gradient vectors. Moreover, we give a characterization of BS-convex functions by means of the Freudenthal simplicial divisions and the Union-Jack simplicial divisions of the dual space in Section 4. Some concluding remarks are given in Section 5.

2. Bisubmodular polyhedra

Let $V$ be a finite nonempty set and $3^V$ be the set of ordered pairs $(X, Y)$ of disjoint subsets $X, Y \subseteq V$. Denote by $\mathbf{Z}$ and $\mathbf{R}$ the set of integers and that of reals, respectively. Also define $\frac{1}{2}\mathbf{Z} = \{ \frac{k}{2} \mid k \in \mathbf{Z} \}$. Any element in $\frac{1}{2}\mathbf{Z}$ is called half-integral and is called a half-integer if it is not an integer. Any vector $x$ in $\left( \frac{1}{2}\mathbf{Z} \right)^V$ is called half-integral and is called integral if $x(v)$ is an integer for each $v \in V$. For any $X \subseteq V$ define $\chi_X \in \{0, 1\}^V$ to be the characteristic vector of $X$, i.e., $\chi_X(v) = 1$ for $v \in X$ and $\chi_X(v) = 0$ for $v \in V \setminus X$. When $X$ is a singleton $\{w\}$, we also write $\chi_w$ as $\chi_{\{w\}}$. For any $x \in \mathbf{R}^V$ and $X \subseteq V$ define $x(X) = \sum_{v \in X} x(v)$, where $x(\emptyset) = 0$.

Let $f : 3^V \rightarrow \mathbf{R}$ be a bisubmodular function, i.e., for every $(X, Y), (W, Z) \in 3^V$ we have

$$f(X, Y) + f(W, Z) \geq f((X, Y) \sqcup (W, Z)) + f((X, Y) \sqcap (W, Z)), \tag{1}$$

where $(X, Y) \sqcup (W, Z) = ((X \sqcup W) \setminus (Y \sqcup Z), (Y \sqcup Z) \setminus (X \sqcup W))$ and $(X, Y) \sqcap (W, Z) = (X \cap W, Y \cap Z)$. We assume $f(\emptyset, \emptyset) = 0$. Define

$$P(f) = \{ x \in \mathbf{R}^V \mid \forall (X, Y) \in 3^V : x(X) - x(Y) \leq f(X, Y) \}, \tag{2}$$
which is called the bisubmodular polyhedron associated with \( f \). When \( f \) is integer-valued, we call the set \( P_\mathbb{Z}(f) \) of all the integral points of \( P(f) \) a BS-convex set (BS stands for ‘bisubmodular’). Note that the convex hull of \( P_\mathbb{Z}(f) \) is equal to \( P(f) \) (see [4, 5] and [10, Sect. 3.5.(b)]). Occasionally we identify a BS-convex set with its corresponding bisubmodular polyhedron.

Now consider an integer-valued function \( g : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \) on the integer lattice \( \mathbb{Z}^V \). Suppose that for every vector \( \mu : V \to \mathbb{R} \) the convex hull of the affinity (or linearity) domain given by

\[
\text{Argmin}\{g(x) - \langle \mu, x \rangle \mid x \in \mathbb{Z}^V\},
\]

if nonempty, is a BS-convex set. Then we call \( g \) a BS-convex function. Note that every face of a bisubmodular polyhedron (or a BS-convex set) is a bisubmodular polyhedron (or a BS-convex set).

We have the following theorem, which can be shown by using characterizations of base polyhedra due to Tomizawa [10, Th. 17.1] and of bisubmodular polyhedra due to Ando and Fujishige [1]. We define an edge vector to be an edge-direction vector identified up to non-zero scalar multiplication.

**Theorem 1.** A pointed polyhedron \( Q \) is a bisubmodular polyhedron if and only if every edge vector of \( Q \) has at most two nonzero components that are equal to 1 or \(-1\).

### 3. BS-convex functions

Now, let us examine the combinatorial structures of BS-convex functions. Let \( g : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \) be a BS-convex function. In the sequel we suppose that the effective domain of BS-convex function \( g \) is full-dimensional and every affinity domain of \( g \) is pointed.

Consider an affinity domain \( Q \), of \( g \), of full dimension and suppose that the affine function supporting \( g \) on \( Q \) is given by

\[
y = \langle \mu, x \rangle + \alpha.
\]

Note that \( \mu \) is the gradient vector of \( g \) on \( Q \).

