Bisubmodular Polyhedra, Simplicial Divisions, and Discrete Convexity

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Abstract

We consider a class of integer-valued discrete convex functions, called BS-convex functions, defined on integer lattices whose affinity domains are sets of integral points of integral bisubmodular polyhedra. We examine discrete structures of BS-convex functions and give a characterization of BS-convex functions in terms of their convex conjugate functions by means of (discordant) Freudenthal simplicial divisions of the dual space.

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1. Introduction

Kazuo Murota [13] has developed the theory of discrete convex functions such as M- and M$^\natural$-convex functions and L- and L$^\natural$-convex functions (also see [10, Chapter VII]). The class of integer-valued such discrete convex functions defined on integer lattices is the most fundamental, where M$^\natural$-convex functions have generalized polymatroids [8, 12] as their affinity domains and L$^\natural$-convex functions have convex extensions with respect to the Freudenthal simplicial divisions.

Murota's M- and M$^\natural$-convex functions and L- and L$^\natural$-convex functions arise in many discrete optimization problems that have efficient solution al-

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gorithms (see, e.g., [15, 18]). Nice combinatorial structures of such discrete convex functions come from a kind of local submodularity or matroidal structure, i.e., the greediness property of Jack Edmonds [7] or the structures of affinity domains, of the conjugate functions, composed by the Freudenthal simplices (the details will be discussed in Sections 3 and 4).

In this paper we consider a class of integer-valued discrete convex functions, called BS-convex functions, which are defined on integer lattices and whose affinity domains are sets of integral points of integral bisubmodular polyhedra. As shown in [1, 3, 4, 5, 10, 16], (integral) bisubmodular polyhedra have a signed greediness property and their conjugate functions have affinity domains composed of reflected Freudenthal simplices. The class of BS-convex functions has not yet been fully investigated in the paradigm of Murota's discrete convex analysis.

We introduce, in Section 2, the concept of BS-convex function. In Section 3 we examine the combinatorial structures of BS-convex functions in detail, especially the half-integrality property of their gradient vectors. Moreover, we give a characterization of BS-convex functions by means of the Freudenthal simplicial divisions and the Union-Jack simplicial divisions of the dual space in Section 4. Some concluding remarks are given in Section 5.

2. Bisubmodular polyhedra

Let V be a finite nonempty set and 3^V be the set of ordered pairs (X, Y) of disjoint subsets $X, Y \subseteq V$. Denote by $\mathbf{Z}$ and $\mathbf{R}$ the set of integers and that of reals, respectively. Also define $\frac{1}{2}\mathbf{Z} = \{\frac{k}{2} \mid k \in \mathbf{Z}\}$. Any element in $\frac{1}{2}\mathbf{Z}$ is called *half-integral* and is called a *half-integer* if it is not an integer. Any vector x in $(\frac{1}{2}\mathbf{Z})^V$ is called *half-integral* and is called *integral* if $x(v)$ is an integer for each $v \in V$. For any $X \subseteq V$ define $\chi_X \in \{0,1\}^V$ to be the characteristic vector of X, i.e., $\chi_X(v) = 1$ for $v \in X$ and $\chi_X(v) = 0$ for $v \in V \setminus X$. When X is a singleton $\{w\}$, we also write χ_w as $\chi_{\{w\}}$. For any $x \in \mathbf{R}^V$ and $X \subseteq V$ define $x(X) = \sum_{v \in X} x(v)$, where $x(\emptyset) = 0$.

