Non-mean-field critical exponent in a mean-field model: Dynamics versus statistical mechanics

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Mean-field theory tells us that the classical critical exponent of susceptibility is twice that of magnetization. However, linear response theory based on the Vlasov equation, which is naturally introduced by the mean-field nature, makes the former exponent half of the latter for families of quasistationary states having second order phase transitions in the Hamiltonian mean-field model and its variances, in the low-energy phase. We clarify that this strange exponent is due to the existence of Casimir invariants which trap the system in a quasistationary state for a time scale diverging with the system size. The theoretical prediction is numerically confirmed by N-body simulations for the equilibrium states and a family of quasistationary states.

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I. INTRODUCTION

Are critical exponents of an isolated dynamical system the same as those computed via the statistical mechanics? We tackle this question by dealing with a ferromagneticlike model in the mean-field universality class and considering the critical exponents of the zero-field susceptibility. Isothermal susceptibility χ^{T} can be obtained by using standard methods of statistical mechanics, while the susceptibility of an isolated system χ^{I} can be derived from linear response theory [1]. These two susceptibilities satisfy the inequality $\chi^{\mathrm{I}} \leqslant \chi^{\mathrm{T}}$ [2,3] which is derived considering the existence of invariants [4,5]. This implies that the exponents γ^{T} and γ^{I} with which the two susceptibilities diverge at the critical point satisfy $\gamma^{\mathrm{I}} \leqslant \gamma^{\mathrm{T}}$. Is it possible that γ^{I} is strictly smaller than γ^{T}? A difficulty in answering this question is that the susceptibility of an isolated system can not be easily evaluated. In this article, we show how kinetic theory [6,7] can effectively answer the initial question in systems of the mean-field type using a recently developed version of linear response theory [8,9] based on the Vlasov equation.

Many different physical systems can be described by kinetic theory, including self-gravitating systems, plasmas, and fluids [6,7,10]. For N-particle systems with long-range interactions [11], both perturbative approaches [6] and the rigorous mean-field limit [12,13] lead to a description of the system in the continuum $N \to \infty$ limit in terms of the Vlasov equation. This equation rules the time evolution of the single-particle distribution function and has an infinity of stationary solutions. For instance, all distribution functions that depend on phase-space variables only through the single-particle energy do not evolve in time, as proven by Jeans [14]. On the long time scale, the system is described by appropriate kinetic equations which include "collisional" (finite N) effects, such as the Landau and Balescu-Lenard equation, and evolves towards Boltzmann-Gibbs (BG) equilibrium. However, since the relaxation time scale diverges with N [15], the early evolution of the system is well described by the Vlasov equation. Therefore, the use of the linear response theory developed in [8,9] is appropriate in the large N limit.

In order to perform explicit calculations of susceptibilities, it is convenient to consider the so-called Hamiltonian mean-field (HMF) model [11,16–18]. This model describes the motion of N particles on a circle interacting with an attractive cosine potential. The BG equilibrium solution of this model displays a high-energy phase where the particles are uniformly distributed on the circle and a low-energy phase where the particles form a cluster. The two phases are separated by a second-order phase transition point at which susceptibility diverges with the classical mean-field exponents. On the other hand, in the mean-field limit, the time evolution of the single-particle distribution function of the HMF model is exactly described by the Vlasov equation. Moreover, a BG homogeneous state is a stationary solution of this equation which loses its stability at an energy which coincides with the second-order phase transition energy [17]. Below this energy, the BG inhomogeneous state is also a stable stationary state of the Vlasov equation.

Stable stationary states almost do not evolve even in the system with finite but large N, and are called quasistationary states (QSSs) [15,19]. The long-lasting QSSs, therefore, show nonequilibrium phase transitions, and a phase diagram is theoretically drawn for a set of initial states with the aid of a nonequilibrium statistical mechanics [20]. In this article, we perform a detailed analysis of the scaling laws of susceptibility around the critical point of (non)equilibrium phase transitions for quasistationary states. We remark that the BG equilibrium states are kinds of QSSs, and hence the obtained scaling laws are valid even for the BG equilibrium states.

This article is constructed as follows. We introduce the HMF model and the corresponding Vlasov system in Sec. II. The scaling of the Vlasov susceptibility is analyzed in Sec. III, and the theoretical prediction is numerically confirmed in Sec. IV. A generalization from the HMF model is discussed in Sec. V. Section VI is devoted to summary and discussions.

