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PBW bases and KLR algebras∗

Syu Kato †‡

May 19, 2014

Abstract

We generalize Lusztig’s geometric construction of the PBW bases of finite quantum groups of type ADE under the framework of [Varagnolo-Vasserot, J. reine angew. Math. 659 (2011)]. In particular, every PBW basis of such quantum groups is proven to yield a semi-orthogonal collection in the module category of the KLR-algebras. This enables us to prove Lusztig’s conjecture on the positivity of the canonical (lower global) bases in terms of the (lower) PBW bases. In addition, we verify Kashiwara’s problem on the finiteness of the global dimensions of the KLR-algebras of type ADE.

Introduction

Canonical/global bases of quantum groups, defined by Lusztig [Lu90a] and Kashiwara [Kas91] subsequently, open up scenery in many areas of mathematics which are visible only through quantum groups [Ari05, Lus08, Nak06]. They are certain bases of quantum groups different from the natural quantum analogue of the classical Poincaré-Birkhoff-Witt theorem (that are usually referred to as the PBW bases).

Among these, the interaction between canonical/global bases of quantum groups and affine Hecke algebras of type A (and their cyclotomic quotients) yields many representation-theoretic consequences [Ari96, Ari05]. It is generalized to more general quantum groups and their representations by Khovanov-Lauda, Rouquier, Varagnolo-Vasserot, Zheng, Webster, and Kang-Kashiwara [KL09, Rou08, VV11, Zhe08, Web10, KK12] as a categorical counter-part of the theory of canonical/global bases.

More precisely, to each symmetric Kac-Moody algebra g, they introduced a series of algebras R_β (that we call the KLR-algebras) whose simple/projective modules give rise to the upper/lower global bases of the corresponding positive half of the quantum group of g. There the emphasis is on the categorification of quantum groups, and their results are strong enough to generalize and categorify Ariki’s result [Ari96] in these cases (Lauda-Vazirani [LV11] and [VV11, KK12]).

This story is sufficient to recover deep representation-theoretic properties, without the PBW bases. The main observation of this paper is that the PBW

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bases still exist in the world of KLR-algebras, with essential new features which are visible only with the KLR-algebras.

To see what we mean by this, we prepare some notations: Let \( A := \mathbb{Z}[t^\pm 1] \). Let \( \mathfrak{g} \) be a simple Lie algebra of type \text{ADE}, and let \( U^+ \) be the positive half of the \( A \)-integral version of the quantum group of \( \mathfrak{g} \) (see e.g. Lusztig [Lus93] §1). Let \( Q^+ := \mathbb{Z}_{\geq 0}I \), where \( I \) is the set of positive simple roots. We have a weight space decomposition \( U^+ = \bigoplus_{\beta \in Q^+} U^+_\beta \). We have the Weyl group \( W \) of \( \mathfrak{g} \) with its set of simple reflections \( \{ s_i \}_{i \in I} \) and the longest element \( w_0 \). For each \( \beta \in Q^+ \), we have a finite set \( B(\infty)_\beta \) which parameterizes a pair of distinguished bases \( \{ G^{up}(b) \}_{b \in B(\infty)_\beta} \) and \( \{ G^{low}(b) \}_{b \in B(\infty)_\beta} \) of \( \mathbb{Q}(t) \otimes_A U^+_\beta \). The Khovanov-Lauda-Rouquier algebra \( R_\beta \) is a certain graded algebra whose grading is bounded from below with the following properties:

- The set of isomorphism classes of simple graded \( R_\beta \)-modules (up to grading shifts) is also parameterized by \( B(\infty)_\beta \);
- For each \( b \in B(\infty)_\beta \), we have a simple graded \( R_\beta \)-module \( L_b \) and its projective cover \( P_b \). Let \( L_b^\vee(k) \) be the \( k \)-shift of \( L_b \), and let \( [ P_b : L_b^\vee(k) ]_0 \) be the multiplicity of \( L_b^\vee(k) \) in \( P_b \) (that is finite). Then, we have
  \[ G^{low}(b) = \sum_{b' \in B(\infty)_\beta, k \in \mathbb{Z}} t^k [ P_b : L_b^\vee(k) ]_0 G^{up}(b') \];
- For each \( \beta, \beta' \in Q^+ \), there exists an induction functor
  \( \star : R_\beta \text{-gmod} \times R_{\beta'} \text{-gmod} \ni (M, N) \mapsto M \star N \in R_{\beta + \beta'} \text{-gmod} \);
- \( K := \bigoplus_{\beta \in Q^+} \mathbb{Q}(t) \otimes_A K(R_\beta \text{-gmod}) \) is an associative algebra isomorphic to \( \mathbb{Q}(t) \otimes_A U^+ \) with its product inherited from \( \star \) (and the \( t \)-action is a grading shift).

As mentioned earlier, Lusztig [Lus90a] studied the geometric side of the story. By applying the results in [K12a], we first observe the following:

**Theorem A** (Kashiwara’s problem = Corollary 2.9). For every \( \beta \in Q^+ \), the algebra \( R_\beta \) has finite global dimension.

This problem is raised by Kashiwara several times in his lectures on KLR algebras. We remark that in type A case, Theorem A follows through a Morita equivalence with an affine Hecke algebra of type A (see e.g. Opdam-Solleveld [OS09]).

For quantum groups, a way to construct a (nice) PBW basis depends on an arbitrary sequence \( i := (i_1, i_2, \cdots, i_\ell) \in I^\ell \) corresponding to a reduced expression of \( w_0 \). Associated to \( i \), we have a total order \( <_i \) on each \( B(\infty)_\beta \) (see §4).

We define two collections of graded \( R_\beta \)-modules \( \{ E^i_b \}_{b \in B(\infty)_\beta} \) and \( \{ F^i_b \}_{b \in B(\infty)_\beta} \) as follows (cf. Corollary 4.14): 1) \( E^i_b \) is obtained from \( P_b \) by annihilating all \( L_{b'}(k) \) with \( b' <_i b \) and \( k \geq 0 \), and 2) \( F^i_b \) is obtained from \( E^i_b \) by annihilating all \( L_b(k) \) with \( k > 0 \).

Since \( R_\beta \) is a graded algebra with finite global dimension, we set

\[
(M, N)_{\text{gEB}} := \sum_{i \geq 0} (-1)^i \text{gdim} \text{ext}_{R_\beta}^i(M, N) \in \mathbb{Q}(t) \quad \text{for} \quad M, N \in R_\beta \text{-gmod},
\]
where \( \text{hom}_{R_{\beta}}(M, N) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{R_{\beta} \text{-mod}}(M \langle k \rangle, N) \).

By construction, we deduce that the graded character expansion coefficient \([M : \widetilde{E}_{b'}^i] \in \mathbb{Z}(t)\) is well-defined for every \( M \in R_{\beta} \text{-mod} \).

The above definitions of \( \widetilde{E}_{b}^i \) and \( E_b^i \) look natural, but not apparently related to a PBW basis of \( U^+ \).

**Theorem B** (Orthogonality relation = Theorem 4.13 and its corollaries). In the above setting, we have:

1. For \( b <_1 b' \), we have \( \text{ext}^i_{R_{\beta}}(E^i_b, E^i_{b'}) = \{0\} \);
2. We have
   \[
   \text{ext}^i_{R_{\beta}}(\widetilde{E}_{b}^i, (E_{b'}^i)^*) = \begin{cases} 
   \mathcal{C} & (b \neq b', i = 0) \\
   \{0\} & \text{(otherwise)}
   \end{cases}, \quad \text{and} \quad \langle \widetilde{E}_{b}^i, (E_{b'}^i)^* \rangle_{\text{ep}} = \delta_{b,b'};
   \]
3. The graded \( R_{\beta} \text{-mod} \) module \( \widetilde{E}_{b}^i \) is a self-extension of \( E^i_b \) in the sense that there exists a separable decreasing filtration of \( \widetilde{E}_{b}^i \) whose associated graded is a direct sum of grading shifts of \( E^i_b \).

Since we have \( \langle P_b, L_b \rangle_{\text{ep}} = \delta_{b,b'} \) by definition, the pairing \( (\bullet, \bullet)_{\text{ep}} \) is essentially the Lusztig inner form (cf. [Lus93] 1.2.10–1.2.11). Therefore, Theorem B guarantees that our \( \{\widetilde{E}_{b}^i\}_b \) and \( \{E^i_b\}_b \) must be the categorifications of the lower/upper PBW bases by their characterization. We remark that some of these modules seem to coincide with those obtained by Kleshchev-Ram [KR11], Webster [Web10], and Benkart-Kang-Oh-Park [BKOP].

**Theorem C** (Lusztig’s conjecture = Theorem 4.17). We have \( [P_b : \widetilde{E}_{b'}^i] = [E^i_{b'} : L_b] \) for each \( b, b' \in B(\infty)_{\beta} \). In particular, we have

\( [P_b : \widetilde{E}_{b'}^i] \in \mathbb{N}[t] \) for every \( b, b' \in B(\infty)_{\beta} \).

Theorem C is conjectured by Lusztig as his comment on [Lu90a] in his webpage. Note that Theorem C is established in Lusztig [Lu90a] Corollary 10.7 when the reduced expression \( i \) satisfies the condition so-called “adapted” (see §3).

**Example D** (\( q = s_0 \)). We have \( I = \{\alpha_1, \alpha_2\} \). The standard generators \( E_1, E_2 \) of \( U^+ \) correspond to projective modules \( P_1 \) and \( P_2 \) of \( R_{\alpha_1} \) and \( R_{\alpha_2} \), respectively. Then, one series of the (lower) PBW basis \( \{E^i_b\}_b \) are:

\[
P_1^{(c_1)} \ast Q_{21}^{(c_2)} \ast P_2^{(c_3)} \text{ for } c_1, c_2, c_3 \geq 0.
\]

Here \( X^{(c)} \) denotes a direct factor of \( X \ast X \ast \cdots \ast X \) (\( c \) times). Note that \( P_1^{(c_1)}, Q_{21}^{(c_2)} \), and \( P_2^{(c_3)} \) are maximal self-extensions of simple modules (this is a general phenomenon). We have a short exact sequence

\[
0 \rightarrow P_1 \ast P_2 \langle 2 \rangle \rightarrow P_2 \ast P_1 \rightarrow Q_{21} \rightarrow 0,
\]

which is a categorical version of \( E_2 E_1 - t^2 E_1 E_2 \).
The organization of this paper is as follows: In the first section, we collect several results from [K12a] needed in the sequel. The second section is the preliminary on the KLR algebra. In the third section, we abstract and categorify Lusztig’s arguments in the setting of the Hall algebras [Lus98] to the KLR algebras by utilizing the results of [K12a] and the induction theorem imported from [KL87, Lus02, K09]. This includes categorifications of Saito’s reflection actions [Sai94] that we call the Saito reflection functors. In the fourth section, we depart from geometry and utilize the properties of the Saito reflection functors established in the earlier sections to deduce Theorem B and Theorem C.

After submitted the initial version this paper, there appeared another (algebraic) proofs of the main results of this paper by McNamara [Mac12] and Brundan-Kleshchev-McNamara [BKM12], which also covers non-simply laced cases. Thier approach is quite different from that of ours, and one merit of our approach is that it provides a bridge between geometric/algebraic view points, typically seen in the Saito reflection functor used in the proof.

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Convention

An algebra $R$ is a (not necessarily commutative) unital $\mathbb{C}$-algebra. A variety $X$ is a separated reduced scheme $X_0$ of finite type over some localization $\mathbb{Z}_S$ of $\mathbb{Z}$ specialized to $\mathbb{C}$. It is called a $G$-variety if we have an action of a connected affine algebraic group scheme $G$ flat over $\mathbb{Z}_S$ on $X_0$ (specialized to $\mathbb{C}$). As in [BBD82] §6 and [BL94], we transplant the notion of weights to the derived category of $(G$-equivariant) constructible sheaves with finite monodromy on $X$. Let us denote by $D^b(X)$ (resp. $D^+(X)$) the bounded (resp. bounded from the below) derived category of the category of constructible sheaves on $X$, and denote by $D^b_G(X)$ the $G$-equivariant derived category of $X$. We have a natural forgetful functor $D^b_G(X) \to D^+(X)$, whose preimage of $D^b(X)$ is denoted by $D^b_G(X)$. For an object of $D^b_G(X)$, we may denote its image in $D^b(X)$ by the same letter.

Let $\text{vec}$ be the category of $\mathbb{Z}$-graded vector spaces (over $\mathbb{C}$) bounded from the below so that its objects have finite-dimensional graded pieces. In particular, for $V = \bigoplus_{i > -\infty} V_i \in \text{vec}$, its graded dimension $g\dim V := \sum_i t^i \dim V_i \in \mathbb{Z}[[t]]$ makes sense (with $t$ being indeterminant). We define $V \langle m \rangle$ by setting $(V \langle m \rangle)_i := V_{i-m}$.

In this paper, a graded algebra $A$ is always a $\mathbb{C}$-algebra whose underlying space is in $\text{vec}$. Let $A\text{-gmod}$ be the category of finitely generated graded $A$-modules. For $E, F \in A\text{-gmod}$, we define $\text{hom}_A(E, F)$ to be the direct sum of graded $A$-module homomorphisms $\text{hom}_A(E, F)^j$ of degree $j = \text{Hom}_{A\text{-gmod}}(E \langle j \rangle, F)$. We employ the same notation for extensions (i.e. $\text{ext}_A^j(E, F) = \bigoplus_{j \in \mathbb{Z}} \text{ext}_A^j(E, F)$).

We denote by $\text{Irr} A$ the set of isomorphism classes of graded simple modules of $A$, and denote by $\text{Irr}_0 A$ the set of isomorphism classes of graded simple modules of $A$ up to grading shifts. Two graded algebras are said to be Morita equivalent if their graded module categories are equivalent. For a graded $A$-module $E$, we denote its head by $\text{hd} E$, and its socle by $\text{soc} E$.

For $Q(t) \in \mathbb{Q}(t)$, we set $Q(t) := Q(t^{-1})$. For derived functors $R F$ or $\Pi F$ of some functor $F$, we represent its arbitrary graded piece (of its homology
complex) by $R^*F$ or $L^*F$, and the direct sum of whole graded pieces by $R^*F$ or $L^*F$. For example, $R^iF \cong R^iG$ means that $R^iF \cong R^iG$ for every $i \in \mathbb{Z}$, while $R^*F \cong R^*G$ means that $\bigoplus_i R^iF \cong \bigoplus_i R^iG$.

When working on some sort of derived category, we suppress $R$ or $L$, or the category from the notation for simplicity when there is only small risk of confusion.

1 Recollection from [K12a]

Let $G$ be a connected reductive algebraic group. Let $\mathfrak{X}$ be a $G$-variety. Let $\Lambda$ be the labelling set of $G$-orbits of $\mathfrak{X}$. For $\lambda \in \Lambda$, we denote the corresponding $G$-orbit by $\Omega_\lambda$. For $\lambda, \mu \in \Lambda$, we write $\lambda \leq \mu$ if $\Omega_\lambda \subset \Omega_\mu$. We assume the following property ($\spadesuit$):

($\spadesuit)_1$ The set $\Lambda$ is finite. For each $\lambda \in \Lambda$, we fix $x_\lambda \in \Omega_\lambda(\mathbb{C})$;

($\spadesuit)_2$ For each $\lambda \in \Lambda$, the group $\text{Stab}_G(x_\lambda)$ is connected.

We have a (relative) dualizing complex $\omega_\mathfrak{X} := p^! \mathbb{C} \in D^b_G(\mathfrak{X})$, where $p : \mathfrak{X} \to \{\text{pt}\}$ is the $G$-equivariant structure map. We have a dualizing functor

$$D : D^b_G(\mathfrak{X})^{\text{op}} \ni C^* \rightarrow \text{Hom}^* (C^*, \omega_\mathfrak{X}) \in D^b_G(\mathfrak{X}).$$

We have a $\mathbb{D}$-auto-dual $t$-structure of $D^b_G(\mathfrak{X})$ whose truncation functor and perverse cohomology functor are denoted by $\tau$ and $p^! H$, respectively.

For each $\lambda \in \Lambda$, we have a constant local system $\underline{C}_\lambda$ on $\Omega_\lambda$. We have inclusions $i_\lambda : \{x_\lambda\} \hookrightarrow \mathfrak{X}$ and $j_\lambda : \Omega_\lambda \hookrightarrow \mathfrak{X}$. Let $\mathcal{C}_\lambda : = (j_\lambda)_! \underline{C}_\lambda[\dim \Omega_\lambda]$ and $\mathcal{IC}_\lambda : = (j_\lambda)_! \underline{C}_\lambda[\dim \Omega_\lambda]$, which we regard as objects of $D^b_G(\mathfrak{X})$. We denote by

$$\text{Ext}^*_G (\bullet, \bullet) : D^b_G(\mathfrak{X})^{\text{op}} \times D^b_G(\mathfrak{X}) \rightarrow D^+(\text{pt})$$

$$\text{Ext}^*_G (\bullet, \bullet) : D^b(\mathfrak{X})^{\text{op}} \times D^b(\mathfrak{X}) \rightarrow D^b(\mathfrak{pt})$$

the Ext (as bifunctors) of $D^b_G(\mathfrak{X})$ and $D^b(\mathfrak{X})$, respectively.

For each $\lambda \in \Lambda$, we fix $L_\lambda \in D^b(\text{pt})$ as a non-zero graded vector space with a trivial differential which satisfies the self-duality condition $L_\lambda \cong L_\lambda^*$. We set

$$\mathcal{L} : = \bigoplus_{\lambda \in \Lambda} L_\lambda \boxtimes \mathcal{IC}_\lambda \in D^b_G(\mathfrak{X}).$$

By construction, we find an isomorphism $\mathcal{L} \cong \mathbb{D} \mathcal{L}$.