Let $f : 3^V \to \mathbf{R}$ be a *bisubmodular function*, i.e., for every $(X, Y), (W, Z) \in 3^V$ we have

$$f(X,Y) + f(W,Z) \geq f((X,Y) \sqcup (W,Z)) + f((X,Y) \sqcap (W,Z)), \qquad (1)$$

where $(X,Y) \sqcup (W,Z) = ((X \cup W) \setminus (Y \cup Z), (Y \cup Z) \setminus (X \cup W))$ and $(X,Y) \sqcap (W,Z) = (X \cap W, Y \cap Z)$. We assume $f(\emptyset, \emptyset) = 0$. Define

$$\mathrm{P}(f) = \{x \in \mathbf{R}^V \mid \forall (X,Y) \in 3^V : x(X) - x(Y) \leq f(X,Y)\}, \qquad (2)$$

which is called the *bisubmodular polyhedron* associated with f. When f is integer-valued, we call the set $\mathrm{P}_{\mathbf{Z}}(f)$ of all the integral points of $\mathrm{P}(f)$ a *BS-convex set* (BS stands for 'bisubmodular'). Note that the convex hull of $\mathrm{P}_{\mathbf{Z}}(f)$ is equal to $\mathrm{P}(f)$ (see [4, 5] and [10, Sect. 3.5.(b)]). Occasionally we identify a BS-convex set with its corresponding bisubmodular polyhedron.

Now consider an integer-valued function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ on the integer lattice $\mathbf{Z}^V$. Suppose that for every vector $\mu : V \to \mathbf{R}$ the convex hull of the affinity (or linearity) domain given by

$$\mathrm{Argmin}\{g(x) - \langle \mu, x \rangle \mid x \in \mathbf{Z}^V\}, \tag{3}$$

if nonempty, is a BS-convex set. Then we call g a *BS-convex function*. Note that every face of a bisubmodular polyhedron (or a BS-convex set) is a bisubmodular polyhedron (or a BS-convex set).

We have the following theorem, which can be shown by using characterizations of base polyhedra due to Tomizawa [10, Th. 17.1] and of bisubmodular polyhedra due to Ando and Fujishige [1]. We define an *edge vector* to be an edge-direction vector identified up to non-zero scalar multiplication.

Theorem 1. *A pointed polyhedron Q is a bisubmodular polyhedron if and only if every edge vector of Q has at most two nonzero components that are equal to* 1 *or* -1.

3. BS-convex functions

Now, let us examine the combinatorial structures of BS-convex functions. Let $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be a BS-convex function. In the sequel we suppose that the effective domain of BS-convex function g is full-dimensional and every affinity domain of g is pointed.

Consider an affinity domain Q, of g, of full dimension and suppose that the affine function supporting g on Q is given by

$$y = \langle \mu, x \rangle + \alpha. \tag{4}$$

Note that μ is the gradient vector of g on Q.

Let q be an extreme point of Q. Then we have a signed poset $\mathcal{P}(q) = (V, A(q))$ that expresses the signed exchangeability associated with q for Q (see [1, 2, 9]). Signed poset $\mathcal{P}(q)$ has possible bidirected arcs a as follows:

(a) $a = u{+}{-}v$ for distinct vertices $u, v \in V$, which means that $q+\chi_u-\chi_v \in Q$.

(b) $a = u{+}{+}v$ for vertices $u, v \in V$, which means that $q + \chi_u + \chi_v \in Q$ if $u \neq v$, and $q + \chi_u \in Q$ if $u = v$.

(c) $a = u{-}{-}v$ for vertices $u, v \in V$, which means that $q - \chi_u - \chi_v \in Q$ if $u \neq v$, and $q - \chi_u \in Q$ if $u = v$.

For any arc $a = u{\pm}{\pm}v$ define $\partial a = \pm\chi_u \pm \chi_v$ if $u \neq v$, and $\partial a = \pm\chi_u$ if $u = v$. Note that (a), (b), and (c) mean that for any arc $a \in A(q)$ we have $q + \partial a \in Q$.

For a half-integral vector $x \in (\frac{1}{2}\mathbf{Z})^V$ we call $U_0 = \{v \in V \mid x(v) \in \mathbf{Z}\}$ the *integer support* of x and $U_1 = V \setminus U_0$ the *half-integer support* of x, respectively.

