PBW bases and KLR algebras*

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Abstract

We generalize Lusztig's geometric construction of the PBW bases of finite quantum groups of type ADE under the framework of [Varagnolo-Vasserot, J. reine angew. Math. 659 (2011)]. In particular, every PBW basis of such quantum groups is proven to yield a semi-orthogonal collection in the module category of the KLR-algebras. This enables us to prove Lusztig's conjecture on the positivity of the canonical (lower global) bases in terms of the (lower) PBW bases. In addition, we verify Kashiwara's problem on the finiteness of the global dimensions of the KLR-algebras of type ADE.

Introduction

Canonical/global bases of quantum groups, defined by Lusztig [Lu90a] and Kashiwara [Kas91] subsequently, open up scenery in many areas of mathematics which are visible only through quantum groups [Ari05, Lus08, Nak06]. They are certain bases of quantum groups different from the natural quantum analogue of the classical Poincaré-Birkhoff-Witt theorem (that are usually referred to as the PBW bases).

Among these, the interaction between canonical/global bases of quantum groups and affine Hecke algebras of type A (and their cyclotomic quotients) yields many representation-theoretic consequences [Ari96, Ari05]. It is generalized to more general quantum groups and their representations by Khovanov-Lauda, Rouquier, Varagnolo-Vasserot, Zheng, Webster, and Kang-Kashiwara [KL09, Rou08, VV11, Zhe08, Web10, KK12] as a categorical counter-part of the theory of canonical/global bases.

More precisely, to each symmetric Kac-Moody algebra $\mathfrak{g}$, they introduced a series of algebras R_β (that we call the KLR-algebras) whose simple/projective modules give rise to the upper/lower global bases of the corresponding positive half of the quantum group of $\mathfrak{g}$. There the emphasis is on the categorification of quantum groups, and their results are strong enough to generalize and categorify Ariki's result [Ari96] in these cases (Lauda-Vazirani [LV11] and [VV11, KK12]).

This story is sufficient to recover deep representation-theoretic properties, without the PBW bases. The main observation of this paper is that the PBW

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bases still exist in the world of KLR-algebras, with essential new features which are visible only with the KLR-algebras.

To see what we mean by this, we prepare some notations: Let $\mathcal{A} := \mathbb{Z}[t^{\pm 1}]$. Let $\mathfrak{g}$ be a simple Lie algebra of type ADE, and let U^+ be the positive half of the $\mathcal{A}$-integral version of the quantum group of $\mathfrak{g}$ (see e.g. Lusztig [Lus93] §1). Let $Q^+ := \mathbb{Z}_{\geq 0} I$, where I is the set of positive simple roots. We have a weight space decomposition $U^+ = \bigoplus_{\beta \in Q^+} U^+_\beta$. We have the Weyl group W of $\mathfrak{g}$ with its set of simple reflections $\{s_i\}_{i \in I}$ and the longest element w_0. For each $\beta \in Q^+$, we have a finite set $B(\infty)_\beta$ which parameterizes a pair of distinguished bases $\{G^{up}(b)\}_{b \in B(\infty)_\beta}$ and $\{G^{low}(b)\}_{b \in B(\infty)_\beta}$ of $\mathbb{Q}(t) \otimes_{\mathcal{A}} U^+_\beta$. The Khovanov-Lauda-Rouquier algebra R_β is a certain graded algebra whose grading is bounded from below with the following properties:

- The set of isomorphism classes of simple graded R_β-modules (up to grading shifts) is also parameterized by $B(\infty)_\beta$;
- For each $b \in B(\infty)_\beta$, we have a simple graded R_β-module L_b and its projective cover P_b. Let $L_{b'} \langle k \rangle$ be the grade k shift of $L_{b'}$, and let $[P_b : L_{b'} \langle k \rangle]_0$ be the multiplicity of $L_{b'} \langle k \rangle$ in P_b (that is finite). Then, we have
$$G^{low}(b) = \sum_{b' \in B(\infty)_\beta, k \in \mathbb{Z}} t^k [P_b : L_{b'} \langle k \rangle]_0 G^{up}(b');$$
- For each $\beta, \beta' \in Q^+$, there exists an induction functor
$$\star : R_\beta\text{-}\mathsf{gmod} \times R_{\beta'}\text{-}\mathsf{gmod} \ni (M, N) \mapsto M \star N \in R_{\beta+\beta'}\text{-}\mathsf{gmod};$$
- $\mathbf{K} := \bigoplus_{\beta \in Q^+} \mathbb{Q}(t) \otimes_{\mathcal{A}} K(R_\beta\text{-}\mathsf{gmod})$ is an associative algebra isomorphic to $\mathbb{Q}(t) \otimes_{\mathcal{A}} U^+$ with its product inherited from $\star$ (and the t-action is a grading shift).

As mentioned earlier, Lusztig [Lu90a] studied the geometric side of the story. By applying the results in [K12a], we first observe the following:

Theorem A (Kashiwara's problem = Corollary 2.9)**.** *For every $\beta \in Q^+$, the algebra R_β has finite global dimension.*

This problem is raised by Kashiwara several times in his lectures on KLR algebras. We remark that in type A case, Theorem A follows through a Morita equivalence with an affine Hecke algebra of type A (see e.g. Opdam-Solleveld [OS09]).

For quantum groups, a way to construct a (nice) PBW basis depends on an arbitrary sequence $\mathbf{i} := (i_1, i_2, \cdots, i_\ell) \in I^\ell$ corresponding to a reduced expression of w_0. Associated to $\mathbf{i}$, we have a total order $<_{\mathbf{i}}$ on each $B(\infty)_\beta$ (see §4). We define two collections of graded R_β-modules $\{\widetilde{E}^{\mathbf{i}}_b\}_{b \in B(\infty)_\beta}$ and $\{E^{\mathbf{i}}_b\}_{b \in B(\infty)_\beta}$ as follows (cf. Corollary 4.14): **1)** $\widetilde{E}^{\mathbf{i}}_b$ is obtained from P_b by annihilating all $L_{b'} \langle k \rangle$ with $b' <_{\mathbf{i}} b$ and $k \geq 0$, and **2)** $E^{\mathbf{i}}_b$ is obtained from $\widetilde{E}^{\mathbf{i}}_b$ by annihilating all $L_b \langle k \rangle$ with $k > 0$.

Since R_β is a graded algebra with finite global dimension, we set
$$\langle M, N \rangle_{\mathsf{gEP}} := \sum_{i \geq 0} (-1)^i \mathsf{gdim}\, \mathrm{ext}^i_{R_\beta}(M, N) \in \mathbb{Q}(t) \quad \text{for} \quad M, N \in R_\beta\text{-}\mathsf{gmod},$$

where $\hom_{R_\beta}(M,N) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{R_\beta\text{-}\mathsf{gmod}}(M\langle k\rangle, N)$.

By construction, we deduce that the graded character expansion coefficient $[M : \widetilde{E}^{\mathbf{i}}_{b'}] \in \mathbb{Z}(\!(t)\!)$ is well-defined for every $M \in R_\beta\text{-}\mathsf{gmod}$.

The above definitions of $\widetilde{E}^{\mathbf{i}}_b$ and $E^{\mathbf{i}}_b$ look natural, but not apparently related to a PBW basis of U^+.

Theorem B (Orthogonality relation = Theorem 4.13 and its corollaries)**.** *In the above setting, we have:*

1. *For $b <_{\mathbf{i}} b'$, we have $\mathrm{ext}^\bullet_{R_\beta}(E^{\mathbf{i}}_b, E^{\mathbf{i}}_{b'}) = \{0\}$;*
2. *We have*
$$\mathrm{ext}^i_{R_\beta}(\widetilde{E}^{\mathbf{i}}_b, (E^{\mathbf{i}}_{b'})^*) = \begin{cases} \mathbb{C} & (b \neq b', i = 0) \\ \{0\} & (\mathit{otherwise}) \end{cases}, \quad \mathit{and} \quad \left\langle \widetilde{E}^{\mathbf{i}}_b, (E^{\mathbf{i}}_{b'})^* \right\rangle_{\mathsf{gEP}} = \delta_{b,b'};$$
3. *The graded R_β-module $\widetilde{E}^{\mathbf{i}}_b$ is a self-extension of $E^{\mathbf{i}}_b$ in the sense that there exists a separable decreasing filtration of $\widetilde{E}^{\mathbf{i}}_b$ whose associated graded is a direct sum of grading shifts of $E^{\mathbf{i}}_b$.*

Since we have $\langle P_b, L_{b'}\rangle_{\mathsf{gEP}} = \delta_{b,b'}$ by definition, the pairing $\langle \bullet, \bullet\rangle_{\mathsf{gEP}}$ is essentially the Lusztig inner form (cf. [Lus93] 1.2.10–1.2.11). Therefore, Theorem B guarantees that our $\{\widetilde{E}^{\mathbf{i}}_b\}_b$, and $\{E^{\mathbf{i}}_b\}_b$ must be the categorifications of the lower/upper PBW bases by their characterization. We remark that some of these modules seem to coincide with those obtained by Kleshchev-Ram [KR11], Webster [Web10], and Benkart-Kang-Oh-Park [BKOP].

Theorem C (Lusztig's conjecture = Theorem 4.17)**.** *We have $[P_b : \widetilde{E}^{\mathbf{i}}_{b'}] = [E^{\mathbf{i}}_{b'} : L_b]$ for each $b, b' \in B(\infty)_\beta$. In particular, we have*
$$[P_b : \widetilde{E}^{\mathbf{i}}_{b'}] \in \mathbb{N}[t] \quad \textit{for every} \quad b, b' \in B(\infty)_\beta.$$

Theorem C is conjectured by Lusztig as his comment on [Lu90a] in his webpage. Note that Theorem C is established in Lusztig [Lu90a] Corollary 10.7 when the reduced expression $\mathbf{i}$ satisfies the condition so-called "adapted" (see §3).

Example D ($\mathfrak{g} = \mathfrak{sl}_3$). We have $I = \{\alpha_1, \alpha_2\}$. The standard generators E_1, E_2 of U^+ correspond to projective modules P_1 and P_2 of R_{α_1} and R_{α_2}, respectively. Then, one series of the (lower) PBW basis $\{\widetilde{E}^{\mathbf{i}}_b\}_b$ are:
$$P_1^{(c_1)} \star Q_{21}^{(c_2)} \star P_2^{(c_3)} \quad \text{for} \quad c_1, c_2, c_3 \geq 0.$$

Here $X^{(c)}$ denotes a direct factor of $X \star X \star \cdots \star X$ (c times). Note that $P_1^{(c_1)}$, $Q_{21}^{(c_2)}$, and $P_2^{(c_3)}$ are maximal self-extensions of simple modules (this is a general phenomenon). We have a short exact sequence
$$0 \to P_1 \star P_2 \langle 2\rangle \longrightarrow P_2 \star P_1 \longrightarrow Q_{21} \to 0,$$

which is a categorical version of $E_2E_1 - t^2E_1E_2$.

The organization of this paper is as follows: In the first section, we collect several results from [K12a] needed in the sequel. The second section is the preliminary on the KLR algebra. In the third section, we abstract and categorify Lusztig's arguments in the setting of the Hall algebras [Lus98] to the KLR algebras by utilizing the results of [K12a] and the induction theorem imported from [KL87, Lus02, K09]. This includes categorifications of Saito's reflection actions [Sai94] that we call the Saito reflection functors. In the fourth section, we depart from geometry and utilize the properties of the Saito reflection functors established in the earlier sections to deduce Theorem B and Theorem C.

After submitted the initial version this paper, there appeared another (algebraic) proofs of the main results of this paper by McNamara [Mac12] and Brundan-Kleshchev-McNamara [BKM12], which also covers non-simply laced cases. Thier approach is quite different from that of ours, and one merit of our approach is that it provides a bridge between geometric/algebraic view points, typically seen in the Saito reflection functor used in the proof.

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Convention

An algebra R is a (not necessarily commutative) unital $\mathbb{C}$-algebra. A variety $\mathfrak{X}$ is a separated reduced scheme $\mathfrak{X}_0$ of finite type over some localization $\mathbb{Z}_S$ of $\mathbb{Z}$ specialized to $\mathbb{C}$. It is called a G-variety if we have an action of a connected affine algebraic group scheme G flat over $\mathbb{Z}_S$ on $\mathfrak{X}_0$ (specialized to $\mathbb{C}$). As in [BBD82] §6 and [BL94], we transplant the notion of weights to the derived category of (G-equivariant) constructible sheaves with finite monodromy on $\mathfrak{X}$. Let us denote by $D^b(\mathfrak{X})$ (resp. $D^+(\mathfrak{X})$) the bounded (resp. bounded from the below) derived category of the category of constructible sheaves on $\mathfrak{X}$, and denote by $D^+_G(\mathfrak{X})$ the G-equivariant derived category of $\mathfrak{X}$. We have a natural forgetful functor $D^+_G(\mathfrak{X}) \to D^+(\mathfrak{X})$, whose preimage of $D^b(\mathfrak{X})$ is denoted by $D^b_G(\mathfrak{X})$. For an object of $D^b_G(\mathfrak{X})$, we may denote its image in $D^b(\mathfrak{X})$ by the same letter.

Let vec be the category of $\mathbb{Z}$-graded vector spaces (over $\mathbb{C}$) bounded from the below so that its objects have finite-dimensional graded pieces. In particular, for $V = \oplus_{i \gg -\infty} V^i \in \mathsf{vec}$, its graded dimension $\mathsf{gdim}\, V := \sum_i t^i \dim V_i \in \mathbb{Z}(\!(t)\!)$ makes sense (with t being indeterminant). We define $V \langle m \rangle$ by setting $(V \langle m \rangle)_i := V_{i-m}$.

In this paper, a graded algebra A is always a $\mathbb{C}$-algebra whose underlying space is in vec. Let A-gmod be the category of finitely generated graded A-modules. For $E, F \in A$-gmod, we define $\hom_A(E, F)$ to be the direct sum of graded A-module homomorphisms $\hom_A(E, F)^j$ of degree j ($= \mathrm{Hom}_{A\text{-}\mathsf{gmod}}(E \langle j \rangle, F)$). We employ the same notation for extensions (i.e. $\mathrm{ext}^i_A(E, F) = \oplus_{j \in \mathbb{Z}} \mathrm{ext}^i_A(E, F)^j$). We denote by $\mathsf{Irr}\, A$ the set of isomorphism classes of graded simple modules of A, and denote by $\mathsf{Irr}_0\, A$ the set of isomorphism classes of graded simple modules of A up to grading shifts. Two graded algebras are said to be Morita equivalent if their graded module categories are equivalent. For a graded A-module E, we denote its head by $\mathsf{hd}\, E$, and its socle by $\mathsf{soc}\, E$.

For $Q(t) \in \mathbb{Q}(t)$, we set $\overline{Q}(t) := Q(t^{-1})$. For derived functors $\mathbb{R}F$ or $\mathbb{L}F$ of some functor F, we represent its arbitrary graded piece (of its homology complex) by $\mathbb{R}^*F$ or $\mathbb{L}^*F$, and the direct sum of whole graded pieces by $\mathbb{R}^\bullet F$ or $\mathbb{L}^\bullet F$. For example, $\mathbb{R}^*F \cong \mathbb{R}^*G$ means that $\mathbb{R}^iF \cong \mathbb{R}^iG$ for every $i \in \mathbb{Z}$, while $\mathbb{R}^\bullet F \cong \mathbb{R}^\bullet G$ means that $\bigoplus_i \mathbb{R}^iF \cong \bigoplus_i \mathbb{R}^iG$.

When working on some sort of derived category, we suppress $\mathbb{R}$ or $\mathbb{L}$, or the category from the notation for simplicity when there is only small risk of confusion.

1 Recollection from [K12a]

Let G be a connected reductive algebraic group. Let $\mathfrak{X}$ be a G-variety. Let Λ be the labelling set of G-orbits of $\mathfrak{X}$. For $\lambda \in \Lambda$, we denote the corresponding G-orbit by $\mathbb{O}_\lambda$. For $\lambda, \mu \in \Lambda$, we write $\lambda \preceq \mu$ if $\mathbb{O}_\lambda \subset \overline{\mathbb{O}_\mu}$. We assume the following property ($\spadesuit$):

($\spadesuit$)$_1$ The set Λ is finite. For each $\lambda \in \Lambda$, we fix $x_\lambda \in \mathbb{O}_\lambda(\mathbb{C})$;

($\spadesuit$)$_2$ For each $\lambda \in \Lambda$, the group $\mathsf{Stab}_G(x_\lambda)$ is connected.

We have a (relative) dualizing complex $\omega_{\mathfrak{X}} := p^! \underline{\mathbb{C}} \in D^b_G(\mathfrak{X})$, where $p : \mathfrak{X} \to \{\mathrm{pt}\}$ is the G-equivariant structure map. We have a dualizing functor

$$\mathbb{D} : D^b_G(\mathfrak{X})^{op} \ni C^\bullet \mapsto \mathcal{H}om^\bullet(C^\bullet, \omega_{\mathfrak{X}}) \in D^b_G(\mathfrak{X}).$$

We have a $\mathbb{D}$-autodual t-structure of $D^b_G(\mathfrak{X})$ whose truncation functor and perverse cohomology functor are denoted by τ and ${}^pH$, respectively.

For each $\lambda \in \Lambda$, we have a constant local system $\underline{\mathbb{C}}_\lambda$ on $\mathbb{O}_\lambda$. We have inclusions $i_\lambda : \{x_\lambda\} \hookrightarrow \mathfrak{X}$ and $j_\lambda : \mathbb{O}_\lambda \hookrightarrow \mathfrak{X}$. Let $\mathbb{C}_\lambda := (j_\lambda)_! \underline{\mathbb{C}}_\lambda[\dim \mathbb{O}_\lambda]$ and $\mathsf{IC}_\lambda := (j_\lambda)_{!*} \underline{\mathbb{C}}_\lambda[\dim \mathbb{O}_\lambda]$, which we regard as objects of $D^b_G(\mathfrak{X})$. We denote by

$$\mathrm{Ext}^\bullet_G(\bullet, \bullet) : D^b_G(\mathfrak{X})^{op} \times D^b_G(\mathfrak{X}) \longrightarrow D^+(\mathrm{pt})$$
$$\mathrm{Ext}^\bullet(\bullet, \bullet) : D^b(\mathfrak{X})^{op} \times D^b(\mathfrak{X}) \longrightarrow D^b(\mathrm{pt})$$

the Ext (as bifunctors) of $D^b_G(\mathfrak{X})$ and $D^b(\mathfrak{X})$, respectively.

For each $\lambda \in \Lambda$, we fix $L_\lambda \in D^b(\mathrm{pt})$ as a non-zero graded vector space with a trivial differential which satisfies the self-duality condition $L_\lambda \cong L_\lambda^*$. We set

$$\mathcal{L} := \bigoplus_{\lambda \in \Lambda} L_\lambda \boxtimes \mathsf{IC}_\lambda \in D^b_G(\mathfrak{X}).$$

By construction, we find an isomorphism $\mathcal{L} \cong \mathbb{D}\mathcal{L}$.

We form a graded Yoneda algebra

$$A_{(G,\mathfrak{X})} = \bigoplus_{i \in \mathbb{Z}} A^i_{(G,\mathfrak{X})} := \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^i_G(\mathcal{L}, \mathcal{L})$$

whose degree is the cohomological degree. We denote by $B_{(G,\mathfrak{X})}$ the algebra $A_{(G,\mathfrak{X})}$ by taking $\mathcal{L} = \bigoplus_{\lambda \in \Lambda} \mathsf{IC}_\lambda$ (and call it the basic ring of $A_{(G,\mathfrak{X})}$). The algebra $B_{(G,\mathfrak{X})}$ is Morita equivalent to $A_{(G,\mathfrak{X})}$, and hence all the arguments in the below are independent of the choice of $\mathcal{L}$, which we suppress for simplicity. We also drop $(G, \mathfrak{X})$ in case the meaning is clear from the context. It is standard that $\{L_\lambda\}_{\lambda \in \Lambda}$ forms a complete collection of graded simple A-modules up to grading shifts.

Lemma 1.1 (see [K12a] 1.2)**.** *For a graded A-module M, its graded dual M^* is again a graded A-module.* □

For each $\lambda \in \Lambda$, we set

$$P_\lambda := \mathrm{Ext}^\bullet_G(\mathsf{IC}_\lambda, \mathcal{L}) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^i_G(\mathsf{IC}_\lambda, \mathcal{L}).$$

Each P_λ is a graded projective left A-module. By construction, we have

$$A \cong \bigoplus_{\lambda \in \Lambda} L^*_\lambda \boxtimes \mathrm{Ext}^\bullet_G(\mathsf{IC}_\lambda, \mathcal{L}) = \bigoplus_{\lambda \in \Lambda} P_\lambda \boxtimes L^*_\lambda$$

as left A-modules. It is standard that P_λ is an indecomposable A-module whose head is isomorphic to L_λ (cf. [CG97] §8.7). We have an idempotent $e_\lambda \in A$ so that $P_\lambda \cong A e_\lambda$ as left graded A-modules (up to a grading shift).

For each $\lambda \in \Lambda$, we set

$$\widetilde{K}_\lambda := \mathrm{Ext}^\bullet_G(\mathbb{C}_\lambda, \mathcal{L}) \text{ and } K_\lambda := H^\bullet i^!_\lambda \mathcal{L}[\dim \mathbb{O}_\lambda].$$

We call K_λ a standard module, and $\widetilde{K}_\lambda$ a dual standard module of A.