We form a graded Yoneda algebra

$$A_{(G, \mathfrak{X})} = \bigoplus_{i \in \mathbb{Z}} A^i_{(G, \mathfrak{X})} := \bigoplus_{i \in \mathbb{Z}} \text{Ext}^i_G(\mathcal{L}, \mathcal{L})$$

whose degree is the cohomological degree. We denote by $B_{(G, \mathfrak{X})}$ the algebra $A_{(G, \mathfrak{X})}$ by taking $\mathcal{L} = \bigoplus_{\lambda \in \Lambda} \mathcal{IC}_\lambda$ (and call it the basic ring of $A_{(G, \mathfrak{X})}$). The algebra $B_{(G, \mathfrak{X})}$ is Morita equivalent to $A_{(G, \mathfrak{X})}$, and hence all the arguments in the below are independent of the choice of $\mathcal{L}$, which we suppress for simplicity. We also drop $(G, \mathfrak{X})$ in case the meaning is clear from the context. It is standard that $\{L_\lambda\}_{\lambda \in \Lambda}$ forms a complete collection of graded simple $A$-modules up to grading shifts.
Lemma 1.1 (see [K12a] 1.2). For a graded $A$-module $M$, its graded dual $M^*$ is again a graded $A$-module. □

For each $\lambda \in \Lambda$, we set

$$P_{\lambda} := \text{Ext}_G^\bullet (\mathcal{IC}_{\lambda}, \mathcal{L}) = \bigoplus_{i \in \mathbb{Z}} \text{Ext}_G^i (\mathcal{IC}_{\lambda}, \mathcal{L}).$$

Each $P_{\lambda}$ is a graded projective left $A$-module. By construction, we have

$$A \cong \bigoplus_{\lambda \in \Lambda} L_{\lambda} \otimes \text{Ext}_G^\bullet (\mathcal{IC}_{\lambda}, \mathcal{L}) = \bigoplus_{\lambda \in \Lambda} P_{\lambda} \otimes L_{\lambda}^*$$

as left $A$-modules. It is standard that $P_{\lambda}$ is an indecomposable $A$-module whose head is isomorphic to $L_{\lambda}$ (cf. [CG97] §8.7). We have an idempotent $e_{\lambda} \in A$ so that $P_{\lambda} \cong A e_{\lambda}$ as left graded $A$-modules (up to a grading shift).

We consider the following property ($\spadesuit$):

($\spadesuit$)$_1$ The algebra $A$ is pure of weight 0;

($\spadesuit$)$_2$ For each $\lambda \in \Lambda$, the perverse sheaf $\mathcal{IC}_{\lambda}$ is pointwise pure;

Theorem 1.2 ([K12a] 3.5). Assume the properties ($\spadesuit$) and ($\clubsuit$). Then, the algebra $A$ has finite global dimension. □

For $M \in A\text{-gmod}$ and $i \in \mathbb{Z}$, we define

$$[M : L_{\lambda} (i)]_0 := \dim \text{Hom}_{A \text{-gmod}} (P_{\lambda} (i), M) \in \mathbb{Z} \quad \text{and} \quad [M : L_{\lambda}] := \text{gdim} \text{Hom}_{A} (P_{\lambda}, M) \in \mathbb{Z}(t).$$

We have $[M : L_{\lambda}] = \sum_{i \in \mathbb{Z}} [M : L_{\lambda} (i)]_0 t^i \in \mathbb{Z}(t)$.

Theorem 1.3 ([K12a] 1.6). Assume the properties ($\spadesuit$) and ($\clubsuit$):

1. We have

$$[\tilde{K}_{\lambda} : L_{\mu}] = 0 = [K_{\lambda} : L_{\mu}] \quad \text{for} \lambda \not\succeq \mu \quad \text{and} \quad [K_{\lambda} : L_{\lambda}] = 1;$$

2. For each $\mu \not\succeq \lambda$, we have

$$\text{ext}_A^\bullet (\tilde{K}_{\lambda}, \tilde{K}_{\mu}) = \{0\} \quad \text{and} \quad \text{ext}_A^\bullet (K_{\lambda}, K_{\mu}) = \{0\};$$
3. For each $\lambda \in \Lambda$, we have
\[
\tilde{K}_\lambda \cong P_\lambda / \left( \sum_{\mu \prec \lambda} Ae_\mu P_\lambda \right);
\]

4. Each $\tilde{K}_\lambda$ is a successive self-extension of $K_\lambda$. In addition, we have
\[
[\tilde{K}_\lambda : L_\lambda] = \text{gdim} H^0_{\text{Stab}_G(x_\lambda)}(\text{pt}).
\]

For $M \in A\text{-gmod}$ and $N \in A\text{-gmod}$, we define its graded Euler-Poincaré characteristic as:
\[
\langle M, N \rangle_{\text{gEP}} := \sum_{i \geq 0} (-1)^i \text{gdim} \text{ext}^i_A(M, N) \in \mathbb{Z}(t).
\]

Let $j : \mathcal{Y} \hookrightarrow \mathcal{X}$ be the inclusion of an open $G$-stable subvariety. We form a graded algebra
\[
A_{(G,\mathcal{Y})} := \text{Ext}^*_G(j^* \mathcal{L}, j^* \mathcal{L}).
\]

**Lemma 1.4** ([K12a] 4.4). Let $j : \mathcal{Y} \hookrightarrow \mathcal{X}$ be the inclusion of an open $G$-stable subvariety. Then, $\mathcal{Y}$ satisfies the conditions (♠) and (♣) if $\mathcal{X}$ does. \hfill $\square$

**Proposition 1.5** ([K12a] 4.3, 4.5). Let $i : Q_\lambda \hookrightarrow \mathcal{X}$ be the inclusion of a closed $G$-orbit (with $\lambda \in \Lambda$), and let $j : \mathcal{Y} \hookrightarrow \mathcal{X}$ be its complement. Then, we have an isomorphism $A_{(G,X)}/(A_{(G,X)} \cdot \lambda A_{(G,X)}) \xrightarrow{\cong} A_{(G,\mathcal{Y})}$. \hfill $\square$

**Corollary 1.6** ([K12a] 4.2, 4.3, 4.5). Let $j : \mathcal{Y} \hookrightarrow \mathcal{X}$ be the inclusion of an open $G$-stable subvariety. We have
\[
\text{ext}^*_A(A, L_\mu) \cong \text{ext}^*_A(A_{(G,\mathcal{Y})}, L_\mu)
\]
for every $\mu \in \Lambda$ so that $Q_\mu \subset \mathcal{Y}$. \hfill $\square$

## 2 Quivers and the KLR algebras

Let $\Gamma = (I, \Omega)$ be an oriented graph with the set of its vertex $I$ and the set of its oriented edges $\Omega$. Here $I$ is fixed, and $\Omega$ might change so that the underlying graph $\Gamma_0$ of $\Gamma$ is a fixed Dynkin diagram of type ADE. We refer $\Omega$ as the orientation of $\Gamma$. We form a path algebra $\mathbb{C}[[\Gamma]]$ of $\Gamma$.

For $h \in \Omega$, we define $h' \in I$ to be the source of $h$ and $h'' \in I$ to be the sink of $h$. We denote $i \leftrightarrow j$ for $i, j \in I$ if and only if there exists $h \in \Omega$ such that $\{h', h''\} = \{i, j\}$. A vertex $i \in I$ is called a sink of $\Gamma$ if $h' \neq i$ for every $h \in \Omega$. A vertex $i \in I$ is called a source of $\Gamma$ if $h'' \neq i$ for every $h \in \Omega$.

Let $Q^+$ be the free abelian semi-group generated by $\{\alpha_i\}_{i \in I}$, and let $Q^+ \subset Q$ be the free abelian group generated by $\{\alpha_i\}_{i \in I}$. We sometimes identify $Q$ with the root lattice of type $\Gamma_0$ with a set of its simple roots $\{\alpha_i\}_{i \in I}$. Let $W = W(\Gamma_0)$ denote the Weyl group of type $\Gamma_0$ with a set of its simple reflections $\{s_i\}_{i \in I}$. The group $W$ acts on $Q$ via the above identification. Let $R^+ := W \{\alpha_i\}_{i \in I} \cap Q^+$ be the set of positive roots of a simple Lie algebra with its Dynkin diagram $\Gamma_0$.

An $I$-graded vector space $V$ is a vector space over $\mathbb{C}$ equipped with a direct sum decomposition $V = \bigoplus_{i \in I} V_i$. 


Let $V$ be an $I$-graded vector space. For $\beta \in Q^+$, we declare $\dim V = \beta$ if and only if $\beta = \sum_{i \in I} (\dim V_\alpha) \alpha_i$. We call $\dim V$ the dimension vector of $V$.

Form a vector space $E^\Omega_V := \bigoplus_{h \in \Omega} \Hom_{\mathbb{C}}(V_h, V_{h^\sigma})$.

We set $G_V := \prod_{i \in I} GL(V_i)$. The group $G_V$ acts on $E^\Omega_V$ through its natural action on $V$. The space $E^\Omega_V$ can be identified with the based space of $\mathbb{C}[\Gamma]$-modules with its dimension vector $\beta$. Let $\mathbb{M}_\beta$ be a unique $\mathbb{C}[\Gamma]$-module (up to an isomorphism) with $\dim \mathbb{M}_\beta = \beta$.

For each $k \geq 0$, we consider a sequence $m = (m_1, m_2, \ldots, m_k) \in I^k$. We abbreviate this as $ht(m) = k$. We set $wt(m) := \sum_{j=1}^k \alpha_{m_j} \in Q^+$. For $\beta = wt(m) \in Q^+$, we set $ht \beta = k$. For a sequence $m' = (m'_1, \ldots, m'_{k'}) \in I^{k'}$, we set $m + m' := (m_1, \ldots, m_k, m'_1, \ldots, m'_{k'}) \in I^{k+k'}$.

For $i \in I$ and $k \geq 0$, we understand that $k i = (i, \ldots, i) \in I^k$.

For each $\beta \in Q^+$, we set $Y^\beta$ to be the set of all sequences $m$ such that $wt(m) = \beta$. For each $\beta \in Q^+$ with $ht \beta = n$ and $1 \leq i < n$, we define an action of $\{ \sigma_i \}_{i=1}^{n-1}$ on $Y^\beta$ as follows: For each $1 \leq i < n$ and $m = (m_1, \ldots, m_n) \in Y^\beta$, we set $\sigma_i m := (m_1, \ldots, m_{i-1}, m_{i+1}, m_i, m_{i+2}, \ldots, m_n)$.

It is clear that $\{ \sigma_i \}_{i=1}^{n-1}$ generates a $\mathfrak{S}_n$-action on $Y^\beta$. In addition, $\mathfrak{S}_n$ naturally acts on a set of integers $\{1, 2, \ldots, n\}$.

**Definition 2.1** (Khovanov-Lauda [KL09], Rouquier [Rou08]). Let $\beta \in Q^+$ so that $n = ht \beta$. We define the KLR algebra $R_\beta$ as a unital algebra generated by the elements $\kappa_1, \ldots, \kappa_n$, $\tau_1, \ldots, \tau_{n-1}$, and $e(m)$ ($m \in Y^\beta$) subject to the following relations:

1. $\deg \kappa_i e(m) = 2$ for every $i$, and

$$\deg \tau_i e(m) = \begin{cases} -2 & (m_i = m_{i+1}) \\ 1 & (m_i = m_{i+1}) \\ 0 & (\text{otherwise}) \end{cases}$$

2. $[\kappa_i, \kappa_j] = 0$, $e(m) e(m') = \delta_{m,m'} e(m)$, and $\sum_{m \in Y^\beta} e(m) = 1$;

3. $\tau_i e(m) = e(\sigma_i m) \tau_i e(m)$, and $\tau_i \tau_j e(m) = \tau_j \tau_i e(m)$ for $|i - j| > 1$;

4. $\tau_i^2 e(m) = Q_{m,i}(\kappa_i, \kappa_{i+1}) e(m)$;

5. For each $1 \leq i < n$, we have

$$\tau_{i+1} \tau_i \tau_{i+1} e(m) \tau_i e(m) = \begin{cases} Q_{m,i}(\kappa_{i+2}, \kappa_{i+1}) - Q_{m,i}(\kappa_i, \kappa_{i+1}) e(m) & (m_{i+2} = m_i) \\ 0 & (\text{otherwise}) \end{cases};$$

6. $\tau_i \kappa_k e(m) = \kappa_{\sigma_i(k)} \tau_i e(m) = \begin{cases} -e(m) & (i = k, m_i = m_{i+1}) \\ e(m) & (i = k - 1, m_i = m_{i+1}) \\ 0 & (\text{otherwise}) \end{cases}$. 

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Here we set \( h_{m,i} := \# \{ h \in \Omega \mid h' = m_i, h'' = m_{i+1} \} \) and

\[
Q_{m,i}(u, v) = \begin{cases} 
1 & (m_i \neq m_{i+1}, m_i \neq m_{i+1}) \\
-1 & (m_i = m_{i+1}) \\
0 & \text{(otherwise)}
\end{cases}
\]

where \( u, v \) are indeterminants. \( \square \)

Remark 2.2. Note that the algebra \( R_\beta \) a priori depends on the orientation \( \Omega \) through \( Q_{m,i}(u, v) \). Since the graded algebras \( R_\beta \) are known to be mutually isomorphic for any two choices of \( \Omega \) (cf. Theorem 2.3), we suppress this dependence in the below.

For an \( I \)-graded vector space \( V \) with \( \dim V = \beta \), we define

\[
F^\Omega_\beta := \left\{ \{ F_j \}_{j=0}^{ht_\beta} \mid \dim F_j/F_j+1 = m_j \quad \forall j \right\} \subset F^\Omega, \\
B^\Omega_\beta := \left\{ \{ F_j \}_{j=0}^{ht_\beta} \mid F_j \subset V \text{ is an } I \text{-graded vector subspace s.t. } F_j+1 \subset F_j \right\}.
\]

We have a projection

\[
\varpi^\Omega_\beta : F^\Omega_\beta \ni \{ F_j \}_{j=0}^{ht_\beta} \mapsto \{ F_j \}_{j=0}^{ht_\beta} \in B^\Omega_\beta,
\]

which is \( G_V \)-equivariant. For each \( \mathbf{m} \in Y_\beta \), we have a connected component

\[
F^\Omega_\mathbf{m} := \left\{ \{ F_j \}_{j=0}^{ht_\beta} \mid \dim F_j/F_j+1 = m_j \quad \forall j \right\} \subset F^\Omega_\beta,
\]

that is smooth of dimension \( d^\Omega_\mathbf{m} \). We set \( B^\Omega_\mathbf{m} := \varpi^\Omega_\beta(F^\Omega_\mathbf{m}) \), that is an irreducible component of \( B^\Omega_\beta \). Let

\[
\pi^\Omega_\mathbf{m} : F^\Omega_\mathbf{m} \ni \{ F_j \}_{j=0}^{ht_\beta} \mapsto x \in E^\Omega_V
\]

be the second projection that is also \( G_V \)-equivariant. The map \( \pi^\Omega_\mathbf{m} \) is projective, and hence

\[
L^\Omega_{\mathbf{m}} := (\pi^\Omega_\mathbf{m})_! \mathbb{C}[d^\Omega_\mathbf{m}]
\]

decomposes into a direct sum of (shifted) irreducible perverse sheaves with their coefficients in \( D^b(\text{pt}) \) (Gabber’s decomposition theorem, [BBD82] 6.2.5). We set \( L^\Omega_\beta := \bigoplus_{\mathbf{m} \in Y_\beta} L^\Omega_\mathbf{m} \). Let \( \epsilon(\mathbf{m}) \) be the idempotent in \( \text{End}(L^\Omega_\beta) \) so that \( \epsilon(\mathbf{m}) L^\Omega_\beta = L^\Omega_\mathbf{m} \). Since \( \pi^\Omega_\mathbf{m} \) is projective, we conclude that \( D L^\Omega_{\mathbf{m}} \cong L^\Omega_{\mathbf{m}} \) for each \( \mathbf{m} \in Y_\beta \), and hence

\[
D L^\Omega_\beta \cong L^\Omega_\beta.
\]  

**Theorem 2.3** (Varagnolo-Vasserot [VV11]). Under the above settings, we have an isomorphism of graded algebras:

\[
R_\beta \cong \bigoplus_{i \in \mathbb{Z}} \text{Ext}^i_{G_V}(L^\Omega_\beta, L^\Omega_\beta).
\]

In particular, the RHS does not depend on the choice of an orientation \( \Omega \) of \( \Gamma_0 \).
For each $\mathbf{m}, \mathbf{m}' \in Y^\beta$, we set
\[
R_{\mathbf{m}, \mathbf{m}'} := e(\mathbf{m})R_\beta e(\mathbf{m}') = \bigoplus_{\epsilon \in \mathbb{Z}} \text{Ext}^\epsilon_{G_V}(L^\mathcal{O}_{\mathbf{m}'}, L^\mathcal{O}_{\mathbf{m}}).
\]

We set $S_\beta \subset R_\beta$ to be a subalgebra which is generated by $e(\mathbf{m})$ ($\mathbf{m} \in Y^\beta$) and $\kappa_1, \ldots, \kappa_n$.