Then we have the following.

Theorem 2. *Let $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be a BS-convex function. For every affinity domain Q of g of full dimension the gradient vector μ of g on Q and the constant α in* (4) *are half-integral, and for the half-integer support U_1 of μ we have even $z(U_1)$ for all $z \in Q$ or odd $z(U_1)$ for all $z \in Q$ according as α is an integer or a half-integer.*

Proof: Since Q is full-dimensional, letting q be an extreme point of Q, the gradient vector μ is the unique solution of the following system of linear equations with integral right-hand sides:

$$\langle \partial a, \mu \rangle = g(q + \partial a) - g(q) \qquad (\forall a \in A(q)), \tag{5}$$

which has a half-integral solution.

Moreover, it follows from the above argument that μ is expressed as $\mu_0 + \frac{1}{2}\chi_{U_1}$, where $\mu_0 = \lfloor \mu \rfloor$, the integral vector obtained from μ by rounding $\mu(v)$ $(v \in V)$ downward to the nearest integers. Then we have $g(z) = \langle \mu_0, z \rangle + \frac{1}{2}z(U_1) + \alpha$, which is an integer. Hence, α is half-integral, from which the latter part of the present theorem easily follows. $\square$

Example 3. *The set of four points*

$$Q = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$$

in $\mathbf{Z}^3$ *is a BS-convex set due to Theorem* 1*. A linear function*

$$y = \frac{1}{2}\{x(1) + x(2) + x(3)\}$$

with a half-integer gradient takes on integers on Q *since* $x(1)+x(2)+x(3)$ *is even for all* $x \in Q$*. Actually* Q *is an even-parity delta-matroid* (*see* [4, 11]).

A BS-convex set $Q \subseteq \mathbf{Z}^V$ is said to have *constant parity* if $x(V)$ for all $x \in Q$ are even or are odd.

Conjecture 4. *Every constant-parity BS-convex set of full dimension is a translation of a delta-matroid.*

Note that BS-convex sets are exactly jump systems without any hole ([4, 11]) and that all the points of every constant-parity BS-convex set Q of full dimension lie on the boundary of the convex hull of Q.

4. BS-convex functions and Freudenthal simplicial divisions

For the unit hypercube $[0,1]^V$ a *Freudenthal cell* is defined as follows. Let $\lambda = (v_1, \cdots, v_n)$ be a permutation of V, where $n = |V|$. For each $i = 0, 1, \cdots, n$ denote by S_i the set of the first i elements of λ. Then the simplex formed by χ_{S_i} $(i = 0, 1, \cdots, n)$ is a Freudenthal cell. The collection of $n!$ such Freudenthal cells corresponding to permutations of V gives us the (*standard*) *Freudenthal simplicial division* of the unit hypercube $[0,1]^V$.

For any $S \subseteq V$, transforming the standard Freudenthal simplicial division of $[0,1]^V$ by making points χ_X correspond to points $\chi_{(X \setminus S) \cup (S \setminus X)}$ for all $X \subseteq V$, we get another simplicial division of $[0,1]^V$, which we call the *Freudenthal simplicial division reflected by* S and each cell of it a *Freudenthal cell reflected by* S.

The (*standard*) *Freudenthal simplicial division of* $\mathbf{R}^V$ is obtained by translations of the standard Freudenthal simplicial division of $[0,1]^V$ to translated unit hypercubes $[0,1]^V + z$ $(= [z, z + \chi_V])$ by all integral $z \in \mathbf{Z}^V$ (see Figure 1).