II. HAMILTONIAN MEAN-FIELD MODEL

The Hamiltonian function of the HMF model reads as

$$H_N = \sum_{i=1}^{N} \left[\frac{p_i^2}{2} + \sum_{j=1}^{N} \frac{1-\cos(q_i-q_j)}{2N} - h\Theta(t)\cos(q_i-\phi) \right], \tag{1}$$

where h and ϕ are, respectively, the modulus and the phase of the external magnetic vector $(h\cos\phi, h\sin\phi)$, and $\Theta(t)$ is the Heaviside step function. The magnetization vector $(\langle M_x\rangle_N, \langle M_y\rangle_N)$ is defined by

$$(\langle M_x\rangle_N, \langle M_y\rangle_N) = \frac{1}{N}\sum_{j=1}^{N}(\cos q_j, \sin q_j), \quad (2)$$

where $\langle\ldots\rangle_N$ represents the average over N particles. The isolated system ($h=0$) has the rotational symmetry, therefore, we consider $\phi=0$ and $\langle M_y\rangle_N = 0$ without loss of generality. As a consequence, we call the x axis the direction of the spontaneous magnetization.

The corresponding effective one-particle Hamiltonian of HMF is

$$H_h[f_h](q,p,t) = \frac{p^2}{2} - \langle M\rangle_h \cos q - h\Theta(t)\cos q, \quad (3)$$

where the magnetization observable is $M(q) = \cos q$ and the brackets $\langle\ldots\rangle_h$ mean the average respects to the single-particle distribution f_h. The distribution f_h evolves following the Vlasov equation

$$\frac{\partial f_h}{\partial t} + \{H_h[f_h], f_h\} = 0, \quad (4)$$

where $\{\ldots,\ldots\}$ is the Poisson bracket defined by

$$\{a,b\} = \frac{\partial a}{\partial p}\frac{\partial b}{\partial q} - \frac{\partial a}{\partial q}\frac{\partial b}{\partial q}. \quad (5)$$

We note that the magnetization $\langle M\rangle_h$ appearing in H_h [Eq. (3)] is determined self-consistently to satisfy the equation

$$\langle M\rangle_h = \iint \cos q f_h dq\, dp. \quad (6)$$

Let us consider the case in which the external field is turned off. In that case, all the stationary states are in the Jeans' class, and are functions of H_0 specified by

$$f_0(q,p;b,a_1,\ldots,a_n) = \frac{F(H_0(q,p);b,a_1,\ldots,a_n)}{\langle\langle F(H_0(q,p);b,a_1,\ldots,a_n)\rangle\rangle}, \quad (7)$$

where

$$\langle\langle \varphi(q,p)\rangle\rangle = \iint \varphi(q,p) dq\, dp. \quad (8)$$

For instance, in the canonical equilibrium, the function F of energy is $F(E;b) = e^{-bE}$, where b is the inverse temperature. For the sake of simplicity, we consider only one independent parameter b, and other parameters $a_1,\ldots,a_n$ depend on b. The effective Hamiltonian of any one-dimensional system in a stationary state is integrable, and the angle-action variables (θ, J) [21] can be introduced accordingly. The Hamiltonian $H_0[f_0]$ and the distribution function f_0 depend only on the action J.

III. SCALING OF THE VLASOV SUSCEPTIBILITY

The Vlasov susceptibility is given by the linear response theory [8,9], and reads as

$$\chi^{\mathrm{V}}(b) = \frac{1 - D^{\mathrm{V}}(b)}{D^{\mathrm{V}}(b)}, \quad (9)$$

where D^{V} is the stability functional, and $D^{\mathrm{V}}(b) > 0$ implies the stability of the state [15,22]. This functional can be decomposed in two terms

$$D^{\mathrm{V}}(b) = D_1^{\mathrm{V}}(b) + D_2^{\mathrm{V}}(b), \quad (10)$$

where the first one is

$$D_1^{\mathrm{V}}(b) = 1 + \frac{\langle\langle F'(H_0(J);b)\langle\cos^2 q\rangle_J\rangle\rangle}{\langle\langle F(H_0(J);b)\rangle\rangle}, \quad (11)$$

while the second one is

$$D_2^{\mathrm{V}}(b) = -\frac{\langle\langle F'(H_0(J);b)\langle\cos q\rangle_J^2\rangle\rangle}{\langle\langle F(H_0(J);b)\rangle\rangle}. \quad (12)$$

The prime means the derivative $F' = dF/dE$ and $\langle\ldots\rangle_J$ represents the average with fixed J, i.e.,

$$\langle\varphi(\theta,J)\rangle_J = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(\theta,J)d\theta. \quad (13)$$

For homogeneous distribution, we have $q=\theta$ and D_2^{V} vanishes. The stability functional in this case is

$$D_{\mathrm{homo}}^{\mathrm{V}}(b) = 1 + \frac{1}{2}\frac{\langle\langle F'(p^2/2;b)\rangle\rangle}{\langle\langle F(p^2/2;b)\rangle\rangle}. \quad (14)$$

For instance, using the canonical equilibrium, we obtain that the susceptibility is $\chi^{\mathrm{V}} = b/(b-2)$ and its critical point is $b_{\mathrm{c}}^{\mathrm{cano}} = 2$.