For each $\beta_1, \beta_2 \in Q_+$ with $\text{ht} \beta_1 = n_1$ and $\text{ht} \beta_2 = n_2$, we have a natural inclusion:
\[
\begin{align*}
R_{\beta_1} \boxtimes R_{\beta_2} &\ni e(\mathbf{m}) \boxtimes e(\mathbf{m}') \mapsto e(\mathbf{m} + \mathbf{m}') \in R_{\beta_1 + \beta_2}. \\
R_{\beta_1} \boxtimes 1 &\ni \kappa_1 \boxtimes 1, \tau_1 \boxtimes 1 \mapsto \kappa_1, \tau_1 \in R_{\beta_1 + \beta_2} \\
1 \boxtimes R_{\beta_2} &\ni 1 \boxtimes \kappa_1, 1 \boxtimes \tau_1 \mapsto \kappa_1 + \tau_1, \tau_1 + \kappa_1 \in R_{\beta_1 + \beta_2}
\end{align*}
\]

This defines an exact functor
\[
*: R_{\beta_1} \boxtimes R_{\beta_2} \text{-gmod} \ni M_1 \boxtimes M_2 \mapsto R_{\beta_1 + \beta_2} \otimes_{R_{\beta_1} \otimes R_{\beta_2}} (M_1 \boxtimes M_2) \in R_{\beta_1 + \beta_2} \text{-gmod}.
\]

It is straight-forward to see that $*$ restricts to an exact functor in the category of graded projective modules:
\[
*: R_{\beta_1} \boxtimes R_{\beta_2} \text{-proj} \ni M_1 \boxtimes M_2 \mapsto R_{\beta_1 + \beta_2} \otimes_{R_{\beta_1} \otimes R_{\beta_2}} (M_1 \boxtimes M_2) \in R_{\beta_1 + \beta_2} \text{-proj}.
\]

If $i \in I$ is a source of $\Gamma$ and $f = (f_h)_{h \in \Omega} \in \mathbb{P}^I$, then we define
\[
\epsilon^*_i(f) := \dim \ker \bigoplus_{h \in \Omega, h' = i} f_h \leq \dim V_i.
\]

If $i \in I$ is a sink of $\Gamma$ and $f = (f_h)_{h \in \Omega} \in \mathbb{Q}^I$, then we define
\[
\epsilon_i(f) := \dim \text{coker} \bigoplus_{h \in \Omega, h' = i} f_h \leq \dim V_i.
\]

Each of $\epsilon^*_i(f)$ or $\epsilon_i(f)$ do not depend on the choice of a point in a $G_V$-orbit. Hence, $\epsilon_i$ or $\epsilon^*_i$ induces a function on $\mathbb{P}^I$ that is constant on each $G_V$-orbit, and a function on the set of isomorphism classes of simple $G_V$-equivariant perverse sheaves on $E^\mathcal{O}_V$ through a unique open dense $G_V$-orbit of its support whenever $i$ is a source or a sink.

**Proposition 2.4** (Lusztig [Lus91] 6.6). *For each $i \in I$, the functions $\epsilon_i$ and $\epsilon^*_i$ descend to functions on the set of isomorphism classes of simple graded $R_\beta$-modules (up to degree shift).*

**Proof.** Note that [Lus91] 6.6 considers only $\epsilon_i$, but $\epsilon^*_i$ is obtained by swapping the order of the convolution operation. \qed

**Theorem 2.5** (Khovanov-Lauda [KL09], Rouquier [Rou08], Varagnolo-Vasserot [VV11]). *In the above setting, we have:*

1. *For each $i \in I$ and $n \geq 0$, $R_{\alpha_i}$ has a unique indecomposable projective module $P_{n_i}$ up to grading shifts;*
2. The functor \( \star \) induces a \( \mathbb{Z}[t^{\pm 1}] \)-algebra structure on
\[
K := \bigoplus_{\beta \in Q^+} K(\mathbf{R}_\beta\text{-proj});
\]

3. The algebra \( K \) is isomorphic to the integral form \( U^+ \) of the positive part of the quantized enveloping algebra of type \( \Gamma_0 \) by identifying \([P_n]\) with the \( n \)-th divided power of a Chevalley generator of \( U^+ \);

4. The above isomorphism identifies the classes of indecomposable graded projective \( \mathbf{R}_\beta \)-modules (\( \beta \in Q^+ \)) with an element of the lower global basis of \( U^+ \) in the sense of [Kas91];

5. There exists a set \( B(\infty) = \bigsqcup_{\beta \in Q^+} B(\infty)_\beta \) that parameterizes indecomposable projective modules of \( \bigoplus_{\beta \in Q^+} \mathbf{R}_\beta\text{-gmod} \). This identifies the functions \( \epsilon_i, \epsilon_i^\ast \) (\( i \in I \)) with the corresponding functions on \( B(\infty) \).

**Proof.** The first assertion is [KL09] 2.2 3), the second and the third assertions are [KL09] 3.4, and the fourth assertion is [VV11] 4.4. Based on this, the fifth follows from Proposition 2.4. See also Theorem 2.12 in the below.

**Remark 2.6.** The coincidence of the lower global basis and the canonical basis is proved by Lusztig [Lu90a] and Grojnowski-Lusztig [GL93]. We freely utilize this identification in the below.

**Proposition 2.7.** In the above setting, the conditions (\( \spadesuit \)) and (\( \clubsuit \)) are satisfied.

**Proof.** The condition (\( \spadesuit \)) is the Gabriel theorem (on the classification of finite algebras, applied to \( \mathbb{C}[\Gamma] \)). The condition (\( \clubsuit \)) follows by the fact that \( \text{Stab}_G(x_\lambda) \) is the automorphism group of a \( \mathbb{C}[\Gamma] \)-module \( \mathcal{M} \), which must be an open dense part of a linear subspace.

We set \( Z^\alpha_\beta := E^\alpha_\beta \times_\mathbb{P} F^\alpha_\beta \). By [VV11] 1.8 (b) and 2.23 (or [CG97] 8.6.7), we have an isomorphism \( H^G_{\mathcal{V}}(Z^\alpha_\beta) \cong \text{Ext}^\ast_{\mathcal{V}}(L^\alpha_\beta, L^\alpha_\beta) \) as graded algebras (here we warn that the grading on the LHS is imported from the RHS, and is not the standard one; cf. [VV11] 1.9). Since \( G_\mathcal{V} \) is a reductive group, we know that each \( G_\mathcal{V} \)-orbit of \( (B^\alpha_\beta)^2 \) is an affine bundle over a connected component of \( B^\alpha_\beta \) (see e.g. [CG97] §3.4). By [VV11] 2.11, each fiber of the \( G_\mathcal{V} \)-equivariant map \( Z^\alpha_\beta \rightarrow (B^\alpha_\beta)^2 \) induced from \( \psi^\alpha_\beta \) is a vector space. Therefore, we conclude that \( Z^\alpha_\beta \) is a union of finite increasing sequence of closed subvarieties
\[
\emptyset = Z^\alpha_{\beta,0} \subsetneq Z^\alpha_{\beta,1} \subsetneq Z^\alpha_{\beta,2} \subsetneq \cdots \subsetneq Z^\alpha_{\beta,t} = Z^\alpha_{\beta},
\]
where each \( Z^\alpha_{\beta,j} \setminus Z^\alpha_{\beta,j-1} \) is an affine bundle over a connected component of \( B^\alpha_\beta \). This implies the purity of \( H^G_{\mathcal{V}}(Z^\alpha_\beta) \), and hence (\( \spadesuit \)) follows.

The condition (\( \clubsuit \)) is Lusztig [Lu90a] 10.6.

**Corollary 2.8** (Lusztig [Lu90a]). Every simple \( G_\mathcal{V} \)-equivariant perverse sheaf on \( E^\alpha_\beta \) appears as a non-zero direct summand of \( L^\alpha_\beta \) up to a degree shift.

**Proof.** By Proposition 2.7 and Theorem 2.3, we deduce that the assertion is equivalent to \( \# \text{Ir}_{\mathbf{R}_\beta} = \#(G_\mathcal{V} \backslash E^\alpha_\beta) \). This follows from a standard bijection between the set of isomorphism classes of indecomposable \( \mathbb{C}[\Gamma] \)-modules and a basis of \( U^+ \) à la Ringel [Rin90] (or a consequence of the Gabriel theorem). □
The algebra $R_\beta$ has finite global dimension.

Proof. Apply Theorem 1.2 to (2.1), Proposition 2.7, and Corollary 2.8.

Thanks to Corollary 2.8 and Theorem 2.5 5), we have an identification $B(\infty)_\beta \cong G V \setminus E^0 V$, where $V$ is an $I$-graded vector space with $\dim V = \beta$. By regarding $G V \setminus E^0 V$ as the space of $\mathbb{C}[\Gamma]$-modules with its dimension vector $\beta$, each $b \in B(\infty)_\beta$ gives rise to (an isomorphism class of) a $\mathbb{C}[\Gamma]$-module $\mathcal{M}_b$. Let us denote by $\mathcal{O}_b^0$ the $G V$-orbit of $E^0 V$ corresponding to $b \in B(\infty)_\beta$. Each $b \in B(\infty)_\beta$ defines an indecomposable graded projective module $P_b$ of $R_\beta$ with simple head $L_b$ that is isomorphic to its graded dual $L^*_b$ (see §1).

The standard module $K_b$ and the dual standard module $K^*_b$ in §1 depends on the choice of $\Omega$ since the Fourier transform interchanges the closure relations. Therefore, we denote by $K^0_b$ (resp. $K^*_0$) the standard module (resp. the dual standard module) of $L_b$ arising from $E^0 V$.

Example 2.10. If $\beta = m\alpha_i$ for $m \geq 1$ and $i \in I$, then the set $B(\infty)_{m\alpha_i}$ is a singleton. Let $L_{m\alpha_i}$ and $P_{m\alpha_i}$ be unique simple and projective graded modules of $R_{m\alpha_i}$, up to grading shifts, respectively. The standard module $K_{m\alpha_i}$ and the dual standard module $K^*_{m\alpha_i}$ do not depend on the choice of $\Omega$ in this case. We have $L_{m\alpha_i} \cong K_{m\alpha_i}$ and $P_{m\alpha_i} \cong K^*_{m\alpha_i}$, and

$$[K_{m\alpha_i} : K_{m\alpha_i}] = \text{gdim} \mathbb{C}[x_1, \ldots, x_m]^\oplus_m.$$  

Let $Q^0_\beta$ be the full subcategory of $D^b_{G V}(E^0 V)$ consisting all complexes whose direct summands are degree shifts of that of $E^0 V$.

Let $\beta \in Q^+$ with $\text{ht} \beta = n$. Let $\leq_B$ be the Bruhat order of $\mathfrak{S}_n$ with respect to the set of simple reflections $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$. For each $w \in \mathfrak{S}_n$ and its reduced expression

$$w = \sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_L},$$

we set $\tau_w := \tau_{j_1} \tau_{j_2} \cdots \tau_{j_L}$. Note that $\tau_w$ depends on the choice of a reduced expression.

Theorem 2.11 (Poincaré-Birkoff-Witt theorem [KL09] 2.7). We have equalities as vector spaces:

$$R_\beta = \bigoplus_{w \in \mathfrak{S}_n, m \in Y^\beta} \tau_w S_\beta e(m) = \bigoplus_{w \in \mathfrak{S}_n, m \in Y^\beta} S_\beta \tau_w e(m),$$

regardless the choices of $\tau_w$.

Let $\beta \in Q^+$ so that $\text{ht} \beta = n$. For each $i \in I$ and $k \geq 0$, we set

$$Y^\beta_{k,i} := \{m = (m_j) \in Y^\beta \mid m_1 = \cdots = m_k = i\}$$

and

$$Y^\beta_{k,i}^* := \{m = (m_j) \in Y^\beta \mid m_n = \cdots = m_{n-k+1} = i\}.$$  

In addition, we define two idempotents of $R_\beta$ as:

$$e_i(k) := \sum_{m \in Y^\beta_{k,i}} e(m), \text{ and } e_i^*(k) := \sum_{m \in Y^\beta_{k,i}^*} e(m).$$

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Theorem 2.12 (Lusztig [Lus91] §6, Lauda-Vazirani [LV11] 2.5.1). Let $\beta \in Q_+$ and $i \in I$. For each $b \in B(\infty)_\beta$ and $i \in I$, we have

$$e_i(b) = \max\{k \mid e_i(k) b \neq 0\} \quad \text{and} \quad e_i^*(b) = \max\{k \mid e_i^*(k) b \neq 0\}.$$  

Moreover, $e_i(e_i(b))$ and $e_i^*(e_i^*(b))$ are irreducible $R_i(b)\otimes R_i(-e_i(b))_\beta$-module and $R_i(-e_i^*(b))\otimes R_i(e_i^*(b))_\beta$-module, respectively. In addition, if we have distinct $b, b' \in B(\infty)_\beta$ so that $e_i(b) = e_i(b')$ with $k \geq 0$, then $e_i(b)$ and $e_i(b')$ are not isomorphic as an $R_{i\alpha_i} \otimes R_{-i\alpha_i}$-module.

$\Box$

3 Saito reflection functors

Keep the setting of the previous section. Let $\Omega_i$ be the set of edges $h \in \Omega$ with $h'' = i$ or $h' = i$. Let $s_i\Omega_i$ be a collection of edges obtained from $h \in \Omega_i$ by setting $(s_i h)' = h''$ and $(s_i h)'' = h'$. We define $s_i \Omega := (\Omega \setminus \Omega_i) \cup s_i \Omega_i$ and set $s_i \Gamma := (I, s_i \Omega)$. Note that $\Gamma_0 = (s_i \Gamma)_0$.

Let $w_0 \in W$ be the longest element. Choose a reduced expression

$$w_0 = s_1 s_2 \cdots s_t.$$

We denote by $I := (i_1, \ldots, i_t) \in I^t$ the data recording this reduced expression. We say $I$ is adapted to $\Omega$ (or $\Gamma$) if each $i_k$ is a sink of $s_{i_{k-1}} \cdots s_i \Gamma$.

Let $V$ be an $I$-graded vector space with $\dim V = \beta$. For a sink $i$ of $\Gamma$, we define

$$i^! E^\Omega_V := \{ (f_h)_{h \in \Omega} \in E^\Omega_V \mid \ker(\bigoplus_{h \in \Omega, h'' = i} f_h : \bigoplus_{h'} V_{h'} \to V_i) = \{0\} \}.$$  

For a source $i$ of $\Gamma$, we define

$$i^! E^\Omega_V := \{ (f_h)_{h \in \Omega} \in E^\Omega_V \mid \ker(\bigoplus_{h \in \Omega, h' = i} f_h : \bigoplus_{h''} V_{h''} \to V_i) = \{0\} \}.$$  

Let $\Omega$ be an orientation of $\Gamma$ so that $i \in I$ is a sink. Let $\beta \in Q^+ \cap s_i Q^+$. Let $V$ and $V'$ be $I$-graded vector spaces with $\dim V = \beta$ and $\dim V' = s_i \beta$, respectively. We fix an isomorphism $\phi : \bigoplus_{j \neq i} V_j \xrightarrow{\cong} \bigoplus_{j \neq i} V'_j$ as $I$-graded vector spaces. We define:

$$i^! Z^\Omega_{V, V'} := \{ (f_h)_{h \in \Omega}, (f'_h)_{h \in s_i \Omega}, \psi \mid (f_h) \in i^! E^\Omega_V, (f'_h) \in i^! E^\Omega_{V'}, \phi f_h = f'_h \phi \text{ for } h \notin \Omega_i, \phi \psi : V'_i \xrightarrow{\cong} \ker(\bigoplus_{h \in \Omega_i} f_h : \bigoplus_{h'} V_{h'} \to V_i) \}.$$  

We have a diagram:

$$E^\Omega_V \xrightarrow{j_V} E^\Omega_{V'} \xrightarrow{q_V} i^! Z^\Omega_{V, V'} \xrightarrow{p_V^\prime} E^\Omega_{V'} \xrightarrow{i^! \phi} E^\Omega_{V', V'}.$$  

If we set

$$G_{V, V'} := GL(V_i) \times GL(V'_i) \times \prod_{j \neq i} GL(V_j) \cong GL(V_i) \times GL(V'_i) \times \prod_{j \neq i} GL(V'_j),$$

then the maps $p'_V$ and $q'_V$ are $G_{V, V'}$-equivariant.
The morphisms

First, note that the maps

We have an algebra isomorphism

transforms are induced by the pairing between direct summands

for the time being (see Corollary 3.4).

Lemma 3.2. We have an algebra isomorphism

Similarly, the algebra

Proof. The maps

and

are

are

 equivariant open embeddings, respectively. Therefore, we apply Lemma 1.4 and Proposition 1.5 repeatedly to deduce

and

where

are again direct sums of shifted equivariant perverse sheaves. In the setting of Proposition 3.1, two graded algebras

are Morita equivalent to each other. In addition, this Morita equivalence is independent of the choice of

Proposition 3.5. In the setting of Proposition 3.1, two graded algebras

are Morita equivalent to each other. In addition, this Morita equivalence is independent of the choice of

Proof. First, note that the maps

are open embeddings. In particular,

are again direct sums of shifted equivariant perverse sheaves. By Proposition 3.1 and [BL94] 2.2.5, we have equivalences

In addition, a simple

equivariant perverse sheaf

admits isomorphisms

where

and

are simple

 and

 equivariant perverse sheaves on

 respectively. These induce isomorphisms of algebras:

Therefore,

and

are Morita equivalent to the algebras in the assertion by Corollary 2.8, which implies the first assertion.

We prove the second assertion. For any two orientations

 which have

 as a common sink, we have Fourier transforms

 and

 so that

 and

. Since

, these two Fourier transforms are induced by the pairing between direct summands

 and

 which can be identified with those of

 and

 in (3.1) via

 Proposition 3.1 (Lusztig [Lus98]). The morphisms

 and

 in (3.1) are

 for

 and

 respectively.