We regard each IC_λ as a simple mixed perverse sheaf (of weight zero) in the category of mixed sheaves on $\mathfrak{X}$ via [BBD82] §5 and §6, and each L_λ as a mixed (complex of) vector space of weight zero. I.e. each L^i_λ is pure of weight i in the sense that the geometric Frobenius acts by $q^{i/2}\mathsf{id}$ if we switch the base field to the algebraic closure of a finite field of cardinality q. It follows that the algebra A acquires a (mixed) weight structure.

We consider the following property (♣):

(♣)$_1$ The algebra A is pure of weight 0;

(♣)$_2$ For each $\lambda \in \Lambda$, the perverse sheaf IC_λ is pointwise pure;

Theorem 1.2 ([K12a] 3.5)**.** *Assume the properties* (♠) *and* (♣)*. Then, the algebra A has finite global dimension.* □

For $M \in A\text{-}\mathsf{gmod}$ and $i \in \mathbb{Z}$, we define

$$[M : L_\lambda \langle i \rangle]_0 := \dim \mathrm{Hom}_{A\text{-}\mathsf{gmod}}(P_\lambda \langle i \rangle, M) \in \mathbb{Z} \quad \text{and}$$
$$[M : L_\lambda] := \mathsf{gdim}\, \mathrm{hom}_A(P_\lambda, M) \in \mathbb{Z}(\!(t)\!).$$

We have $[M : L_\lambda] = \sum_{i \in \mathbb{Z}} [M : L_\lambda \langle i \rangle]_0 t^i \in \mathbb{Z}(\!(t)\!)$.

Theorem 1.3 ([K12a] 1.6)**.** *Assume the properties* (♠) *and* (♣)*:*

1. *We have*

$$[\widetilde{K}_\lambda : L_\mu] = 0 = [K_\lambda : L_\mu] \quad \textit{for } \lambda \not\leq \mu \quad \textit{and} \quad [K_\lambda : L_\lambda] = 1;$$

2. *For each $\mu \not\leq \lambda$, we have*

$$\mathrm{ext}^\bullet_A(\widetilde{K}_\lambda, \widetilde{K}_\mu) = \{0\} \quad \textit{and} \quad \mathrm{ext}^\bullet_A(K_\lambda, K_\mu) = \{0\};$$

3. For each $\lambda \in \Lambda$, we have

$$\widetilde{K}_\lambda \cong P_\lambda / \Big(\sum_{\mu \prec \lambda} A e_\mu P_\lambda \Big);$$

4. Each $\widetilde{K}_\lambda$ is a successive self-extension of K_λ. In addition, we have

$$[\widetilde{K}_\lambda : L_\lambda] = \mathsf{gdim}\, H^\bullet_{\mathsf{Stab}_G(x_\lambda)}(\mathrm{pt}).$$

For $M \in A\text{-}\mathsf{gmod}$ and $N \in A\text{-}\mathsf{gmod}$, we define its graded Euler-Poincaré characteristic as:

$$\langle M, N \rangle_{\mathsf{gEP}} := \sum_{i \geq 0} (-1)^i \mathsf{gdim}\, \mathrm{ext}^i_A(M, N) \in \mathbb{Z}((t)). \tag{1.1}$$

Let $j : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ be the inclusion of an open G-stable subvariety. We form a graded algebra

$$A_{(G,\mathfrak{Y})} := \mathrm{Ext}^\bullet_G(j^* \mathcal{L}, j^* \mathcal{L}).$$

Lemma 1.4 ([K12a] 4.4)**.** *Let $j : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ be the inclusion of an open G-stable subvariety. Then, $\mathfrak{Y}$ satisfies the conditions (♠) and (♣) if $\mathfrak{X}$ does.* □

Proposition 1.5 ([K12a] 4.3, 4.5)**.** *Let $i : \mathbb{O}_\lambda \hookrightarrow \mathfrak{X}$ be the inclusion of a closed G-orbit (with $\lambda \in \Lambda$), and let $j : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ be its complement. Then, we have an isomorphism $A_{(G,\mathfrak{X})}/(A_{(G,\mathfrak{X})} e_\lambda A_{(G,\mathfrak{X})}) \xrightarrow{\cong} A_{(G,\mathfrak{Y})}$.* □

Corollary 1.6 ([K12a] 4.2, 4.3, 4.5)**.** *Let $j : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ be the inclusion of an open G-stable subvariety. We have*

$$\mathrm{ext}^*_A(A, L_\mu) \cong \mathrm{ext}^*_A(A_{(G,\mathfrak{Y})}, L_\mu)$$

for every $\mu \in \Lambda$ so that $\mathbb{O}_\mu \subset \mathfrak{Y}$. □

2 Quivers and the KLR algebras

Let $\Gamma = (I, \Omega)$ be an oriented graph with the set of its vertex I and the set of its oriented edges Ω. Here I is fixed, and Ω might change so that the underlying graph Γ_0 of Γ is a fixed Dynkin diagram of type ADE. We refer Ω as the orientation of Γ. We form a path algebra $\mathbb{C}[\Gamma]$ of Γ.

For $h \in \Omega$, we define $h' \in I$ to be the source of h and $h'' \in I$ to be the sink of h. We denote $i \leftrightarrow j$ for $i, j \in I$ if and only if there exists $h \in \Omega$ such that $\{h', h''\} = \{i, j\}$. A vertex $i \in I$ is called a sink of Γ if $h' \neq i$ for every $h \in \Omega$. A vertex $i \in I$ is called a source of Γ if $h'' \neq i$ for every $h \in \Omega$.

Let Q^+ be the free abelian semi-group generated by $\{\alpha_i\}_{i \in I}$, and let $Q^+ \subset Q$ be the free abelian group generated by $\{\alpha_i\}_{i \in I}$. We sometimes identify Q with the root lattice of type Γ_0 with a set of its simple roots $\{\alpha_i\}_{i \in I}$. Let $W = W(\Gamma_0)$ denote the Weyl group of type Γ_0 with a set of its simple reflections $\{s_i\}_{i \in I}$. The group W acts on Q via the above identification. Let $R^+ := W\{\alpha_i\}_{i \in I} \cap Q^+$ be the set of positive roots of a simple Lie algebra with its Dynkin diagram Γ_0.

An I-graded vector space V is a vector space over $\mathbb{C}$ equipped with a direct sum decomposition $V = \bigoplus_{i \in I} V_i$.

Let V be an I-graded vector space. For $\beta \in Q^+$, we declare $\underline{\dim} V = \beta$ if and only if $\beta = \sum_{i \in I} (\dim V_i) \alpha_i$. We call $\underline{\dim} V$ the dimension vector of V. Form a vector space

$$E_V^\Omega := \bigoplus_{h \in \Omega} \mathrm{Hom}_{\mathbb{C}}(V_{h'}, V_{h''}).$$

We set $G_V := \prod_{i \in I} GL(V_i)$. The group G_V acts on E_V^Ω through its natural action on V. The space E_V^Ω can be identified with the based space of $\mathbb{C}[\Gamma]$-modules with its dimension vector β. Let $\mathtt{M}_i$ be a unique $\mathbb{C}[\Gamma]$-module (up to an isomorphism) with $\underline{\dim}\, \mathtt{M}_i = \alpha_i$.

For each $k \geq 0$, we consider a sequence $\mathbf{m} = (m_1, m_2, \ldots, m_k) \in I^k$. We abbreviate this as $\mathsf{ht}(\mathbf{m}) = k$. We set $\mathsf{wt}(\mathbf{m}) := \sum_{j=1}^k \alpha_{m_j} \in Q^+$. For $\beta = \mathsf{wt}(\mathbf{m}) \in Q^+$, we set $\mathsf{ht}\, \beta = k$. For a sequence $\mathbf{m}' := (m'_1, \ldots, m'_{k'}) \in I^{k'}$, we set

$$\mathbf{m} + \mathbf{m}' := (m_1, \ldots, m_k, m'_1, \ldots, m'_{k'}) \in I^{k+k'}.$$

For $i \in I$ and $k \geq 0$, we understand that $ki = (i, \ldots, i) \in I^k$.

For each $\beta \in Q^+$, we set Y^β to be the set of all sequences $\mathbf{m}$ such that $\mathsf{wt}(\mathbf{m}) = \beta$. For each $\beta \in Q^+$ with $\mathsf{ht}\, \beta = n$ and $1 \leq i < n$, we define an action of $\{\sigma_i\}_{i=1}^{n-1}$ on Y^β as follows: For each $1 \leq i < n$ and $\mathbf{m} = (m_1, \ldots, m_n) \in Y^\beta$, we set

$$\sigma_i \mathbf{m} := (m_1, \ldots, m_{i-1}, m_{i+1}, m_i, m_{i+2}, \ldots, m_n).$$

It is clear that $\{\sigma_i\}_{i=1}^{n-1}$ generates a $\mathfrak{S}_n$-action on Y^β. In addition, $\mathfrak{S}_n$ naturally acts on a set of integers $\{1, 2, \ldots, n\}$.

Definition 2.1 (Khovanov-Lauda [KL09], Rouquier [Rou08])**.** Let $\beta \in Q^+$ so that $n = \mathsf{ht}\, \beta$. We define the KLR algebra R_β as a unital algebra generated by the elements $\kappa_1, \ldots, \kappa_n$, $\tau_1, \ldots, \tau_{n-1}$, and $e(\mathbf{m})$ ($\mathbf{m} \in Y^\beta$) subject to the following relations:

1. $\deg \kappa_i e(\mathbf{m}) = 2$ for every i, and
$$\deg \tau_i e(\mathbf{m}) = \begin{cases} -2 & (m_i = m_{i+1}) \\ 1 & (m_i \leftrightarrow m_{i+1}) \\ 0 & (\textit{otherwise}) \end{cases};$$
2. $[\kappa_i, \kappa_j] = 0$, $e(\mathbf{m}) e(\mathbf{m}') = \delta_{\mathbf{m},\mathbf{m}'} e(\mathbf{m})$, and $\sum_{\mathbf{m} \in Y^\beta} e(\mathbf{m}) = 1$;
3. $\tau_i e(\mathbf{m}) = e(\sigma_i \mathbf{m}) \tau_i e(\mathbf{m})$, and $\tau_i \tau_j e(\mathbf{m}) = \tau_j \tau_i e(\mathbf{m})$ for $|i - j| > 1$;
4. $\tau_i^2 e(\mathbf{m}) = Q_{\mathbf{m},i}(\kappa_i, \kappa_{i+1}) e(\mathbf{m})$;
5. For each $1 \leq i < n$, we have
$$\tau_{i+1}\tau_i\tau_{i+1} e(\mathbf{m}) - \tau_i \tau_{i+1} \tau_i e(\mathbf{m}) = \begin{cases} \frac{Q_{\mathbf{m},i}(\kappa_{i+2},\kappa_{i+1}) - Q_{\mathbf{m},i}(\kappa_i,\kappa_{i+1})}{\kappa_{i+2}-\kappa_i} e(\mathbf{m}) & (m_{i+2} = m_i) \\ 0 & (\text{otherwise}) \end{cases};$$
6. $$\tau_i \kappa_k e(\mathbf{m}) - \kappa_{\sigma_i(k)} \tau_i e(\mathbf{m}) = \begin{cases} -e(\mathbf{m}) & (i = k, m_i = m_{i+1}) \\ e(\mathbf{m}) & (i = k-1, m_i = m_{i+1}) \\ 0 & (\text{otherwise}) \end{cases}.$$

Here we set $h_{\mathbf{m},i} := \#\{h \in \Omega \mid h' = m_i, h'' = m_{i+1}\}$ and

$$Q_{\mathbf{m},i}(u,v) = \begin{cases} 1 & (m_i \neq m_{i+1}, m_i \not\leftrightarrow m_{i+1}) \\ (-1)^{h_{\mathbf{m},i}}(u-v) & (m_i \leftrightarrow m_{i+1}) \\ 0 & (\text{otherwise}) \end{cases},$$

where u, v are indeterminants. □

Remark 2.2. Note that the algebra R_β a priori depends on the orientation Ω through $Q_{\mathbf{m},i}(u,v)$. Since the graded algebras R_β are known to be mutually isomorphic for any two choices of Ω (cf. Theorem 2.3), we suppress this dependence in the below.

For an I-graded vector space V with $\underline{\dim}\, V = \beta$, we define

$$F_\beta^\Omega := \left\{ (\{F_j\}_{j=0}^{\mathsf{ht}\beta}, x) \,\middle|\, \begin{array}{c} x \in E_V^\Omega. \text{ For each } 0 < j \leq \mathsf{ht}\beta, \\ F_j \subset V \text{ is an } I\text{-graded vector subspace,} \\ F_{j+1} \subsetneq F_j, \text{ and satisfies } xF_j \subset F_{j+1}. \end{array} \right\} \quad \text{and}$$

$$\mathcal{B}_\beta^\Omega := \left\{ \{F_j\}_{j=0}^{\mathsf{ht}\beta} \,\middle|\, F_j \subset V \text{ is an } I\text{-graded vector subspace s.t. } F_{j+1} \subsetneq F_j. \right\}.$$

We have a projection

$$\varpi_\beta^\Omega : F_\beta^\Omega \ni (\{F_j\}_{j=0}^{\mathsf{ht}\beta}, x) \mapsto \{F_j\}_{j=0}^{\mathsf{ht}\beta} \in \mathcal{B}_\beta^\Omega,$$

which is G_V-equivariant. For each $\mathbf{m} \in Y^\beta$, we have a connected component

$$F_{\mathbf{m}}^\Omega := \{(\{F_j\}_{j=0}^{\mathsf{ht}\beta}, x) \in F_\beta^\Omega \mid \underline{\dim}\, F_j/F_{j+1} = \alpha_{m_{j+1}} \ \ \forall j\} \subset F_\beta^\Omega,$$

that is smooth of dimension $d_{\mathbf{m}}^\Omega$. We set $\mathcal{B}_{\mathbf{m}}^\Omega := \varpi_\beta^\Omega(F_{\mathbf{m}}^\Omega)$, that is an irreducible component of $\mathcal{B}_\beta^\Omega$. Let

$$\pi_{\mathbf{m}}^\Omega : F_{\mathbf{m}}^\Omega \ni (\{F_j\}_{j=0}^{\mathsf{ht}\beta}, x) \mapsto x \in E_V^\Omega$$

be the second projection that is also G_V-equivariant. The map $\pi_{\mathbf{m}}^\Omega$ is projective, and hence

$$\mathcal{L}_{\mathbf{m}}^\Omega := (\pi_{\mathbf{m}}^\Omega)_! \, \underline{\mathbb{C}}\, [d_{\mathbf{m}}^\Omega]$$

decomposes into a direct sum of (shifted) irreducible perverse sheaves with their coefficients in $D^b(\mathrm{pt})$ (Gabber's decomposition theorem, [BBD82] 6.2.5). We set $\mathcal{L}_\beta^\Omega := \bigoplus_{\mathbf{m} \in Y^\beta} \mathcal{L}_{\mathbf{m}}^\Omega$. Let $e(\mathbf{m})$ be the idempotent in $\mathrm{End}(\mathcal{L}_\beta^\Omega)$ so that $e(\mathbf{m})\mathcal{L}_\beta^\Omega = \mathcal{L}_{\mathbf{m}}^\Omega$. Since $\pi_{\mathbf{m}}^\Omega$ is projective, we conclude that $\mathbb{D}\mathcal{L}_{\mathbf{m}}^\Omega \cong \mathcal{L}_{\mathbf{m}}^\Omega$ for each $\mathbf{m} \in Y^\beta$, and hence

$$\mathbb{D}\mathcal{L}_\beta^\Omega \cong \mathcal{L}_\beta^\Omega. \tag{2.1}$$

Theorem 2.3 (Varagnolo-Vasserot [VV11])**.** *Under the above settings, we have an isomorphism of graded algebras:*

$$R_\beta \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}_{G_V}^i(\mathcal{L}_\beta^\Omega, \mathcal{L}_\beta^\Omega).$$

In particular, the RHS does not depend on the choice of an orientation Ω of Γ_0.

For each $\mathbf{m}, \mathbf{m}' \in Y^\beta$, we set

$$R_{\mathbf{m},\mathbf{m}'} := e(\mathbf{m}) R_\beta e(\mathbf{m}') = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^i_{G_V}(\mathcal{L}^\Omega_{\mathbf{m}'}, \mathcal{L}^\Omega_{\mathbf{m}}).$$

We set $S_\beta \subset R_\beta$ to be a subalgebra which is generated by $e(\mathbf{m})$ ($\mathbf{m} \in Y^\beta$) and $\kappa_1, \ldots, \kappa_n$.

For each $\beta_1, \beta_2 \in Q_+$ with $\mathsf{ht}\,\beta_1 = n_1$ and $\mathsf{ht}\,\beta_2 = n_2$, we have a natural inclusion:

$$\begin{array}{ccccccc} R_{\beta_1} \boxtimes R_{\beta_2} & \ni & e(\mathbf{m}) \boxtimes e(\mathbf{m}') & \mapsto & e(\mathbf{m} + \mathbf{m}') & \in & R_{\beta_1+\beta_2} \\ R_{\beta_1} \boxtimes 1 & \ni & \kappa_i \boxtimes 1, \tau_i \boxtimes 1 & \mapsto & \kappa_i, \tau_i & \in & R_{\beta_1+\beta_2} \\ 1 \boxtimes R_{\beta_2} & \ni & 1 \boxtimes \kappa_i, 1 \boxtimes \tau_i & \mapsto & \kappa_{i+n_1}, \tau_{i+n_1} & \in & R_{\beta_1+\beta_2} \end{array}.$$

This defines an exact functor

$$\star : R_{\beta_1} \boxtimes R_{\beta_2}\text{-}\mathsf{gmod} \ni M_1 \boxtimes M_2 \mapsto R_{\beta_1+\beta_2} \otimes_{R_{\beta_1} \boxtimes R_{\beta_2}} (M_1 \boxtimes M_2) \in R_{\beta_1+\beta_2}\text{-}\mathsf{gmod}.$$

It is straight-forward to see that $\star$ restricts to an exact functor in the category of graded projective modules:

$$\star : R_{\beta_1} \boxtimes R_{\beta_2}\text{-}\mathsf{proj} \ni M_1 \boxtimes M_2 \mapsto R_{\beta_1+\beta_2} \otimes_{R_{\beta_1} \boxtimes R_{\beta_2}} (M_1 \boxtimes M_2) \in R_{\beta_1+\beta_2}\text{-}\mathsf{proj}.$$

If $i \in I$ is a source of Γ and $f = (f_h)_{h \in \Omega} \in E^\Omega_V$, then we define

$$\epsilon^*_i(f) := \dim \ker \bigoplus_{h \in \Omega, h' = i} f_h \le \dim V_i.$$

If $i \in I$ is a sink of Γ and $f = (f_h)_{h \in \Omega} \in E^\Omega_V$, then we define

$$\epsilon_i(f) := \dim \mathrm{coker} \bigoplus_{h \in \Omega, h'' = i} f_h \le \dim V_i.$$

Each of $\epsilon^*_i(f)$ or $\epsilon_i(f)$ do not depend on the choice of a point in a G_V-orbit. Hence, ϵ_i or ϵ^*_i induces a function on E^Ω_V that is constant on each G_V-orbit, and a function on the set of isomorphism classes of simple G_V-equivariant perverse sheaves on E^Ω_V through a unique open dense G_V-orbit of its support whenever i is a source or a sink.

Proposition 2.4 (Lusztig [Lus91] 6.6)**.** *For each $i \in I$, the functions ϵ_i and ϵ^*_i descend to functions on the set of isomorphism classes of simple graded R_β-modules (up to degree shift).*

Proof. Note that [Lus91] 6.6 considers only ϵ_i, but ϵ^*_i is obtained by swapping the order of the convolution operation. □

Theorem 2.5 (Khovanov-Lauda [KL09], Rouquier [Rou08], Varagnolo-Vasserot [VV11])**.** *In the above setting, we have:*

1. *For each $i \in I$ and $n \ge 0$, $R_{n\alpha_i}$ has a unique indecomposable projective module P_{ni} up to grading shifts;*

2. *The functor $\star$ induces a $\mathbb{Z}[t^{\pm 1}]$-algebra structure on*

$$\mathbf{K} := \bigoplus_{\beta \in Q^+} K(R_\beta\text{-}\mathsf{proj});$$

3. *The algebra $\mathbf{K}$ is isomorphic to the integral form U^+ of the positive part of the quantized enveloping algebra of type Γ_0 by identifying $[P_{ni}]$ with the n-th divided power of a Chevalley generator of U^+;*

4. *The above isomorphism identifies the classes of indecomposable graded projective R_β-modules ($\beta \in Q^+$) with an element of the lower global basis of U^+ in the sense of [Kas91];*

5. *There exists a set $B(\infty) = \bigsqcup_{\beta \in Q^+} B(\infty)_\beta$ that parameterizes indecomposable projective modules of $\bigoplus_{\beta \in Q^+} R_\beta$-$\mathsf{gmod}$. This identifies the functions ϵ_i, ϵ_i^* ($i \in I$) with the corresponding functions on $B(\infty)$.*

Proof. The first assertion is [KL09] 2.2 3), the second and the third assertions are [KL09] 3.4, and the fourth assertion is [VV11] 4.4. Based on this, the fifth follows from Proposition 2.4. See also Theorem 2.12 in the below. □

Remark 2.6. The coincidence of the lower global basis and the canonical basis is proved by Lusztig [Lu90b] and Grojnowski-Lusztig [GL93]. We freely utilize this identification in the below.

