Abstract

We consider a class of integer-valued discrete convex functions, called BS-
convex functions, defined on integer lattices whose affinity domains are sets
of integral points of integral bisubmodular polyhedra. We examine discrete
structures of BS-convex functions and give a characterization of BS-convex
functions in terms of their convex conjugate functions by means of (discord-
ant) Freudenthal simplicial divisions of the dual space.

Keywords: BS-convex functions, bisubmodular polyhedra, the Freudenthal
simplicial divisions, discrete convexity

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1. Introduction

Kazuo Murota [13] has developed the theory of discrete convex functions
such as M- and M♮-convex functions and L- and L♮-convex functions (also
see [10, Chapter VII]). The class of integer-valued such discrete convex func-
tions defined on integer lattices is the most fundamental, where M♮-convex
functions have generalized polymatroids [8, 12] as their affinity domains and
L♮-convex functions have convex extensions with respect to the Freudenthal
simplicial divisions.

Murota’s M- and M♮-convex functions and L- and L♮-convex functions
arise in many discrete optimization problems that have efficient solution al-

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gorithms (see, e.g., [15, 18]). Nice combinatorial structures of such discrete convex functions come from a kind of local submodularity or matroidal structure, i.e., the greediness property of Jack Edmonds [7] or the structures of affinity domains, of the conjugate functions, composed by the Freudenthal simplices (the details will be discussed in Sections 3 and 4).

In this paper we consider a class of integer-valued discrete convex functions, called BS-convex functions, which are defined on integer lattices and whose affinity domains are sets of integral points of integral bisubmodular polyhedra. As shown in [1, 3, 4, 5, 10, 16], (integral) bisubmodular polyhedra have a signed greediness property and their conjugate functions have affinity domains composed of reflected Freudenthal simplices. The class of BS-convex functions has not yet been fully investigated in the paradigm of Murota’s discrete convex analysis.

We introduce, in Section 2, the concept of BS-convex function. In Section 3 we examine the combinatorial structures of BS-convex functions in detail, especially the half-integrality property of their gradient vectors. Moreover, we give a characterization of BS-convex functions by means of the Freudenthal simplicial divisions and the Union-Jack simplicial divisions of the dual space in Section 4. Some concluding remarks are given in Section 5.

2. Bisubmodular polyhedra

Let $V$ be a finite nonempty set and $3^V$ be the set of ordered pairs $(X, Y)$ of disjoint subsets $X, Y \subseteq V$. Denote by $\mathbb{Z}$ and $\mathbb{R}$ the set of integers and that of reals, respectively. Also define $\frac{1}{2}\mathbb{Z} = \{\frac{k}{2} \mid k \in \mathbb{Z}\}$. Any element in $\frac{1}{2}\mathbb{Z}$ is called half-integer and is called a half-integer if it is not an integer. Any vector $x$ in $(\frac{1}{2}\mathbb{Z})^V$ is called half-integral and is called integral if $x(v)$ is an integer for each $v \in V$. For any $X \subseteq V$ define $\chi_X \in \{0, 1\}^V$ to be the characteristic vector of $X$, i.e., $\chi_X(v) = 1$ for $v \in X$ and $\chi_X(v) = 0$ for $v \in V \setminus X$. When $X$ is a singleton $\{w\}$, we also write $\chi_w$ as $\chi_{\{w\}}$. For any $x \in \mathbb{R}^V$ and $X \subseteq V$ define $x(X) = \sum_{v \in X} x(v)$, where $x(\emptyset) = 0$. Let $f : 3^V \to \mathbb{R}$ be a bisubmodular function, i.e., for every $(X, Y), (W, Z) \in 3^V$ we have

$$f(X, Y) + f(W, Z) \geq f((X, Y) \cup (W, Z)) + f((X, Y) \cap (W, Z)),$$

(1)

where $(X, Y) \cup (W, Z) = ((X \cup W) \setminus (Y \cup Z), (Y \cup Z) \setminus (X \cup W))$ and $(X, Y) \cap (W, Z) = (X \cap W, Y \cap Z)$. We assume $f(\emptyset, \emptyset) = 0$. Define

$$P(f) = \{x \in \mathbb{R}^V \mid \forall (X, Y) \in 3^V : x(X) - x(Y) \leq f(X, Y)\},$$