 Corollary 3.3. The set of isomorphism classes of graded simple modules of

 and

 are

 and

 respectively. Therefore, we apply Lemma

 and Proposition 1.5 repeatedly to deduce

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 respectively. Therefore, we apply Lemma 1.4 and Proposition 1.5 repeatedly to deduce

 and
Let $\mathcal{F}_2$ and $\mathcal{F}_3$ be each a functor of the choices involved. By the latter part of Proposition 3.5, we see that these functors (cf. [Sai94]). By the latter part of Proposition 3.5, we have $\epsilon_i(T_i(b)) = 0$ if $T_i(b) \not= 0$. We set $T_i^{-1}(b') := b$ if $b' = T_i(b) \not= 0$.

Thanks to Corollary 3.4, we can drop $\Omega$ or $s_i \Omega$ from $i^* R_{\beta,\Omega}^1$ and $i^* R_{\beta,\Omega}^1$. We define a left exact functor $\mathcal{T}_i : R_{\delta,\cdot} \rightarrow i^* R_{\delta,\cdot} \rightarrow i^* R_{\delta,\cdot} \rightarrow R_{\delta,\cdot}$, where the first functor is $\text{Hom}_{R_{\delta,\cdot}}(i R_{\delta,\cdot}, \bullet)$, the second functor is Proposition 3.5, and the third functor is the pullback. Similarly, we define a right exact functor $\mathcal{T}_i : R_{\delta,\cdot} \rightarrow i^* R_{\delta,\cdot} \rightarrow i^* R_{\delta,\cdot} \rightarrow R_{\delta,\cdot}$, where the first functor is $i^* R_{\delta,\cdot} \otimes R_{\delta,\cdot}$.

We call these functors the Saito reflection functors (cf. [Sai94]). By the latter part of Proposition 3.5, we see that these functors are independent of the choices involved.

Let $i \in I$. We define $R_{\delta,\cdot}$ (resp. $R_{\delta,-\cdot}$) to be the fullsubcategory of $R_{\delta,\cdot}$ so that each simple subquotient is of the form $L_b(k)$ ($k \in \mathbb{Z}$) with $b \in B(\infty)_{\delta}$ that satisfies $\epsilon_i(b) = 0$ (resp. $\epsilon_i(b) = 0$). In addition, for each $i \not= j \in I$, we define $R_{\delta,-\cdot} := R_{\delta,-\cdot} \cap R_{\delta,-\cdot}$.

**Theorem 3.6 (Saito reflection functors).** Let $i \in I$. We have:

1. Assume that $i$ is a source of $\Omega$. For each $b \in B(\infty)_{\delta}$, we have

$$T_i K_{b}^\Omega = \begin{cases} K_{b}^\Omega & (\epsilon_i^*(b) = 0) \\ \{0\} & (\epsilon_i^*(b) > 0) \end{cases} ;$$

2. For each $b \in B(\infty)_{\delta}$, we have

$$T_i L_b = \begin{cases} L_{T_i(b)} & (\epsilon_i^*(b) = 0) \\ \{0\} & (\epsilon_i^*(b) > 0) \end{cases} \text{ and } T_i^* L_b = \begin{cases} L_{T_i^{-1}(b)} & (\epsilon_i(b) = 0) \\ \{0\} & (\epsilon_i(b) > 0) \end{cases} ;$$

3. The functors $(T_i, T_i^*)$ form an adjoint pair;

4. For each $M \in R_{\delta,-\cdot}$ and $N \in R_{\delta,-\cdot}$, we have

$$\text{ext}^{i}_{R_{\delta,-\cdot}}(T_i M, N) \cong \text{ext}^{i}_{R_{\delta,-\cdot}}(M, T_i^* N);$$

5. Let $i \not= j \in I$. For each $\beta \in Q^+$ and $m \geq 0$, we have

$$T_i (P_{mj} \star M) \cong (T_i p_{mj}) \star T_i M$$

as graded $R_{\delta, R_{\delta,\cdot} + m \alpha_j}$-modules for every $M \in R_{\delta,-\cdot}$. 

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The proof of Theorem 3.6 is given by two parts, namely 1)–4) and 5). We warn that the proof of the latter part rests on the earlier part.

Proof of Theorem 3.6 1)–4). We prove the first assertion. The subset \( E_{i}^{\Omega} \subset E_{i}^{\Omega} \) (with \( \dim V = \beta \)) is open. Therefore, Theorem 1.3 1) asserts that \( [K_{i}^{\Omega} : L_{0}] \neq 0 \) only if \( \epsilon_{i}(b') = 0 \) whenever \( \epsilon_{i}(b) = 0 \). It follows that \( R_{\beta} \otimes R_{\beta} K_{i}^{\Omega} \cong K_{b}^{\Omega} \) as a vector space if \( \epsilon_{i}(b) = 0 \), and \( \{0\} \) otherwise. This gives rise to a standard module of \( R_{\beta} \) by Lemma 1.4, and thus it gives a standard module of \( iR_{\beta} \) by Proposition 3.5. Note that the subset \( E_{i}^{\Omega} \subset E_{i}^{\Omega} \) (with \( \dim V' = s_{i} \beta \)) is also open. Therefore, we use Lemma 3.2 to deduce the first assertion.

The second assertion is immediate from the first assertion and the construction of \( T_{i} \) and \( T_{i}^{*} \).

We prove the third assertion. By Lemma 3.2, we know that \( T_{i} \) factors through the functor giving the maximal quotient which is a \( iR_{\beta} \)-module, while \( T_{i}^{*} \) factors through the functor giving the maximal submodule which is an \( iR_{\beta} \)-module. Therefore, the third assertion follows by the Morita equivalence \( iR_{\beta} \cdot \text{gmod} \cong iR_{s_{i} \beta} \cdot \text{gmod} \) for every \( \beta \in Q^{+} \cap s_{i}Q^{+} \).

For the fourth assertion, notice that \( R_{\beta} \cdot \text{gmod} \) and \( R_{s_{i} \beta} \cdot \text{gmod} \) respectively. It follows that \( R_{\beta} \otimes R_{\beta} M \cong M \), \( R_{s_{i} \beta} \otimes R_{s_{i} \beta} T_{i} M \cong T_{i} M \), and \( \text{Hom}_{R_{\beta}}(s_{i} \beta, N) \cong N \). By Lemma 3.2 and Corollary 1.6, we deduce that each indecomposable projective \( iR_{s_{i} \beta} \)-module \( iP \) admits an \( R_{s_{i} \beta} \)-graded projective resolution

\[
\cdots \to P_{2} \to P_{1} \to P_{0} \to iP \to 0
\]

so that \( P_{0} \) is indecomposable and \( iR_{s_{i} \beta} \otimes R_{s_{i} \beta} M_{k} = \{0\} \) for \( k \geq 1 \). Therefore, we have

\[
\text{ext}_{R_{s_{i} \beta}}^{*}(M, N) = \text{ext}_{R_{s_{i} \beta}}^{*}(M, N),
\]

where we regard \( M, N \) as \( iR_{s_{i} \beta} \)-modules via Proposition 3.5 (here we treat the Morita equivalence as an isomorphism for simplicity). Applying the same argument for \( R_{s_{i} \beta} \) (again for \( M \)), we conclude the result.

Lemma 3.8. Let \( i \in I \). For each \( \beta \in Q^{+}, m \geq 0 \), and an indecomposable graded projective \( iR_{\beta} \)-module \( P \), the module \( P_{m_{i}} \cdot P \) is an \( R_{\beta_{1} + m_{i}} \cdot \text{module with simple head.} \)

Proof. By the Frobenius reciprocity, we have

\[
\text{hom}_{R_{\beta_{1} + m_{i}}}(P_{m_{i}} \cdot P, L_{b}) = \text{hom}_{R_{\beta_{1} + m_{i}}}(P_{m_{i}} \cdot P, L_{b}) \tag{3.4}
\]

for every \( b \in B(\infty)_{\beta_{1} + m_{i}} \). Assume that the above space is non-zero to deduce the uniqueness of \( b \) and the one-dimensionality of (3.4). Choose \( d \in B(\infty)_{\beta} \) so that \( L_{d} \) is the unique simple quotient of \( P \). We have \( \epsilon_{i}(d) = 0 \) by assumption. By Theorem 2.11, we have \( e(m)(P_{m_{i}} \cdot P) \neq \{0\} \) only if there exist a minimal length representative \( w \in S_{N_{i} \beta_{1} + m} / S_{N_{i} \beta_{1} + m} \) and \( m' \in Y_{\beta} \) so that \( e(m')P \neq \{0\} \) and \( m = w(m_{i} + m') \). Since \( m' \notin Y_{\beta} \), we deduce \( \epsilon_{i}(b) \leq m \). Thus, if (3.4) is non-trivial, then we have \( \epsilon_{i}(b) = m \) and \( w = 1 \). Now Theorem 2.12 forces \( \epsilon_{i}(m)L_{0} \cong L_{m_{i}} \otimes L_{d} \). Therefore, \( P_{m_{i}} \cdot P \) has at most one quotient, which completes the proof. \( \square \)
Theorem 3.9 (Induction theorem). Let $V(i)$ be 1-graded vector spaces with $\dim V(i) = \beta_i$, and $b_i \in B(\infty)_{\beta_i}$ for $i = 1, 2$. Let $b \in B(\infty)_{\beta_1 + \beta_2}$ so that $M_b \cong M_{b_1} \oplus M_{b_2}$ as $C[\Gamma]$-modules. Assume the following condition (\ast):

\((\ast)_0\) $M_{b_i}$ is not a quotient of $M_b$ for every $b_1 \neq b_1' \in B(\infty)_{\beta_1}$, and $M_{b_2}$ is not a submodule of $M_b$ for every $b_2 \neq b_2' \in B(\infty)_{\beta_2}$;

\((\ast)_1\) $\text{Ext}^1_{C[\Gamma]}(M_{b_1}, M_{b_2}) = \{0\}$.

We have an isomorphism $K_\beta^{\Omega} \star K_\beta^{\Omega} \cong K_\beta^{\Omega}$ as an ungraded $R_{\beta_1 + \beta_2}$-module.

In addition, if $M_b$ canonically determines the factor $M_{b_2}$ as a vector subspace, then $(\ast)_0$ and $(\ast)_1$ implies

$$K_\beta^{\Omega} \star K_\beta^{\Omega} \cong K_\beta^{\Omega}$$

as a graded $R_{\beta_1 + \beta_2}$-module.

Before proving Theorem 3.9, we present some of its consequences. The proof of Theorem 3.9 itself is given at the end of this section.

Corollary 3.10. Suppose that $i$ is a sink of $\Omega$. Let $m \geq 0$. For each $\beta \in Q^+$ and $b \in B(\infty)_{\beta}$ with $\epsilon_i(b) = 0$, the module $K_{m_\beta} \star K_\beta^{\Omega}$ is an indecomposable graded $R_{\beta + m_\beta}$-module isomorphic to $K_\beta^{\Omega}$ with $M_{m_\beta} \cong M_{m_\beta}^{\mathbb{Z}^m} \oplus M_b$.

Proof. By Example 2.10, we deduce that the first part of $(\ast)_0$ is a void condition. Every irreducible subquotient of a $C[\Gamma]$-module isomorphic to $M_i$ is in its socle. Hence, the second part of $(\ast)_0$ follows by the comparison of the socles. Since $i$ is a sink, we have no extension of $M_i^{\mathbb{Z}^m}$ by $M_b$, which is $(\ast)_1$. We write $\beta = k\alpha_i + \sum_{j \neq i} k_j \alpha_j$. Since $\epsilon_i(b) = 0$, $M_i$ is not a direct summand of $M_b$. In particular, $M_b$ is canonically determined by $M_{m_\beta}$ as its direct factor. Applying Theorem 3.9 yields the result. \(\square\)

Corollary 3.11. Assume that $i$ is a source and $j$ is a sink of $\Omega$. Let $\beta \in Q^+$.

For each $m \geq 0$ and $b \in B(\infty)_{\beta}$ such that $\epsilon_j(b) = 0$, we have

$$T_i(K_{m_\beta} \star K_\beta^{\Omega}) \cong (T_i K_{m_\beta}) \star T_i K_\beta^{\Omega}$$

as graded $R_{s_i(\beta + m_\beta)}$-modules.

Proof. By Corollary 3.10, we see that $K_{m_\beta} \star K_\beta^{\Omega} \cong K_\beta^{\Omega}$, where $M_{m_\beta} \cong M_{m_\beta}^{\mathbb{Z}^m} \oplus M_b$. By [Lu90a] 4.4 (c), we deduce that $T_i(b') \neq 0$ if and only if $T_i(b) \neq 0$. Since a standard module is generated by its simple head, we deduce that $T_i(K_{m_\beta} \star K_\beta^{\Omega}) = \{0\}$ if $\epsilon_j^*(b) > 0$, and it is isomorphic to $K_{T_i(b')}^{\Omega}$ if $\epsilon_j^*(b) = 0$.

Since $i \neq j$, we always have $T_i K_{m_\beta} \neq \{0\}$. Therefore, we conclude that the RHS is non-zero if and only if the LHS is non-zero. Thus, it suffices to show that the RHS is isomorphic to $K_{T_i(b')}^{\Omega}$.

If we have $i \not\leftrightarrow j$, then $j$ is a sink of $s_i \Gamma$. By $\epsilon_j(b) = 0$ and the assumption, we deduce that $M_{T_i(b)}$ also do not contain $M_i$ in this case. Hence, we deduce $\epsilon_j(T_i(b)) = 0$. In addition, we have $T_i K_{m_\beta} \cong K_{m_\beta}^{\Omega}$. Therefore, we apply Corollary 3.10 to deduce that the RHS is $K_{T_i(b')}^{\Omega}$.

Assume that we have $i \leftrightarrow j$. Let $M_{i,j}$ be a unique indecomposable $C[s_i \Gamma]$-module with $\dim M_{i,j} = \alpha_i + \alpha_j$ (up to an isomorphism). By $\epsilon_j(b) = 0$ and loc. cit. 4.4 (c), we conclude that $M_{T_i(b)}$ does not contain $M_i, M_{i,j}$ as its direct
factor. By assumption, \( i \) is a sink of \( s_i \Gamma \) and \( j \) is a source of an edge from \( j \) to \( i \), but is a source of no other edges. This particularly implies that \( M_i \) is the socle of \( M_{i,j} \). Therefore, we conclude the first half of (\( \ast \)_0) in Theorem 3.9. If an indecomposable \( \mathbb{C}[s_i \Gamma] \)-module contains \( M_i \) or \( M_{i,j} \) as its subquotient, then it must be a submodule. If an indecomposable \( \mathbb{C}[s_i \Gamma] \)-module has a non-zero homomorphism to \( M_i \) or \( M_{i,j} \), then it must be isomorphic to either \( M_i \) or \( M_{i,j} \). These imply the latter half of (\( \ast \)_0) in Theorem 3.9. In addition, we have

\[
\text{Ext}^1_{\mathbb{C}[s_i \Gamma]}(M_{i,j}, M_{\alpha(b)}) = \{0\}.
\]

Therefore, we conclude (\( \ast \)) in Theorem 3.9. Let \( h_s \in s_i \Omega \) be the unique edge so that \( h_s^i = j, h_s^j = i \). For a representation \((f_h)_{h \in s, \Omega}\) on \( V = \bigoplus_{i,j} V_i \) isomorphic to \( M_{i,j} \oplus M_{\alpha(b)} \), we set

\[
V'_i := \begin{cases} 
V_i & (k = i, j) \\
\text{Im}(\bigoplus_{h \in s, \Omega, h^s = i} f_h \oplus \bigoplus_{h \in s, \Omega, h^s = j} f_h, f_j) & (k \neq i, j).
\end{cases}
\]

Then, the space \( V' \subset V \) defines a canonical \( \mathbb{C}[s_i \Gamma] \)-submodule \( V' \) on \( V' \) so that \( V' \cong M_{\alpha(b)} \). Therefore, we conclude that \((T_i K_m) \times T_i K_{\Omega} \cong K_{\alpha(b)} \) as required.

**Lemma 3.12.** Let \( i, j \in I \) be distinct vertices, \( m \geq 0 \), and \( \beta \in Q^+ \). For each \( b \in B(\infty) \) so that \( \epsilon_i(b) > 0 \), the module \( T_i K_{\alpha(b)} \times L_b \) has simple head that is isomorphic to \( L'_b \) with \( \epsilon_i(b') > 0 \) up to grading shifts.

**Proof.** We first consider the case \( i \neq j \). We assume that both \( i \) and \( j \) are sink. We have \( T_i K_{m,j} \cong K_{m,j} \). By Theorem 3.9, we further deduce an isomorphism \( K_{m,j} \times K_{i,j} \cong K_{(m+n),j} \) \( \forall \geq 0 \). Together with Corollary 3.10 and the induction-by-stage argument, we conclude that \( K_{m,j} \times L_b \) has simple head \( L'_b \). Moreover, we have \( M_0 \cong M_{i,j} \) \( \oplus \) \( \oplus \). Therefore, we have \( \epsilon_i(b) > 0 \) if and only if \( \epsilon_i(b') > 0 \) since \( \epsilon_i \) counts the number of direct summand isomorphic to \( M_i \) by our assumption on \( \Omega \).