Proposition 2.7. *In the above setting, the conditions ($\spadesuit$) and ($\clubsuit$) are satisfied.*

Proof. The condition $(\spadesuit)_1$ is the Gabriel theorem (on the classification of finite algebras, applied to $\mathbb{C}[\Gamma]$). The condition $(\spadesuit)_2$ follows by the fact that $\mathsf{Stab}_G(x_\lambda)$ is the automorphism group of a $\mathbb{C}[\Gamma]$-module $\mathtt{M}$, which must be an open dense part of a linear subspace.

We set $Z_\beta^\Omega := F_\beta^\Omega \times_{E_V^\Omega} F_\beta^\Omega$. By [VV11] 1.8 (b) and 2.23 (or [CG97] 8.6.7), we have an isomorphism $H_\bullet^{G_V}(Z_\beta^\Omega) \cong \mathrm{Ext}^\bullet_{G_V}(\mathcal{L}_\beta^\Omega, \mathcal{L}_\beta^\Omega)$ as graded algebras (here we warn that the grading on the LHS is imported from the RHS, and is *not* the standard one; cf. [VV11] 1.9). Since G_V is a reductive group, we know that each G_V-orbit of $(\mathcal{B}_\beta^\Omega)^2$ is an affine bundle over a connected component of $\mathcal{B}_\beta^\Omega$ (see e.g. [CG97] §3.4). By [VV11] 2.11, each fiber of the G_V-equivariant map $Z_\beta^\Omega \to (\mathcal{B}_\beta^\Omega)^2$ induced from ϖ_β^Ω is a vector space. Therefore, we conclude that Z_β^Ω is a union of finite increasing sequence of closed subvarieties

$$\emptyset = Z_{\beta,0}^\Omega \subsetneq Z_{\beta,1}^\Omega \subsetneq Z_{\beta,2}^\Omega \subsetneq \cdots \subsetneq Z_{\beta,\ell}^\Omega = Z_\beta^\Omega,$$

where each $Z_{\beta,j}^\Omega \backslash Z_{\beta,j-1}^\Omega$ is an affine bundle over a connected component of $\mathcal{B}_\beta^\Omega$. This implies the purity of $H_\bullet^{G_V}(Z_\beta^\Omega)$, and hence $(\clubsuit)_1$ follows.

The condition $(\clubsuit)_2$ is Lusztig [Lu90a] 10.6. □

Corollary 2.8 (Lusztig [Lu90a])**.** *Every simple G_V-equivariant perverse sheaf on E_V^Ω appears as a non-zero direct summand of $\mathcal{L}_\beta^\Omega$ up to a degree shift.*

Proof. By Proposition 2.7 and Theorem 2.3, we deduce that the assertion is equivalent to $\#\mathsf{Irr}_0 R_\beta = \#(G_V \backslash E_V^\Omega)$. This follows from a standard bijection between the set of isomorphism classes of indecomposable $\mathbb{C}[\Gamma]$-modules and a basis of U^+ à la Ringel [Rin90] (or a consequence of the Gabriel theorem). □

Theorem 2.9 (Kashiwara's problem)**.** *The algebra R_β has finite global dimension.*

Proof. Apply Theorem 1.2 to (2.1), Proposition 2.7, and Corollary 2.8. $\square$

Thanks to Corollary 2.8 and Theorem 2.5 5), we have an identification $B(\infty)_\beta \cong G_V \backslash E_V^\Omega$, where V is an I-graded vector space with $\underline{\dim} V = \beta$. By regarding $G_V \backslash E_V^\Omega$ as the space of $\mathbb{C}[\Gamma]$-modules with its dimension vector β, each $b \in B(\infty)_\beta$ gives rise to (an isomorphism class of) a $\mathbb{C}[\Gamma]$-module $\mathtt{M}_b$. Let us denote by $\mathbb{O}_b^\Omega$ the G_V-orbit of E_V^Ω corresponding to $b \in B(\infty)_\beta$. Each $b \in B(\infty)_\beta$ defines an indecomposable graded projective module P_b of R_β with simple head L_b that is isomorphic to its graded dual L_b^* (see §1).

The standard module K_b and the dual standard module $\widetilde{K}_b$ in §1 depends on the choice of Ω since the Fourier transform interchanges the closure relations. Therefore, we denote by K_b^Ω (resp. $\widetilde{K}_b^\Omega$) the standard module (resp. the dual standard module) of L_b arising from E_V^Ω.

Example 2.10. If $\beta = m\alpha_i$ for $m \geq 1$ and $i \in I$, then the set $B(\infty)_{m\alpha_i}$ is a singleton. Let L_{mi} and P_{mi} be unique simple and projective graded modules of $R_{m\alpha_i}$ up to grading shifts, respectively. The standard module K_{mi} and the dual standard module $\widetilde{K}_{mi}$ do not depend on the choice of Ω in this case. We have $L_{mi} \cong K_{mi}$ and $P_{mi} \cong \widetilde{K}_{mi}$, and

$$[\widetilde{K}_{mi} : K_{mi}] = \mathsf{gdim}\, \mathbb{C}[x_1, \cdots, x_m]^{\mathfrak{S}_m}.$$

Let $\mathcal{Q}_\beta^\Omega$ be the fullsubcategory of $D^b_{G_V}(E_V^\Omega)$ consisting all complexes whose direct summands are degree shifts of that of $\mathcal{L}_\beta^\Omega$.

Let $\beta \in Q^+$ with $\mathsf{ht}\, \beta = n$. Let $\leq_B$ be the Bruhat order of $\mathfrak{S}_n$ with respect to the set of simple reflections $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$. For each $w \in \mathfrak{S}_n$ and its reduced expression

$$w = \sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_L},$$

we set $\tau_w := \tau_{j_1} \tau_{j_2} \cdots \tau_{j_L}$. Note that τ_w *depends* on the choice of a reduced expression.

Theorem 2.11 (Poincaré-Birkoff-Witt theorem [KL09] 2.7)**.** *We have equalities as vector spaces:*

$$R_\beta = \bigoplus_{w \in \mathfrak{S}_n, \mathbf{m} \in Y^\beta} \tau_w S_\beta e(\mathbf{m}) = \bigoplus_{w \in \mathfrak{S}_n, \mathbf{m} \in Y^\beta} S_\beta \tau_w e(\mathbf{m}),$$

regardless the choices of τ_w. $\square$

Let $\beta \in Q^+$ so that $\mathsf{ht}\, \beta = n$. For each $i \in I$ and $k \geq 0$, we set

$$Y_{k,i}^\beta := \{\mathbf{m} = (m_j) \in Y^\beta \mid m_1 = \cdots = m_k = i\} \text{ and}$$
$$Y_{k,i}^{\beta;*} := \{\mathbf{m} = (m_j) \in Y^\beta \mid m_n = \cdots = m_{n-k+1} = i\}.$$

In addition, we define two idempotents of R_β as:

$$e_i(k) := \sum_{\mathbf{m} \in Y_{k,i}^\beta} e(\mathbf{m}), \quad \text{and} \quad e_i^*(k) := \sum_{\mathbf{m} \in Y_{k,i}^{\beta;*}} e(\mathbf{m}).$$

Theorem 2.12 (Lusztig [Lus91] §6, Lauda-Vazirani [LV11] 2.5.1)**.** *Let $\beta \in Q_+$ and $i \in I$. For each $b \in B(\infty)_\beta$ and $i \in I$, we have*

$$\epsilon_i(b) = \max\{k \,|\, e_i(k)L_b \neq \{0\}\} \text{ and}$$
$$\epsilon_i^*(b) = \max\{k \,|\, e_i^*(k)L_b \neq \{0\}\}.$$

Moreover, $e_i(\epsilon_i(b))L_b$ and $e_i^*(\epsilon_i^*(b))L_b$ are irreducible $R_{\epsilon_i(b)\alpha_i} \boxtimes R_{\beta-\epsilon_i(b)\alpha_i}$-module and $R_{\beta-\epsilon_i^*(b)\alpha 1} \boxtimes R_{\epsilon_i^*(b)\alpha_i}$-module, respectively. In addition, if we have distinct $b, b' \in B(\infty)_\beta$ so that $\epsilon_i(b) = k = \epsilon_i(b')$ with $k \geq 0$, then $e_i(k)L_b$ and $e_i(k)L_{b'}$ are not isomorphic as an $R_{k\alpha_i} \boxtimes R_{\beta-k\alpha_i}$-module. □

3 Saito reflection functors

Keep the setting of the previous section. Let Ω_i be the set of edges $h \in \Omega$ with $h'' = i$ or $h' = i$. Let $s_i\Omega_i$ be a collection of edges obtained from $h \in \Omega_i$ by setting $(s_ih)' = h''$ and $(s_ih)'' = h'$. We define $s_i\Omega := (\Omega\backslash\Omega_i) \cup s_i\Omega_i$ and set $s_i\Gamma := (I, s_i\Omega)$. Note that $\Gamma_0 = (s_i\Gamma)_0$.

Let $w_0 \in W$ be the longest element. Choose a reduced expression

$$w_0 = s_{i_1}s_{i_2}\cdots s_{i_\ell}.$$

We denote by $\mathbf{i} := (i_1, \ldots, i_\ell) \in I^\ell$ the data recording this reduced expression. We say $\mathbf{i}$ is adapted to Ω (or Γ) if each i_k is a sink of $s_{i_{k-1}}\cdots s_{i_1}\Gamma$.

Let V be an I-graded vector space with $\underline{\dim}\, V = \beta$. For a sink i of Γ, we define

$${}_iE_V^\Omega := \big\{(f_h)_{h\in\Omega} \in E_V^\Omega \mid \operatorname{coker}(\bigoplus_{h\in\Omega, h''=i} f_h : \bigoplus_{h'} V_{h'} \to V_i) = \{0\}\big\}.$$

For a source i of Γ, we define

$${}^iE_V^\Omega := \big\{(f_h)_{h\in\Omega} \in E_V^\Omega \mid \ker(\bigoplus_{h\in\Omega, h'=i} f_h : V_i \to \bigoplus_{h''} V_{h''}) = \{0\}\big\}.$$

Let Ω be an orientation of Γ so that $i \in I$ is a sink. Let $\beta \in Q^+ \cap s_iQ^+$. Let V and V' be I-graded vector spaces with $\underline{\dim}\, V = \beta$ and $\underline{\dim}\, V' = s_i\beta$, respectively. We fix an isomorphism $\phi : \oplus_{j\neq i}V_j \xrightarrow{\cong} \oplus_{j\neq i}V'_j$ as I-graded vector spaces. We define:

$${}_iZ_{V,V'}^\Omega := \left\{ \{(f_h)_{h\in\Omega}, (f'_h)_{h\in s_i\Omega}, \psi\} \middle| \begin{array}{c} (f_h) \in {}_iE_V^\Omega, (f'_h) \in {}^iE_{V'}^{s_i\Omega}, \\ \phi f_h = f'_h\phi \text{ for } h \notin \Omega_i \\ \psi : V'_i \xrightarrow{\cong} \ker(\bigoplus_{h\in\Omega_i} f_h : \bigoplus_h V_{h'} \to V_i) \end{array} \right\}.$$

We have a diagram:

$$E_V^\Omega \xleftarrow{j_V} {}_iE_V^\Omega \xleftarrow{q_V^i} {}_iZ_{V,V'}^\Omega \xrightarrow{p_{V'}^i} {}^iE_{V'}^{s_i\Omega} \xrightarrow{\jmath_{V'}} E_{V'}^{s_i\Omega} \quad . \tag{3.1}$$

If we set

$$G_{V,V'} := GL(V_i) \times GL(V'_i) \times \prod_{j\neq i} GL(V_j) \cong GL(V_i) \times GL(V'_i) \times \prod_{j\neq i} GL(V'_j),$$

then the maps $p_{V'}^i$ and q_V^i are $G_{V,V'}$-equivariant.

Proposition 3.1 (Lusztig [Lus98])**.** *The morphisms p_V^i and q_V^i in* (3.1) *are* $\mathrm{Aut}(V_i)$*-torsor and* $\mathrm{Aut}(V_i')$*-torsor, respectively.* □

When $\beta = \underline{\dim} V$, we set

$${}_iR_\beta^\Omega := \mathrm{Ext}^\bullet_{G_V}(j_V^* \mathcal{L}_V^\Omega, j_V^* \mathcal{L}_V^\Omega) \quad \text{and} \quad {}^iR_{s_i\beta}^{s_i\Omega} := \mathrm{Ext}^\bullet_{G_{V'}}(\jmath_{V'}^* \mathcal{L}_{V'}^{s_i\Omega}, \jmath_{V'}^* \mathcal{L}_{V'}^{s_i\Omega})$$

for the time being (see Corollary 3.4).

Lemma 3.2. *We have an algebra isomorphism ${}_iR_\beta^\Omega \cong R_\beta/(R_\beta e_i(1) R_\beta)$. Similarly, the algebra ${}^iR_{s_i\beta}^{s_i\Omega}$ is isomorphic to $R_{s_i\beta}/(R_{s_i\beta} e_i^*(1) R_{s_i\beta})$.*

Proof. The maps j_V and $\jmath_{V'}$ are G_V- and $G_{V'}$-equivariant open embeddings, respectively. Therefore, we apply Lemma 1.4 and Proposition 1.5 repeatedly to deduce ${}_iR_\beta^\Omega \cong R_\beta/(R_\beta e R_\beta)$, where $e \in R_\beta$ is a degree zero idempotent so that $eL_b = L_b$ ($\mathbb{O}_b^\Omega \not\subset \mathrm{Im}\, j_V$) or $\{0\}$ ($\mathbb{O}_b^\Omega \subset \mathrm{Im}\, j_V$). By Proposition 2.4, Theorem 2.5 5), and Theorem 2.12, we conclude $R_\beta e R_\beta = R_\beta e_i(1) R_\beta$, which proves the first assertion. The case of ${}^iR_{s_i\beta}^{s_i\Omega}$ is similar, and we omit the detail. □

Corollary 3.3. *The set of isomorphism classes of graded simple modules of ${}_iR_\beta^\Omega$ and ${}^iR_{s_i\beta}^{s_i\Omega}$ are $\{L_b \langle j \rangle\}_{\epsilon_i(b)=0, j\in\mathbb{Z}}$ and $\{L_b \langle j \rangle\}_{\epsilon_i^*(b)=0, j\in\mathbb{Z}}$, respectively.* □

Corollary 3.4. *The algebras ${}_iR_\beta^\Omega$ and ${}^iR_{s_i\beta}^{s_i\Omega}$ do not depend on the choice of Ω.* □

Proposition 3.5. *In the setting of Proposition 3.1, two graded algebras ${}_iR_\beta^\Omega$ and ${}^iR_{s_i\beta}^{s_i\Omega}$ are Morita equivalent to each other. In addition, this Morita equivalence is independent of the choice of Ω (as long as i is a sink).*

Proof. First, note that the maps $j_V, \jmath_{V'}$ are open embeddings. In particular, $j_V^* \mathcal{L}_V^\Omega$ and $\jmath_{V'}^* \mathcal{L}_{V'}^{s_i\Omega}$ are again direct sums of shifted equivariant perverse sheaves. By Proposition 3.1 and [BL94] 2.2.5, we have equivalences

$$D^b_{G_V}({}_iE_V^\Omega) \xrightarrow{(q_V^i)^*} D^b_{G_{V,V'}}({}_iZ_{V,V'}^\Omega) \xleftarrow{(p_{V'}^i)^*} D^b_{G_{V'}}({}^iE_{V'}^{s_i\Omega}).$$

In addition, a simple $G_{V,V'}$-equivariant perverse sheaf $\mathcal{L}$ on ${}_iZ_{V,V'}^\Omega$ admits isomorphisms

$$(q_V^i)^* \left({}_i\mathcal{L}\,[\dim GL(V_i')]\right) \cong \mathcal{L} \cong (p_{V'}^i)^* \left({}^i\mathcal{L}\,[\dim GL(V_i)]\right), \tag{3.2}$$

where ${}_i\mathcal{L}$ and ${}^i\mathcal{L}$ are simple G_V- and $G_{V'}$-equivariant perverse sheaves on ${}_iE_V^\Omega$ and ${}^iE_{V'}^{s_i\Omega}$, respectively. These induce isomorphisms of algebras:

$$B_{(G_V, {}_iE_V^\Omega)} \cong B_{(G_{V,V'}, {}_iZ_{V,V'}^\Omega)} \cong B_{(G_{V'}, {}^iE_{V'}^{s_i\Omega})}. \tag{3.3}$$

Therefore, $B_{(G_V, {}_iE_V^\Omega)}$ and $B_{(G_{V'}, {}^iE_{V'}^{s_i\Omega})}$ are Morita equivalent to the algebras in the assertion by Corollary 2.8, which implies the first assertion.

We prove the second assertion. For any two orientations Ω and Ω' which have i as a common sink, we have Fourier transforms $\mathcal{F}^\Omega$ and $\mathcal{F}^{s_i\Omega}$ so that $\mathcal{F}^\Omega(\mathcal{L}_\beta^\Omega) = \mathcal{L}_\beta^{\Omega'}$ and $\mathcal{F}^{s_i\Omega}(\mathcal{L}_{s_i\beta}^{s_i\Omega}) = \mathcal{L}_{s_i\beta}^{s_i\Omega'}$. Since $\Omega_i = \Omega_i'$, these two Fourier transforms are induced by the pairing between direct summands $E \subset E_V^\Omega$ and $E^* \subset E_V^{\Omega'}$ which can be identified with those of $E_{V'}^{s_i\Omega}$ and $E_{V'}^{s_i\Omega'}$ in (3.1) via ϕ.

Since the diagram (3.1) is the product of a vector space and the contribution from Ω_i, we conclude that two pairs of sheaves $(\mathcal{L}_\beta^\Omega, \mathcal{L}_{s_i\beta}^{s_i\Omega})$ and $(\mathcal{L}_\beta^{\Omega'}, \mathcal{L}_{s_i\beta}^{s_i\Omega'})$ are exchanged by $\mathcal{F}^\Omega$ and $\mathcal{F}^{s_i\Omega}$ commuting with the diagram (3.1). This identifies the Morita equivalences obtained by Ω and Ω' as required. □

The maps q_V^i and $p_{V'}^i$ give rise to a correspondence between orbits. For each $b \in B(\infty)_{s_i\beta}$, we denote by $T_i(b) \in B(\infty)_\beta \sqcup \{\emptyset\}$ the element so that $(p_{V'}^i)^{-1}(\mathbb{O}_b^{s_i\Omega}) \cong (q_V^i)^{-1}(\mathbb{O}_{T_i(b)}^\Omega)$ (we understand that $T_i(b) = \emptyset$ if $\mathbb{O}_b^{s_i\Omega} \not\subset \operatorname{Im} p_{V'}^i$). Note that $T_i(b) = \emptyset$ if and only if $\epsilon_i^*(b) > 0$. In addition, we have $\epsilon_i(T_i(b)) = 0$ if $T_i(b) \neq \emptyset$. We set $T_i^{-1}(b') := b$ if $b' = T_i(b) \neq \emptyset$.

Thanks to Corollary 3.4, we can drop Ω or $s_i\Omega$ from ${}_iR_\beta^\Omega$ and ${}^iR_\beta^{s_i\Omega}$. We define a left exact functor

$$\mathbb{T}_i^* : R_\beta\text{-}\mathsf{gmod} \longrightarrow\!\!\!\!\rightarrow {}_iR_\beta\text{-}\mathsf{gmod} \xrightarrow{\cong} {}^iR_{s_i\beta}\text{-}\mathsf{gmod} \hookrightarrow R_{s_i\beta}\text{-}\mathsf{gmod},$$

where the first functor is $\operatorname{Hom}_{R_\beta}({}_iR_\beta, \bullet)$, the second functor is Proposition 3.5, and the third functor is the pullback. Similarly, we define a right exact functor

$$\mathbb{T}_i : R_\beta\text{-}\mathsf{gmod} \longrightarrow\!\!\!\!\rightarrow {}^iR_\beta\text{-}\mathsf{gmod} \xrightarrow{\cong} {}_iR_{s_i\beta}\text{-}\mathsf{gmod} \hookrightarrow R_{s_i\beta}\text{-}\mathsf{gmod},$$

where the first functor is ${}^iR_\beta \otimes_{R_\beta} \bullet$. We call these functors the Saito reflection functors (cf. [Sai94]). By the latter part of Proposition 3.5, we see that these functors are independent of the choices involved.

Let $i \in I$. We define $R_\beta\text{-}\mathsf{gmod}_i$ (resp. $R_\beta\text{-}\mathsf{gmod}^i$) to be the fullsubcategory of $R_\beta\text{-}\mathsf{gmod}$ so that each simple subquotient is of the form $L_b\langle k\rangle$ ($k \in \mathbb{Z}$) with $b \in B(\infty)_\beta$ that satisfies $\epsilon_i(b) = 0$ (resp. $\epsilon_i^*(b) = 0$). In addition, for each $i \neq j \in I$, we define $R_\beta\text{-}\mathsf{gmod}_j^i := R_\beta\text{-}\mathsf{gmod}^i \cap R_\beta\text{-}\mathsf{gmod}_j$.