Let \( q \) be an extreme point of \( Q \). Then we have a signed poset \( \mathcal{P}(q) = (V, A(q)) \) that expresses the signed exchangeability associated with \( q \) for \( Q \) (see [1, 2, 9]). Signed poset \( \mathcal{P}(q) \) has possible bidirected arcs \( a \) as follows:
(a) \(a = u+v\) for distinct vertices \(u, v \in V\), which means that \(q+\chi_u-\chi_v \in Q\).

(b) \(a = u+v\) for vertices \(u, v \in V\), which means that \(q + \chi_u + \chi_v \in Q\) if \(u \neq v\), and \(q + \chi_u \in Q\) if \(u = v\).

(c) \(a = u-v\) for vertices \(u, v \in V\), which means that \(q - \chi_u - \chi_v \in Q\) if \(u \neq v\), and \(q - \chi_u \in Q\) if \(u = v\).

For any arc \(a = u \pm v\) define \(\partial a = \pm \chi_u \pm \chi_v\) if \(u \neq v\), and \(\partial a = \pm \chi_u\) if \(u = v\). Note that (a), (b), and (c) mean that for any arc \(a \in \mathcal{A}(q)\) we have \(q + \partial a \in Q\).

For a half-integral vector \(x \in \left(\frac{1}{2}\mathbb{Z}\right)^V\) we call \(U_0 = \{v \in V \mid x(v) \in \mathbb{Z}\}\) the integer support of \(x\) and \(U_1 = V \setminus U_0\) the half-integer support of \(x\), respectively.

Then we have the following.

**Theorem 2.** Let \(g : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\}\) be a BS-convex function. For every affinity domain \(Q\) of \(g\) of full dimension the gradient vector \(\mu\) of \(g\) on \(Q\) and the constant \(\alpha\) in (4) are half-integral, and for the half-integer support \(U_1\) of \(\mu\) we have even \(z(U_1)\) for all \(z \in Q\) or odd \(z(U_1)\) for all \(z \in Q\) according as \(\alpha\) is an integer or a half-integer.

**Proof:** Since \(Q\) is full-dimensional, letting \(q\) be an extreme point of \(Q\), the gradient vector \(\mu\) is the unique solution of the following system of linear equations with integral right-hand sides:

\[
\langle \partial a, \mu \rangle = g(q + \partial a) - g(q) \quad (\forall a \in \mathcal{A}(q)),
\]

which has a half-integral solution.

Moreover, it follows from the above argument that \(\mu\) is expressed as \(\mu_0 + \frac{1}{2}\chi_{U_1}\), where \(\mu_0 = [\mu]\), the integral vector obtained from \(\mu\) by rounding \(\mu(v)\) (\(v \in V\)) downward to the nearest integers. Then we have \(g(z) = \langle \mu_0, z \rangle + \frac{1}{2}z(U_1) + \alpha\), which is an integer. Hence, \(\alpha\) is half-integral, from which the latter part of the present theorem easily follows. \(\square\)

**Example 3.** The set of four points

\[Q = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}\]
in \( \mathbb{Z}^3 \) is a BS-convex set due to Theorem 1. A linear function

\[
y = \frac{1}{2}(x(1) + x(2) + x(3))
\]

with a half-integer gradient takes on integers on \( Q \) since \( x(1) + x(2) + x(3) \) is even for all \( x \in Q \). Actually \( Q \) is an even-parity delta-matroid (see [4, 11]).

A BS-convex set \( Q \subseteq \mathbb{Z}^V \) is said to have \textit{constant parity} if \( x(V) \) for all \( x \in Q \) are even or are odd.

\textbf{Conjecture 4.} Every constant-parity BS-convex set of full dimension is a translation of a delta-matroid.

Note that BS-convex sets are exactly jump systems without any hole ([4, 11]) and that all the points of every constant-parity BS-convex set \( Q \) of full dimension lie on the boundary of the convex hull of \( Q \).