Now we consider the case \( i \leftrightarrow j \). We rearrange \( \Omega \) so that \( j \) is a sink of \( \Omega \) and \( i \) is sink of \( s_i \Omega \), and employ the same notation as in the proof of Corollary 3.11. We have a decomposition

\[
M_0 \cong M_{\alpha} \oplus M_{i,j} \oplus M_{d} \text{ with } d \in B(\infty)_{\beta - \alpha_i + \alpha_j + \alpha_0},
\]

as \( \mathbb{C}[s_i \Gamma] \)-modules so that \( d \) does not contain \( M_0 \) or \( M_{i,j} \) as its direct summand. Here we have \( d = T_i(f) \) with \( \epsilon_j(f) = 0 \) by [Lu90a] 4.4 (c). We set \( d' \in B(\infty)_{m \alpha + \alpha_j + \beta} \) so that \( M_0 \cong M_{i,j} \oplus M_0 \). Thanks to Corollary 3.10 and Corollary 3.11, we have

\[
K_{\alpha_i, \Omega} \cong K_{p_1} \times (T_i K_{q_1}) \times K_{d'}.\]

By Corollary 3.10, we deduce that \( K_i \times T_i K_j \) is isomorphic to a standard module of \( R_{2 \alpha_i + \alpha_j} \). Since the orbit corresponding to \( K_i \times T_i K_j \) is open dense, we deduce that \( K_i \times T_i K_j \) is simple. By inspection, we find that \# \( \text{Hom} R_{2 \alpha_i + \alpha_j} = 2 \) and each of simple graded \( R_{2 \alpha_i + \alpha_j} \)-module has dimension 3. Hence, \( T_i K_j \times K_i \) must be simple. By a weight comparison argument, we deduce that \( K_i \times T_i K_j \cong T_i K_j \times K_i(1) \). By Theorem 3.9, we deduce that

\[
(T_i K_{r+j}) \times (T_i K_{s+j}) \cong T_i K_{(r+s)j} \text{ for every } r, s \geq 0.
\]
Hence, we deduce $K_pl \times T_i K_{mj} \cong T_i K_{mj} \ast K_{pl}$ up to grading shifts by induction.

Therefore, the induction-by-stage implies that the ungraded $R_{\beta + ms, \alpha_j}$-module $T_i K_{mj} \ast K_p^{\ast \Omega} \cong T_i K_{mj} \times (T_i K_{qj}) \ast K_p^{\ast \Omega} \cong K_p^{\ast \Omega}$ has simple head $L_{\beta}^{\ast \Omega}$ with $\epsilon(b) = p > 0$ as desired.

**Proof of Theorem 3.6 5.** We choose an orientation $\Omega$ so that $i$ is a source and $j$ is a sink. Let $F_1 := (T_i P_{mj}) \ast (T_i \bullet)$ and $F_2 := T_i(P_{mj} \ast \bullet)$ be two functors from $R_{\beta}$-$\text{gmod}_j$ to $R_{\beta}$-$\text{gmod}_j$. Both of them are exact on $R_{\beta}$-$\text{mod}$. Therefore, taking successive quotients of the isomorphisms in Corollary 3.11 (cf. Theorem 3.6 1)) yield

$$T_i(K_{mj} \ast L_{\beta}) \cong (T_i K_{mj}) \ast T_i L_{\beta}$$

as a graded $R_{\beta + ms, \alpha_j}$-module for every $b \in B(\infty)_{\beta}$ such that $\epsilon_j(b) = 0$. Consider the composition map

$$\eta : R_{ms, \alpha_j} \otimes R_{s, \beta} \rightarrow R_{ms, \alpha_j + s, \beta} \rightarrow iR_{ms, \alpha_j + s, \beta}.$$ 

By the construction of the induction isomorphism (3.5), the map $\eta$ induces an inclusion $T_i K_{mj} \otimes T_i L_{\beta} \rightarrow T_i(K_{mj} \ast L_{\beta})$ if $\epsilon_j(b) = 0$. We have a two-sided ideal $J := \ker(R_{ms, \alpha_j} \otimes R_{s, \beta} \rightarrow iR_{ms, \alpha_j} \otimes iR_{s, \beta})$. We have

$$\text{hom}_{R_{ms, \alpha_j} \otimes R_{s, \beta}}(J, L_{b_1}' \otimes L_{b_2}') \neq \{0\}$$

only if $b_1' \in B(\infty)_{ms, \alpha_j}$ satisfies $\epsilon_i(b_1') > 0$, or $b_2' \in B(\infty)_{s, \beta}$ satisfies $\epsilon_i(b_2') > 0$. In the former case, it is straightforward to verify that its image to $iR_{ms, \alpha_j + s, \beta}$ must be zero by Lemma 3.2. Thus, we consider the case $\epsilon_i(b_1') = 0$ and $\epsilon_i(b_2') > 0$. As $b_1' \in B(\infty)_{m(\alpha_j + s, \beta)}$, it is standard to see $L_{b_1}' \cong T_i K_{mj}$ by rank two inspection (see Theorem 3.6 1)).

The image of $L_{b_1}' \otimes L_{b_2}'$ in $R_{ms, \alpha_j + s, \beta}$ factors through $L_{b_1}' \ast L_{b_2}'$ by construction. By Lemma 3.12, the unique simple quotient $L_{\beta}'$ of $L_{b_1}' \ast L_{b_2}'$ satisfies $\epsilon_i(b') > 0$. Hence, the map $\eta$ factors through $iR_{ms, \alpha_j} \otimes iR_{s, \beta}$. In other words, we have a graded algebra map

$$T_i R_{ms, \alpha_j} \otimes T_i R_{s, \beta} \rightarrow T_i R_{ms, \alpha_j + s, \beta}.$$ 

Therefore, we have a natural transformation

$$F_1 = (T_i P_{mj}) \ast T_i \bullet \rightarrow T_i(P_{mj} \ast \bullet) = F_2.$$

Thanks to Lemma 3.8, we see that $F_2$ sends an indecomposable projective module of $iR_{\beta}$ (regarded as an $R_{\beta}$-module) to a module with simple head (or zero). The image of this natural transformation surjects onto this simple head by (3.5). This forces two functors $F_1$ and $F_2$ to be isomorphic on projective objects of $R_{\beta}$-$\text{gmod}_j$ by the comparison of their graded characters. Therefore, we conclude that they are isomorphic.

The rest of this section is devoted to the proof of Theorem 3.9. During the proof of Theorem 3.9, we omit $\Omega$ from the notation. We set $\beta := \beta_1 + \beta_2$, and $V := V(1) \oplus V(2)$. We set $n := \text{ht } \beta$, and $n_i := \text{ht } \beta_i$ for $i = 1, 2$. We write $\beta_i = \sum_{j \in I} d_i(j) \alpha_j$ for $i = 0, 1, 2$. 19
In the above settings, we have:

\[\text{Gr}_{V(1), V(2)}(V) := \left\{(F, x, \psi_1, \psi_2) \mid F \subset V : I\text{-graded vector subspace} \right\},\]

\[\text{Gr}_{\beta_1, \beta_2}(V) := \left\{(F, x) \mid F \subset V : \dim F = \beta_2 \right\}.\]

We have a \(G_{V(1)} \times G_{V(2)}\)-torsor structure \(\vartheta : \text{Gr}_{V(1), V(2)}(V) \to \text{Gr}_{\beta_1, \beta_2}(V)\) given by forgetting \(\psi_1\) and \(\psi_2\). We have two maps

\[p : \text{Gr}_{\beta_1, \beta_2}(V) \ni (F, x) \mapsto x \in EV\]

\[q : \text{Gr}_{V(1), V(2)}(V) \ni (F, x, \psi_1, \psi_2) \mapsto (\psi_1(x \mod F), \psi_2(x|F)) \in E_{V(1)} \oplus E_{V(2)}.\]

Notice that \(\vartheta\) and \(q\) are smooth of relative dimensions \(\dim G_{V(1)} + \dim G_{V(2)}\) and \(\frac{1}{2}(\dim G_V + \dim G_{V(1)} + \dim G_{V(2)}) + \sum_{h \in O} d_1(h')d_2(h'')\), respectively. The map \(p\) is projective. We set \(N_{\beta_1, \beta_2} := \frac{1}{2}(\dim G_{V(1)} + \dim G_{V(2)}) + \sum_{h \in O} d_1(h')d_2(h'')\). For \(G_{V(i)}\)-equivariant constructible sheaves \(\mathcal{F}_i\) on \(E_{V(i)}\) for \(i = 1, 2\), we define their convolution products as

\[\mathcal{F}_1 \circ \mathcal{F}_2 := p_! f_{12}[N_{\beta_1, \beta_2}], \text{ where } \vartheta^* \mathcal{F}_{12} \cong q^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \text{ in } D^b_{G_V}(\text{Gr}_{V(1), V(2)}(V)).\]

We return to the proof of Theorem 3.9. Let us fix objects \(O_{b_1}, O_{b_2}, L_{m_1, m_2}\) of \(D^b_{G_V}(\text{Gr}_{\beta_1, \beta_2}(V))\) \((m_1 \in Y_{\beta_1} \text{ and } m_2 \in Y_{\beta_2})\) so that we have isomorphisms

\[\vartheta^*(O_{b_1}) \cong q^*(O_{b_1} \boxtimes O_{b_2})[N_{\beta_1, \beta_2}] \text{ and } \vartheta^*L_{m_1, m_2} \cong q^*(L_{m_1} \boxtimes L_{m_2})[N_{\beta_1, \beta_2}].\]

**Lemma 3.13.** In the above settings, we have:

1. the variety \(p^{-1}(\emptyset_b)\) is a single \(G_V\)-orbit;

2. the map \(p : p^{-1}(\emptyset_b) \to \emptyset_b\) is a \(P\)-fibration, where \(P\) is a suitable partial flag variety.

**Proof.** We have \(M_b \cong \mathfrak{m}_{b_1} \oplus \mathfrak{m}_{b_2}\) by assumption. The condition (**) asserts that the image of every two inclusions \(\mathfrak{m}_{b_1} \subset \mathfrak{m}_{b_2}\) are transformed by \(\text{Aut}_{G_V}(\mathfrak{m}_b)\). Hence we have \(\text{Aut}_{G_V}(\mathfrak{m}_{b}) \cong \text{Stab}_{G_V}(x_b)\) for \(x_b \in \emptyset_b(\mathbb{C})\). Therefore, \(p^{-1}(\emptyset_b)\) is a single \(G_V\)-orbit, which is the first assertion. Since \(p\) is projective, we conclude that \(p^{-1}(\emptyset_b) \to \emptyset_b\) is projective. By (**), the group \(\text{Stab}_{G_V}(x_b)\) is connected. Let \(U_b\) denote the unipotent radical of \(\text{Stab}_{G_V}(x_b)\). Since we have \(p^{-1}(\emptyset_b) \cong G_V/U_b\) with \(H_b \subset \text{Stab}_{G_V}(x_b)\), the fiber of \(p\) is isomorphic to \(\text{Stab}_{G_V}(x_b)/H_b\), that is projective. Therefore, we deduce \(U_b \subset H_b\) and the inclusion

\[H_b/U_b \subset \text{Stab}_{G_V}(x_b)/U_b\]

must be a parabolic subgroup (of a connected reductive group). Therefore, we set \(P\) to be their quotient to deduce the second part of the result.

**Corollary 3.14.** We have

\[C_{b_1}[\dim \emptyset_{b_1}] \circ C_{b_2}[\dim \emptyset_{b_2}] \cong D[d] \boxtimes C_b[\dim \emptyset_b],\]

where \(D \cong H^*(P, \mathbb{C})\) by a suitable partial flag variety \(P\) with its dimension \(d\).
Proof. Thanks to (∗)1, we deduce that \( \vartheta(q^{-1}(\mathcal{O}_{b_1} \times \mathcal{O}_{b_2})) \) is contained in a single \( G_V \)-orbit. This, together with Lemma 3.13, implies that the stalk of the LHS vanishes outside of \( \mathcal{O}_b \). In addition, every direct summand of \( \mathcal{p}_* \mathcal{C}_{b_1, b_2} \big|_{\mathcal{O}_b} \), viewed as a shifted \( G_V \)-equivariant local system (which in turn follows by [BBD82] 5.4.5 or 6.2.5), must be a trivial local system by (△)2. Therefore, we conclude that \( \mathcal{C}_{b_1}[\dim \mathcal{O}_{b_1}] \oplus \mathcal{C}_{b_2}[\dim \mathcal{O}_{b_2}] \cong \mathcal{D}' \boxtimes \mathcal{C}_0[\dim \mathcal{O}_b] \) with a graded vector space \( \mathcal{D}' \). The isomorphism \( \mathcal{D}' \cong H^*(\mathcal{P}, \mathbb{C})[d] \) is by Lemma 3.13 2).

We return to the proof of Theorem 3.9. In the below (during this section), we freely use the notation from Corollary 3.14.

Thanks to Corollary 2.8, \( \mathcal{L}_{\beta_1} \) and \( \mathcal{L}_{\beta_2} \) contains \( \mathcal{C}_{b_1} \) and \( \mathcal{C}_{b_2} \), respectively. We have

\[ \mathcal{L}_{m^1} \oplus \mathcal{L}_{m^2} \cong \mathcal{L}_{m^1 + m^2} \]

by construction. Thanks to (∗) and [BBD82] 5.4.5 or 6.2.5, \( \mathcal{C}_b \) appears in \( \mathcal{L}_{m^1 + m^2} \) up to a grading shift if the following condition (○) hold:

(○) \( \mathcal{C}_b \) appears in \( \mathcal{L}_{m^i} \) for \( i = 1, 2 \).

We set \( m := m^1 + m^2 \). Let \( x_b \in \mathcal{O}_b \) be a point and let \( i_b : \{x_b\} \hookrightarrow E_V \) be the inclusion.

Lemma 3.15. Assume that (○) holds. Then, the subspace

\[ i_b^! \mathcal{E}xt^*_{\mathcal{D}^b(E_V)}(\mathcal{C}_{b_1} \circ \mathcal{C}_{b_2}, \mathcal{L}_{m^1 + m^2}) \subset i_b^! \mathcal{E}xt^*_{\mathcal{D}^b(E_V)}(\mathcal{C}_{b_1} \circ \mathcal{C}_{b_2}, \mathcal{L}_{\beta}) \]

\[ \cong \mathcal{E}xt^*_{\mathcal{D}^b(E_V)}(i_b^! \mathcal{C}_{b_1} \circ \mathcal{C}_{b_2}, i_b^! \mathcal{L}_{\beta}) \cong \mathcal{D}^*[-d] \boxtimes K_b(2 \dim \mathcal{O}_b) \]

is a generating subspace as an \( R_\beta \)-module.

Proof. The isomorphism parts of the assertion follow by [KS90] 3.1.13 and Corollary 3.14. By (○) and (∗)1, we conclude that \( \mathcal{L}_{b_1, b_2} \) contains an irreducible perverse sheaf supported on \( \text{Supp} \mathcal{C}_{b_1, b_2} \). Thanks to [BBD82] 5.4.5 or 6.2.5, we conclude that \( \mathcal{L}_{m^1} \oplus \mathcal{L}_{m^2} \) contains \( \mathcal{C}_b \). Therefore, the head \( L_b \) of \( K_b \) satisfies \( c(m) L_b \neq \{0\} \), which proves the assertion.

We set \( \mathcal{O}_b \) to be the union of \( G_V \)-orbits which contains \( \mathcal{O}_b \) in its closure. Let \( j_b^! : \mathcal{O}_b^! \hookrightarrow E_V \) be its inclusion.

Proposition 3.16. We have a canonical isomorphism

\[ \mathcal{E}xt^*_{\mathcal{D}^b(E_V)}(\mathcal{C}_{b_1} \circ \mathcal{C}_{b_2}, \mathcal{L}_{m^1 + m^2}) \cong \mathcal{p}_* \mathcal{E}xt^*_{\mathcal{D}^b(\mathcal{G}_{b_1, b_2}(V))}(\mathcal{C}_{b_1, b_2}, \mathcal{D}^* \boxtimes \mathcal{L}_{m^1, m^2}) \]

in the bounded derived category of constructible sheaves on \( E_V \).

Proof. During this proof, we repeatedly use the local form of the Verdier duality (see e.g. [KS90] 3.1.10, or [SGA4] Exposé XVIII 3.1.10). We have

\[ \mathcal{E}xt^*_{\mathcal{D}^b(E_V)}(\mathcal{C}_{b_1} \circ \mathcal{C}_{b_2}, \mathcal{L}_{m^1 + m^2}) \cong \mathcal{p}_* \mathcal{E}xt^*_{\mathcal{D}^b(\mathcal{G}_{b_1, b_2}(V))}(\mathcal{C}_{b_1, b_2}, \mathcal{p}^! \mathcal{L}_{m^1 + m^2}) \]

Consider the Cartesian diagram

\[
\begin{array}{ccc}
\text{Gr}_{b_1, b_2}(V) & \overset{p}{\longrightarrow} & E_V \\
\downarrow{j_b^!} & & \\
\mathcal{O}_b & \overset{p}{\longrightarrow} & \mathcal{O}_b^!
\end{array}
\]
Note that $j^!_b$ is an open embedding since $p$ is continuous. It follows that
\[
\mathfrak{p}_!\mathcal{E}xt^{\bullet}_{\mathcal{D}(\overline{V}(\mathcal{G}))}((\mathcal{C}_b, b_2, p^!\mathcal{L}_m^1 + m^2)) \\
\cong (j^!_b)_!\mathfrak{p}_!\mathcal{E}xt^{\bullet}_{\mathcal{D}(\mathcal{G})}((j^!_b)^*\mathcal{C}_b, b_2, p^!\mathcal{L}_m^1 + m^2) \\
\cong (j^!_b)_!\mathfrak{p}_!\mathcal{E}xt^{\bullet}_{\mathcal{D}(\mathcal{G})}((j^!_b)^*\mathcal{C}_b, b_2, p^!\mathcal{L}_m^1 + m^2)
\]
\[
\cong (j^!_b)_!\mathfrak{p}_!\mathcal{E}xt^{\bullet}_{\mathcal{D}(\mathcal{G})}((j^!_b)^*\mathcal{C}_b, b_2, p^!\mathcal{L}_m^1 + m^2)
\]
In addition, $(j^!_b)^*\mathcal{C}_b, b_2$ is a local system supported on the closed $G_U$-orbit $\mathcal{O}_b$ of $\mathcal{G}$. Let us denote by $j_b : \mathcal{O}_b \hookrightarrow \mathcal{G}$ the inclusion. We have $(j^!_b)^*\mathcal{C}_b, b_2 \cong (j_b)_!\mathcal{C}[[\mathfrak{m}]]$. Thus, we deduce
\[
(j^!_b)_!\mathfrak{p}_!\mathcal{E}xt^{\bullet}_{\mathcal{D}(\mathcal{G})}((j^!_b)^*\mathcal{C}_b, b_2, p^!\mathcal{L}_m^1 + m^2) \\
\cong (j^!_b)_!\mathfrak{p}_!\mathcal{E}xt^{\bullet}_{\mathcal{D}(\mathcal{O}_b)}((j^!_b)^*\mathcal{C}_b, b_2, p^!\mathcal{L}_m^1 + m^2) \\
\cong (j^!_b)_!\mathfrak{p}_!\mathcal{E}xt^{\bullet}_{\mathcal{D}(\mathcal{O}_b)}((j^!_b)^*\mathcal{C}_b, b_2, D^* \otimes (j^!_b)^!\mathcal{L}_m^1 + m^2) \\
\cong (j^!_b)_!\mathfrak{p}_!\mathcal{E}xt^{\bullet}_{\mathcal{D}(\mathcal{O}_b)}((j^!_b)^*\mathcal{C}_b, b_2, D^* \otimes \mathcal{L}_m^1 + m^2).
\]
where $p^! : \mathcal{O}_b \twoheadrightarrow \mathcal{O}_b$ is the restriction of $p$. Since all the maps are canonically defined, composing all the isomorphisms yield the result.