Theorem 3.6 (Saito reflection functors)**.** *Let $i \in I$. We have:*

1. *Assume that i is a source of Ω. For each $b \in B(\infty)_\beta$, we have*

$$\mathbb{T}_i K_b^\Omega = \begin{cases} K_{T_i(b)}^{s_i\Omega} & (\epsilon_i^*(b) = 0) \\ \{0\} & (\epsilon_i^*(b) > 0) \end{cases};$$

2. *For each $b \in B(\infty)_\beta$, we have*

$$\mathbb{T}_i L_b = \begin{cases} L_{T_i(b)} & (\epsilon_i^*(b) = 0) \\ \{0\} & (\epsilon_i^*(b) > 0) \end{cases}, \text{ and } \quad \mathbb{T}_i^* L_b = \begin{cases} L_{T_i^{-1}(b)} & (\epsilon_i(b) = 0) \\ \{0\} & (\epsilon_i(b) > 0) \end{cases};$$

3. *The functors $(\mathbb{T}_i, \mathbb{T}_i^*)$ form an adjoint pair;*

4. *For each $M \in R_\beta\text{-}\mathsf{gmod}^i$ and $N \in R_{s_i\beta}\text{-}\mathsf{gmod}_i$, we have*

$$\operatorname{ext}^*_{R_{s_i\beta}}(\mathbb{T}_i M, N) \cong \operatorname{ext}^*_{R_\beta}(M, \mathbb{T}_i^* N);$$

5. *Let $i \neq j \in I$. For each $\beta \in Q^+$ and $m \geq 0$, we have*

$$\mathbb{T}_i(P_{mj} \star M) \cong (\mathbb{T}_i P_{mj}) \star \mathbb{T}_i M$$

as graded $R_{s_i(\beta + m\alpha_j)}$-modules for every $M \in R_\beta\text{-}\mathsf{gmod}_j$.

Remark 3.7. The proof of Theorem 3.6 is given by two parts, namely 1)–4) and 5). We warn that the proof of the latter part rests on the earlier part.

Proof of Theorem 3.6 1)–4). We prove the first assertion. The subset ${}^iE_V^\Omega \subset E_V^\Omega$ (with $\underline{\dim} V = \beta$) is open. Therefore, Theorem 1.3 1) asserts that $[K_b^\Omega : L_{b'}] \neq 0$ only if $\epsilon_i^*(b') = 0$ whenever $\epsilon_i^*(b) = 0$. It follows that ${}^iR_\beta \otimes_{R_\beta} K_b^\Omega \cong K_b^\Omega$ as a vector space if $\epsilon_i^*(b) = 0$, and $\{0\}$ otherwise. This gives rise to a standard module of ${}^iR_\beta$ by Lemma 1.4, and thus it gives a standard module of ${}_iR_{s_i\beta}$ by Proposition 3.5. Note that the subset ${}_iE_{V'}^{s_i\Omega} \subset E_{V'}^{s_i\Omega}$ (with $\underline{\dim} V' = s_i\beta$) is also open. Therefore, we use Lemma 3.2 to deduce the first assertion.

The second assertion is immediate from the first assertion and the construction of $\mathbb{T}_i$ and $\mathbb{T}_i^*$.

We prove the third assertion. By Lemma 3.2, we know that $\mathbb{T}_i$ factors through the functor giving the maximal quotient which is a ${}^iR_\beta$-module, while $\mathbb{T}_i^*$ factors through the functor giving the maximal submodule which is an ${}_iR_\beta$-module. Therefore, the third assertion follows by the Morita equivalence ${}^iR_\beta\text{-}\mathsf{gmod} \cong {}_iR_{s_i\beta}\text{-}\mathsf{gmod}$ for every $\beta \in Q^+ \cap s_iQ^+$.

For the fourth assertion, notice that R_β- and $R_{s_i\beta}$-action on M and N factors through ${}^iR_\beta$ and ${}_iR_{s_i\beta}$, respectively. It follows that ${}^iR_\beta \otimes_{R_\beta} M \cong M$, ${}_iR_{s_i\beta} \otimes_{R_{s_i\beta}} \mathbb{T}_iM \cong \mathbb{T}_iM$, and $\mathrm{Hom}_{R_{s_i\beta}}({}_iR_{s_i\beta}, N) \cong N$. By Lemma 3.2 and Corollary 1.6, we deduce that each indecomposable projective ${}_iR_{s_i\beta}$-module ${}_iP$ admits an $R_{s_i\beta}$-graded projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to {}_iP \to 0$$

so that P_0 is indecomposable and ${}_iR_{s_i\beta} \otimes_{R_{s_i\beta}} P_k = \{0\}$ for $k \geq 1$. Therefore, we have

$$\mathrm{ext}^*_{R_{s_i\beta}}(M, N) \cong \mathrm{ext}^*_{{}_iR_{s_i\beta}}(M, N),$$

where we regard M, N as ${}_iR_{s_i\beta}$-modules via Proposition 3.5 (here we treat the Morita equivalence as an isomorphism for simplicity). Applying the same argument for ${}^iR_\beta$ (again for M), we conclude the result. □

Lemma 3.8. *Let $i \in I$. For each $\beta \in Q^+$, $m \geq 0$, and an indecomposable graded projective ${}_iR_\beta$-module P, the module $P_{mi} \star P$ is an $R_{\beta+m\alpha_i}$-module with simple head.*

Proof. By the Frobenius reciprocity, we have

$$\mathrm{hom}_{R_{\beta+m\alpha_i}}(P_{mi} \star P, L_b) \cong \mathrm{hom}_{R_{m\alpha_i} \boxtimes R_\beta}(P_{mi} \boxtimes P, L_b) \tag{3.4}$$

for every $b \in B(\infty)_{\beta+m\alpha_i}$. Assume that the above space is non-zero to deduce the uniqueness of b and the one-dimensionality of (3.4). Choose $d \in B(\infty)_\beta$ so that L_d is the unique simple quotient of P. We have $\epsilon_i(d) = 0$ by assumption. By Theorem 2.11, we have $e(\mathbf{m})(P_{mi} \star P) \neq \{0\}$ only if there exist a minimal length representative $w \in \mathfrak{S}_{(\mathsf{ht}\,\beta+m)}/(\mathfrak{S}_m \times \mathfrak{S}_{\mathsf{ht}\,\beta})$ and $\mathbf{m}' \in Y^\beta$ so that $e(\mathbf{m}')P \neq \{0\}$ and $\mathbf{m} = w(mi + \mathbf{m}')$. Since $\mathbf{m}' \notin Y^\beta_{1,i}$, we deduce $\epsilon_i(b) \leq m$. Thus, if (3.4) is non-trivial, then we have $\epsilon_i(b) = m$ and $w = 1$. Now Theorem 2.12 forces $e_i(m)L_b \cong L_{mi} \boxtimes L_d$. Therefore, $P_{mi} \star P$ has at most one quotient, which completes the proof. □

Theorem 3.9 (Induction theorem). *Let $V(i)$ be I-graded vector spaces with $\underline{\dim}\, V(i) = \beta_i$, and $b_i \in B(\infty)_{\beta_i}$ for $i = 1, 2$. Let $b \in B(\infty)_{\beta_1+\beta_2}$ so that $\mathtt{M}_b \cong \mathtt{M}_{b_1} \oplus \mathtt{M}_{b_2}$ as $\mathbb{C}[\Gamma]$-modules. Assume the following condition $(\star)$:*

$(\star)_0$ *$M_{b_1'}$ is not a quotient of M_b for every $b_1 \neq b_1' \in B(\infty)_{\beta_1}$, and $M_{b_2'}$ is not a submodule of M_b for every $b_2 \neq b_2' \in B(\infty)_{\beta_2}$;*

$(\star)_1$ $\mathrm{Ext}^1_{\mathbb{C}[\Gamma]}(\mathtt{M}_{b_1}, \mathtt{M}_{b_2}) = \{0\}$.

We have an isomorphism $K^{\Omega}_{b_1} \star K^{\Omega}_{b_2} \cong K^{\Omega}_{b}$ as an ungraded $R_{\beta_1+\beta_2}$-module.

In addition, if $\mathtt{M}_b$ canonically determines the factor $\mathtt{M}_{b_2}$ as a vector subspace, then $(\star)_0$ and $(\star)_1$ implies

$$K^{\Omega}_{b_1} \star K^{\Omega}_{b_2} \cong K^{\Omega}_{b}$$

as a graded $R_{\beta_1+\beta_2}$-module.

Before proving Theorem 3.9, we present some of its consequences. The proof of Theorem 3.9 itself is given at the end of this section.

Corollary 3.10. *Suppose that i is a sink of Ω. Let $m \geq 0$. For each $\beta \in Q^+$ and $b \in B(\infty)_\beta$ with $\epsilon_i(b) = 0$, the module $K_{mi} \star K^{\Omega}_b$ is an indecomposable graded $R_{\beta+m\alpha_i}$-module isomorphic to $K^{\Omega}_{b'}$ with $\mathtt{M}_{b'} \cong \mathtt{M}_i^{\oplus m} \oplus \mathtt{M}_b$.*

Proof. By Example 2.10, we deduce that the first part of $(\star)_0$ is a void condition. Every irreducible subquotient of a $\mathbb{C}[\Gamma]$-module isomorphic to $\mathtt{M}_i$ is in its socle. Hence, the second part of $(\star)_0$ follows by the comparison of the socles. Since i is a sink, we have no extension of $\mathtt{M}_i^{\oplus m}$ by $\mathtt{M}_b$, which is $(\star)_1$. We write $\beta = k\alpha_i + \sum_{j \neq i} k_j \alpha_j$. Since $\epsilon_i(b) = 0$, $\mathtt{M}_i$ is not a direct summand of $\mathtt{M}_b$. In particular, M_b is canonically determined by $M_{b'}$ as its direct factor. Applying Theorem 3.9 yields the result. □

Corollary 3.11. *Assume that i is a source and j is a sink of Ω. Let $\beta \in Q^+$. For each $m \geq 0$ and $b \in B(\infty)_\beta$ such that $\epsilon_j(b) = 0$, we have*

$$\mathbb{T}_i(K_{mj} \star K^{\Omega}_b) \cong (\mathbb{T}_i K_{mj}) \star \mathbb{T}_i K^{\Omega}_b$$

as graded $R_{s_i(\beta+m\alpha_j)}$-modules.

Proof. By Corollary 3.10, we see that $K_{mj} \star K^{\Omega}_b \cong K^{\Omega}_{b'}$, where $\mathtt{M}_{b'} \cong \mathtt{M}_j^{\oplus m} \oplus \mathtt{M}_b$. By [Lu90a] 4.4 (c), we deduce that $T_i(b') \neq \emptyset$ if and only if $T_i(b) \neq \emptyset$. Since a standard module is generated by its simple head, we deduce that $\mathbb{T}_i(K_{mj} \star K^{\Omega}_b) = \{0\}$ if $\epsilon_i^*(b) > 0$, and it is isomorphic to $K^{s_i\Omega}_{T_i(b')}$ if $\epsilon_i^*(b) = 0$.

Since $i \neq j$, we always have $\mathbb{T}_i K_{mj} \neq \{0\}$. Therefore, we conclude that the RHS is non-zero if and only if the LHS is non-zero. Thus, it suffices to show that the RHS is isomorphic to $K^{s_i\Omega}_{T_i(b')}$.

If we have $i \not\leftrightarrow j$, then j is a sink of $s_i\Gamma$. By $\epsilon_j(b) = 0$ and the assumption, we deduce that $\mathtt{M}_{T_i(b)}$ also do not contain $\mathtt{M}_j$ in this case. Hence, we deduce $\epsilon_j(T_i(b)) = 0$. In addition, we have $\mathbb{T}_i K^{\Omega}_{mj} \cong K^{s_i\Omega}_{mj}$. Therefore, we apply Corollary 3.10 to deduce that the RHS is $K^{s_i\Omega}_{T_i(b')}$.

Assume that we have $i \leftrightarrow j$. Let $\mathtt{M}_{i,j}$ be a unique indecomposable $\mathbb{C}[s_i\Gamma]$-module with $\underline{\dim}\, \mathtt{M}_{i,j} = \alpha_i + \alpha_j$ (up to an isomorphism). By $\epsilon_j(b) = 0$ and *loc. cit.* 4.4 (c), we conclude that $\mathtt{M}_{T_i(b)}$ does not contain $\mathtt{M}_i, \mathtt{M}_{i,j}$ as its direct factor. By assumption, i is a sink of $s_i\Gamma$ and j is a source of an edge from j to i, but is a source of no other edges. This particularly implies that $\mathtt{M}_i$ is the socle of $\mathtt{M}_{i,j}$. Therefore, we conclude the first half of $(\star)_0$ in Theorem 3.9. If an indecomposable $\mathbb{C}[s_i\Gamma]$-module contains $\mathtt{M}_i$ or $\mathtt{M}_{i,j}$ as its subquotient, then it must be a submodule. If an indecomposable $\mathbb{C}[s_i\Gamma]$-module has a non-zero homomorphism to $\mathtt{M}_i$ or $\mathtt{M}_{i,j}$, then it must be isomorphic to either $\mathtt{M}_i$ or $\mathtt{M}_{i,j}$. These imply the latter half of $(\star)_0$ in Theorem 3.9. In addition, we have

$$\mathrm{Ext}^1_{\mathbb{C}[s_i\Gamma]}(\mathtt{M}_{i,j}^{\oplus m}, \mathtt{M}_{T_i(b)}) = \{0\}.$$

Therefore, we conclude $(\star)_1$ in Theorem 3.9. Let $h_* \in s_i\Omega$ be the unique edge so that $h'_* = j, h''_* = i$. For a representation $(f_h)_{h\in s_i\Omega}$ on $V = \bigoplus_{i\in I} V_i$ isomorphic to $\mathtt{M}_{i,j}^{\oplus m} \oplus \mathtt{M}_{T_i(b)}$, we set

$$V'_k := \begin{cases} V_k & (k \neq i,j) \\ \mathrm{Im}\big(\bigoplus_{h\in s_i\Omega, h''=i} f_h \oplus \bigoplus_{h\in s_i\Omega, h''=j} f_{h_*} f_h\big) & (k = i) \\ \mathrm{Im} \bigoplus_{h\in s_i\Omega, h''=j} f_h + f_{h_*}^{-1}(V'_i) + \ker f_{h_*} & (k=j) \end{cases}.$$

Then, the space $V' \subset V$ defines a canonical $\mathbb{C}[s_i\Gamma]$-submodule $\mathtt{M}'$ on V' so that $\mathtt{M}' \cong \mathtt{M}_{T_i(b)}$. Therefore, we conclude that $(\mathbb{T}_i K_{mj}) \star \mathbb{T}_i K_b^{\Omega} \cong K_{T_i(b')}^{s_i\Omega}$ as required. □

Lemma 3.12. *Let $i,j \in I$ be distinct vertices, $m \geq 0$, and $\beta \in Q^+$. For each $b \in B(\infty)$ so that $\epsilon_i(b) > 0$, the module $\mathbb{T}_i K_{mj} \star L_b$ has simple head that is isomorphic to $L_{b'}$ with $\epsilon_i(b') > 0$ up to grading shifts.*

Proof. We first consider the case $i \not\leftrightarrow j$. We assume that both i and j are sink. We have $\mathbb{T}_i K_{mj} \cong K_{mj}$. By Theorem 3.9, we further deduce an isomorphism $K_{mj} \star K_{pj} \cong K_{(m+p)j}$ for $p \geq 0$. Together with Corollary 3.10 and the induction-by-stage argument, we conclude that $K_{mj} \star L_b$ has simple head $L_{b'}$. Moreover, we have $\mathtt{M}_{b'} \cong \mathtt{M}_j^{\oplus m} \oplus \mathtt{M}_b$. Therefore, we have $\epsilon_i(b) > 0$ if and only if $\epsilon_i(b') > 0$ since ϵ_i counts the number of direct summand isomorphic to $\mathtt{M}_i$ by our assumption on Ω.

Now we consider the case $i \leftrightarrow j$. We rearrange Ω so that j is a sink of Ω and i is sink of $s_i\Omega$, and employ the same notation as in the proof of Corollary 3.11. We have a decomposition

$$\mathtt{M}_b \cong \mathtt{M}_i^{\oplus p} \oplus \mathtt{M}_{i,j}^{\oplus q} \oplus \mathtt{M}_d \quad \text{with} \quad d \in B(\infty)_{\beta - p\alpha_i - q s_i \alpha_j}$$

as $\mathbb{C}[s_i\Gamma]$-modules so that $\mathtt{M}_d$ does not contain $\mathtt{M}_i$ or $\mathtt{M}_{i,j}$ as its direct summand. Here we have $d = T_i(f)$ with $\epsilon_j(f) = 0$ by [Lu90a] 4.4 (c). We set $d' \in B(\infty)_{m s_i \alpha_j + \beta}$ so that $\mathtt{M}_{d'} \cong \mathtt{M}_{i,j}^{\oplus m} \oplus \mathtt{M}_b$. Thanks to Corollary 3.10 and Corollary 3.11, we have

$$K_b^{s_i\Omega} \cong K_{pi} \star (\mathbb{T}_i K_{qj}) \star K_d^{s_i\Omega}.$$

By Corollary 3.10, we deduce that $K_i \star \mathbb{T}_i K_j$ is isomorphic to a standard module of $R_{2\alpha_i+\alpha_j}$. Since the orbit corresponding to $K_i \star \mathbb{T}_i K_j$ is open dense, we deduce that $K_i \star \mathbb{T}_i K_j$ is simple. By inspection, we find that $\#\mathsf{Irr}_0 R_{2\alpha_i+\alpha_j} = 2$ and each of simple graded $R_{\alpha_i+2\alpha_j}$-module has dimension 3. Hence, $\mathbb{T}_i K_j \star K_i$ must be simple. By a weight comparison argument, we deduce that $K_i \star \mathbb{T}_i K_j \cong \mathbb{T}_i K_j \star K_i \langle 1 \rangle$. By Theorem 3.9, we deduce that

$$(\mathbb{T}_i K_{rj}) \star (\mathbb{T}_i K_{sj}) \cong \mathbb{T}_i K_{(r+s)j} \qquad \text{for every } r,s \geq 0.$$

Hence, we deduce $K_{pi} \star \mathbb{T}_i K_{mj} \cong \mathbb{T}_i K_{mj} \star K_{pi}$ up to grading shifts by induction.

Therefore, the induction-by-stage implies that the ungraded $R_{\beta + m s_i \alpha_j}$-module

$$\mathbb{T}_i K_{mj} \star K_b^{s_i \Omega} \cong \mathbb{T}_i K_{mj} \star K_{pi} \star (\mathbb{T}_i K_{qj}) \star K_d^{s_i \Omega} \cong K_{pi} \star (\mathbb{T}_i K_{(m+q)j}) \star K_d^{s_i \Omega} \cong K_{d'}^{s_i \Omega}$$

has simple head $L_{b'}$ with $\epsilon_i(b') = p > 0$ as desired. □

Proof of Theorem 3.6 5). We choose an orientation Ω so that i is a source and j is a sink. Let $F_1 := (\mathbb{T}_i P_{mj}) \star (\mathbb{T}_i \bullet)$ and $F_2 := \mathbb{T}_i (P_{mj} \star \bullet)$ be two functors from $R_\beta\text{-}\mathsf{gmod}_j$ to $R_{s_i(\beta + m\alpha_j)}\text{-}\mathsf{gmod}$. Both of them are exact on $R_\beta\text{-}\mathsf{gmod}_j^i$. Therefore, taking successive quotients of the isomorphisms in Corollary 3.11 (cf. Theorem 3.6 1)) yield

$$\mathbb{T}_i (K_{mj} \star L_b) \cong (\mathbb{T}_i K_{mj}) \star \mathbb{T}_i L_b \tag{3.5}$$

as a graded $R_{s_i(\beta + m\alpha_j)}$-module for every $b \in B(\infty)_\beta$ such that $\epsilon_j(b) = 0$. Consider the composition map

$$\eta : R_{m s_i \alpha_j} \boxtimes R_{s_i \beta} \hookrightarrow R_{m s_i \alpha_j + s_i \beta} \longrightarrow {}_i R_{m s_i \alpha_j + s_i \beta}.$$

By the construction of the induction isomorphism (3.5), the map η induces an inclusion $\mathbb{T}_i K_{mj} \boxtimes \mathbb{T}_i L_b \hookrightarrow \mathbb{T}_i (K_{mj} \star L_b)$ if $\epsilon_j(b) = 0$. We have a two-sided ideal $J := \ker(R_{m s_i \alpha_j} \boxtimes R_{s_i \beta} \to {}_i R_{m s_i \alpha_j} \boxtimes {}_i R_{s_i \beta})$. We have

$$\hom_{R_{m s_i \alpha_j} \boxtimes R_{s_i \beta}} (J, L_{b'_1} \boxtimes L_{b'_2}) \neq \{0\}$$

only if $b'_1 \in B(\infty)_{m s_i \alpha_j}$ satisfies $\epsilon_i(b'_1) > 0$, or $b'_2 \in B(\infty)_{s_i \beta}$ satisfies $\epsilon_i(b'_2) > 0$. In the former case, it is straightforward to verity that its image to ${}_i R_{m s_i \alpha_j + s_i \beta}$ must be zero by Lemma 3.2. Thus, we consider the case $\epsilon_i(b'_1) = 0$ and $\epsilon_i(b'_2) > 0$. As $b'_1 \in B(\infty)_{m(\alpha_i + \alpha_j)}$, it is standard to see $L_{b'_1} \cong \mathbb{T}_i K_{mj}$ by rank two inspection (see Theorem 3.6 1)).