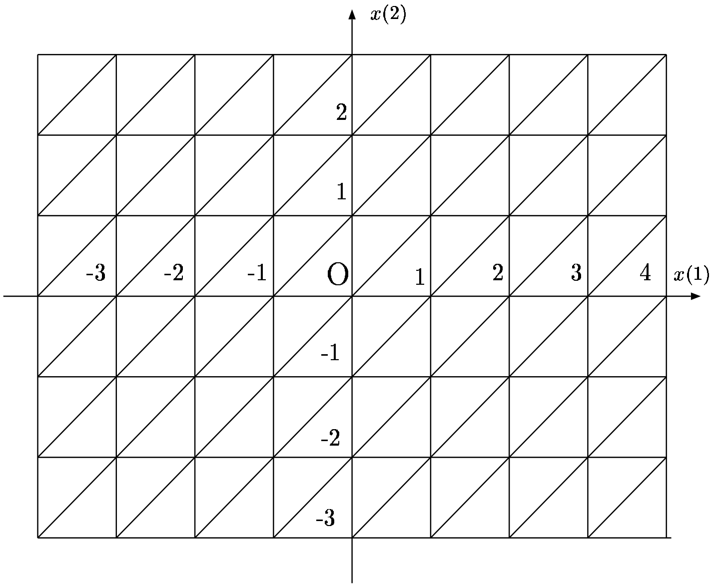

Figure 1. The Freudenthal simplicial division.

For each integral point $z \in \mathbf{Z}^V$ let us consider a Freudenthal simplicial division of $[0,1]^V + z$ reflected by a set (depending on z) in such a way that it gives us a simplicial division of $\mathbf{R}^V$. We call such a simplicial division of $\mathbf{R}^V$ a *discordant Freudenthal simplicial division* of $\mathbf{R}^V$ (see Figure 2). Given a discordant Freudenthal simplicial division D of $\mathbf{R}^V$, we call $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ a *D-convex function* if the extension, denoted by $\hat{f}$, of f with respect to simplicial division D is convex on $\mathbf{R}^V$. The convex conjugate $f^\bullet : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ of f is defined by

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^V\} \qquad (\forall p \in \mathbf{R}^V). \tag{6}$$

The restriction of $f^\bullet$ on the integer lattice $\mathbf{Z}^V$ is denoted by $f^\bullet_{\mathbf{Z}}$.

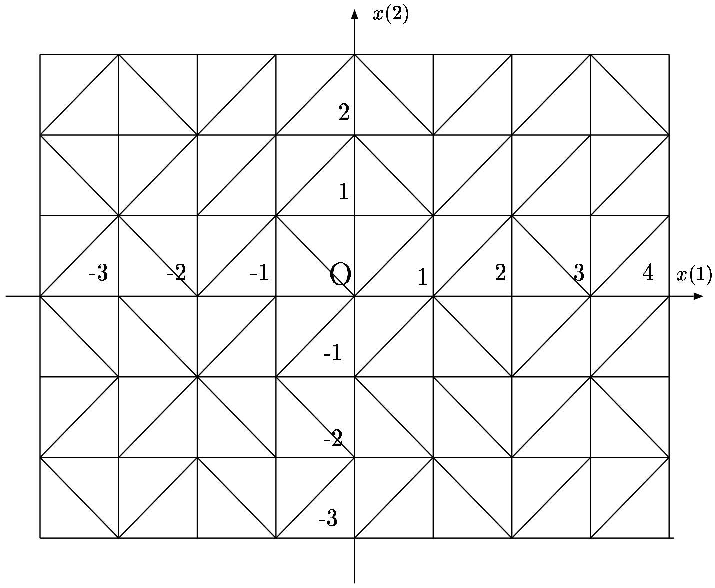

Figure 2. A discordant Freudenthal simplicial division D.