We return to the proof of Theorem 3.9. Taking account into the fact $p^{-1}(x_b) \cong \mathcal{P}$, we have an isomorphism
\[
D^* \langle d \rangle \boxtimes \mathcal{E}(m^1 + m^2)K_b \cong \mathcal{H}^\bullet j^!_b\mathcal{E}xt^{\bullet}_{\mathcal{D}(\mathcal{G})}((\mathcal{C}_b \oplus \mathcal{C}_b, D^* \boxtimes \mathcal{L}_m^1 + m^2))[2 \dim \mathcal{O}_b]
\]
and a spectral sequence arising from the base change (applied to $i_b$ and $p$)
\[
E_2 := D^* \otimes \mathcal{H}^\bullet(\mathcal{P}) \otimes \mathcal{E}(m^1)K_b \boxtimes \mathcal{E}(m^2)K_b \\
\Rightarrow \mathcal{H}^\bullet j^!_b\mathfrak{p}_!\mathcal{E}xt^{\bullet}_{\mathcal{D}(\overline{V}(\mathcal{G}))}((\mathcal{C}_b, b_2, D^* \boxtimes \mathcal{L}_m^1 + m^2))[2 \dim \mathcal{O}_b],
\]
where we used the fact that $\dim \mathfrak{p}^{-1}(\mathcal{O}_b) - \dim \mathfrak{p}^{-1}(x_b) = \dim \mathcal{O}_b$ in the degree shift of the second spectral sequence. Here the modules $K_{b_1}, K_{b_2}$, and $K_b$ are pure of weight 0 by [Lu90a] 10.6 (see the proof of Proposition 2.7 for a bit precise account). By Lemma 3.13 2), we deduce that $\mathcal{H}^\bullet(\mathcal{P})$ is also pure. Therefore, the spectral sequence $E_2$ degenerates at the $E_2$-stage. By factoring out the effect of $D^*$, we conclude that
\[
\mathcal{E}(m^1 + m^2)K_b \cong \mathcal{H}^\bullet(\mathcal{P}) \boxtimes (\mathcal{E}(m^1)K_b \boxtimes (\mathcal{E}(m^2)K_b) \langle -d \rangle.
\]
This induces an inclusion as $R_{m^1} \boxtimes R_{m^2}$-modules
\[
\varphi_{m^1, m^2} : (\mathcal{E}(m^1)K_b \boxtimes (\mathcal{E}(m^2)K_b) \langle d \rangle \hookrightarrow \mathcal{E}(m^1 + m^2)K_b.
\]
The module $\mathcal{E}(m^1 + m^2)K_b$ admits an $R_{m^1 + m^2}$-$\mathcal{O}_b$-module structure with simple head thanks to Theorem 1.3 3). This extends the $R_{m^1} \boxtimes R_{m^2}$-$\mathcal{O}_b$-module structure. Recall that for each $i = 0, 1, 2$, the simple head of $K_{b_i}$ as an irreducible $R_{b_i}$-module is realized as the coefficient vector space of $IC_{b_i}$ inside $\mathcal{L}_{\beta_i}$ (see §1), and its weight $\mathcal{E}(m^i)$-part is that of $\mathcal{L}_{m^i}$ (see §2). (Note that this sheaf-theoretic interpretation gives a splitting of $L_{b_i}$ to $K_{b_i}$ as vector spaces for each $i = 0, 1, 2$.) By [BBD82] 5.4.5 or 6.2.5 and Corollary 3.14, the
complex $H^\bullet (\mathcal{P})[d] \otimes IC_b$ is a direct summand of $IC_{b_1} \otimes IC_{b_2}$. Therefore, the above interpretation implies that the unique simple quotients $L_{b_1}$ and $L_{b_2}$ of $K_{b_1}$ and $K_{b_2}$ satisfy

$$\varphi_{m_1, m_2} (H^\bullet (\mathcal{P}) \otimes (e(m_1^1)L_{b_1} \otimes e(m_2^2)L_{b_2})) \langle -d \rangle \subset e(m)L_b \subset e(m)K_b$$

as vector subspaces, where $L_b$ is the simple head of $K_b$. In addition, these inclusions are non-zero if $m_1$ and $m_2$ satisfies $\text{(\dagger)}$. Since we can choose $m_1$ and $m_2$ so that $\text{(\dagger)}$ is satisfied, we have a surjective map of graded $R_{\beta}$-modules:

$$K_{b_1} \ast K_{b_2} \langle d \rangle \longrightarrow K_b.$$

**Lemma 3.17.** In the above settings, we have

$$\dim K_b = \dim (K_{b_1} \ast K_{b_2}).$$

**Proof.** In this proof, $i$ denotes either $\emptyset, 1, \text{ or } 2$. Let us choose a point $x_{b_i} \in \mathcal{O}_{b_i} \subset E_{V(i)}$. Let $T_i$ be a maximal torus of $\text{Stab}_{G_{V(i)}} x_{b_i}$. Choose $m_i \in Y_{\beta_i}$. Thanks to the purity of each module (Lusztig [Lu90a] 10.6), we deduce that the spectral sequence

$$H^\bullet (T_i \text{ (pt)} \otimes H^\bullet (\pi_{m_1}^{-1}(x_{b_i}))) \Rightarrow H^\bullet (\pi_{m_1}^{-1}(x_{b_i})).$$

degenerates at the $E_2$-stage. Here the RHS have the same $H^\bullet (T_i \text{ (pt)}$-rank as that of $H^\bullet (\pi_{m_1}^{-1}(x_{b_i})).$ Therefore, we have

$$\dim H^\bullet (\pi_{m_1}^{-1}(x_{b_i})) = \dim H^\bullet (\pi_{m_1}^{-1}(x_{b_i})).$$

By Theorem 2.11, we deduce that $R_{\beta}$ is a free $R_{\beta_1} \otimes R_{\beta_2}$-module of rank $\frac{n_1!}{n_1!n_2!}$. Hence, it is enough to show

$$\sum_{m \in Y_{\beta}} \dim H^\bullet (\pi_m^{-1}(x_b)) = \frac{n_1!}{n_1!n_2!} \sum_{\begin{subarray}{c} m_1 \in Y_{\beta_1} \\ m_2 \in Y_{\beta_2} \end{subarray}} \left( \dim H^\bullet (\pi_{m_1}^{-1}(x_{b_1})) \right) \left( \dim H^\bullet (\pi_{m_2}^{-1}(x_{b_2})) \right).$$

This follows by a simple counting since $E_{V(\emptyset)}$ decomposes into the product of varieties corresponding to each indecomposable module. \hfill \Box

We return to the proof of Theorem 3.9. Lemma 3.17 asserts that

$$K_{b_1} \ast K_{b_2} \langle d \rangle \cong K_b$$

as graded $R_{\beta}$-modules. This completes the proof of Theorem 3.9 except for the last assertion. The last assertion follows since the assumption implies that $p_{b_1, b_2}$ is birational onto its image, and hence $d = 0$. 23
4 Characterization of the PBW bases

Keep the setting of the previous section. For a reduced expression $i$ of $w_0$ and a sequence of non-negative integers $c := (c_1, c_2, \ldots, c_l) \in \mathbb{Z}_{\geq 0}^l$, we call the pair $(i, c)$ a Lusztig datum, and we call $c$ an $i$-Lusztig datum. For a Lusztig datum $(i, c)$, we define

$$\text{wt}(i, c) := \sum_{k=1}^{l} c_k \gamma_i^{(k)},$$

where $\gamma_i^{(k)} := s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$.

For two $i$-Lusztig data $c$ and $c'$, we define $c < c'$ as: There exists $0 \leq k < \ell$ so that

$$c_1 = c'_1, c_2 = c'_2, \ldots, c_k = c'_k \quad \text{and} \quad c_{k+1} > c'_{k+1}.$$ 

Associated to each Lusztig datum $(i, c)$, we define the lower PBW-module $E^i_c$ as:

$$E^i_c := P_{c_1} \star \mathbb{T}_{i_1} (P_{c_2} \star \mathbb{T}_{i_2} (P_{c_3} \star \mathbb{T}_{i_3} (P_{c_4} \star \mathbb{T}_{i_4} \cdots ) \cdots ) \cdots ). \quad (4.1)$$

Similarly, we define the corresponding upper PBW-module $E^i_c$ as:

$$E^i_c := L_{c_1} \star \mathbb{T}_{i_1} (L_{c_2} \star \mathbb{T}_{i_2} (L_{c_3} \star \mathbb{T}_{i_3} \cdots ) \cdots ) \cdots . \quad (4.2)$$

By construction, it is clear that $E^i_c$ is a quotient of $E^i_c$.

Remark 4.1. The modules $E^i_c$ and $E^i_c$ are identified with $\tilde{K}_c^\Omega$ and $K_c^\Omega$ from §2 when the reduced expression $i$ is adapted to $\Omega$ (cf. Corollary 4.14). In addition, we have $P_{c_l} = \tilde{K}_{c_l}$ and $L_{c_l} = K_{c_l}$ for each $c \in \mathbb{Z}_{\geq 0}$ and $i \in I$ by Example 2.10.

Lemma 4.2. For each Lusztig datum $(i, c)$, we have:

1. $E^i_c$ and $E^i_c$ are $R_{\text{wt}(i,e)}$-modules;
2. There exist isomorphisms as graded $R_{\text{wt}(i,e)}$-modules:

$$E^3_c \cong P_{c_1 \star} \mathbb{T}_{i_1} (P_{c_2 \star} \mathbb{T}_{i_2} (P_{c_3 \star} \mathbb{T}_{i_3} \cdots ) \cdots ) \cong L_{c_1 \star} \mathbb{T}_{i_1} (L_{c_2 \star} \mathbb{T}_{i_2} \cdots ) \cdots.$$ 

3. $E^i_c$ is a successive self-extension of $E^i_c$;
4. $E^i_c$ and $E^i_c$ are modules with simple heads if they are non-zero;
5. $E^i_c \neq \{0\}$ if and only if $E^i_c \neq \{0\}$.

Proof. Since $\mathbb{T}_i$ is a functor sending an $R_\beta$-module to an $R_{\alpha \beta}$-module (possibly zero), the first assertion is immediate. Notice that each part in (4.1) grouped as $\mathbb{T}_{i_k} (P_{c_{k+1}} \cdots )$ belongs to $R_{-\gamma \text{mod}_{i_k}}$ for some $\gamma \in Q^+$ by Theorem 3.6 2). Therefore, we apply Theorem 3.6 5) repeatedly to deduce the second assertion from (4.1). In addition, each $\mathbb{T}_{i_k} \mathbb{T}_{i_{k+1}} \cdots L_{c_{k+1}}$ is simple (unless it is zero) and $\mathbb{T}_{i_k} \mathbb{T}_{i_{k+1}} \cdots L_{c_{k+1}} \cdot P_{c_k}$ is a successive self-extension of $\mathbb{T}_{i_k} \mathbb{T}_{i_{k+1}} \cdots L_{c_{k+1}}$.

Therefore, the third assertion follows by the second assertion. The functor $\mathbb{T}_i$ also preserves the simple head property (provided if it does not annihilate the whole module) by construction. Therefore, we apply Lemma 3.8 repeatedly to deduce the simple head property of $E^i_c$ from that of $P_{c_k}$ ($1 \leq k \leq \ell$), which is the fourth assertion. By the third assertion, $E^i_c$ contains the head of $E^i_c$, and hence the fifth assertion. 

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Theorem 4.3 [Lusztig [Lu90a]]. Assume that the reduced expression $i$ is adapted to $\Omega$. Then, we have $E_i^c \neq \{0\}$ for every $i$-Lusztig datum. Moreover, the set of $i$-Lusztig data is in bijection with $B(\infty)$ as:

$$c \mapsto \text{hd } E_i^c \cong L_b \quad \text{for } b \in B(\infty).$$

Proof. Since $i$ is adapted, we deduce that a module $T_i, T_{i_2} \cdots T_{i_{k-1}} L_{c_1} \cdots L_{c_k}$ is simple and it corresponds to an indecomposable $C[R]$-module $M_{c_k}$ with $\dim M_{c_k} = \gamma_i^{(k)}$ (Lusztig [Lu90a] 4.7). We apply Corollary 3.10 and Theorem 3.6 1) repeatedly to construct a module with its simple head corresponding to the quiver representation $M_{c_k}^{(1)} \oplus \cdots \oplus M_{c_1}^{(1)}$. Now the Gabriel theorem yields the result. □

Definition 4.4 (2-move, 3-move, [Lu90a] 2.3). We say that two Lusztig data $(i, c)$ and $(i', c')$ are connected by a 2-move if

1. there exists $1 \leq k < \ell$ so that $i_k = i_k' = i_{k+1} \neq i_{k+1}$, and $i_l = i_l'$ for every $l \neq k, k + 1$;
2. we have $c_k = c_k', c_{k+1} = c_{k+1}'$, and $c_l = c_l'$ for every $l \neq k, k + 1$.

We say that $(i, c)$ and $(i', c')$ are connected by a 3-move if

1. there exists $1 < k < \ell$ so that $i_{k-1} = i_{k+1} = i_k = i_{k+1}' = i_{k+1}'$, $i_k \leftrightarrow i_{k+1}$, and $i_l = i_l'$ for every $l \neq k-1, k, k + 1$;
2. we have $c_l = c_l'$ for every $l \neq k-1, k, k + 1$, and

$$(c_{k-1}', c_{k}', c_{k+1}') = (c_k + c_{k+1} - c_0, c_0, c_{k-1} + c_k - c_0)$$ for $c_0 := \min\{c_{k-1}, c_{k+1}\}$.

Lemma 4.5. For two Lusztig data $(i, c)$ and $(i', c')$ which are connected by a 2-move, we have $E_i^c \cong E_i'^{c'}$.

Proof. Find a unique $1 \leq k < \ell$ so that $i_k = i_k' \neq i_{k+1} = i_k'$. By Lemma 4.2 2), it suffices to prove

$$L_{c_{k-1}} L_{c_{k-1}+c_{k+1}} T_{i_{k-1}} L_{c_{k+1}} T_{i_{k+1}} \cong L_{c_{k-1}+c_{k+1}} T_{i_{k-1}} L_{c_{k+1}} T_{i_{k+1}} \quad (4.3)$$

and $T_{i_k} \cong T_{i_k'}$. Here we realize $T_{i_k}$ and $T_{i_{k+1}}$ by choosing the orientation $\Omega$ so that the both of $i_k, i_{k+1}$ are source (which is in turn possible since $i_k \neq i_{k+1}$). Then, we have $T_{i_k} L_{c_{k-1}+c_{k+1}} \cong L_{c_{k-1}+c_{k+1}} T_{i_{k-1}} L_{c_{k+1}} T_{i_{k+1}}$ and $L_{c_{k-1}} L_{c_{k-1}+c_{k+1}} \cong L_{c_{k-1}+c_{k+1}} T_{i_{k-1}} L_{c_{k+1}} T_{i_{k+1}}$ since $R_{p+q} R_q R_p$ is Morita equivalent to $R_{p+q} R_q R_p$ for each $p, q \geq 0$ by the product decomposition of $(GV, E^2)$. This shows (4.3).

We have $T_{i_k} T_{i_{k+1}}(b) = T_{i_{k+1}'} T_{i_k}(b)$, $e_{i_k}(b) = e_{i_k}(T_{i_{k+1}}(b))$, and $e_{i_{k+1}}(b) = e_{i_{k+1}}(T_{i_{k+1}}(b))$ by inspection. The essential image of the functor $T_i$ (applied to $R_{\beta}$-gmod for some $\beta \in Q^+$) is equivalent to $R_{\beta}$-gmod by construction. We have $e_{i_k}(1) e_{i_{k+1}}(1) = 0 = e_{i_{k+1}'}(1) e_{i_k}(1)$ by definition. Therefore, we deduce that the essential image of each of the functors $T_{i_k} T_{i_{k+1}}$ and $T_{i_{k+1}'} T_{i_k}$ is equivalent to the graded module category of

$$(R_{\beta}/(R_{\beta} e_{i_k}(1) R_{\beta})) \otimes R_{\beta} (R_{\beta}/(R_{\beta} e_{i_{k+1}}(1) R_{\beta})) \cong R_{\beta}/(R_{\beta} e R_{\beta})$$ for some $\beta \in Q^+$,

where $e := e_{i_k}(1) + e_{i_{k+1}}(1)$ is the minimal idempotent so that $e e_{i_k}(1) = e_{i_k}(1)$ and $e e_{i_{k+1}}(1) = e_{i_{k+1}}(1)$. Hence, Theorem 3.6 2) guarantees that $T_{i_k} T_{i_{k+1}} \cong T_{i_{k+1}} T_{i_k}$ as functors, which completes the proof. □
For a reduced expression $i = (i_1, \ldots, i_t)$ of $w_0$, we have a unique reduced expression of the form $i^\# := (i_2, i_3, \ldots, i_t, i_1')$. (Namely $s_i' := w_0 s_i w_0^{-1}$.)