The image of $L_{b'_1} \boxtimes L_{b'_2}$ in $R_{m s_i \alpha_j + s_i \beta}$ factors through $L_{b'_1} \star L_{b'_2}$ by construction. By Lemma 3.12, the unique simple quotient $L_{b'}$ of $L_{b'_1} \star L_{b'_2}$ satisfies $\epsilon_i(b') > 0$. Hence, the map η factors through ${}_i R_{m s_i \alpha_j} \boxtimes {}_i R_{s_i \beta}$. In other words, we have a graded algebra map

$$\mathbb{T}_i R_{m \alpha_j} \boxtimes \mathbb{T}_i R_\beta \longrightarrow \mathbb{T}_i R_{m \alpha_j + \beta}.$$

Therefore, we have a natural transformation

$$F_1 = (\mathbb{T}_i P_{mj}) \star \mathbb{T}_i \bullet \longrightarrow \mathbb{T}_i (P_{mj} \star \bullet) = F_2.$$

Thanks to Lemma 3.8, we see that F_2 sends an indecomposable projective module of ${}_j R_\beta$ (regarded as an R_β-module) to a module with simple head (or zero). The image of this natural transformation surjects onto this simple head by (3.5). This forces two functors F_1 and F_2 to be isomorphic on projective objects of $R_\beta\text{-}\mathsf{gmod}_j$ by the comparison of their graded characters. Therefore, we conclude that they are isomorphic. □

The rest of this section is devoted to the proof of Theorem 3.9. During the proof of Theorem 3.9, we omit Ω from the notation. We set $\beta := \beta_1 + \beta_2$, and $V := V(1) \oplus V(2)$. We set $n = \mathsf{ht}\, \beta$, and $n_i := \mathsf{ht}\, \beta_i$ for $i = 1, 2$. We write $\beta_i = \sum_{j \in I} d_i(j) \alpha_j$ for $i = \emptyset, 1, 2$.

We recall the convolution operation from Lusztig [Lu90a].

We consider two varieties with natural G_V-actions:

$$\mathrm{Gr}_{V(1),V(2)}(V) := \left\{ (F, x, \psi_1, \psi_2) \middle| \begin{matrix} F \subset V : I\text{-graded vector subspace} \\ x \in E_V, \text{ s.t. } xF \subset F \\ \psi_1 : V/F \cong V(1), \psi_2 : F \cong V(2) \end{matrix} \right\},$$

$$\mathrm{Gr}_{\beta_1,\beta_2}(V) := \left\{ (F, x) \middle| \begin{matrix} F \subset V : I\text{-graded vector subspace} \\ x \in E_V, \text{ s.t. } xF \subset F \\ \underline{\dim} F = \beta_2 \end{matrix} \right\}.$$

We have a $G_{V(1)} \times G_{V(2)}$-torsor structure $\vartheta : \mathrm{Gr}_{V(1),V(2)}(V) \longrightarrow \mathrm{Gr}_{\beta_1,\beta_2}(V)$ given by forgetting ψ_1 and ψ_2. We have two maps

$\mathsf{p} : \mathrm{Gr}_{\beta_1,\beta_2}(V) \ni (F, x) \mapsto x \in E_V$ and

$\mathsf{q} : \mathrm{Gr}_{V(1),V(2)}(V) \ni (F, x, \psi_1, \psi_2) \mapsto (\psi_1(x \mod F), \psi_2(x|_F)) \in E_{V(1)} \oplus E_{V(2)}$.

Notice that ϑ and q are smooth of relative dimensions $\dim G_{V(1)} + \dim G_{V(2)}$ and $\frac{1}{2}(\dim G_V + \dim G_{V(1)} + \dim G_{V(2)}) + \sum_{h \in \Omega} d_1(h')d_2(h'')$, respectively. The map p is projective. We set $N^{\beta}_{\beta_1,\beta_2} := \frac{1}{2}(\dim G_V - \dim G_{V(1)} - \dim G_{V(2)}) + \sum_{h \in \Omega} d_1(h')d_2(h'')$. For $G_{V(i)}$-equivariant constructible sheaves $\mathcal{F}_i$ on $E_{V(i)}$ for $i = 1, 2$, we define their convolution products as

$$\mathcal{F}_1 \odot \mathcal{F}_2 := \mathsf{p}_! \mathcal{F}_{12}[N^{\beta}_{\beta_1,\beta_2}], \text{ where } \vartheta^* \mathcal{F}_{12} \cong \mathsf{q}^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \text{ in } D^b_{G_V}(\mathrm{Gr}_{V(1),V(2)}(V)).$$

We return to the proof of Theorem 3.9. Let us fix objects $\mathbb{C}_{b_1,b_2}, \mathcal{L}_{\mathbf{m}^1,\mathbf{m}^2}$ of $D^b_{G_V}(\mathrm{Gr}_{\beta_1,\beta_2}(V))$ ($\mathbf{m}^1 \in Y^{\beta_1}$ and $\mathbf{m}^2 \in Y^{\beta_2}$) so that we have isomorphisms

$$\vartheta^*(\mathbb{C}_{b_1,b_2}) \cong \mathsf{q}^*(\mathbb{C}_{b_1} \boxtimes \mathbb{C}_{b_2})[N^{\beta}_{\beta_1,\beta_2}] \text{ and } \vartheta^* \mathcal{L}_{\mathbf{m}^1,\mathbf{m}^2} \cong \mathsf{q}^*(\mathcal{L}_{\mathbf{m}^1} \boxtimes \mathcal{L}_{\mathbf{m}^2})[N^{\beta}_{\beta_1,\beta_2}].$$

Lemma 3.13. *In the above settings, we have:*

1. *the variety $\mathsf{p}^{-1}(\mathbb{O}_b)$ is a single G_V-orbit;*
2. *the map $\mathsf{p} : \mathsf{p}^{-1}(\mathbb{O}_b) \to \mathbb{O}_b$ is a $\mathcal{P}$-fibration, where $\mathcal{P}$ is a suitable partial flag variety.*

Proof. We have $\mathtt{M}_b \cong \mathtt{M}_{b_1} \oplus \mathtt{M}_{b_2}$ by assumption. The condition $(\star)_0$ asserts that the image of every two inclusions $\mathtt{M}_{b_2} \subset \mathtt{M}_b$ are transformed by $\mathrm{Aut}_{\mathbb{C}[\Gamma]}(\mathtt{M}_b)$. Here we have $\mathrm{Aut}_{\mathbb{C}[\Gamma]}(\mathtt{M}_b) \cong \mathsf{Stab}_{G_V}(x_b)$ for $x_b \in \mathbb{O}_b(\mathbb{C})$. Therefore, $\mathsf{p}^{-1}(\mathbb{O}_b)$ is a single G_V-orbit, which is the first assertion. Since p is projective, we conclude that $\mathsf{p}^{-1}(\mathbb{O}_b) \to \mathbb{O}_b$ is projective. By $(\spadesuit)_2$, the group $\mathsf{Stab}_{G_V}(x_b)$ is connected. Let U_b denote the unipotent radical of $\mathsf{Stab}_{G_V}(x_b)$. Since we have $\mathsf{p}^{-1}(\mathbb{O}_b) \cong G_V/H_b$ with $H_b \subset \mathsf{Stab}_{G_V}(x_b)$, the fiber of p is isomorphic to $\mathsf{Stab}_{G_V}(x_b)/H_b$, that is projective. Therefore, we deduce $U_b \subset H_b$ and the inclusion

$$H_b/U_b \subset \mathsf{Stab}_{G_V}(x_b)/U_b$$

must be a parabolic subgroup (of a connected reductive group). Therefore, we set $\mathcal{P}$ to be their quotient to deduce the second part of the result. □

Corollary 3.14. *We have*

$$\mathbb{C}_{b_1}[\dim \mathbb{O}_{b_1}] \odot \mathbb{C}_{b_2}[\dim \mathbb{O}_{b_2}] \cong D[d] \boxtimes \mathbb{C}_b[\dim \mathbb{O}_b],$$

where $D \cong H^\bullet(\mathcal{P}, \mathbb{C})$ by a suitable partial flag variety $\mathcal{P}$ with its dimension d.

Proof. Thanks to $(\star)_1$, we deduce that $\vartheta(\mathsf{q}^{-1}(\mathbb{O}_{b_1} \times \mathbb{O}_{b_2}))$ is contained in a single G_V-orbit. This, together with Lemma 3.13, implies that the stalk of the LHS vanishes outside of $\mathbb{O}_b$. In addition, every direct summand of $\mathsf{p}_* \mathbb{C}_{b_1,b_2} |_{\mathbb{O}_b}$, viewed as a shifted G_V-equivariant local system (which in turn follows by [BBD82] 5.4.5 or 6.2.5), must be a trivial local system by $(\spadesuit)_2$. Therefore, we conclude that $\mathbb{C}_{b_1}[\dim \mathbb{O}_{b_1}] \odot \mathbb{C}_{b_2}[\dim \mathbb{O}_{b_2}] \cong D' \boxtimes \mathbb{C}_b[\dim \mathbb{O}_b]$ with a graded vector space D'. The isomorphism $D' \cong H^\bullet(\mathcal{P}, \mathbb{C})[d]$ is by Lemma 3.13 2). $\square$

We return to the proof of Theorem 3.9. In the below (during this section), we freely use the notation from Corollary 3.14.

Thanks to Corollary 2.8, $\mathcal{L}_{\beta_1}$ and $\mathcal{L}_{\beta_2}$ contains IC_{b_1} and IC_{b_2}, respectively. We have

$$\mathcal{L}_{\mathbf{m}^1} \odot \mathcal{L}_{\mathbf{m}^2} \cong \mathcal{L}_{\mathbf{m}^1 + \mathbf{m}^2}$$

by construction. Thanks to $(\star)_1$ and [BBD82] 5.4.5 or 6.2.5, IC_b appears in $\mathcal{L}_{\mathbf{m}^1 + \mathbf{m}^2}$ up to a grading shift if the following condition $(\diamond)$ hold:

$(\diamond)$ IC_{b_i} appears in $\mathcal{L}_{\mathbf{m}^i}$ for $i = 1, 2$.

We set $\mathbf{m} := \mathbf{m}^1 + \mathbf{m}^2$. Let $x_b \in \mathbb{O}_b$ be a point and let $i_b : \{x_b\} \hookrightarrow E_V$ be the inclusion.

Lemma 3.15. *Assume that* $(\diamond)$ *holds. Then, the subspace*

$$\begin{aligned} & i_b^! \mathcal{E}xt^\bullet_{D^b(E_V)}(\mathbb{C}_{b_1} \odot \mathbb{C}_{b_2}, \mathcal{L}_{\mathbf{m}^1 + \mathbf{m}^2}) \subset i_b^! \mathcal{E}xt^\bullet_{D^b(E_V)}(\mathbb{C}_{b_1} \odot \mathbb{C}_{b_2}, \mathcal{L}_\beta) \\ & \cong \mathcal{E}xt^\bullet_{D^b(\mathrm{pt})}(i_b^*(\mathbb{C}_{b_1} \odot \mathbb{C}_{b_2}), i_b^! \mathcal{L}_\beta) \cong D^*[-d] \boxtimes K_b \langle 2 \dim \mathbb{O}_b \rangle \end{aligned}$$

is a generating subspace as an R_β*-module.*

Proof. The isomorphism parts of the assertion follow by [KS90] 3.1.13 and Corollary 3.14. By $(\diamond)$ and $(\star)_1$, we conclude that $\mathcal{L}_{b_1,b_2}$ contains an irreducible perverse sheaf supported on $\mathrm{Supp}\, \mathbb{C}_{b_1,b_2}$. Thanks to [BBD82] 5.4.5 or 6.2.5, we conclude that $\mathcal{L}_{\mathbf{m}^1} \odot \mathcal{L}_{\mathbf{m}^2}$ contains IC_b. Therefore, the head L_b of K_b satisfies $e(\mathbf{m}) L_b \neq \{0\}$, which proves the assertion. $\square$

We set $\mathbb{O}_b^\uparrow \subset E_V$ to be the union of G_V-orbits which contains $\mathbb{O}_b$ in its closure. Let $j_b^\uparrow : \mathbb{O}_b^\uparrow \hookrightarrow E_V$ be its inclusion.

Proposition 3.16. *We have a canonical isomorphism*

$$\mathcal{E}xt^\bullet_{D^b(E_V)}(\mathbb{C}_{b_1} \odot \mathbb{C}_{b_2}, \mathcal{L}_{\mathbf{m}^1 + \mathbf{m}^2}) \cong \mathsf{p}_* \mathcal{E}xt^\bullet_{D^b(\mathrm{Gr}_{\beta_1,\beta_2}(V))}(\mathbb{C}_{b_1,b_2}, D^* \boxtimes \mathcal{L}_{\mathbf{m}^1,\mathbf{m}^2})$$

in the bounded derived category of constructible sheaves on E_V*.*

Proof. During this proof, we repeatedly use the local form of the Verdier duality (see e.g. [KS90] 3.1.10, or [SGA4] Exposé XVIII 3.1.10). We have

$$\mathcal{E}xt^\bullet_{D^b(E_V)}(\mathbb{C}_{b_1} \odot \mathbb{C}_{b_2}, \mathcal{L}_{\mathbf{m}^1 + \mathbf{m}^2}) \cong \mathsf{p}_* \mathcal{E}xt^\bullet_{D^b(\mathrm{Gr}_{\beta_1,\beta_2}(V))}(\mathbb{C}_{b_1,b_2}, \mathsf{p}^! \mathcal{L}_{\mathbf{m}^1 + \mathbf{m}^2}).$$

Consider the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Gr}_{\beta_1,\beta_2}(V) & \xrightarrow{\ \mathsf{p}\ } & E_V \\ \uparrow{\scriptstyle j_b^\uparrow} & & \uparrow{\scriptstyle j_b^\uparrow} \\ \mathcal{G} & \xrightarrow{\ p\ } & \mathbb{O}_b^\uparrow \end{array} .$$

Note that $\jmath_b^{\uparrow}$ is an open embedding since p is continuous. It follows that

$$\begin{aligned}
&\mathsf{p}_* \mathcal{E}xt^{\bullet}_{D^b(\mathrm{Gr}_{\beta_1,\beta_2}(V))}(\mathbb{C}_{b_1,b_2}, \mathsf{p}^! \mathcal{L}_{\mathbf{m}^1+\mathbf{m}^2}) \\
&\cong (\jmath_b^{\uparrow})_* p_* \mathcal{E}xt^{\bullet}_{D^b(\mathcal{G})}((\jmath_b^{\uparrow})^* \mathbb{C}_{b_1,b_2}, (\jmath_b^{\uparrow})^! \mathsf{p}^! \mathcal{L}_{\mathbf{m}^1+\mathbf{m}^2}) \quad (\mathbb{C}_{b_1,b_2} \cong (\jmath_b^{\uparrow})_! (\jmath_b^{\uparrow})^* \mathbb{C}_{b_1,b_2}) \\
&\cong (\jmath_b^{\uparrow})_* p_* \mathcal{E}xt^{\bullet}_{D^b(\mathcal{G})}((\jmath_b^{\uparrow})^* \mathbb{C}_{b_1,b_2}, p^! (\jmath_b^{\uparrow})^! \mathcal{L}_{\mathbf{m}^1+\mathbf{m}^2}) \qquad (\jmath_b^{\uparrow} \circ p = \mathsf{p} \circ \jmath_b^{\uparrow}).
\end{aligned}$$

In addition, $(\jmath_b^{\uparrow})^* \mathbb{C}_{b_1,b_2}$ is a local system supported on the closed G_V-orbit $\mathcal{O}_b$ of $\mathcal{G}$. Let us denote by $\jmath_b : \mathcal{O}_b \hookrightarrow \mathcal{G}$ the inclusion. We have $(\jmath_b^{\uparrow})^* \mathbb{C}_{b_1,b_2} \cong (\jmath_b)_! \mathbb{C}[\dim \mathcal{O}_b]$. Thus, we deduce

$$\begin{aligned}
&(\jmath_b^{\uparrow})_* p_* \mathcal{E}xt^{\bullet}_{D^b(\mathcal{G})}((\jmath_b^{\uparrow})^* \mathbb{C}_{b_1,b_2}, p^! (\jmath_b^{\uparrow})^! \mathcal{L}_{\mathbf{m}^1+\mathbf{m}^2}) \\
&\cong (\jmath_b^{\uparrow})_* p'_* \mathcal{E}xt^{\bullet}_{D^b(\mathcal{O}_b)}(\mathbb{C}[\dim \mathcal{O}_b], \jmath_b^! p^! (\jmath_b^{\uparrow})^! \mathcal{L}_{\mathbf{m}^1+\mathbf{m}^2}) \qquad ((\jmath_b^{\uparrow})^* \mathbb{C}_{b_1,b_2} \cong (\jmath_b)_! \mathbb{C}[\bullet]) \\
&\cong (\jmath_b^{\uparrow})_* p'_* \mathcal{E}xt^{\bullet}_{D^b(\mathcal{O}_b)}(\mathbb{C}[\dim \mathcal{O}_b], D^* \boxtimes \jmath_b^! (\jmath_b^{\uparrow})^! \mathcal{L}_{\mathbf{m}^1,\mathbf{m}^2}) \qquad \text{(Corollary 3.14)} \\
&\cong \mathsf{p}_* \mathcal{E}xt^{\bullet}_{D^b(\mathrm{Gr}_{\beta_1,\beta_2}(V))}(\mathbb{C}_{b_1,b_2}, D^* \boxtimes \mathcal{L}_{\mathbf{m}^1,\mathbf{m}^2}),
\end{aligned}$$

where $p' : \mathcal{O}_b \to \mathbb{O}_b^{\uparrow}$ is the restriction of p. Since all the maps are canonically defined, composing all the isomorphisms yield the result. □

We return to the proof of Theorem 3.9. Taking account into the fact $\mathsf{p}^{-1}(x_b) \cong \mathcal{P}$, we have an isomorphism

$$D^* \langle d \rangle \boxtimes e(\mathbf{m}^1 + \mathbf{m}^2) K_b \cong \mathbb{H}^{\bullet} i_b^! \mathcal{E}xt^{\bullet}_{D^b(E_V)}(\mathbb{C}_{b_1} \odot \mathbb{C}_{b_2}, \mathcal{L}_{\mathbf{m}^1+\mathbf{m}^2})[2 \dim \mathbb{O}_b]$$

and a spectral sequence arising from the base change (applied to i_b and p)

$$\begin{aligned}
E_2 := D^* \otimes H^{\bullet}(\mathcal{P}) \otimes \big(e(\mathbf{m}^1) K_{b_1} \boxtimes e(\mathbf{m}^2) K_{b_2}\big) \\
\Rightarrow \mathbb{H}^{\bullet} i_b^! \mathsf{p}_* \mathcal{E}xt^{\bullet}_{D^b(\mathrm{Gr}_{\beta_1,\beta_2}(V))}(\mathbb{C}_{b_1,b_2}, D^* \boxtimes \mathcal{L}_{\mathbf{m}^1,\mathbf{m}^2})[2 \dim \mathbb{O}_b],
\end{aligned}$$

where we used the fact that $\dim \mathsf{p}^{-1}(\mathbb{O}_b) - \dim \mathsf{p}^{-1}(x_b) = \dim \mathbb{O}_b$ in the degree shift of the second spectral sequence. Here the modules K_{b_1}, K_{b_2}, and K_b are pure of weight 0 by [Lu90a] 10.6 (see the proof of Proposition 2.7 for a bit precise account). By Lemma 3.13 2), we deduce that $H^{\bullet}(\mathcal{P})$ is also pure. Therefore, the spectral sequence E_2 degenerates at the E_2-stage. By factoring out the effect of D^*, we conclude that

$$e(\mathbf{m}^1 + \mathbf{m}^2) K_b \cong H^{\bullet}(\mathcal{P}) \boxtimes \big(e(\mathbf{m}^1) K_{b_1} \boxtimes e(\mathbf{m}^2) K_{b_2}\big) \langle -d \rangle .$$

This induces an inclusion as $R_{\mathbf{m}^1,\mathbf{m}^1} \boxtimes R_{\mathbf{m}^2,\mathbf{m}^2}$-modules

$$\varphi_{\mathbf{m}^1,\mathbf{m}^2} : \big(e(\mathbf{m}^1) K_{b_1} \boxtimes e(\mathbf{m}^2) K_{b_2}\big) \langle d \rangle \hookrightarrow e(\mathbf{m}^1 + \mathbf{m}^2) K_b.$$

The module $e(\mathbf{m}^1 + \mathbf{m}^2) K_b$ admits an $R_{\mathbf{m}^1+\mathbf{m}^2,\mathbf{m}^1+\mathbf{m}^2}$-module structure with simple head thanks to Theorem 1.3 3). This extends the $R_{\mathbf{m}^1,\mathbf{m}^1} \boxtimes R_{\mathbf{m}^2,\mathbf{m}^2}$-module structure. Recall that for each $i = \emptyset, 1, 2$, the simple head of K_{b_i} as an irreducible R_{β_i}-module is realized as the coefficient vector space of IC_{b_i} inside $\mathcal{L}_{\beta_i}$ (see §1), and its weight $e(\mathbf{m}^i)$-part is that of $\mathcal{L}_{\mathbf{m}^i}$ (see §2). (Note that this sheaf-theoretic interpretation gives a splitting of L_{b_i} to K_{b_i} as vector spaces for each $i = \emptyset, 1, 2$.) By [BBD82] 5.4.5 or 6.2.5 and Corollary 3.14, the complex $H^\bullet(\mathcal{P})[d] \boxtimes \mathsf{IC}_b$ is a direct summand of $\mathsf{IC}_{b_1} \odot \mathsf{IC}_{b_2}$. Therefore, the above interpretation implies that the unique simple quotients L_{b_1} and L_{b_2} of K_{b_1} and K_{b_2} satisfy

$$\varphi_{\mathbf{m}^1,\mathbf{m}^2}(H^\bullet(\mathcal{P}) \otimes \big(e(\mathbf{m}^1)L_{b_1} \boxtimes e(\mathbf{m}^2)L_{b_2}\big))\,\langle -d\rangle \subset e(\mathbf{m})L_b \subset e(\mathbf{m})K_b$$

as vector subspaces, where L_b is the simple head of K_b. In addition, these inclusions are non-zero if $\mathbf{m}^1$ and $\mathbf{m}^2$ satisfies $(\diamond)$. Since we can choose $\mathbf{m}^1$ and $\mathbf{m}^2$ so that $(\diamond)$ is satisfied, we have a surjective map of graded R_β-modules:

$$K_{b_1} \star K_{b_2}\,\langle d\rangle \longrightarrow\!\!\!\!\!\rightarrow K_b.$$