4. BS-convex functions and Freudenthal simplicial divisions

For the unit hypercube \([0,1]^V\) a \textit{Freudenthal cell} is defined as follows. Let \( \lambda = (v_1, \ldots, v_n) \) be a permutation of \( V \), where \( n = |V| \). For each \( i = 0, 1, \ldots, n \) denote by \( S_i \) the set of the first \( i \) elements of \( \lambda \). Then the simplex formed by \( \chi_{S_i} \) \( (i = 0, 1, \ldots, n) \) is a Freudenthal cell. The collection of \( n! \) such Freudenthal cells corresponding to permutations of \( V \) gives us the \textit{(standard) Freudenthal simplicial division} of the unit hypercube \([0,1]^V\).

For any \( S \subseteq V \), transforming the standard Freudenthal simplicial division of \([0,1]^V\) by making points \( \chi_X \) correspond to points \( \chi_{(X \setminus S) \cup (S \setminus X)} \) for all \( X \subseteq V \), we get another simplicial division of \([0,1]^V\), which we call the \textit{Freudenthal simplicial division reflected by} \( S \) and each cell of it a \textit{Freudenthal cell reflected by} \( S \).

The \textit{(standard) Freudenthal simplicial division} of \( \mathbb{R}^V \) is obtained by translations of the standard Freudenthal simplicial division of \([0,1]^V\) to translated unit hypercubes \([0,1]^V + z (= [z, z + \chi_V])\) by all integral \( z \in \mathbb{Z}^V \) (see Figure 1).
For each integral point $z \in \mathbb{Z}^V$ let us consider a Freudenthal simplicial division of $[0, 1]^V + z$ reflected by a set (depending on $z$) in such a way that it gives us a simplicial division of $\mathbb{R}^V$. We call such a simplicial division of $\mathbb{R}^V$ a discordant Freudenthal simplicial division of $\mathbb{R}^V$ (see Figure 2). Given a discordant Freudenthal simplicial division $D$ of $\mathbb{R}^V$, we call $f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\}$ a $D$-convex function if the extension, denoted by $\bar{f}$, of $f$ with respect to simplicial division $D$ is convex on $\mathbb{R}^V$. The convex conjugate $f^\star : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\}$ of $f$ is defined by

$$f^\star(p) = \sup\{ \langle p, x \rangle - f(x) \mid x \in \mathbb{Z}^V \} \quad (\forall p \in \mathbb{R}^V).$$

The restriction of $f^\star$ on the integer lattice $\mathbb{Z}^V$ is denoted by $f^\star_{\mathbb{Z}}$. 

Figure 1. The Freudenthal simplicial division.
Theorem 5. Given a discordant Freudenthal simplicial division $D$ of $\mathbb{R}^V$, let $f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\}$ be a $D$-convex function having full-dimensional pointed affinity domains. Then $f^*_\mathbb{Z}$ is a BS-convex function. Moreover, the gradient of $f^*_\mathbb{Z}$ on every full-dimensional affinity domain is an integral vector.

Proof: Since facets of any (standard) Freudenthal cell have normal vectors of form $\chi_u - \chi_v$ for $u, v \in V$ with $u \neq v$ and $\pm \chi_v$ for $v \in V$ and since $f$ has an integral gradient on every reflected Freudenthal cell, the present theorem follows from Theorem 1 and the definitions of $f^*$ and $f^*_\mathbb{Z}$. □

Now, for a discordant Freudenthal simplicial division $D$ for integer lattice $\mathbb{Z}^V$ let us consider the simplicial division $\frac{1}{2}D$ for the half-integral lattice $(\frac{1}{2}\mathbb{Z})^V$. Then, Theorem 5 leads us to the following.

Corollary 6. Consider any $\frac{1}{2}D$-convex function $f : (\frac{1}{2}\mathbb{Z})^V \to \frac{1}{2}\mathbb{Z} \cup \{+\infty\}$ having full-dimensional pointed affinity domains. Let $Q$ be an affinity domain (a BS-convex set), of $f^*$, of full dimension that corresponds to a point $p \in (\frac{1}{2}\mathbb{Z})^V$ giving a vertex of the epi-graph of $f$. Then, the subdifferential $\partial f(p)$ of $f$ at $p$ (the affinity domain $Q$ of $f^*_\mathbb{Z}$ corresponding to $p$) is a BS-convex set.
It should be noted that for any $\frac{1}{2}D$-convex function $f$ (in Corollary 6) $f^*_{Z}$ defined on $Z^V$ takes on values in $\frac{1}{2}Z$, possibly half-integers.