**Lemma 4.6.** Assume that the set of modules $\{hd E^{i_1}_{c'}\}_{c' \in \mathbb{Z}_{\geq 0}^t}$ is in bijection with $\bigcup_{\beta \in Q^+} \text{Irr}_0 R_\beta$. Then, so is $\{hd E_{c'}^{i_1}\}_{c' \in \mathbb{Z}_{\geq 0}^t}$.

**Proof.** By Theorem 3.6 2), we deduce that

$$\{\mathbb{T}_k (hd E^{i_1}_{c'})\}_{c' \in \mathbb{Z}_{\geq 0}^t} = \{L_b \in \bigcup_{\beta \in Q^+} \text{Irr}_0 R_\beta \mid \epsilon_i(b) = 0\} \cup \{\{0\}\}.$$  

For every $\beta \in Q^+$, we have $w_0 \beta \notin Q^+$. It follows that

$$\mathbb{T}_1 \mathbb{T}_2 \cdots \mathbb{T}_m L_{mi} = \{0\} \quad \text{for each} \quad m > 0.$$  

Therefore, Lemma 4.2 2) asserts that $\mathbb{T}_1 E^{i_1}_{c'} \neq \{0\}$ implies that the $i^\#$-Lusztig datum $c' = (c'_1, \ldots, c'_{t-1})$ satisfies $c'_t = 0$, and we have

$$\mathbb{T}_1 E^{i_1}_{c'} \cong E_{c}^{i_1}$$

for $c = (0, c'_1, c'_2, \ldots, c'_{t-1})$. Thanks to Theorem 2.12 and Corollary 3.10, we deduce that

$$\{hd (L_{mi} \ast \mathbb{T}_1 E^{i_1}_{c'})\}_{c' \in \mathbb{Z}_{\geq 0}^t, m \geq 0} = \{hd E_{c'}^{i_1}\}_{c' \in \mathbb{Z}_{\geq 0}^t} \cup \{\{0\}\}.$$  

However, as in [Lu90a] 2.4 (a), the number of Lusztig data of each $B(\infty)_\beta$ is constant for every reduced expression. Therefore, we must have a bijection as required. \hfill \Box

**Lemma 4.7.** Let $i$ and $i'$ be two reduced expressions which are connected by a 3-move as $(i_{k-1}, i_k, i_{k+1}) = (i'_k, i'_{k+1}, i'_k)$ for some $k$. If the set of modules $\{hd E_{c'}^{i_1}\}_{c' \in \mathbb{Z}_{\geq 0}^t}$ is in bijection with $\bigcup_{\beta \in Q^+} \text{Irr}_0 R_\beta$, then so is $\{hd E_{c'}^{i_1}\}_{c' \in \mathbb{Z}_{\geq 0}^t}$.

**Proof.** Applying Lemma 4.6 repeatedly, we assume that $k = \ell - 1$. We set $i := i_{k-1} = i_\ell, j := i_k = i_{\ell-1}$ for simplicity.

In order to analyze the effect of $\mathbb{T}_i$ or $\mathbb{T}_j^\ast$, we assume that $i \in I$ is a sink or a source by applying a Fourier transform if necessary. Since a Fourier transform is defined by pairing a direct factor $E$ of $E^\Omega_V$ with a direct factor $E^\ast$ of $E^\Omega'_V$ (where $\Omega'$ is a different orientation of $\Gamma_0$), it must be identity if $E = \{0\}$. Therefore, we can regard as if $\Gamma_0$ is a quiver $\{(i,j), \{h\}\}$ of type $A_2$ provided if $\beta \in Z_{\geq 0}\alpha_1 \oplus Z_{\geq 0}\alpha_2$. Here $h \in \{h_\pm\}$, where $h'_+ = j, h'_- = i$ or $h'_1 = i, h'_2 = j$. In this case, the both sequences $(i, j, i)$ and $(j, i, j)$ are adapted to the definitions $\{h_+\}$ and $\{h_-\}$, respectively.

Thanks to Theorem 4.3 and the definition of $\ast$, it follows that each of the two sets

$$\{hd (L_{c_1} \ast (\mathbb{T}_1 L_{c_2})) \ast (\mathbb{T}_1 L_{c_3}))\} \text{ and } \{hd (L_{c_1} \ast (\mathbb{T}_j L_{c_2})) \ast (\mathbb{T}_j L_{c_3}))\}$$

exhausts $\bigcup_{\beta \in Z_{\geq 0}^t} \text{Irr}_0 R_{\alpha_b + q_\alpha} \ast q_\alpha$ (and hence they are identical). By applying Lemma 4.2 2), we conclude the result since the other factors are in common. \hfill \Box

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Corollary 4.8. The module $E^i_c$ is non-zero for every Lusztig datum $(i, c)$, and the map

$$c \mapsto \text{hd } E^i_c \cong L_b$$

for $b \in B(\infty)$ sets up a bijection between the set of $i$-Lusztig data and $B(\infty)$.

Proof. Every two reduced expressions of $w_0 \in W(\Gamma_0)$ are connected by a repeated use of two moves and three moves ([Lu90a] 2.1 (c)). Therefore, we apply Lemma 4.5 and Lemma 4.7 repeatedly to deduce the assertion.

Corollary 4.9. For any reduced expression $i$, $1 \leq k \leq \ell$, and $m \geq 0$, the module

$$T_{i_1}T_{i_2} \cdots T_{i_{k-1}}L_{m_i}$$

is a non-zero simple $R_{\gamma_{i_k}(\ell)}$-module.

Proof. Apply Theorem 3.6 2) and Corollary 4.8 to the Lusztig datum $c = (0, \ldots, 0, m, 0, \ldots, 0)$, where the unique $m$ is sitting at the $k$-th place.

Proposition 4.10. Let $(i, c)$ and $(i', c')$ be two Lusztig data which are connected by a 3-move as $(i_{k-1}, i_k, i_{k+1}) = (i'_{k-1}, i'_{k+1}, i'_k)$ for some $k$. Then, we have

$$\text{hd } E^i_c \cong \text{hd } E^{i'}_{c'}.$$

Proof. Let $i_k = i, i_{k+1} = i$. By an explicit calculation (which reduces to the rank two case), we see that

$$\text{hd } (L_{c_{1-i-j}} \ast T_j L_{c_{j-i}} \ast T_i T_j L_{c_{k+i}}) \cong \text{hd } (L_{c'_{1-i-j}} \ast T_j L_{c'_{j-i}} \ast T_i T_j L_{c'_{k+i+1}}).$$

(4.4)

Thanks to Corollary 4.9 and Lemma 4.2 2), it suffices to show

$$\text{hd } E^i_c \cong \text{hd } E^{i'}_{c'}$$

for every $i$-Lusztig datum $c$ so that $c_1 = \cdots = c_{k+1} = 0$ (and its counterpart $i'$-Lusztig datum $c'$). We set

$$L_{s,c} := T_{i_1} \cdots T_{i_{s-1}} L_{c_{i_1}}$$

and

$$L'_{s,c} := T'_{i_1} \cdots T'_{i_{s-1}} L_{c'_{i_1}}$$

for every $1 \leq s \leq \ell$ and $c \geq 0$. We have $L_{s,c} \cong L'_{s,c}$ for $1 \leq s \leq k-1$. The both $L_{s,c}$ and $L'_{s,c}$ are non-zero simple $R_{\gamma_{i_k}(\ell)}$-modules for $k+1 < s \leq \ell$.

Claim A. We have $L_{s,c} \cong L'_{s,c}$ for every $k+1 < s \leq \ell$ and $c \geq 1$.

Proof. We prove the assertion by induction on $\text{ht}(c_{\gamma_{i_k}(\ell)}^{(s)})$. The case $\text{ht}(c_{\gamma_{i_k}(\ell)}^{(s)}) = \text{cht } \gamma_{i_k}^{(s)} = c$ is trivial since $\gamma_{i_k}^{(s)}$ is a simple root (Example 2.10). Fix $s$ and $c$ so that the assertion is true for $\text{ht } \beta < \text{ht}(c_{\gamma_{i_k}(\ell)}^{(s)})$. Taking Corollary 4.8 into account, we conclude that all but one simple $R_{\gamma_{i_k}(\ell)}$-modules are realized by the heads of the modules of the form (4.2) constructed from simple modules of $R_{\beta}$ with $\text{ht } \beta < \text{ht}(c_{\gamma_{i_k}(\ell)}^{(s)})$. Thanks to (4.4) and the fact that all the other relevant simple modules are ordered in the common way, this simple module must be in common between $i$ and $i'$. Therefore, the induction proceeds and we conclude the result.
We return to the proof of Proposition 4.10. By (4.4), Claim A, and Lemma 4.2 2), we conclude the result.

**Corollary 4.11.** For Lusztig data \((i, c)\) and \((i', c')\), we have \(hd E^i_c \cong hd E^{i'}_{c'}\) if and only if \((i, c)\) and \((i', c')\) are linked by a successive application of two-moves and three-moves.

**Proof.** Every two reduced expressions of \(w_0 \in W(\Gamma_0)\) are connected by a repeated use of two moves and three moves ([Lu90a] 2.1 (c)). Therefore, we enrich Corollary 4.8 by applying Proposition 4.10 instead of Lemma 4.7 in its proof to deduce the result.

Thanks to Corollary 4.8, we often write \(\tilde{E}^i_b\) and \(E^i_b\) instead of \(\tilde{E}^i_c\) and \(E^i_c\). Since \(R_\beta\) is a finitely generated algebra free over a polynomial ring (by Theorem 2.11) with finite global dimension (by Theorem 2.9), it follows that

\[ [M : L_b], \langle M, N \rangle_{gEP} \in \mathbb{Z}(t), \text{ and } gch M \in \bigoplus_{b \in B(\infty)_\beta} \mathbb{Z}(t)[L_b] \]

for \(M, N \in R_\beta\text{-gmod}\). (See (1.1) and Theorem 2.3 for \((\bullet, \bullet)_{gEP}\).)

**Corollary 4.12.** Fix a reduced expression \(i\) and let \(\beta \in Q^+\). Then, two sets \(\{gch \tilde{E}^i_b\}_{b \in B(\infty)_\beta}\) and \(\{gch E^i_b\}_{b \in B(\infty)_\beta}\) are \(\mathbb{Z}(t)\)-bases of \(\bigoplus_{b \in B(\infty)_\beta} \mathbb{Z}(t)[L_b]\), respectively.

**Proof.** Thanks to Corollary 4.8 and Lemma 4.2 4), we deduce

\[ gch \tilde{E}^i_b (c_b) \in [L_b] + \bigoplus_{b' \in B(\infty)_\beta} t\mathbb{Z}[t][L_{b'}] \text{ for some } c_b \in \mathbb{Z}. \]

This is enough to see the first assertion. (In fact, we can show \(c_b = 0\) by a standard argument, or a consequence of Theorem 4.13 3).) The second assertion is similar.

Thanks to Corollary 4.12, we define \([M : \tilde{E}^i_b], [M : E^i_b] \in \mathbb{Z}(t)\) for every \(M \in R_\beta\text{-gmod}\) as:

\[ gch M = \sum_{b \in B(\infty)_\beta} [M : \tilde{E}^i_b] \text{gch } \tilde{E}^i_b \text{ and } gch M = \sum_{b \in B(\infty)_\beta} [M : E^i_b] \text{gch } E^i_b. \]

**Theorem 4.13.** Fix a reduced expression \(i\) and \(\beta \in Q^+\). We have:

1. For every \(b < b'\), it holds \(\text{ext}^*_R(\tilde{E}^i_b, \tilde{E}^i_{b'}) = \{0\};\)
2. For each \(b \in B(\infty)_\beta\), we have
   \[ \text{ext}^*_R(\tilde{E}^i_b, E^i_b) = \text{hom}_R(\tilde{E}^i_b, E^i_b) = \mathbb{C}; \]
3. For each \(b \in B(\infty)_\beta\), we have
   \[ [E^i_b : L_{b'}] = \begin{cases} 0 & (b \not< b') \\ 1 & \text{if } b = b' \end{cases} \text{ and } [\tilde{E}^i_b : L_{b'}] = 0 \text{ if } (b \not< b'). \]
4. For every \( b \leq i \) \( b' \), it holds
\[
\text{ext}^\bullet_{R_b} \left( \tilde{E}_{b}^i, (E_{b'}^1) \right) \cong \text{hom}_{R_b} \left( \tilde{E}_{b}^1, (E_{b'}^1) \right) \cong C^\oplus b, \nu'.
\]

**Proof.** We fix two elements \( b < i \) \( b' \) which correspond to \( i \)-Lusztig data \( c \) and \( c' \), respectively. Let \( m \) be the smallest number so that \( c_m \neq 0 \). (Note that \( c'_1 = \ldots = c'_{m-1} = 0 \).

We prove these assertions by the downward induction on \( m \). In particular, we assume all the assertions if \( c_1 = \ldots = c_m = 0 \). The base case \( m = \ell \) is examined in Example 2.10 (since \( \gamma_i^{(\ell)} \) is a simple root).

We set \( w_m := s_{i_{m-1}} s_{i_{m-2}} \ldots s_{i_1} \). Put \( \beta_1 := w_m \beta - c_m \alpha_{i_m} \) and \( \beta'_1 := w_m \beta' - c'_m \alpha_{i_m} \). By the \((m - 1)\)-times repeated application of the construction \( i \rightarrow \nu' \), we obtain
\[
\nu := (i_m, \ldots, i_\ell, i'_2, \ldots, i'_{m-1}) \in I^\ell.
\]

Let \( d \) and \( d' \) be the \( i \)-Lusztig data given by \( d_1 = c_m, d_2 = c_{m+1}, \ldots, d_{\ell-m+1} = c_\ell, d_{\ell-m+2} = \ldots = d_\ell = 0 \) and \( d'_1 = c'_m, d'_2 = c'_{m+1}, \ldots, d'_{\ell-m+1} = c'_\ell, d'_{\ell-m+2} = \ldots = d'_{\ell} = 0 \), respectively. We also set \( d[1] \) and \( d'[1] \) as sequences defined as: \( d_j[1] = 0 \) \( (j = 1) \) or \( d_j \) \( (1 < j \leq \ell) \), and \( d'_j[1] = 0 \) \( (j = 1) \) or \( d'_j \) \( (1 < j \leq \ell) \). We have
\[
\tilde{E}_c^i \cong T_{i_1} \ldots T_{i_{m-1}} \tilde{E}_c^p \quad \text{and} \quad E_c^i \cong T_{i_1} \ldots T_{i_{m-1}} E_c^p.
\]

**Claim B.** We have
\[
\tilde{E}_d^p \cong T_{i_{m-1}}^* \ldots T_{i_1}^* \tilde{E}_c^p \quad \text{and} \quad E_d^p \cong T_{i_{m-1}}^* \ldots T_{i_1}^* E_c^p.
\]

**Proof.** By Corollary 4.9 and Theorem 3.6 2), we have necessarily
\[
T_{i_j} \ldots \tilde{T}_{i_{j-1}} L_{c_i k} \in R_{s_{i_j-1} \ldots s_{i_1}} \beta \mod^{i-1} \quad \text{for each} \ 1 < j < k.
\]

Applying this to Lemma 4.2 2) and use Theorem 3.6 2), we deduce
\[
E_d^p \cong T_{i_{m-1}}^* \ldots T_{i_1}^* E_c^i.
\]

In particular, the lengths of the \( R_\beta \)-module \( E_d^p \) and \( R_{w_m \beta} \)-module \( E_c^i \) are the same. Since each \( T_{i_j} \) annihilates simple modules which are not irreducible constituents of \( T_{i_{j+1}} \ldots T_{i_{m-1}} E_d^p \), we conclude
\[
\tilde{E}_d^p \cong T_{i_{m-1}}^* \ldots T_{i_1}^* \tilde{E}_c^p
\]
by Lemma 4.2 3). The case of \( E_c^i \) is similar.

**Claim C.** The vanishing of \( \text{ext}^\bullet_{R_b} \left( \tilde{E}_c^i, \tilde{E}_c^i \right) \) follows from the vanishing of
\[
\text{ext}^\bullet_{R_b} \left( \tilde{E}_c^i, E_c^i \right) \cong \text{ext}^\bullet_{R_{c_m \mu m} \otimes R_b} \left( P_{c_m \mu m} \otimes \tilde{E}_d^p, E_d^p \right).
\]

**Proof.** Thanks to Claim B, a repeated use of Theorem 3.6 4) implies a sequence of isomorphisms
\[
\text{ext}^\bullet_{R_b} \left( \tilde{E}_c^i, E_c^i \right) \cong \text{ext}^\bullet_{R_{c_m \mu m} \otimes R_b} \left( \tilde{T}_1 \ldots \tilde{T}_{m-1} \tilde{E}_d^p, T_1 \ldots T_{m-1} E_d^p \right) \cong \text{ext}^\bullet_{R_{c_{m+1} \mu m+1}} \left( \tilde{T}_2 \ldots \tilde{T}_{m-1} \tilde{E}_d^p, T_2 \ldots T_{m-1} E_d^p \right) \ldots \cong \text{ext}^\bullet_{R_{c_m \mu m}} \left( \tilde{E}_d^p, E_d^p \right).
\]

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Since \( \star \) preserves the projectivity (see e.g. [KL09] 2.16) and \( \tilde{E}_d^\nu \cong P_{c_{m; d}} \star P_{d[1]} \), we conclude an isomorphism

\[
\text{ext}_{R_{\alpha}}^\bullet(\tilde{E}_d^\nu, E_d^\nu) \cong \text{ext}_{R_{\alpha}}^\bullet(P_{c_{m; d}} \otimes \tilde{E}_d^\nu, E_d^\nu).
\]

It remains to deduce \( \text{ext}_{R_{\beta}}^\bullet(\tilde{E}_c^\nu, E_c^\nu) = \{0\} \) from the vanishing of (4.5).