Lemma 3.17. *In the above settings, we have*

$$\dim K_b = \dim\left(K_{b_1} \star K_{b_2}\right).$$

Proof. In this proof, i denotes either $\emptyset, 1$, or 2. Let us choose a point $x_{b_i} \in \mathbb{O}_{b_i} \subset E_{V(i)}$. Let T_i be a maximal torus of $\mathsf{Stab}_{G_{V(i)}} x_{b_i}$. Choose $\mathbf{m}^i \in Y^{\beta_i}$. Thanks to the purity of each module (Lusztig [Lu90a] 10.6), we deduce that the spectral sequence

$$H^\bullet_{T_i}(\mathrm{pt}) \otimes H_\bullet(\pi^{-1}_{\mathbf{m}^i}(x_{b_i})) \Rightarrow H^{T_i}_\bullet(\pi^{-1}_{\mathbf{m}^i}(x_{b_i}))$$

degenerates at the E_2-stage. Here the RHS have the same $H^\bullet_{T_i}(\mathrm{pt})$-rank as that of $H^{T_i}_\bullet(\pi^{-1}_{\mathbf{m}^i}(x_b)^{T_i})$. Therefore, we have

$$\dim H_\bullet(\pi^{-1}_{\mathbf{m}^i}(x_b)) = \dim H_\bullet(\pi^{-1}_{\mathbf{m}^i}(x_{b_i})^{T_i}).$$

By Theorem 2.11, we deduce that R_β is a free $R_{\beta_1} \boxtimes R_{\beta_2}$-module of rank $\frac{n!}{n_1!n_2!}$. Hence, it is enough to show

$$\sum_{\mathbf{m}\in Y^\beta} \dim H_\bullet(\pi^{-1}_{\mathbf{m}}(x_b)^T) = \frac{n!}{n_1!n_2!} \sum_{\substack{\mathbf{m}^1 \in Y^{\beta_1} \\ \mathbf{m}^2 \in Y^{\beta_2}}} (\dim H_\bullet(\pi^{-1}_{\mathbf{m}^1}(x_{b_1})^{T_1}))(\dim H_\bullet(\pi^{-1}_{\mathbf{m}^2}(x_{b_2})^{T_2})).$$

This follows by a simple counting since $E^{T_i}_{V(i)}$ decomposes into the product of varieties corresponding to each indecomposable module. □

We return to the proof of Theorem 3.9. Lemma 3.17 asserts that

$$K_{b_1} \star K_{b_2}\,\langle d\rangle \cong K_b$$

as graded R_β-modules. This completes the proof of Theorem 3.9 except for the last assertion. The last assertion follows since the assumption implies that p_{b_1,b_2} is birational onto its image, and hence $d = 0$.

4 Characterization of the PBW bases

Keep the setting of the previous section. For a reduced expression $\mathbf{i}$ of w_0 and a sequence of non-negative integers $\mathbf{c} := (c_1, c_2, \dots, c_\ell) \in \mathbb{Z}_{\geq 0}^{\ell}$, we call the pair $(\mathbf{i}, \mathbf{c})$ a Lusztig datum, and we call $\mathbf{c}$ an $\mathbf{i}$-Lusztig datum. For a Lusztig datum $(\mathbf{i}, \mathbf{c})$, we define

$$\mathsf{wt}(\mathbf{i}, \mathbf{c}) := \sum_{k=1}^{\ell} c_k \gamma_{\mathbf{i}}^{(k)}, \quad \text{where} \quad \gamma_{\mathbf{i}}^{(k)} := s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}.$$

For two $\mathbf{i}$-Lusztig data $\mathbf{c}$ and $\mathbf{c}'$, we define $\mathbf{c} <_{\mathbf{i}} \mathbf{c}'$ as: There exists $0 \leq k < \ell$ so that

$$c_1 = c_1', c_2 = c_2', \dots, c_k = c_k' \quad \text{and} \quad c_{k+1} > c_{k+1}'.$$

Associated to each Lusztig datum $(\mathbf{i}, \mathbf{c})$, we define the lower PBW-module $\widetilde{E}_{\mathbf{c}}^{\mathbf{i}}$ as:

$$\widetilde{E}_{\mathbf{c}}^{\mathbf{i}} := P_{c_1 i_1} \star \mathbb{T}_{i_1} \left(P_{c_2 i_2} \star \mathbb{T}_{i_2} \left(P_{c_3 i_3} \star \cdots \mathbb{T}_{i_{\ell-1}} P_{c_\ell i_\ell} \right) \cdots \right). \tag{4.1}$$

Similarly, we define the corresponding upper PBW-module $E_{\mathbf{c}}^{\mathbf{i}}$ as:

$$E_{\mathbf{c}}^{\mathbf{i}} := L_{c_1 i_1} \star \mathbb{T}_{i_1} \left(L_{c_2 i_2} \star \mathbb{T}_{i_2} \left(L_{c_3 i_3} \star \cdots \mathbb{T}_{i_{\ell-1}} L_{c_\ell i_\ell} \right) \cdots \right). \tag{4.2}$$

By construction, it is clear that $E_{\mathbf{c}}^{\mathbf{i}}$ is a quotient of $\widetilde{E}_{\mathbf{c}}^{\mathbf{i}}$.

Remark 4.1. The modules $E_{\mathbf{c}}^{\mathbf{i}}$ and $\widetilde{E}_{\mathbf{c}}^{\mathbf{i}}$ are identified with $\widetilde{K}_b^{\Omega}$ and K_b^{Ω} from §2 when the reduced expression $\mathbf{i}$ is adapted to Ω (cf. Corollary 4.14). In addition, we have $P_{ci} = \widetilde{K}_{ci}$ and $L_{ci} = K_{ci}$ for each $c \in \mathbb{Z}_{\geq 0}$ and $i \in I$ by Example 2.10.

Lemma 4.2. *For each Lusztig datum* $(\mathbf{i}, \mathbf{c})$*, we have:*

1. $\widetilde{E}_{\mathbf{c}}^{\mathbf{i}}$ *and* $E_{\mathbf{c}}^{\mathbf{i}}$ *are* $R_{\mathsf{wt}(\mathbf{i},\mathbf{c})}$*-modules;*
2. *There exist isomorphisms as graded* $R_{\mathsf{wt}(\mathbf{i},\mathbf{c})}$*-modules:*

$$\widetilde{E}_{\mathbf{c}}^{\mathbf{i}} \cong P_{c_1 i_1} \star (\mathbb{T}_{i_1} P_{c_2 i_2}) \star (\mathbb{T}_{i_1} \mathbb{T}_{i_2} P_{c_3 i_3}) \star \cdots \star (\mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{\ell-1}} P_{c_\ell i_\ell})$$
$$E_{\mathbf{c}}^{\mathbf{i}} \cong L_{c_1 i_1} \star (\mathbb{T}_{i_1} L_{c_2 i_2}) \star (\mathbb{T}_{i_1} \mathbb{T}_{i_2} L_{c_3 i_3}) \star \cdots \star (\mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{\ell-1}} L_{c_\ell i_\ell}).$$

3. $\widetilde{E}_{\mathbf{c}}^{\mathbf{i}}$ *is a successive self-extension of* $E_{\mathbf{c}}^{\mathbf{i}}$*;*
4. $\widetilde{E}_{\mathbf{c}}^{\mathbf{i}}$ *and* $E_{\mathbf{c}}^{\mathbf{i}}$ *are modules with simple heads if they are non-zero;*
5. $\widetilde{E}_{\mathbf{c}}^{\mathbf{i}} \neq \{0\}$ *if and only if* $E_{\mathbf{c}}^{\mathbf{i}} \neq \{0\}$*.*

Proof. Since $\mathbb{T}_i$ is a functor sending an R_β-module to an $R_{s_i\beta}$-module (possibly zero), the first assertion is immediate. Notice that each part in (4.1) grouped as "$\mathbb{T}_{i_k}(P_{ci_{k+1}} \star \cdots)$" belongs to $R_\gamma\text{-}\mathsf{gmod}_{i_k}$ for some $\gamma \in Q^+$ by Theorem 3.6 2). Therefore, we apply Theorem 3.6 5) repeatedly to deduce the second assertion from (4.1). In addition, each $\mathbb{T}_{i_1}\mathbb{T}_{i_2} \cdots \mathbb{T}_{i_{k-1}} L_{c_k i_k}$ is simple (unless it is zero) and $\mathbb{T}_{i_1}\mathbb{T}_{i_2} \cdots \mathbb{T}_{i_{k-1}} P_{c_k i_k}$ is a successive self-extension of $\mathbb{T}_{i_1}\mathbb{T}_{i_2} \cdots \mathbb{T}_{i_{k-1}} L_{c_k i_k}$. Therefore, the third assertion follows by the second assertion. The functor $\mathbb{T}_i$ also preserves the simple head property (provided if it does not annihilate the whole module) by construction. Therefore, we apply Lemma 3.8 repeatedly to deduce the simple head property of $\widetilde{E}_{\mathbf{c}}^{\mathbf{i}}$ from that of $P_{c_k i_k}$ $(1 \leq k \leq \ell)$, which is the fourth assertion. By the third assertion, $E_{\mathbf{c}}^{\mathbf{i}}$ contains the head of $\widetilde{E}_{\mathbf{c}}^{\mathbf{i}}$, and hence the fifth assertion. □

Theorem 4.3 (Lusztig [Lu90a])**.** *Assume that the reduced expression* $\mathbf{i}$ *is adapted to* Ω*. Then, we have* $E^{\mathbf{i}}_{\mathbf{c}} \neq \{0\}$ *for every* $\mathbf{i}$*-Lusztig datum. Moreover, the set of* $\mathbf{i}$*-Lusztig data is in bijection with* $B(\infty)$ *as:*

$$\mathbf{c} \mapsto \mathsf{hd}\, E^{\mathbf{i}}_{\mathbf{c}} \cong L_b \quad \textit{for} \quad b \in B(\infty).$$

Proof. Since $\mathbf{i}$ is adapted, we deduce that a module $\mathbb{T}_{i_1}\mathbb{T}_{i_2}\cdots\mathbb{T}_{i_{k-1}}L_{c_k i_k}$ is simple and it corresponds to an indecomposable $\mathbb{C}[\Gamma]$-module $\mathtt{M}_{(k)}$ with $\underline{\dim}\,\mathtt{M}_{(k)} = \gamma^{(k)}_{\mathbf{i}}$ ([Lu90a] 4.7). We apply Corollary 3.10 and Theorem 3.6 1) repeatedly to construct a module with its simple head corresponding to the quiver representation $\mathtt{M}^{\oplus c_1}_{(1)} \oplus \cdots \oplus \mathtt{M}^{\oplus c_\ell}_{(\ell)}$. Now the Gabriel theorem yields the result. □

Definition 4.4 (2-move, 3-move, [Lu90a] 2.3)**.** We say that two Lusztig data $(\mathbf{i}, \mathbf{c})$ and $(\mathbf{i}', \mathbf{c}')$ are connected by a 2-move if

1. there exists $1 \le k < \ell$ so that $i_k = i'_{k+1}, i_{k+1} = i'_k$, $i_k \not\leftrightarrow i_{k+1}$, and $i_l = i'_l$ for every $l \neq k, k+1$;
2. we have $c_k = c'_{k+1}, c_{k+1} = c'_k$, and $c_l = c'_l$ for every $l \neq k, k+1$.

We say that $(\mathbf{i}, \mathbf{c})$ and $(\mathbf{i}', \mathbf{c}')$ are connected by a 3-move if

1. there exists $1 < k < \ell$ so that $i_{k-1} = i_{k+1} = i'_k$, $i_k = i'_{k-1} = i'_{k+1}$, $i_k \leftrightarrow i_{k+1}$, and $i_l = i'_l$ for every $l \neq k-1, k, k+1$;
2. we have $c_l = c'_l$ for every $l \neq k-1, k, k+1$, and

   $(c'_{k-1}, c'_k, c'_{k+1}) = (c_k + c_{k+1} - c_0, c_0, c_{k-1} + c_k - c_0)$ for $c_0 := \min\{c_{k-1}, c_{k+1}\}$.

Lemma 4.5. *For two Lusztig data* $(\mathbf{i}, \mathbf{c})$ *and* $(\mathbf{i}', \mathbf{c}')$ *which are connected by a* 2*-move, we have* $E^{\mathbf{i}}_{\mathbf{c}} \cong E^{\mathbf{i}'}_{\mathbf{c}'}$*.*

Proof. Find a unique $1 \le k < \ell$ so that $i_k = i'_{k+1} \neq i_{k+1} = i'_k$. By Lemma 4.2 2), it suffices to prove

$$L_{c_k i_k} \star \mathbb{T}_{i_k} L_{c_{k+1} i_{k+1}} \cong L_{c_{k+1} i_{k+1}} \star \mathbb{T}_{i_{k+1}} L_{c_k i_k} \tag{4.3}$$

and $\mathbb{T}_{i_k}\mathbb{T}_{i_{k+1}} \cong \mathbb{T}_{i_{k+1}}\mathbb{T}_{i_k}$. Here we realize $\mathbb{T}_{i_k}$ and $\mathbb{T}_{i_{k+1}}$ by choosing the orientation Ω so that the both of i_k, i_{k+1} are source (which is in turn possible since $i_k \not\leftrightarrow i_{k+1}$). Then, we have $\mathbb{T}_{i_k} L_{c_{k+1} i_{k+1}} \cong L_{c_{k+1} i_{k+1}}$, $\mathbb{T}_{i_{k+1}} L_{c_k i_k} \cong L_{c_k i_k}$, and $L_{c_k i_k} \star L_{c_{k+1} i_{k+1}} \cong L_{c_{k+1} i_{k+1}} \star L_{c_k i_k}$ since $R_{p\alpha_{i_k} + q\alpha_{j_k}}$ is Morita equivalent to $R_{p\alpha_{i_k}} \boxtimes R_{q\alpha_{j_k}}$ for each $p, q \ge 0$ by the product decomposition of (G_V, E^{Ω}_V). This shows (4.3).

We have $T_{i_k}T_{i_{k+1}}(b) = T_{i_{k+1}}T_{i_k}(b)$, $\epsilon_{i_k}(b) = \epsilon_{i_k}(T_{i_{k+1}}(b))$, and $\epsilon_{i_{k+1}}(b) = \epsilon_{i_{k+1}}(T_{i_k}(b))$ by inspection. The essential image of the functor $\mathbb{T}_i$ (applied to $R_{s_i\beta}$-gmod for some $\beta \in Q^+$) is equivalent to ${}_iR_\beta$-gmod by construction. We have $e_{i_k}(1)e_{i_{k+1}}(1) = 0 = e_{i_{k+1}}(1)e_{i_k}(1)$ by definition. Therefore, we deduce that the essential image of each of the functors $\mathbb{T}_{i_k}\mathbb{T}_{i_{k+1}}$ and $\mathbb{T}_{i_{k+1}}\mathbb{T}_{i_k}$ is equivalent to the graded module category of

$$(R_\beta/(R_\beta e_{i_k}(1)R_\beta)) \otimes_{R_\beta} \big(R_\beta/(R_\beta e_{i_{k+1}}(1)R_\beta)\big) \cong R_\beta/(R_\beta e R_\beta) \ \text{ for some } \beta \in Q^+,$$

where $e := e_{i_k}(1) + e_{i_{k+1}}(1)$ is the minimal idempotent so that $ee_{i_k}(1) = e_{i_k}(1)$ and $ee_{i_{k+1}}(1) = e_{i_{k+1}}(1)$. Hence, Theorem 3.6 2) guarantees that $\mathbb{T}_{i_k}\mathbb{T}_{i_{k+1}} \cong \mathbb{T}_{i_{k+1}}\mathbb{T}_{i_k}$ as functors, which completes the proof. □

For a reduced expression $\mathbf{i} = (i_1, \ldots, i_\ell)$ of w_0, we have a unique reduced expression of the form $\mathbf{i}^\# := (i_2, i_3, \ldots, i_\ell, i'_1)$. (Namely $s_{i'_1} := w_0 s_{i_1} w_0^{-1}$.)

Lemma 4.6. *Assume that the set of modules* $\{\mathsf{hd}\, E^{\mathbf{i}^\#}_{\mathbf{c}}\}_{\mathbf{c} \in \mathbb{Z}^\ell_{\ge 0}}$ *is in bijection with* $\bigsqcup_{\beta \in Q^+} \mathsf{Irr}_0\, R_\beta$*. Then, so is* $\{\mathsf{hd}\, E^{\mathbf{i}}_{\mathbf{c}}\}_{\mathbf{c} \in \mathbb{Z}^\ell_{\ge 0}}$.

Proof. By Theorem 3.6 2), we deduce that

$$\{\mathbb{T}_{i_1}(\mathsf{hd}\, E^{\mathbf{i}^\#}_{\mathbf{c}'})\}_{\mathbf{c}' \in \mathbb{Z}^\ell_{\ge 0}} = \{L_b \in \bigcup_{\beta \in Q^+} \mathsf{Irr}_0\, R_\beta \mid \epsilon_i(b) = 0\} \cup \{\{0\}\}.$$

For every $\beta \in Q^+$, we have $w_0\beta \notin Q^+$. It follows that

$$\mathbb{T}_{i_1}\mathbb{T}_{i_2} \cdots \mathbb{T}_{i_\ell} L_{m i'_1} = \{0\} \quad \text{for each} \quad m > 0.$$

Therefore, Lemma 4.2 2) asserts that $\mathbb{T}_{i_1} E^{\mathbf{i}^\#}_{\mathbf{c}'} \neq \{0\}$ implies that the $\mathbf{i}^\#$-Lusztig datum $\mathbf{c}' = (c'_1, \ldots, c'_\ell)$ satisfies $c'_\ell = 0$, and we have

$$\mathbb{T}_{i_1} E^{\mathbf{i}^\#}_{\mathbf{c}'} \cong E^{\mathbf{i}}_{\mathbf{c}}$$

for $\mathbf{c} = (0, c'_1, c'_2, \cdots, c'_{\ell-1})$. Thanks to Theorem 2.12 and Corollary 3.10, we deduce that

$$\{\mathsf{hd}\,(L_{m i_1} \star \mathbb{T}_{i_1} E^{\mathbf{i}^\#}_{\mathbf{c}'})\}_{\mathbf{c}' \in \mathbb{Z}^\ell_{\ge 0}, m \ge 0} = \{\mathsf{hd}\, E^{\mathbf{i}}_{\mathbf{c}}\}_{\mathbf{c} \in \mathbb{Z}^\ell_{\ge 0}} \cup \{\{0\}\}.$$

However, as in [Lu90a] 2.4 (a), the number of Lusztig data of each $B(\infty)_\beta$ is constant for every reduced expression. Therefore, we must have a bijection as required. □

Lemma 4.7. *Let* $\mathbf{i}$ *and* $\mathbf{i}'$ *be two reduced expressions which are connected by a 3-move as* $(i_{k-1}, i_k, i_{k+1}) = (i'_k, i'_{k\pm 1}, i'_k)$ *for some* k*. If the set of modules* $\{\mathsf{hd}\, E^{\mathbf{i}}_{\mathbf{c}}\}_{\mathbf{c} \in \mathbb{Z}^\ell_{\ge 0}}$ *is in bijection with* $\bigsqcup_{\beta \in Q^+} \mathsf{Irr}_0\, R_\beta$*, then so is* $\{\mathsf{hd}\, E^{\mathbf{i}'}_{\mathbf{c}}\}_{\mathbf{c} \in \mathbb{Z}^\ell_{\ge 0}}$.

Proof. Applying Lemma 4.6 repeatedly, we assume that $k = \ell - 1$. We set $i := i_{k-1} = i_\ell, j := i_k = i_{\ell-1}$ for simplicity.

In order to analyze the effect of $\mathbb{T}_i$ or $\mathbb{T}^*_i$, we assume that $i \in I$ is a sink or a source by applying a Fourier transform if necessary. Since a Fourier transform is defined by pairing a direct factor E of E^Ω_V with a direct factor E^* of $E^{\Omega'}_V$ (where Ω' is a different orientation of Γ_0), it must be identity if $E = \{0\}$. Therefore, we can regard as if Γ_0 is a quiver $(\{i, j\}, \{h\})$ of type A_2 provided if $\beta \in \mathbb{Z}_{\ge 0}\alpha_i \oplus \mathbb{Z}_{\ge 0}\alpha_j$. Here $h \in \{h_\pm\}$, where $h'_+ = j, h''_+ = i$ or $h'_- = i, h''_- = j$. In this case, the both sequences (i, j, i) and (j, i, j) are adapted to the orientations $\{h_+\}$ and $\{h_-\}$, respectively.