(2)
which is called the *bisubmodular polyhedron* associated with $f$. When $f$ is integer-valued, we call the set $P_Z(f)$ of all the integral points of $P(f)$ a *BS-convex set* (BS stands for ‘bisubmodular’). Note that the convex hull of $P_Z(f)$ is equal to $P(f)$ (see [4, 5] and [10, Sect. 3.5.(b)]). Occasionally we identify a BS-convex set with its corresponding bisubmodular polyhedron.

Now consider an integer-valued function $g : Z^V \rightarrow Z \cup \{+\infty\}$ on the integer lattice $Z^V$. Suppose that for every vector $\mu : V \rightarrow R$ the convex hull of the affinity (or linearity) domain given by

$$\text{Argmin}\{g(x) - \langle \mu, x \rangle \mid x \in Z^V\},$$

if nonempty, is a BS-convex set. Then we call $g$ a *BS-convex function*. Note that every face of a bisubmodular polyhedron (or a BS-convex set) is a bisubmodular polyhedron (or a BS-convex set).

We have the following theorem, which can be shown by using characterizations of base polyhedra due to Tomizawa [10, Th. 17.1] and of bisubmodular polyhedra due to Ando and Fujishige [1]. We define an *edge vector* to be an edge-direction vector identified up to non-zero scalar multiplication.

**Theorem 1.** A pointed polyhedron $Q$ is a bisubmodular polyhedron if and only if every edge vector of $Q$ has at most two nonzero components that are equal to 1 or $-1$.

### 3. BS-convex functions

Now, let us examine the combinatorial structures of BS-convex functions. Let $g : Z^V \rightarrow Z \cup \{+\infty\}$ be a BS-convex function. In the sequel we suppose that the effective domain of BS-convex function $g$ is full-dimensional and every affinity domain of $g$ is pointed.

Consider an affinity domain $Q$, of $g$, of full dimension and suppose that the affine function supporting $g$ on $Q$ is given by

$$y = \langle \mu, x \rangle + \alpha. \quad (4)$$

Note that $\mu$ is the gradient vector of $g$ on $Q$.

Let $q$ be an extreme point of $Q$. Then we have a signed poset $\mathcal{P}(q) = (V, A(q))$ that expresses the signed exchangeability associated with $q$ for $Q$ (see [1, 2, 9]). Signed poset $\mathcal{P}(q)$ has possible bidirected arcs $a$ as follows:
(a) \( a = u + v \) for distinct vertices \( u, v \in V \), which means that \( q + \chi_u - \chi_v \in Q \).

(b) \( a = u + v \) for vertices \( u, v \in V \), which means that \( q + \chi_u + \chi_v \in Q \) if \( u \neq v \), and \( q + \chi_u \in Q \) if \( u = v \).

(c) \( a = u - v \) for vertices \( u, v \in V \), which means that \( q - \chi_u - \chi_v \in Q \) if \( u \neq v \), and \( q - \chi_u \in Q \) if \( u = v \).

For any arc \( a = u \pm v \) define \( \partial a = \pm \chi_u \pm \chi_v \) if \( u \neq v \), and \( \partial a = \pm \chi_u \) if \( u = v \). Note that (a), (b), and (c) mean that for any arc \( a \in A(q) \) we have \( q + \partial a \in Q \).

For a half-integral vector \( x \in \left( \frac{1}{2} \mathbb{Z} \right)^V \) we call \( U_0 = \{ v \in V \mid x(v) \in \mathbb{Z} \} \) the integer support of \( x \) and \( U_1 = V \setminus U_0 \) the half-integer support of \( x \), respectively.

Then we have the following.