Theorem 5. *Given a discordant Freudenthal simplicial division D of $\mathbf{R}^V$, let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be a D-convex function having full-dimensional pointed affinity domains. Then $f^\bullet_{\mathbf{Z}}$ is a BS-convex function. Moreover, the gradient of $f^\bullet_{\mathbf{Z}}$ on every full-dimensional affinity domain is an integral vector.*

PROOF: Since facets of any (standard) Freudenthal cell have normal vectors of form $\chi_u - \chi_v$ for $u, v \in V$ with $u \neq v$ and $\pm\chi_v$ for $v \in V$ and since f has an integral gradient on every reflected Freudenthal cell, the present theorem follows from Theorem 1 and the definitions of $f^\bullet$ and $f^\bullet_{\mathbf{Z}}$. $\square$

Now, for a discordant Freudenthal simplicial division D for integer lattice $\mathbf{Z}^V$ let us consider the simplicial division $\frac{1}{2}D$ for the half-integral lattice $(\frac{1}{2}\mathbf{Z})^V$. Then, Theorem 5 leads us to the following.

Corollary 6. *Consider any $\frac{1}{2}D$-convex function $f : (\frac{1}{2}\mathbf{Z})^V \to \frac{1}{2}\mathbf{Z} \cup \{+\infty\}$ having full-dimensional pointed affinity domains. Let Q be an affinity domain (a BS-convex set), of $f^\bullet$, of full dimension that corresponds to a point $p \in (\frac{1}{2}\mathbf{Z})^V$ giving a vertex of the epi-graph of $\hat{f}$. Then, the subdifferential $\partial f(p)$ of f at p (the affinity domain Q of $f^\bullet_{\mathbf{Z}}$ corresponding to p) is a BS-convex set.*

It should be noted that for any $\frac{1}{2}D$-convex function f (in Corollary 6) $f^\bullet_{\mathbf{Z}}$ defined on $\mathbf{Z}^V$ takes on values in $\frac{1}{2}\mathbf{Z}$, possibly half-integers.

Theorem 7. *Let $f : (\frac{1}{2}\mathbf{Z})^V \to \frac{1}{2}\mathbf{Z} \cup \{+\infty\}$ be a $\frac{1}{2}D$-convex function having full-dimensional pointed affinity domains. Suppose that for every point $p \in \frac{1}{2}\mathbf{Z}$ corresponding to a vertex of the epi-graph of $\hat{f}$, putting $Q = \partial f(p)$ and letting U_1 be the half-integer support of p, $z(U_1)$ is even for all $z \in Q$ or $z(U_1)$ is odd for all $z \in Q$ according as $f(p)$ is an integer or a half-integer. Then, $f^\bullet_{\mathbf{Z}}$ is a BS-convex function.*

Proof: Note that for the affine function (4) that supports $f^\bullet$ on $Q = \partial f(p)$ we have $\mu = p$ and $\alpha = -f(p)$. We can thus see from the assumption that $f^\bullet_{\mathbf{Z}}$ is integer-valued (cf. Theorem 2). Hence the present theorem follows from Corollary 6. □

We call a $\frac{1}{2}D$-convex function f in Theorem 7 a *BS$^\bullet$-convex function.* From Theorems 2 and 7 we now have the following.

Theorem 8. *A function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ is a BS-convex function if and only if we have $g = f^\bullet_{\mathbf{Z}}$ for a BS$^\bullet$-convex function $f : (\frac{1}{2}\mathbf{Z})^V \to \frac{1}{2}\mathbf{Z} \cup \{+\infty\}$.*

Let us denote by UJ the *Union-Jack simplicial division* for $\mathbf{Z}^V$ of $\mathbf{R}^V$. (The Union-Jack simplicial division is a discordant Freudenthal simplicial division obtained in a somewhat concordant way as follows. For each integral point $z \in \mathbf{Z}^V$ z is expressed as $z_0 + \chi_W$ where z_0 has all even values $z_0(v)$ ($v \in V$) and W is a subset of V. Then consider a Freudenthal simplicial division of $[z, z + \chi_V]$ reflected by W.) Also denote by $\frac{1}{2}$UJ the half Union-Jack simplicial division for $(\frac{1}{2}\mathbf{Z})^V$ (see Figure 3). Similarly we define the quarter Union-Jack simplicial division $\frac{1}{4}$UJ for $(\frac{1}{4}\mathbf{Z})^V$. Then we have