Thanks to Theorem 4.3 and the definition of $\star$, it follows that each of the two sets

$$\{\mathsf{hd}\left(L_{c_1 i} \star (\mathbb{T}_i L_{c_2 j}) \star (\mathbb{T}_i \mathbb{T}_j L_{c_3 i})\right)\} \text{ and } \{\mathsf{hd}\left(L_{c'_1 j} \star (\mathbb{T}_j L_{c'_2 i}) \star (\mathbb{T}_j \mathbb{T}_i L_{c'_3 j})\right)\}$$

exhausts $\bigsqcup_{p,q \in \mathbb{Z}_{\ge 0}} \mathsf{Irr}_0\, R_{p\alpha_i + q\alpha_j}$ (and hence they are identical). By applying Lemma 4.2 2), we conclude the result since the other factors are in common. □

Corollary 4.8. *The module $E^{\mathbf{i}}_{\mathbf{c}}$ is non-zero for every Lusztig datum $(\mathbf{i}, \mathbf{c})$, and the map*

$$\mathbf{c} \mapsto \mathsf{hd}\, E^{\mathbf{i}}_{\mathbf{c}} \cong L_b \text{ for } b \in B(\infty)$$

sets up a bijection between the set of $\mathbf{i}$*-Lusztig data and* $B(\infty)$*.*

Proof. Every two reduced expressions of $w_0 \in W(\Gamma_0)$ are connected by a repeated use of two moves and three moves ([Lu90a] 2.1 (c)). Therefore, we apply Lemma 4.5 and Lemma 4.7 repeatedly to deduce the assertion. □

Corollary 4.9. *For any reduced expression* $\mathbf{i}$*,* $1 \le k \le \ell$*, and* $m \ge 0$*, the module*

$$\mathbb{T}_{i_1}\mathbb{T}_{i_2}\cdots\mathbb{T}_{i_{k-1}}L_{mi_k}$$

is a non-zero simple $R_{m\gamma^{(k)}_{\mathbf{i}}}$*-module.*

Proof. Apply Theorem 3.6 2) and Corollary 4.8 to the Lusztig datum $\mathbf{c} = (0, \ldots, 0, m, 0, \ldots, 0)$, where the unique m is sitting at the k-th place. □

Proposition 4.10. *Let* $(\mathbf{i}, \mathbf{c})$ *and* $(\mathbf{i}', \mathbf{c}')$ *be two Lusztig data which are connected by a* 3*-move as* $(i_{k-1}, i_k, i_{k+1}) = (i'_k, i'_{k\pm1}, i'_k)$ *for some* k*. Then, we have* $\mathsf{hd}\, E^{\mathbf{i}}_{\mathbf{c}} \cong \mathsf{hd}\, E^{\mathbf{i}'}_{\mathbf{c}'}$*.*

Proof. Let $i_k = j, i_{k+1} = i$. By an explicit calculation (which reduces to the rank two case), we see that

$$\mathsf{hd}\,\left(L_{c_{k-1}i} \star \mathbb{T}_i L_{c_k j} \star \mathbb{T}_i\mathbb{T}_j L_{c_{k+1}i}\right) \cong \mathsf{hd}\,\left(L_{c'_{k-1}j} \star \mathbb{T}_j L_{c'_k i} \star \mathbb{T}_j\mathbb{T}_i L_{c'_{k+1}j}\right). \quad (4.4)$$

Thanks to Corollary 4.9 and Lemma 4.2 2), it suffices to show

$$\mathsf{hd}\, E^{\mathbf{i}}_{\mathbf{c}} \cong \mathsf{hd}\, E^{\mathbf{i}'}_{\mathbf{c}'}$$

for every $\mathbf{i}$-Lusztig datum $\mathbf{c}$ so that $c_1 = \cdots = c_{k+1} = 0$ (and its counterpart $\mathbf{i}'$-Lusztig datum $\mathbf{c}'$). We set

$$L_{s,c} := \mathbb{T}_{i_1}\cdots\mathbb{T}_{i_{s-1}}L_{ci_s} \text{ and } L'_{s,c} := \mathbb{T}_{i'_1}\cdots\mathbb{T}_{i'_{s-1}}L_{ci'_s}$$

for every $1 \le s \le \ell$ and $c \ge 0$. We have $L_{s,c} \cong L'_{s,c}$ for $1 \le s < k-1$. The both $L_{s,c}$ and $L'_{s,c}$ are non-zero simple $R_{c\gamma^{(s)}_{\mathbf{i}}}$-modules for $k+1 < s \le \ell$.

Claim A. *We have* $L_{s,c} \cong L'_{s,c}$ *for every* $k+1 < s \le \ell$ *and* $c \ge 1$*.*

Proof. We prove the assertion by induction on $\mathsf{ht}(c\gamma^{(s)}_{\mathbf{i}})$. The case $\mathsf{ht}(c\gamma^{(s)}_{\mathbf{i}}) = c\,\mathsf{ht}\,\gamma^{(s)}_{\mathbf{i}} = c$ is trivial since $\gamma^{(s)}_{\mathbf{i}}$ is a simple root (Example 2.10). Fix s and c so that the assertion is true for $\mathsf{ht}\,\beta < \mathsf{ht}(c\gamma^{(s)}_{\mathbf{i}})$. Taking Corollary 4.8 into account, we conclude that all but one simple $R_{c\gamma^{(s)}_{\mathbf{i}}}$-modules are realized by the heads of the modules of the form (4.2) constructed from simple modules of R_β with $\mathsf{ht}\,\beta < \mathsf{ht}(c\gamma^{(s)}_{\mathbf{i}})$. Thanks to (4.4) and the fact that all the other relevant simple modules are ordered in the common way, this simple module must be in common between $\mathbf{i}$ and $\mathbf{i}'$. Therefore, the induction proceeds and we conclude the result. □

We return to the proof of Proposition 4.10. By (4.4), Claim A, and Lemma 4.2 2), 4), we conclude the result. □

Corollary 4.11. *For Lusztig data* $(\mathbf{i}, \mathbf{c})$ *and* $(\mathbf{i}', \mathbf{c}')$*, we have* $\mathsf{hd}\, E^{\mathbf{i}}_{\mathbf{c}} \cong \mathsf{hd}\, E^{\mathbf{i}'}_{\mathbf{c}'}$ *if and only if* $(\mathbf{i}, \mathbf{c})$ *and* $(\mathbf{i}', \mathbf{c}')$ *are linked by a successive application of two-moves and three-moves.*

Proof. Every two reduced expressions of $w_0 \in W(\Gamma_0)$ are connected by a repeated use of two moves and three moves ([Lu90a] 2.1 (c)). Therefore, we enrich Corollary 4.8 by applying Proposition 4.10 instead of Lemma 4.7 in its proof to deduce the result. □

Thanks to Corollary 4.8, we often write $\widetilde{E}^{\mathbf{i}}_b$ and $E^{\mathbf{i}}_b$ instead of $\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}$ and $E^{\mathbf{i}}_{\mathbf{c}}$.

Since R_β is a finitely generated algebra free over a polynomial ring (by Theorem 2.11) with finite global dimension (by Theorem 2.9), it follows that

$$[M : L_b],\ \langle M, N \rangle_{\mathsf{gEP}} \in \mathbb{Z}(\!(t)\!), \quad \text{and} \quad \mathsf{gch}\, M \in \bigoplus_{b \in B(\infty)_\beta} \mathbb{Z}(\!(t)\!)[L_b]$$

for $M, N \in R_\beta$-gmod. (See (1.1) and Theorem 2.3 for $\langle \bullet, \bullet \rangle_{\mathsf{gEP}}$.)

Corollary 4.12. *Fix a reduced expression* $\mathbf{i}$ *and let* $\beta \in Q_+$*. Then, two sets* $\{\mathsf{gch}\, \widetilde{E}^{\mathbf{i}}_b\}_{b \in B(\infty)_\beta}$ *and* $\{\mathsf{gch}\, E^{\mathbf{i}}_b\}_{b \in B(\infty)_\beta}$ *are* $\mathbb{Z}(\!(t)\!)$*-bases of* $\bigoplus_{b \in B(\infty)_\beta} \mathbb{Z}(\!(t)\!)[L_b]$*, respectively.*

Proof. Thanks to Corollary 4.8 and Lemma 4.2 4), we deduce

$$\mathsf{gch}\, \widetilde{E}^{\mathbf{i}}_b \langle c_b \rangle \in [L_b] + \bigoplus_{b' \in B(\infty)_\beta} t\mathbb{Z}[\![t]\!][L_{b'}] \quad \text{for some} \quad c_b \in \mathbb{Z}.$$

This is enough to see the first assertion. (In fact, we can show $c_b = 0$ by a standard argument, or a consequence of Theorem 4.13 3).) The second assertion is similar. □

Thanks to Corollary 4.12, we define $[M : \widetilde{E}^{\mathbf{i}}_b], [M : E^{\mathbf{i}}_b] \in \mathbb{Z}(\!(t)\!)$ for every $M \in R_\beta$-gmod as:

$$\mathsf{gch}\, M = \sum_{b \in B(\infty)_\beta} [M : \widetilde{E}^{\mathbf{i}}_b]\, \mathsf{gch}\, \widetilde{E}^{\mathbf{i}}_b \quad \text{and} \quad \mathsf{gch}\, M = \sum_{b \in B(\infty)_\beta} [M : E^{\mathbf{i}}_b]\, \mathsf{gch}\, E^{\mathbf{i}}_b.$$

Theorem 4.13. *Fix a reduced expression* $\mathbf{i}$ *and* $\beta \in Q^+$*. We have:*

1. *For every* $b <_{\mathbf{i}} b'$*, it holds* $\mathrm{ext}^\bullet_{R_\beta}(\widetilde{E}^{\mathbf{i}}_b, \widetilde{E}^{\mathbf{i}}_{b'}) = \{0\}$*;*
2. *For each* $b \in B(\infty)_\beta$*, we have*
$$\mathrm{ext}^\bullet_{R_\beta}(\widetilde{E}^{\mathbf{i}}_b, E^{\mathbf{i}}_b) = \hom_{R_\beta}(\widetilde{E}^{\mathbf{i}}_b, E^{\mathbf{i}}_b) \cong \mathbb{C};$$
3. *For each* $b \in B(\infty)_\beta$*, we have*
$$[E^{\mathbf{i}}_b : L_{b'}] = \begin{cases} 0 & (b \not\leq_{\mathbf{i}} b') \\ 1 & (b = b') \end{cases} \quad \text{and} \quad [\widetilde{E}^{\mathbf{i}}_b : L_{b'}] = 0 \quad (b \not\leq_{\mathbf{i}} b');$$

4. For every $b \leq_{\mathbf{i}} b'$, it holds

$$\mathrm{ext}^{\bullet}_{R_\beta}(\widetilde{E}^{\mathbf{i}}_b, (E^{\mathbf{i}}_{b'})^*) \cong \hom_{R_\beta}(\widetilde{E}^{\mathbf{i}}_b, (E^{\mathbf{i}}_{b'})^*) \cong \mathbb{C}^{\oplus \delta_{b,b'}}.$$

Proof. We fix two elements $b <_{\mathbf{i}} b' \in B(\infty)_\beta$ which correspond to $\mathbf{i}$-Lusztig data $\mathbf{c}$ and $\mathbf{c}'$, respectively. Let m be the smallest number so that $c_m \neq 0$. (Note that $c'_1 = \cdots = c'_{m-1} = 0$.)

We prove these assertions by the downward induction on m. In particular, we assume all the assertions if $c_1 = \cdots = c_m = 0$. The base case $m = \ell$ is examined in Example 2.10 (since $\gamma^{(\ell)}_{\mathbf{i}}$ is a simple root).

We set $w_m := s_{i_{m-1}} s_{i_{m-2}} \cdots s_{i_1}$. Put $\beta_1 := w_m \beta - c_m \alpha_{i_m}$ and $\beta'_1 := w_m \beta' - c'_m \alpha_{i_m}$. By the $(m-1)$-times repeated application of the construction $\mathbf{i} \mapsto \mathbf{i}^\sharp$, we obtain

$$\mathbf{i}^\flat := (i_m, \ldots, i_\ell, i'_1, i'_2, \ldots, i'_{m-1}) \in I^\ell.$$

Let $\mathbf{d}$ and $\mathbf{d}'$ be the $\mathbf{i}^\flat$-Lusztig data given by $d_1 = c_m, d_2 = c_{m+1}, \ldots, d_{\ell-m+1} = c_\ell, d_{\ell-m+2} = \cdots = d_\ell = 0$ and $d'_1 = c'_m, d'_2 = c'_{m+1}, \ldots, d'_{\ell-m+1} = c'_\ell, d'_{\ell-m+2} = \cdots = d'_\ell = 0$, respectively. We also set $\mathbf{d}[1]$ and $\mathbf{d}'[1]$ as sequences defined as: $d_j[1] = 0$ $(j = 1)$ or d_j $(1 < j \leq \ell)$, and $d'_j[1] = 0$ $(j = 1)$ or d'_j $(1 < j \leq \ell)$. We have

$$\widetilde{E}^{\mathbf{i}}_{\mathbf{c}} \cong \mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{m-1}} \widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}} \quad \text{and} \quad E^{\mathbf{i}}_{\mathbf{c}'} \cong \mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{m-1}} E^{\mathbf{i}^\flat}_{\mathbf{d}'}.$$

Claim B. *We have*

$$\widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}} \cong \mathbb{T}^*_{i_{m-1}} \cdots \mathbb{T}^*_{i_1} \widetilde{E}^{\mathbf{i}}_{\mathbf{c}} \quad \textit{and} \quad E^{\mathbf{i}^\flat}_{\mathbf{d}'} \cong \mathbb{T}^*_{i_{m-1}} \cdots \mathbb{T}^*_{i_1} E^{\mathbf{i}}_{\mathbf{c}'}.$$

Proof. By Corollary 4.9 and Theorem 3.6 2), we have necessarily

$$\mathbb{T}_{i_j} \mathbb{T}_{i_{j+1}} \cdots \mathbb{T}_{i_{k-1}} L_{c_k i_k} \in R_{s_{i_{j-1}} \cdots s_{i_1} \beta}\text{-}\mathsf{gmod}^{i_{j-1}} \qquad \text{for each } 1 < j < k.$$

Applying this to Lemma 4.2 2) and use Theorem 3.6 2), we deduce

$$E^{\mathbf{i}^\flat}_{\mathbf{d}} \cong \mathbb{T}^*_{i_{m-1}} \cdots \mathbb{T}^*_{i_1} E^{\mathbf{i}}_{\mathbf{c}}.$$

In particular, the lengths of the R_β-module $E^{\mathbf{i}^\flat}_{\mathbf{d}}$ and $R_{w_m\beta}$-module $E^{\mathbf{i}}_{\mathbf{c}}$ are the same. Since each $\mathbb{T}_{i_j}$ annihilates simple modules which are not irreducible constituents of $\mathbb{T}_{i_{j+1}} \cdots \mathbb{T}_{i_{m-1}} E^{\mathbf{i}^\flat}_{\mathbf{d}}$, we conclude

$$\widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}} \cong \mathbb{T}^*_{i_{m-1}} \cdots \mathbb{T}^*_{i_1} \widetilde{E}^{\mathbf{i}}_{\mathbf{c}}$$

by Lemma 4.2 3). The case of $E^{\mathbf{i}}_{\mathbf{c}'}$ is similar. □

Claim C. *The vanishing of* $\mathrm{ext}^{\bullet}_{R_\beta}(\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}, \widetilde{E}^{\mathbf{i}}_{\mathbf{c}'})$ *follows from the vanishing of*

$$\mathrm{ext}^{\bullet}_{R_\beta}(\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}, E^{\mathbf{i}}_{\mathbf{c}'}) \cong \mathrm{ext}^{\bullet}_{R_{c_m \alpha_{i_m}} \boxtimes R_{\beta_1}}(P_{c_m i_m} \boxtimes \widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}[1]}, E^{\mathbf{i}^\flat}_{\mathbf{d}'}). \tag{4.5}$$

Proof. Thanks to Claim B, a repeated use of Theorem 3.6 4) implies a sequence of isomorphisms

$$\begin{aligned}\mathrm{ext}^{\bullet}_{R_\beta}(\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}, E^{\mathbf{i}}_{\mathbf{c}'}) &\cong \mathrm{ext}^{\bullet}_{R_\beta}(\mathbb{T}_1 \cdots \mathbb{T}_{m-1} \widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}}, \mathbb{T}_1 \cdots \mathbb{T}_{m-1} E^{\mathbf{i}^\flat}_{\mathbf{d}'}) \\ &\cong \mathrm{ext}^{\bullet}_{R_{s_{i_1}\beta}}(\mathbb{T}_2 \cdots \mathbb{T}_{m-1} \widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}}, \mathbb{T}_2 \cdots \mathbb{T}_{m-1} E^{\mathbf{i}^\flat}_{\mathbf{d}'}) \\ \cdots &\cong \mathrm{ext}^{\bullet}_{R_{w_m\beta}}(\widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}}, E^{\mathbf{i}^\flat}_{\mathbf{d}'}).\end{aligned}$$

Since $\star$ preserves the projectivity (see e.g. [KL09] 2.16) and $\widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}} \cong P_{c_m i_m} \star \widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}[1]}$, we conclude an isomorphism

$$\mathrm{ext}^\bullet_{R_{w_m\beta}}(\widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}}, E^{\mathbf{i}^\flat}_{\mathbf{d}'}) \cong \mathrm{ext}^\bullet_{R_{c_m i_m} \boxtimes R_{\beta_1}}(P_{c_m i_m} \boxtimes \widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}[1]}, E^{\mathbf{i}^\flat}_{\mathbf{d}'}).$$

It remains to deduce $\mathrm{ext}^\bullet_{R_\beta}(\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}, \widetilde{E}^{\mathbf{i}}_{\mathbf{c}'}) = \{0\}$ from the vanishing of (4.5).

Since R_β is a Noetherian ring with finite global dimension, we have a projective resolution $\mathcal{P}^\bullet$ of $\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}$, which consists of finitely many finitely generated projective R_β-modules. In particular, there exists $x \in \mathbb{Z}$ so that the degrees of simple quotients of all R_β-module direct summands of $\mathcal{P}^\bullet$ are $\le x$.

For each $j \in \mathbb{Z}$, we have a (surjective) A-module quotient $\varphi_j : \widetilde{E}^{\mathbf{i}}_{\mathbf{c}'} \to E_j$ so that **a)** $\ker \varphi_j$ is concentrated in degree $> j + x$, and **b)** E_j is a finite successive self-extension of (grading shifts) of $E^{\mathbf{i}}_{\mathbf{c}'}$ by Lemma 4.2 3). Then, $\mathrm{ext}^\bullet_{R_\beta}(\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}, E^{\mathbf{i}}_{\mathbf{c}'}) = \{0\}$ implies

$$\mathrm{ext}^\bullet_{R_\beta}(\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}, \ker \varphi_j)^j = \{0\} = \mathrm{ext}^\bullet_{R_\beta}(\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}, E_j).$$

This yields $\mathrm{ext}^\bullet_{R_\beta}(\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}, \widetilde{E}^{\mathbf{i}}_{\mathbf{c}'})^j = \{0\}$ (for each j) as required. □

We return to the proof of Theorem 4.13.