Since \( R_{\beta} \) is a Noetherian ring with finite global dimension, we have a projective resolution \( \mathcal{P}^\bullet \) of \( \tilde{E}_c^\nu \), which consists of finitely many finitely generated projective \( R_{\beta} \)-modules. In particular, there exists \( x \in \mathbb{Z} \) so that the degrees of simple quotients of all \( R_{\beta} \)-module direct summands of \( \mathcal{P}^\bullet \) are \( \leq x \).

For each \( j \in \mathbb{Z} \), we have a (surjective) \( A \)-module quotient \( \varphi_j : \tilde{E}_c^\nu \to E_j \) so that a) \( \ker \varphi_j \) is concentrated in degree \( > j + x \), and b) \( E_j \) is a finite successive self-extension of (grading shifts) of \( E_c^\nu \) by Lemma 4.2 3). Then, \( \text{ext}_{R_{\beta}}^\bullet(\tilde{E}_c^\nu, E_c^\nu) = \{0\} \) implies

\[
\text{ext}_{R_{\beta}}^\bullet(\tilde{E}_c^\nu, \ker \varphi_j)^{\oplus} = \{0\} = \text{ext}_{R_{\beta}}^\bullet(\tilde{E}_c^\nu, E_j).
\]

This yields \( \text{ext}_{R_{\beta}}^\bullet(\tilde{E}_c^\nu, E_c^\nu) = \{0\} \) (for each \( j \)) as required.

We return to the proof of Theorem 4.13. We have the following two short exact sequences:

\[
0 \to L_{c_{m; d}} \otimes E_d^\nu \to E_d^\nu \to C \to 0 \quad \text{as } R_{c_{m; d}} \otimes R_{\beta_1} \text{-modules, and}
\]
\[
0 \to L_{c_{m; d}} \otimes E_d^\nu \to E_d^\nu \to C' \to 0 \quad \text{as } R_{c_{m; d}} \otimes R_{\beta_2} \text{-modules.}
\]

**Claim D.** We have \( e_{c_{m; d}}(c_{m; d}) C = \{0\} \) and \( e_{c_{m; d}}(c_{m; d}) C' = \{0\} \).

**Proof.** Let \( \Psi \) be the set of \( m \in Y_{w_\beta} \) so that

\[
e_m(L_{c_{m; d}} \otimes E_d^\nu) \neq \{0\}.
\]

Let \( n = \text{ht } w_\beta \), and let \( \mathfrak{S} \) be the set of minimal length representatives of \( \mathfrak{S}_n / (\mathfrak{S} \times \mathfrak{S}_n \setminus \mathfrak{S}_n) \) inside \( \mathfrak{S}_n \). We set \( \mathfrak{S}^* := \mathfrak{S} \setminus \{1\} \). We have \( e_m(C') \neq \{0\} \) only if \( m \in \mathfrak{S}^* \Psi \). Since \( E_d^\nu \) belongs to the (essential) image of \( T_{c_{m; d}} \), we have \( e_{c_{m; d}}(1) E_d^\nu = \{0\} \). On the other hand, we have \( Y_{c_{m; d}} = \{1, \ldots, i_1, \ldots, i_1\} \). Since every element of \( \mathfrak{S}^* \) decreases the number of heading \( i_1, \ldots, i_1 \), we deduce the assertion for \( C' \). The case of \( C \) is the same.

**Claim E.** Every irreducible constituent \( L_{b''} (k) \) of \( E_c^\nu \) \((b'' \in B(\infty) \beta, k > 0)\) satisfies \( b'' >_1 b \).

**Proof.** By Claim B and Theorem 3.6 2), the module \( L_{b''} (k) \) gives rise to an irreducible constituent \( \tilde{T}_{b''} (k) \) of \( E_d^\nu \), that we denote by \( L_{b''} (k) \) with \( b'' \in B(\infty) w_m \beta \). We have \( T_1 \cdots T_i = 0 \) since it must send an \( R_{\gamma} \)-module to an \( R_{\gamma_{m; \gamma}} \)-module, which is inexistent. In particular, the \( \tilde{\Pi} \)-Lusztig datum \( e_m (b'' \nu) \) must be of the form \( d_{\ell-m+2} \cdots d_\ell = 0 \), and the \( \nu \)-Lusztig datum \( e'' \nu \) is given as

\[
e''_1 = \cdots e''_{m; d} = 0, e''_m = d_1, e''_{m+1} = d_2, \ldots, e''_\ell = d_{\ell-m+1}.
\]
Claim D implies that \( \epsilon_{i_m}(b') = d_i^1 \leq c_m \). In case this inequality is strict, we have \( b' > b \). Hence, we assume \( \epsilon_{i_m}(b') = c_m \). Theorem 2.12 asserts that we have \( \epsilon_{i_m}(c_m)L_{b'} = L_{c_{i_m}} \otimes L_{b'} \) for some \( b'_1 \in B(\infty)_{\beta_1} \). In addition, Claim D (and the above arguments) imply

\[
0 \neq [E^p_{d'|[1]} : L_{b'_1} (k)]_0 = [T_{i_1} \cdots T_{i_{m-1}} E^p_{d'|[1]} : T_{i_1} \cdots T_{i_{m-1}} L_{b'_1} (k)]_0.
\]

In case \( k > 0 \), this yields \( d'_1 > c'[1] \) by induction hypothesis, where \( d'_1 \) is the \( \beta \)-Lusztig datum of \( b'_1 \) (that is obtained from \( d' \) by replacing \( d'_1 \) by 0). This implies \( b'' > b \) as required.

We return to the proof of Theorem 4.13. Claim E, together with Lemma 4.2 3), implies the third assertion. In addition, we deduce that the RHS of (4.5) must be zero if \( c_m = c'_m \).

We assume \( c_m = c'_m \). Applying Claim D, we deduce that

\[
\text{ext}_{R_{c_m, \alpha_{i_m}}}^*(P_{c_{i_m}} \otimes \tilde{E}^p_{d'|[1]}; C') = \{0\}.
\]

Therefore, the vector spaces in (4.5) is isomorphic to

\[
\text{ext}_{R_{c_m, \alpha_{i_m}}}^*(P_{c_{i_m}} \otimes \tilde{E}^p_{d'|[1]}; L_{c_{i_m}} \otimes E^p_{d'|[1]}).
\]

Here we have \( \text{ext}_{R_{c_m, \alpha_{i_m}}}(P_{c_{i_m}}, L_{c_{i_m}}) = \text{hom}_{R_{c_m, \alpha_{i_m}}}(P_{c_{i_m}}, L_{c_{i_m}}) = \mathbb{C} \). Therefore, Claim B, Lemma 4.2 2), and Theorem 3.6 4) implies

\[
\text{ext}_{R_{c_m, \alpha_{i_m}}}^*(P_{c_{i_m}} \otimes \tilde{E}^p_{d'|[1]}, L_{c_{i_m}} \otimes E^p_{d'|[1]}) \cong \text{ext}_{R_{\beta_1}}^*(\tilde{E}^p_{d'|[1]}, E^p_{d'|[1]})
\]

\[
\cong \text{ext}_{R_{i_{m-1}} \beta_1}(\tilde{E}^p_{d'|[1]}, T_{i_{m-1}} E^p_{d'|[1]}) \cong \cdots \cong \text{ext}_{R_{c_{i_m}}}^*(\tilde{E}^p_{c'[1]}, E^p_{c'[1]})
\]

(4.6)

where \( c'[1] \) and \( c'_m \) are the \( \beta \)-Lusztig data of \( T_{i_1} \cdots T_{i_{m-1}} \tilde{E}^p_{d'|[1]} \) and \( T_{i_1} \cdots T_{i_{m-1}} E^p_{d'|[1]} \), respectively. Therefore, we deduce the first two assertions by the induction hypothesis and Claim C.

The fourth assertion follows from the vanishing of (4.5) and the middle two assertions by applying long exact sequences repeatedly.

Therefore, the induction proceeds and we conclude the result.

\[ \square \]

**Corollary 4.14.** Fix a reduced expression \( \mathbf{i} \) and \( \beta \in Q^+ \). We have

\[
\tilde{E}^p_b = P_b/(\sum_{f \in \text{hom}_{R_{\beta}}(P_{c'}, P_{b'}), b' < b} \text{Im } f) \quad \text{and} \quad E^p_b = P_b/(\sum_{f \in \text{hom}_{R_{\beta}}(P_{b}, \tilde{E}^p_b), b > b} \text{Im } f),
\]

where \( b \) and \( b' \) runs over \( B(\infty)_{\beta} \).

**Proof.** By Lemma 4.2 4), \( \tilde{E}^p_b \) admits a surjection from \( P_b \). By Theorem 4.13 3), we conclude that all the simple subquotients \( \tilde{E}^p_b \) is of the form \( L_{b'} (k) \) for \( b \leq b' \), and hence the RHS surjects onto \( \tilde{E}^p_b \). By Theorem 4.13 3) and 4), the head of \( \ker (P_b \rightarrow \tilde{E}^p_b) \) must be spanned by \( L_{b'} (k) \) for \( b' < b \) and \( k \in \mathbb{Z} \), and hence the both sides are maximal quotients of \( P_b \) whose simple subquotients are that form.

Therefore, they are isomorphic to each other. This proves the first assertion. The second assertion follows by Lemma 4.2 3) and Theorem 4.13 2).

\[ \square \]
Corollary 4.15. Fix a reduced expression $i$ and $\beta \in Q^+$. Then, we have

$$\text{ext}_{R_{\beta}}^{i}(\tilde{E}_b^i, (E_{b'}^i)^*) = \begin{cases} \mathbb{C} & (b \neq b', i = 0) \\ \{0\} & \text{(otherwise)} \end{cases}, \text{ and } \langle \tilde{E}_b^i, (E_{b'}^i)^* \rangle_{gEP} = \delta_{b,b'}$$

for every $b, b' \in B(\infty)_\beta$.

Proof. Since the first assertion implies the second assertion, we prove only the first assertion. If $b \leq_1 b'$, then the assertion follows from Theorem 4.13 4).

Thanks to Example 2.10, each $L_{\epsilon_i}$ admits a finite resolution by the graded shifts of $P_{\epsilon_i}$ (for each $c \geq 1$ and $i \in I$). By Lemma 4.2 2), we deduce that each $E_b^j$ admits a finite resolution by the graded shifts by $\tilde{E}_b^i$. By taking the spectral sequence of this resolution, we deduce

$$\text{ext}^*_R(E_{b}^i, (E_{b'}^i)^*) = \{0\} \quad (4.7)$$

for each $b <_1 b'$. Since $\ast$ is an exact functor and $\text{ext}_A^i$ is a universal $\delta$-functor, we deduce

$$\text{hom}_{R_{\beta}}(M, N^*) \cong \text{hom}_{R_{\beta}}(N, M^*)$$

for each $M, N \in R_{\beta}g\text{-mod}$. In particular, we conclude (4.7) unless $b = b'$.

Thanks to Theorem 2.3 and the definition of the algebras $A_{(G,X)}$ (and $B_{(G,X)}$) in §1, we can replace $R_{\beta}$ with its basic ring to assume that it is non-negatively graded. Then, we have $(E_{b'}^j)^*_k = \{0\}$ for every $k > 0$. For each $j \in \mathbb{Z}$, we have an $R_{\beta}$-module quotient $\bar{\varphi}_j : \tilde{E}_b^i \to E_j$ so that a) $\ker \bar{\varphi}_j$ is concentrated in degree $>-j$, and b) $E_j$ is a finite successive self-extension of (graded shifts) of $E_b^i$ by Lemma 4.2 3). Then, the minimal projective resolution of $\ker \bar{\varphi}_j$ is concentrated in degree $>-j$. In particular, we have

$$\text{ext}^*_R(\ker \bar{\varphi}_j, (E_{b'}^i)^*) = \{0\} = \text{ext}^*_R(E_j, (E_{b'}^i)^*) \quad \text{for each } b \neq b'.$$

This yields $\text{ext}^*_R(\tilde{E}_b^i, (E_{b'}^i)^*) = \{0\}$ (for each $j$) as required. \hfill $\square$

Remark 4.16. For each $\beta \in Q^+$, the standard normalization

$$\langle P_b, L_b \rangle_{gEP} = g\text{dim} \text{hom}_{R_{\beta}}(P_b, L_b) = \delta_{b,b'},$$

combined with Theorem 2.5, Corollary 4.15 and Theorem 4.13 implies that \{gch $E_b^i$\} and \{gch $E_b^i$\} give rise to the lower/upper PBW bases corresponding to $i$, respectively.

Theorem 4.17 (Lusztig’s conjecture). For every reduced expression $i$ of $w_0$, $\beta \in Q^+$, and $b, b' \in B(\infty)_\beta$, we have an equality

$$[P_b : \tilde{E}_{b'}^i] = [E_{b'}^i : L_b].$$

In particular, the expansion coefficients of the lower global basis in terms of the lower PBW basis are in $\mathbb{N}[t]$.

Remark 4.18. Thanks to [K12a] 1.7 (and Lusztig [Lu90a] 10.6), a projective module $P_b$ admits a filtration by \{\tilde{E}_{b'}^i\} if $i$ is adapted to $\Gamma$.  

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Proof of Theorem 4.17. By Corollary 4.15, we have
\[ \delta_{b,b'} = (P_b, L_{b'})_{\text{gEP}} = \sum_{d,d' \in B(\infty), d} \frac{[P_b : E^d_d][L_{b'} : E^d_{d'}]}{[P_{b'} : E^d_d][L_b : E^d_d]} \left( \tilde{E}^d_d, (E^d_d)^* \right)_{\text{gEP}} \]
\[ = \sum_{d \in B(\infty), d} \frac{[P_b : E^d_d][L_{b'} : E^d_d]}{[P_{b'} : E^d_d][L_b : E^d_d]} \]
By applying the bar involution, this shows that
\[ ((P_b : E^d_d))(L_{b'} : E^d_d))^{-1} = (\delta_{b,b'}) \]
which is equivalent to the assertion. \(\square\)

For each \(\beta, \beta' \in Q^+\), we define the formal expression \(q^\beta\) and \(q^\beta'\) so that \(q^\beta \cdot q^\beta' = q^{\beta + \beta'}\). We define
\[ \text{ep}_i(q^\beta) := \sum_{n \geq 0} q^{n\beta} \frac{(1 - t^2)(1 - t^4) \ldots (1 - t^{2n})}{(1 - t)^n} \in \mathbb{Q}(t)[Q^+] \]

Corollary 4.19 (cf. Problem 2 in Kashiwara [Kas95].) For each \(\beta \in Q^+\), we set
\[ (P : L)_{\beta} := ((P_b : L_{b'}))_{b,b' \in B(\infty), d} = ((P_{b'} : P_b)_{b,b' \in B(\infty), d}) \]
as the square matrix with its determinant \(D_{\beta}\). We have
\[ \sum_{\beta \in Q^+} D_{\beta}q^\beta = \prod_{\alpha \in R^+} \text{ep}_i(\alpha) \]
Proof. As in the proof of Theorem 4.17, we factorize
\[ (P : L)_{\beta} = (P : E^d_d)[E : L]_{\beta}, \]
where the second term is the \#\#-matrix of expansion coefficients between projectives/lower PBWs, lower PBWs/upper PBWs, and upper PBWs/simples, respectively. By Theorem 4.13 3), the determinant of the third matrix is 1. By Theorem 4.17, the determinant of the first matrix is also 1. By Lemma 4.2 3) (cf. Corollary 4.15), we conclude
\[ D_{\beta} = \prod_{b \in B(\infty), d} [\tilde{E}^d_d : E^d_d]. \]
By Lemma 4.2 2) and the construction of \(T_{i,j}\), if we denote \(c\) the \(i\)-Lusztig datum corresponding to \(b\), then we have
\[ [\tilde{E}^d_d : E^d_d] = \prod_{j=1}^{\ell} [P_{c_{i,j}} : L_{c_{i,j}}] = \prod_{j=1}^{\ell} \frac{1}{(1 - t^2)(1 - t^4) \ldots (1 - t^{2n})}. \]
This is equivalent to the assertion by a simple counting. \(\square\)

Remark 4.20. 1) By a formal manipulation, we have
\[ (P_b, P_{b'})_{\text{gEP}} = (F_b, F_{b'})_{\text{gEP}} \quad \text{for every } b, b' \in B(\infty). \]
Since the RHS calculates the Lusztig inner form \(\{\bullet, \bullet\}\) ([Lus93] 1.2.10) of the lower global basis, Corollary 4.19 yields the Sha沙发上 determinant formula of quantum groups of type ADE. 2) The proof of Corollary 4.19 also follows from [K12a] 3.12, but here the proof works also from the PBW bases \(\{E^d_d\}_b\) in which \(\ell\) is not an adapted reduced expression of \(w_0\).
References