**Theorem 7.** Let $f : (\frac{1}{2}Z)^V \to \frac{1}{2}Z \cup \{+\infty\}$ be a $\frac{1}{2}D$-convex function having full-dimensional pointed affinity domains. Suppose that for every point $p \in \frac{1}{2}Z$ corresponding to a vertex of the epi-graph of $f$, putting $Q = \partial f(p)$ and letting $U_1$ be the half-integer support of $p$, $z(U_1)$ is even for all $z \in Q$ or $z(U_1)$ is odd for all $z \in Q$ according as $f(p)$ is an integer or a half-integer. Then, $f^*_{Z}$ is a BS-convex function.

**Proof:** Note that for the affine function (4) that supports $f^*$ on $Q = \partial f(p)$ we have $\mu = p$ and $\alpha = -f(p)$. We can thus see from the assumption that $f^*_{Z}$ is integer-valued (cf. Theorem 2). Hence the present theorem follows from Corollary 6. □

We call a $\frac{1}{2}D$-convex function $f$ in Theorem 7 a $BS^*$-convex function. From Theorems 2 and 7 we now have the following.

**Theorem 8.** A function $g : Z^V \to Z \cup \{+\infty\}$ is a BS-convex function if and only if we have $g = f^*_{Z}$ for a $BS^*$-convex function $f : (\frac{1}{2}Z)^V \to \frac{1}{2}Z \cup \{+\infty\}$.

Let us denote by UJ the Union-Jack simplicial division for $Z^V$ of $R^V$. (The Union-Jack simplicial division is a discordant Freudenthal simplicial division obtained in a somewhat concordant way as follows. For each integral point $z \in Z^V$, $z$ is expressed as $z_0 + x_W$ where $z_0$ has all even values $z_0(v)$ ($v \in V$) and $W$ is a subset of $V$. Then consider a Freudenthal simplicial division of $[z, z + x_V]$ reflected by $W$. Also denote by $\frac{1}{2}$UJ the half Union-Jack simplicial division for $(\frac{1}{2}Z)^V$ (see Figure 3). Similarly we define the quarter Union-Jack simplicial division $\frac{1}{4}$UJ for $(\frac{1}{4}Z)^V$. Then we have

**Theorem 9.** Every discordant Freudenthal simplicial division $D$ for $Z^V$ of $R^V$ is a coarsening of the half Union-Jack simplicial division $\frac{1}{2}$UJ for $(\frac{1}{2}Z)^V$. Hence the class of the convex extensions of BS-convex functions is a subclass of the convex conjugate functions of $\frac{1}{4}Z$-valued $\frac{1}{4}$UJ-convex functions for the fixed quarter Union-Jack simplicial division $\frac{1}{4}$UJ for $(\frac{1}{4}Z)^V$. 

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5. Concluding Remarks

We have examined structures of BS-convex functions, which are integer-valued discrete convex functions having BS-convex sets (sets of integral points in integral bisubmodular polyhedra) as their affinity domains. We have shown the following relations:

\[
\{D\text{-convex functions (}\forall D\}\} \subset \{\text{BS-convex functions}\} \subset \{\frac{1}{2}D\text{-convex functions (}\forall D\}\}
\]

and by duality (or conjugacy)

\[
\{D\text{-convex functions (}\forall D\}\}^\ast \subset \{\text{BS-convex functions}\} \subset \{\frac{1}{2}D\text{-convex functions (}\forall D\}\}^\ast,
\]

where \(\{f, \cdots\}^\ast = \{f^\ast, \cdots\}\). We also have

\[
\{\frac{1}{2}D\text{-convex functions (}\forall D\}\} \subset \{\frac{1}{4}\text{UJ-convex functions}\}.
\]

Murota [14] considered M-convex functions on constant-parity jump systems, which are closely related to BS-convex functions since the convex hulls
of BS-convex sets and of jump systems are both integral bisubmodular polyhedra (see [4, 11]). Domains of M-convex functions on jump systems considered in [14] may have holes. Moreover, the convex extension of such an M-convex function restricted on the underlying integer lattice may take on non-integral values on the holes. A special case of BS-convex functions defined on delta-matroids was also considered in [6, 17].

Since BS-convex functions have combinatorially nice structures, we think that we will find practical problems where BS-convex functions play a fundamental rôle in solving them effectively.