We have the following two short exact sequences:

$$0 \to L_{c_m i_m} \boxtimes E^{\mathbf{i}^\flat}_{\mathbf{d}[1]} \to E^{\mathbf{i}^\flat}_{\mathbf{d}} \to C \to 0 \quad \text{as } R_{c_m\alpha_{i_m}} \boxtimes R_{\beta_1}\text{-modules, and}$$
$$0 \to L_{c'_m i_m} \boxtimes E^{\mathbf{i}^\flat}_{\mathbf{d}'[1]} \to E^{\mathbf{i}^\flat}_{\mathbf{d}'} \to C' \to 0 \quad \text{as } R_{c'_m\alpha_{i_m}} \boxtimes R_{\beta'_1}\text{-modules.}$$

Claim D. *We have $e_{i_m}(c_m)C = \{0\}$ and $e_{i_m}(c'_m)C' = \{0\}$.*

Proof. Let Ψ be the set of $\mathbf{m} \in Y^{w_m\beta}$ so that

$$e(\mathbf{m})(L_{c'_m i_m} \boxtimes E^{\mathbf{i}^\flat}_{\mathbf{d}'[1]}) \neq \{0\}.$$

Let $n = \mathsf{ht}\, w_m\beta$, and let $\mathfrak{S}$ be the set of minimal length representatives of $\mathfrak{S}_n/(\mathfrak{S}_{c'_m} \times \mathfrak{S}_{n-c'_m})$ inside $\mathfrak{S}_n$. We set $\mathfrak{S}^* := \mathfrak{S}\backslash\{1\}$. We have $e(\mathbf{m})C' \neq \{0\}$ only if $\mathbf{m} \in \mathfrak{S}^*\Psi$. Since $E^{\mathbf{i}^\flat}_{\mathbf{d}'[1]}$ belongs to the (essential) image of $\mathbb{T}_{i_m}$, we have $e_{i_m}(1)E^{\mathbf{i}^\flat}_{\mathbf{d}'[1]} = \{0\}$. On the other hand, we have $Y^{c'_1\alpha_{i_1}} = \{(i_1, \ldots, i_1)\}$. Since every element of $\mathfrak{S}^*$ decreases the number of heading $i_1, \ldots, i_1$, we deduce the assertion for C'. The case of C is the same. □

Claim E. *Every irreducible constituent $L_{b''}\langle k\rangle$ of $E^{\mathbf{i}}_{\mathbf{c}}$ ($b'' \in B(\infty)_\beta, k > 0$) satisfies $b'' >_{\mathbf{i}} b$.*

Proof. By Claim B and Theorem 3.6 2), the module $L_{b''}\langle k\rangle$ gives rise to an irreducible constituent $\mathbb{T}^*_{i_{m-1}} \cdots \mathbb{T}^*_{i_1} L_{b''}$ of $E^{\mathbf{i}^\flat}_{\mathbf{d}}$, that we denote by $L_{b^\flat}\langle k\rangle$ with $b^\flat \in B(\infty)_{w_m\beta}$. We have $\mathbb{T}_{i_1} \cdots \mathbb{T}_{i_\ell} = 0$ since it must send an R_γ-module to an $R_{w_0\gamma}$-module, which is inexistent. In particular, the $\mathbf{i}^\flat$-Lusztig datum $\mathbf{d}^\flat$ of $b^\flat$ must be of the form $d^\flat_{\ell-m+2} = \cdots d^\flat_\ell = 0$, and the $\mathbf{i}$-Lusztig datum $\mathbf{c}''$ of b'' is given as

$$c''_1 = \cdots c''_{m-1} = 0, c''_m = d^\flat_1, c''_{m+1} = d^\flat_2, \ldots, c''_\ell = d^\flat_{\ell-m+1}.$$

Claim D implies that $\epsilon_{i_m}(b^\flat) = d_1^\flat \le c_m$. In case this inequality is strict, we have $b'' >_{\mathbf{i}} b$. Hence, we assume $\epsilon_{i_m}(b^\flat) = c_m$. Theorem 2.12 asserts that we have $e_{i_m}(c_m) L_{b''} = L_{c_m i_m} \boxtimes L_{b_1^\flat}$ for some $b_1^\flat \in B(\infty)_{\beta_1}$. In addition, Claim D (and the above arguments) imply

$$0 \neq [E^{\mathbf{i}^\flat}_{\mathbf{d}'[1]} : L_{b_1^\flat} \langle k \rangle]_0 = [\mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{m-1}} E^{\mathbf{i}^\flat}_{\mathbf{d}'[1]} : \mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{m-1}} L_{b_1^\flat} \langle k \rangle]_0.$$

In case $k > 0$, this yields $\mathbf{d}_1^\flat >_{\mathbf{i}^\flat} \mathbf{d}'[1]$ by induction hypothesis, where $\mathbf{d}_1^\flat$ is the $\mathbf{i}^\flat$-Lusztig datum of $b_1^\flat$ (that is obtained from $\mathbf{d}^\flat$ by replacing $d_1^\flat$ by 0). This implies $b'' >_{\mathbf{i}} b$ as required. □

We return to the proof of Theorem 4.13. Claim E, together with Lemma 4.2 3), implies the third assertion. In addition, we deduce that the RHS of (4.5) must be zero if $c_m > c'_m$.

We assume $c_m = c'_m$. Applying Claim D, we deduce that

$$\mathrm{ext}^\bullet_{R_{c_m \alpha_{i_m}} \boxtimes R_{\beta_1}} (P_{c_m i_m} \boxtimes \widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}[1]}, C') = \{0\}.$$

Therefore, the vector spaces in (4.5) is isomorphic to

$$\mathrm{ext}^\bullet_{R_{c_m \alpha_{i_m}} \boxtimes R_{\beta_1}} (P_{c_m i_m} \boxtimes \widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}[1]}, L_{c_m i_m} \boxtimes E^{\mathbf{i}^\flat}_{\mathbf{d}'[1]}).$$

Here we have $\mathrm{ext}^\bullet_{R_{c_m \alpha_{i_m}}} (P_{c_m i_m}, L_{c_m i_m}) = \hom_{R_{c_m \alpha_{i_m}}} (P_{c_m i_m}, L_{c_m i_m}) = \mathbb{C}$. Therefore, Claim B, Lemma 4.2 2), and Theorem 3.6 4) implies

$$\begin{aligned} &\mathrm{ext}^*_{R_{c_m \alpha_{i_m}} \boxtimes R_{\beta_1}} (P_{c_m i_m} \boxtimes \widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}[1]}, L_{c_m i_m} \boxtimes E^{\mathbf{i}^\flat}_{\mathbf{d}'[1]}) \cong \mathrm{ext}^*_{R_{\beta_1}} (\widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}[1]}, E^{\mathbf{i}^\flat}_{\mathbf{d}'[1]}) \\ &\cong \mathrm{ext}^*_{R_{s_{i_{m-1}} \beta_1}} (\mathbb{T}_{i_{m-1}} \widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}[1]}, \mathbb{T}_{i_{m-1}} E^{\mathbf{i}^\flat}_{\mathbf{d}'[1]}) \cong \cdots \cong \mathrm{ext}^*_{R_{w_m \beta_1}} (\widetilde{E}^{\mathbf{i}}_{\mathbf{c}[1]}, E^{\mathbf{i}}_{\mathbf{c}'[1]}), \quad (4.6) \end{aligned}$$

where $\mathbf{c}[1]$ and $\mathbf{c}'[1]$ are the $\mathbf{i}$-Lusztig data of $\mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{m-1}} \widetilde{E}^{\mathbf{i}^\flat}_{\mathbf{d}[1]}$ and $\mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{m-1}} E^{\mathbf{i}^\flat}_{\mathbf{d}'[1]}$, respectively. Therefore, we deduce the first two assertions by the induction hypothesis and Claim C.

The fourth assertion follows from the vanishing of (4.5) and the middle two assertions by applying long exact sequences repeatedly.

Therefore, the induction proceeds and we conclude the result. □

Corollary 4.14. *Fix a reduced expression* $\mathbf{i}$ *and* $\beta \in Q^+$. *We have*

$$\widetilde{E}^{\mathbf{i}}_b = P_b / \Big(\sum_{f \in \hom_{R_\beta}(P_{b'}, P_b), b' <_{\mathbf{i}} b} \mathrm{Im}\, f \Big) \quad \text{and} \quad E^{\mathbf{i}}_b = P_b / \Big(\sum_{f \in \hom_{R_\beta}(P_b, \widetilde{E}^{\mathbf{i}}_b)^{>0}} \mathrm{Im}\, f \Big),$$

where b *and* b' *runs over* $B(\infty)_\beta$.

Proof. By Lemma 4.2 4), $\widetilde{E}^{\mathbf{i}}_b$ admits a surjection from P_b. By Theorem 4.13 3), we conclude that all the simple subquotient $\widetilde{E}^{\mathbf{i}}_b$ is of the form $L_{b'} \langle k \rangle$ for $b \le_{\mathbf{i}} b'$, and hence the RHS surjects onto $\widetilde{E}^{\mathbf{i}}_b$. By Theorem 4.13 3) and 4), the head of $\ker (P_b \to \widetilde{E}^{\mathbf{i}}_b)$ must be spanned by $L_{b'} \langle k \rangle$ for $b' <_{\mathbf{i}} b$ and $k \in \mathbb{Z}$, and hence the both sides are maximal quotients of P_b whose simple subquotients are that form. Therefore, they are isomorphic to each other. This proves the first assertion. The second assertion follows by Lemma 4.2 3) and Theorem 4.13 2). □

Corollary 4.15. *Fix a reduced expression* $\mathbf{i}$ *and* $\beta \in Q^+$. *Then, we have*

$$\mathrm{ext}^i_{R_\beta}(\widetilde{E}^{\mathbf{i}}_b, (E^{\mathbf{i}}_{b'})^*) = \begin{cases} \mathbb{C} & (b \neq b', i = 0) \\ \{0\} & (\textit{otherwise}) \end{cases}, \quad \textit{and} \quad \left\langle \widetilde{E}^{\mathbf{i}}_b, (E^{\mathbf{i}}_{b'})^* \right\rangle_{\mathsf{gEP}} = \delta_{b,b'}$$

for every $b, b' \in B(\infty)_\beta$.

Proof. Since the first assertion implies the second assertion, we prove only the first assertion. If $b \le_{\mathbf{i}} b'$, then the assertion follows from Theorem 4.13 4). Thanks to Example 2.10, each L_{ci} admits a finite resolution by the graded shifts of P_{ci} (for each $c \geq 1$ and $i \in I$). By Lemma 4.2 2), we deduce that each $E^{\mathbf{i}}_{\mathbf{c}}$ admits a finite resolution by the graded shifts by $\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}$. By taking the spectral sequence of this resolution, we deduce

$$\mathrm{ext}^{\bullet}_{R_\beta}(E^{\mathbf{i}}_b, (E^{\mathbf{i}}_{b'})^*) = \{0\} \tag{4.7}$$

for each $b <_{\mathbf{i}} b'$. Since $*$ is an exact functor and ext^i_A is a universal δ-functor, we deduce

$$\hom_{R_\beta}(M, N^*) \cong \hom_{R_\beta}(N, M^*)$$

for each $M, N \in R_\beta\text{-}\mathsf{gmod}$. In particular, we conclude (4.7) unless $b = b'$.

Thanks to Theorem 2.3 and the definition of the algebras $A_{(G,\mathfrak{x})}$ (and $B_{(G,\mathfrak{x})}$) in §1, we can replace R_β with its basic ring to assume that it is non-negatively graded. Then, we have $(E^{\mathbf{i}}_{b'})^*_k = \{0\}$ for every $k > 0$. For each $j \in \mathbb{Z}$, we have an R_β-module quotient $\varphi_j : \widetilde{E}^{\mathbf{i}}_b \to E_j$ so that **a)** ker φ_j is concentrated in degree $> -j$, and **b)** E_j is a finite successive self-extension of (graded shifts) of $E^{\mathbf{i}}_b$ by Lemma 4.2 3). Then, the minimal projective resolution of ker φ_j is concentrated in degree $> -j$. In particular, we have

$$\mathrm{ext}^{\bullet}_{R_\beta}(\ker \varphi_j, (E^{\mathbf{i}}_{b'})^*)^j = \{0\} = \mathrm{ext}^{\bullet}_{R_\beta}(E_j, (E^{\mathbf{i}}_{b'})^*) \quad \text{for each} \quad b \neq b'.$$

This yields $\mathrm{ext}^{\bullet}_{R_\beta}(\widetilde{E}^{\mathbf{i}}_b, (E^{\mathbf{i}}_{b'})^*)^j = \{0\}$ (for each j) as required. □

Remark 4.16. For each $\beta \in Q^+$, the standard normalization

$$\langle P_b, L_b \rangle_{\mathsf{gEP}} = \mathsf{gdim}\, \hom_{R_\beta}(P_b, L_b) = \delta_{b,b'},$$

combined with Theorem 2.5, Corollary 4.15 and Theorem 4.13 implies that $\{\mathsf{gch}\, \widetilde{E}^{\mathbf{i}}_b\}_b$ and $\{\mathsf{gch}\, E^{\mathbf{i}}_b\}_b$ give rise to the lower/upper PBW bases corresponding to $\mathbf{i}$, respectively.

Theorem 4.17 (Lusztig's conjecture)**.** *For every reduced expression* $\mathbf{i}$ *of* w_0, $\beta \in Q^+$, *and* $b, b' \in B(\infty)_\beta$, *we have an equality*

$$[P_b : \widetilde{E}^{\mathbf{i}}_{b'}] = [E^{\mathbf{i}}_{b'} : L_b].$$

In particular, the expansion coefficients of the lower global basis in terms of the lower PBW basis are in $\mathbb{N}[t]$.

Remark 4.18. Thanks to [K12a] 1.7 (and Lusztig [Lu90a] 10.6), a projective module P_b admits a filtration by $\{\widetilde{E}^{\mathbf{i}}_{b'}\}_{b'}$ if $\mathbf{i}$ is adapted to Γ.

Proof of Theorem 4.17. By Corollary 4.15, we have

$$\delta_{b,b'} = \langle P_b, L^*_{b'} \rangle_{\mathsf{gEP}} = \sum_{d,d' \in B(\infty)_\beta} \overline{[P_b : \widetilde{E}^{\mathbf{i}}_d]} \overline{[L_{b'} : E^{\mathbf{i}}_{d'}]} \left\langle \widetilde{E}^{\mathbf{i}}_d, (E^{\mathbf{i}}_{d'})^* \right\rangle_{\mathsf{gEP}}$$
$$= \sum_{d \in B(\infty)_\beta} \overline{[P_b : \widetilde{E}^{\mathbf{i}}_d]} \overline{[L_{b'} : E^{\mathbf{i}}_d]}.$$

By applying the bar involution, this shows that

$$([P_b : \widetilde{E}^{\mathbf{i}}_d])([E^{\mathbf{i}}_{d'} : L_{b'}])^{-1} = (\delta_{b,b'}),$$

which is equivalent to the assertion. □

For each $\beta, \beta' \in Q^+$, we define the formal expression q^β and $q^{\beta'}$ so that $q^\beta \cdot q^{\beta'} = q^{\beta+\beta'}$. We define

$$\mathrm{ep}_t(q^\beta) := \sum_{n \ge 0} \frac{q^{n\beta}}{(1-t^2)(1-t^4)\cdots(1-t^{2n})} \in \mathbb{Q}(t)[\![Q^+]\!].$$

Corollary 4.19 (cf. Problem 2 in Kashiwara [Kas95]). *For each $\beta \in Q^+$, we set*

$$[P : L]_\beta := ([P_b : L_{b'}])_{b,b' \in B(\infty)_\beta} = (\langle P_{b'}, P_b \rangle_{\mathsf{gEP}})_{b,b' \in B(\infty)_\beta}$$

as the square matrix with its determinant D_β. We have

$$\sum_{\beta \in Q^+} D_\beta q^\beta = \prod_{\alpha \in R^+} \mathrm{ep}_t(\alpha).$$

Proof. As in the proof of Theorem 4.17, we factorize

$$[P : L]_\beta = [P : \widetilde{E}]_\beta [\widetilde{E} : E]_\beta [E : L]_\beta,$$

where the second term is the $\# B(\infty)_\beta$-square matrix of expansion coefficients between projectives/lower PBWs, lower PBWs/upper PBWs, and upper PBWs/simples, respectively. By Theorem 4.13 3), the determinant of the third matrix is 1. By Theorem 4.17, the determinant of the first matrix is also 1. By Lemma 4.2 3) (cf. Corollary 4.15), we conclude

$$D_\beta = \prod_{b \in B(\infty)_\beta} [\widetilde{E}^{\mathbf{i}}_b : E^{\mathbf{i}}_b].$$

By Lemma 4.2 2) and the construction of $\mathbb{T}_{i_j}$, if we denote $\mathbf{c}$ the $\mathbf{i}$-Lusztig datum corresponding to b, then we have

$$[\widetilde{E}^{\mathbf{i}}_b : E^{\mathbf{i}}_b] = \prod_{j=1}^{\ell} [P_{c_j i_j} : L_{c_j i_j}] = \prod_{j=1}^{\ell} \frac{1}{(1-t^2)(1-t^4)\cdots(1-t^{2c_j})}.$$

This is equivalent to the assertion by a simple counting. □

Remark 4.20. **1)** By a formal manipulation, we have

$$\langle P_b, P_{b'} \rangle_{\mathsf{gEP}} = \overline{\langle P_b, P^*_{b'} \rangle_{\mathsf{gEP}}} \qquad \text{for every } b, b' \in B(\infty).$$

Since the RHS calculates the Lusztig inner form $\{\bullet, \bullet\}$ ([Lus93] 1.2.10) of the lower global basis, Corollary 4.19 yields the Shapovalev determinant formula of quantum groups of type ADE. **2)** The proof of Corollary 4.19 also follows from [K12a] 3.12, but here the proof works also from the PBW bases $\{E^{\mathbf{i}}_b\}_b$ in which $\mathbf{i}$ is not an adapted reduced expression of w_0.

References

[Ari96] Susumu Ariki, On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$, J. Math. Kyoto Univ. 36, no. 4 (1996), 789–808.

[Ari05] Susumu Ariki. Modular representation theory of Hecke algebras, a survey. Contemp. Math. 392 (2005) 1–14

[BBD82] Alexander Beilinson, Joseph Bernstein, and Pierre Deligne, Faisceaux pervers, Astérisque 100 (1982).

[BKOP] Georgia Benkart, Seok-Jin Kang, Se-jin Oh, Euiyong Park, Construction of irreducible representations over Khovanov-Lauda-Rouquier algebras of finite classical type, arXiv:1108.1048, to appear in Int. Math. Res. Notices

[BL94] Joseph Bernstein, and Varely Lunts, Equivariant sheaves and functors, Lecture Note in Math. 1578, Springer-Verlag 1994.

[BKM12] Jonathan Brundan, Alexander Kleshchev, Peter J. McNamara, Homological properties of finite type Khovanov-Lauda-Rouquier algebras, Duke Math. J. 163 no. 7 1353–1404 (2014)

[CG97] Neil Chriss, and Victor Ginzburg, Representation theory and complex geometry. Birkhäuser Boston, Inc., Boston, MA, 1997. x+495 pp. ISBN 0-8176-3792-3

[GL93] Ian Grojnowski, and George Lusztig, A comparison of bases of quantized enveloping algebras. In Linear algebraic groups and their representations (Los Angeles, CA, 1992), Contemp. Math. 153, pages 11–19. 1993

[KK12] Seok-Jin Kang, and Masaki Kashiwara, Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras, Invent. Math. 190 no.3 (2012) 699–742.

[Kas91] Masaki Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras. Duke Math. J. 63 (1991), 465–516.

[Kas95] Masaki Kashiwara, On crystal bases, Representations of Groups, Proceedings of the 1994 Annual Seminar of the Canadian Math. Soc. Ban 16 (1995) 155–197, Amer. Math. Soc., Providence, RI.

[KS90] Masaki Kashiwara, and Pierre Schapira, Sheaves on Manifolds, Grundlehren Math. Wiss., vol. 292, Springer-Verlag, 1990, x+512 pp.

[K09] Syu Kato, An exotic Deligne-Langlands correspondence for symplectic groups, Duke Math. J. 148 (2009), 305–371.

[K12a] Syu Kato, An algebraic study of extension algebras, arXiv:1207.4640v4.

[KL87] David Kazhdan, and George Lusztig, Proof of the Deligne-Langlands conjecture for Hecke algebras, Invent. Math. 87 (1987), no. 1, 153–215.

[KL09] Mikhail Khovanov, and Aaron Lauda, A diagrammatic approach tocategorification of quantum groups I, Represent. Theory 13 (2009), 309–347.

[KR11] Alexander Kleshchev, and Arun Ram, Representations of Khovanov-Lauda-Rouquier algebras and combinatorics of Lyndon words, Math. Ann. 349 (2011), 943–975

[LV11] Aaron Lauda, and Monica Vazirani, Crystals from categorified quantum groups, Adv. Math., 228, (2011) 803–861.

[Lu90a] George Lusztig, Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. 3 (1990) 447–498

[Lu90b] George Lusztig, Canonical bases arising from quantized enveloping algebras II, in "Common trends in mathematics and quantum field theories", Progr. of Theor. Phys. Suppl. 102 (1990), ed. T.Eguchi & al., 175–201.

[Lus91] George Lusztig, Quivers, perverse sheaves and quantized enveloping algebras, J. Amer. Math. Soc. 4 (1991) 365–421.

[Lus93] George Lusztig, Introduction to quantum groups, Birkhäuser, 1993.

[Lus98] George Lusztig, Canonical bases and Hall algebras, In: Representation theories and algebraic geometry (Montreal, PQ, 1997), 365–399, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Kluwer Acad. Publ., Dordrecht, 1998.

[Lus02] George Lusztig, Cuspidal local systems and graded Hecke algebras, III, Represent. Theory 6 (2002), 202–242.

[Lus08] George Lusztig, A survey of total positivity, Milan J. Math., 76, no. 1, (2008) 125–134

[Mac12] Peter J. McNamara Finite Dimensional Representations of Khovanov-Lauda-Rouquier Algebras I, arXiv:1207.5860, to appear in J. reine angew. Math.

[Nak06] Hiraku Nakajima, Quiver varieties and quantum affine algebras, Sugaku Expositions, 19 (2006), no. 1, 53–78.

[OS09] Eric Opdam, and Maarten Solleveld, Homological algebra for affine Hecke algebras, Adv. Math. 220 (2009), 1549-1601.

[Rin90] Claus Michael Ringel. Hall algebras and quantum groups. Invent. Math., 101 no.3 583–591,1990

[Rou08] Raphaël Rouquier, 2-Kac-Moody algebras, arXiv:0812.5023.

[Sai94] Yoshihisa Saito, PBW basis of quantized universal enveloping algebras. Publ. Res. Inst. Math. Sci. 30 (1994), no. 2, 209–232.

[SGA4] Michel Artin, Alexandre Grothendieck, and Jean-Louis Verdier, eds. Seminaire de Geometrie Algebrique - Theorie des topos et cohomologie etale des schemas vol. 3. Lecture Notes in Math. 305. Springer-Verlag, 1972 vi+640.

[VV99] Michela Varagnolo, and Eric Vasserot, On the decomposition matrices of the quantized Schur algebra, Duke Math. J. 100, (1999) 267–297.

[VV11] Michela Varagnolo, and Eric Vasserot, Canonical bases and KLR-algebras, J. reine angew. Math. 659 67–100 (2011)

[Web10] Ben Webster, Knot invariants and higher representation theory I: diagrammatic and geometric categorification of tensor products, arXiv:1001.2020

[Zhe08] Hao Zheng, Categorification of integrable representations of quantum groups, arXiv:0803.3668