Theorem 9. *Every discordant Freudenthal simplicial division D for $\mathbf{Z}^V$ of $\mathbf{R}^V$ is a coarsening of the half Union-Jack simplicial division $\frac{1}{2}$UJ for $(\frac{1}{2}\mathbf{Z})^V$. Hence the class of the convex extensions of BS-convex functions is a subclass of the convex conjugate functions of $\frac{1}{4}\mathbf{Z}$-valued $\frac{1}{4}$UJ-convex functions for the* fixed *quarter Union-Jack simplicial division $\frac{1}{4}$UJ for $(\frac{1}{4}\mathbf{Z})^V$.*

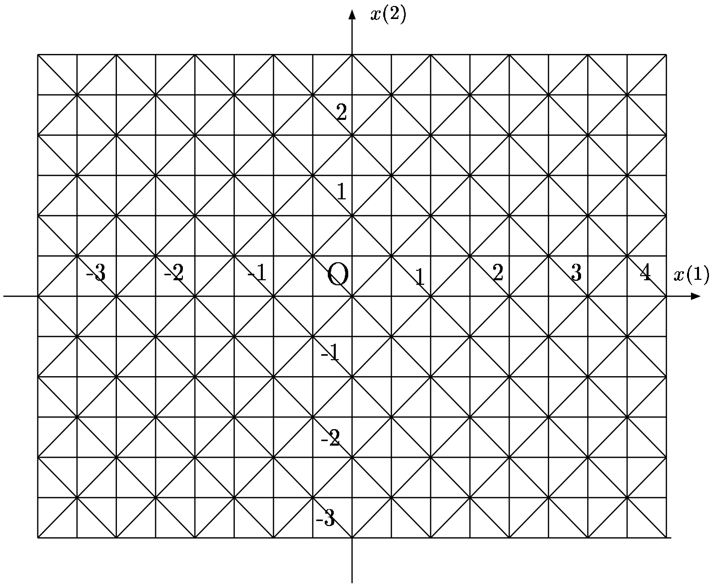

Figure 3. The half Union-Jack simplicial division $\frac{1}{2}$UJ.

5. Concluding Remarks

We have examined structures of BS-convex functions, which are integer-valued discrete convex functions having BS-convex sets (sets of integral points in integral bisubmodular polyhedra) as their affinity domains. We have shown the following relations.

$$\begin{aligned}\{D\text{-convex functions } (\forall D)\} \;&\subset\; \{\text{BS}^\bullet\text{-convex functions}\}\\ &\subset\; \{\tfrac{1}{2}D\text{-convex functions } (\forall D)\}\end{aligned}$$

and by duality (or conjugacy)

$$\begin{aligned}\{D\text{-convex functions } (\forall D)\}^\bullet \;&\subset\; \{\text{BS-convex functions}\}\\ &\subset\; \{\tfrac{1}{2}D\text{-convex functions } (\forall D)\}^\bullet,\end{aligned}$$

where $\{f, \cdots\}^\bullet = \{f^\bullet, \cdots\}$. We also have

$$\{\tfrac{1}{2}D\text{-convex functions } (\forall D)\} \subset \{\tfrac{1}{4}\text{UJ-convex functions}\}.$$

Murota [14] considered M-convex functions on constant-parity jump systems, which are closely related to BS-convex functions since the convex hulls of BS-convex sets and of jump systems are both integral bisubmodular polyhedra (see [4, 11]). Domains of M-convex functions on jump systems considered in [14] may have holes. Moreover, the convex extension of such an M-convex function restricted on the underlying integer lattice may take on non-integral values on the holes. A special case of BS-convex functions defined on delta-matroids was also considered in [6, 17].

Since BS-convex functions have combinatorially nice structures, we think that we will find practical problems where BS-convex functions play a fundamental rôle in solving them effectively.

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