**Theorem 2.** Let \( g : \mathbb{Z}^V \rightarrow \mathbb{Z} \cup \{ +\infty \} \) be a BS-convex function. For every affinity domain \( Q \) of \( g \) of full dimension the gradient vector \( \mu \) of \( g \) on \( Q \) and the constant \( \alpha \) in (4) are half-integral, and for the half-integer support \( U_1 \) of \( \mu \) we have even \( z(U_1) \) for all \( z \in Q \) or odd \( z(U_1) \) for all \( z \in Q \) according as \( \alpha \) is an integer or a half-integer.

**Proof:** Since \( Q \) is full-dimensional, letting \( q \) be an extreme point of \( Q \), the gradient vector \( \mu \) is the unique solution of the following system of linear equations with integral right-hand sides:

\[
\langle \partial a, \mu \rangle = g(q + \partial a) - g(q) \quad (\forall a \in A(q)) ,
\]

which has a half-integral solution.

Moreover, it follows from the above argument that \( \mu \) is expressed as \( \mu_0 + \frac{1}{2} \chi U_1 \), where \( \mu_0 = [\mu] \), the integral vector obtained from \( \mu \) by rounding \( \mu(v) \) \( (v \in V) \) downward to the nearest integers. Then we have \( g(z) = \langle \mu_0, z \rangle + \frac{1}{2} z(U_1) + \alpha \), which is an integer. Hence, \( \alpha \) is half-integral, from which the latter part of the present theorem easily follows. \( \square \)

**Example 3.** The set of four points

\[ Q = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\} \]
in $\mathbb{Z}^3$ is a BS-convex set due to Theorem 1. A linear function

$$y = \frac{1}{2}(x(1) + x(2) + x(3))$$

with a half-integer gradient takes on integers on $Q$ since $x(1) + x(2) + x(3)$ is even for all $x \in Q$. Actually $Q$ is an even-parity delta-matroid (see [4, 11]).

A BS-convex set $Q \subseteq \mathbb{Z}^V$ is said to have constant parity if $x(V)$ for all $x \in Q$ are even or are odd.

**Conjecture 4.** Every constant-parity BS-convex set of full dimension is a translation of a delta-matroid.

Note that BS-convex sets are exactly jump systems without any hole ([4, 11]) and that all the points of every constant-parity BS-convex set $Q$ of full dimension lie on the boundary of the convex hull of $Q$.

**4. BS-convex functions and Freudenthal simplicial divisions**

For the unit hypercube $[0,1]^V$ a Freudenthal cell is defined as follows. Let $\lambda = (v_1, \cdots, v_n)$ be a permutation of $V$, where $n = |V|$. For each $i = 0, 1, \cdots, n$ denote by $S_i$ the set of the first $i$ elements of $\lambda$. Then the simplex formed by $\chi_{S_i}$ ($i = 0, 1, \cdots, n$) is a Freudenthal cell. The collection of $n!$ such Freudenthal cells corresponding to permutations of $V$ gives us the (standard) Freudenthal simplicial division of the unit hypercube $[0,1]^V$.

For any $S \subseteq V$, transforming the standard Freudenthal simplicial division of $[0,1]^V$ by making points $\chi_X$ correspond to points $\chi_{(X \setminus S) \cup (S \setminus X)}$ for all $X \subseteq V$, we get another simplicial division of $[0,1]^V$, which we call the Freudenthal simplicial division reflected by $S$ and each cell of it a Freudenthal cell reflected by $S$.

The (standard) Freudenthal simplicial division of $\mathbb{R}^V$ is obtained by translations of the standard Freudenthal simplicial division of $[0,1]^V$ to translated unit hypercubes $[0,1]^V + z (= [z, z + \chi_V])$ by all integral $z \in \mathbb{Z}^V$ (see Figure 1).
Figure 1. The Freudenthal simplicial division.

For each integral point \( z \in \mathbb{Z}^V \) let us consider a Freudenthal simplicial division of \( [0,1]^V + z \) reflected by a set (depending on \( z \)) in such a way that it gives us a simplicial division of \( \mathbb{R}^V \). We call such a simplicial division of \( \mathbb{R}^V \) a discordant Freudenthal simplicial division of \( \mathbb{R}^V \) (see Figure 2). Given a discordant Freudenthal simplicial division \( D \) of \( \mathbb{R}^V \), we call \( f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \) a \( D \)-convex function if the extension, denoted by \( \hat{f} \), of \( f \) with respect to simplicial division \( D \) is convex on \( \mathbb{R}^V \). The convex conjugate \( f^* : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\} \) of \( f \) is defined by

\[
    f^*(p) = \sup \{ \langle p, x \rangle - f(x) \mid x \in \mathbb{Z}^V \} \quad (\forall p \in \mathbb{R}^V).
\]

(6)

The restriction of \( f^* \) on the integer lattice \( \mathbb{Z}^V \) is denoted by \( f^*_\mathbb{Z} \).
**Theorem 5.** Given a discordant Freudenthal simplicial division $D$ of $\mathbb{R}^V$, let $f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\}$ be a $D$-convex function having full-dimensional pointed affinity domains. Then $f_\mathbb{Z}$ is a BS-convex function. Moreover, the gradient of $f_\mathbb{Z}$ on every full-dimensional affinity domain is an integral vector.

**Proof:** Since facets of any (standard) Freudenthal cell have normal vectors of form $\chi_u - \chi_v$ for $u, v \in V$ with $u \neq v$ and $\pm \chi_v$ for $v \in V$ and since $f$ has an integral gradient on every reflected Freudenthal cell, the present theorem follows from Theorem 1 and the definitions of $f^*$ and $f_\mathbb{Z}^*$.

Now, for a discordant Freudenthal simplicial division $D$ for integer lattice $\mathbb{Z}^V$ let us consider the simplicial division $\frac{1}{2}D$ for the half-integral lattice $\left(\frac{1}{2}\mathbb{Z}\right)^V$. Then, Theorem 5 leads us to the following.

**Corollary 6.** Consider any $\frac{1}{2}D$-convex function $f : \left(\frac{1}{2}\mathbb{Z}\right)^V \to \frac{1}{2}\mathbb{Z} \cup \{+\infty\}$ having full-dimensional pointed affinity domains. Let $Q$ be an affinity domain (a BS-convex set), of $f^*$, of full dimension that corresponds to a point $p \in \left(\frac{1}{2}\mathbb{Z}\right)^V$ giving a vertex of the epi-graph of $f$. Then, the subdifferential $\partial f(p)$ of $f$ at $p$ (the affinity domain $Q$ of $f_\mathbb{Z}$ corresponding to $p$) is a BS-convex set.
It should be noted that for any \( \frac{1}{2}D \)-convex function \( f \) (in Corollary 6) \( f^*_{\mathbf{Z}} \)
defined on \( \mathbf{Z}^V \) takes on values in \( \frac{1}{2}\mathbf{Z} \), possibly half-integers.

**Theorem 7.** Let \( f : \left( \frac{1}{2}\mathbf{Z} \right)^V \to \frac{1}{2}\mathbf{Z} \cup \{+\infty\} \) be a \( \frac{1}{2}D \)-convex function having
full-dimensional pointed affinity domains. Suppose that for every point \( p \in \frac{1}{2}\mathbf{Z} \) corresponding to a vertex of the epi-graph of \( \hat{f} \), putting \( Q = \partial f(p) \) and
letting \( U_1 \) be the half-integer support of \( p \), \( z(U_1) \) is even for all \( z \in Q \) or \( z(U_1) \) is odd for all \( z \in Q \) according as \( f(p) \) is an integer or a half-integer. Then, \( f^*_{\mathbf{Z}} \) is a BS-convex function.

**Proof:** Note that for the affine function (4) that supports \( f^* \) on \( Q = \partial f(p) \)
we have \( \mu = p \) and \( \alpha = -f(p) \). We can thus see from the assumption that \( f^*_{\mathbf{Z}} \) is integer-valued (cf. Theorem 2). Hence the present theorem follows from
Corollary 6. \( \square \)

We call a \( \frac{1}{2}D \)-convex function \( f \) in Theorem 7 a BS\(^*\)-convex function.

From Theorems 2 and 7 we now have the following.

**Theorem 8.** A function \( g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\} \) is a BS-convex function if and only if we have \( g = f^*_{\mathbf{Z}} \) for a BS\(^*\)-convex function \( f : \left( \frac{1}{2}\mathbf{Z} \right)^V \to \frac{1}{2}\mathbf{Z} \cup \{+\infty\} \).

Let us denote by UJ the Union-Jack simplicial division for \( \mathbf{Z}^V \) of \( \mathbf{R}^V \).
(The Union-Jack simplicial division is a discordant Freudenthal simplicial division obtained in a somewhat concordant way as follows. For each integral point \( z \in \mathbf{Z}^V \) \( z \) is expressed as \( z_0 + \chi_W \) where \( z_0 \) has all even values \( z_0(v) \) \( (v \in V) \) and \( W \) is a subset of \( V \). Then consider a Freudenthal simplicial division of \([z, z + \chi_V]\) reflected by \( W \).) Also denote by \( \frac{1}{2}\)UJ the half Union-Jack simplicial division for \( \left( \frac{1}{2}\mathbf{Z} \right)^V \) (see Figure 3). Similarly we define the quarter Union-Jack simplicial division \( \frac{1}{4}\)UJ for \( \left( \frac{1}{4}\mathbf{Z} \right)^V \). Then we have

**Theorem 9.** Every discordant Freudenthal simplicial division \( D \) for \( \mathbf{Z}^V \) of \( \mathbf{R}^V \) is a coarsening of the half Union-Jack simplicial division \( \frac{1}{2}\)UJ for \( \left( \frac{1}{2}\mathbf{Z} \right)^V \). Hence the class of the convex extensions of BS-convex functions is a subclass of the convex conjugate functions of \( \frac{1}{2}\mathbf{Z} \)-valued \( \frac{1}{2}\)UJ-convex functions for the fixed quarter Union-Jack simplicial division \( \frac{1}{4}\)UJ for \( \left( \frac{1}{4}\mathbf{Z} \right)^V \).
5. Concluding Remarks

We have examined structures of BS-convex functions, which are integer-valued discrete convex functions having BS-convex sets (sets of integral points in integral bisubmodular polyhedra) as their affinity domains. We have shown the following relations.

\[
\{ D\text{-convex functions (}\forall D\text{)} \} \subset \{ \text{BS}^\bullet\text{-convex functions} \} \\
\subset \{ \frac{1}{2} D\text{-convex functions (}\forall D\text{)} \}
\]

and by duality (or conjugacy)

\[
\{ D\text{-convex functions (}\forall D\text{)} \}^\bullet \subset \{ \text{BS-convex functions} \} \\
\subset \{ \frac{1}{2} D\text{-convex functions (}\forall D\text{)} \}^\bullet,
\]

where \( \{f, \cdots\}^\bullet = \{f^\bullet, \cdots\} \). We also have

\[
\{ \frac{1}{2} D\text{-convex functions (}\forall D\text{)} \} \subset \{ \frac{1}{4} \text{UJ-convex functions} \}.
\]

Murota [14] considered M-convex functions on constant-parity jump systems, which are closely related to BS-convex functions since the convex hulls
of BS-convex sets and of jump systems are both integral bisubmodular polyhedra (see [4, 11]). Domains of M-convex functions on jump systems considered in [14] may have holes. Moreover, the convex extension of such an M-convex function restricted on the underlying integer lattice may take on non-integral values on the holes. A special case of BS-convex functions defined on delta-matroids was also considered in [6, 17].

Since BS-convex functions have combinatorially nice structures, we think that we will find practical problems where BS-convex functions play a fundamental rôle in solving them effectively.