Let us introduce three assumptions in order to obtain Jeans' distributions (7) which describe continuous phase transitions at $b = b_{\mathrm{c}}$: (i) The states are homogeneously stable for $b < b_{\mathrm{c}}$ and inhomogeneously stable for $b > b_{\mathrm{c}}$. (ii) The magnetization $\langle M\rangle_0$ is a continuous function of b. (iii) The solution of the self-consistency equation (6) gives an unstable homogeneous branch for $b > b_{\mathrm{c}}$. As a consequence, the stability functional (10) is positive for any $b (\neq b_{\mathrm{c}})$, and

$$\langle M\rangle_0 = \begin{cases} 0, & b \leqslant b_{\mathrm{c}} \\ (b-b_{\mathrm{c}})^{\beta}, & b \gtrsim b_{\mathrm{c}}. \end{cases} \quad (15)$$

Moreover, in the homogeneous branches of both sides, the stability functional reads as

$$D_{\mathrm{homo}}^{\mathrm{V}}(b) = \begin{cases} c_+(b)(b_{\mathrm{c}}-b)^{\Gamma_+}, & b \lesssim b_{\mathrm{c}} \\ -c_-(b)(b-b_{\mathrm{c}})^{\Gamma_-}, & b \gtrsim b_{\mathrm{c}} \end{cases} \quad (16)$$

with positive $c_{\pm}(b)$ and $\pm$ discriminates between the two regions overcritical and undercritical. The exponents $\Gamma_{\pm}$ depend on the choice of the parameter b of the distribution. In general, we can consider a parametrization such that $\Gamma_{\pm} = 1$.

The critical exponents $\gamma_{\pm}^{\mathrm{V}}$ of the susceptibility (9) depend on the behavior of the stability functional $D^{\mathrm{V}}(b) \to 0$ close to the critical point b_{c}. Equation (16) gives $\gamma_+^{\mathrm{V}} = \Gamma_+$ when the state of the system is homogeneous, that is equal to the classical exponent. In the following, we show the nonclassical relation $\gamma_-^{\mathrm{V}} = \beta/2$ settled in the inhomogeneous phase, where $\beta = \Gamma_-/2$ in the case of ferromagnetic mean-field systems.

Let us start showing the relation $\beta = \Gamma_-/2$. Around the critical point $b \gtrsim b_{\mathrm{c}}$, the magnetization $\langle M\rangle_0$ is small by assumption (ii), and we expand $F(H_0)$ around the critical point as

$$F(H_0;b) = \sum_{n=0}^{\infty}\frac{(-\langle M\rangle_0\cos q)^n}{n!}F^{(n)}(p^2/2;b), \quad (17)$$

where $F^{(n)}$ is the nth derivative of F and we assumed that f_0 depends on $\langle M\rangle_0$ through H_0 only. For such distributions, the self-consistency equation (6) becomes

$$A_1(b)\langle M\rangle_0 + A_3(b)\langle M\rangle_0^3 + O\big(\langle M\rangle_0^5\big) = 0, \tag{18}$$

where

$$A_1(b) = 1 - \frac{B_1}{B_0} = D^{\mathrm{V}}_{\mathrm{homo}}(b), \qquad A_3(b) = \frac{B_1B_2 - B_0B_3}{B_0^2}, \tag{19}$$

and B_n $(n = 0,1,2,\ldots)$ are defined by

$$B_n(b) = \frac{(-1)^n}{n!}\iint F^{(n)}(p^2/2;b)\cos^{2\lceil n/2\rceil} q\, dq\, dp, \tag{20}$$

with $\lceil x\rceil = \min\{m\in\mathbb{Z} | m \geqslant x\}$. We further assume that $B_n(b_c) \neq 0$ for any n. The nonzero solution of the self-consistency equation gives the scaling

$$\langle M\rangle_0 = \sqrt{\frac{c_-(b)}{A_3(b)}}(b - b_c)^{\Gamma_-/2} \tag{21}$$

whenever $A_3(b) > 0$, which implies existence of Jeans' inhomogeneous states. We, therefore, get the relation $\beta = \Gamma_-/2$.

To prove the main relation $\gamma^{\mathrm{V}}_- = \beta/2$, we separately estimate D^{V}_1 and D^{V}_2. To evaluate the behavior of the first term, we remark that $\langle\langle F'(H_0;b)\langle\cos^2 q\rangle_J\rangle\rangle = \langle\langle F'(H_0;b)\cos^2 q\rangle\rangle$. Using the expansion (17), the first component of the stability functional scales as $D^{\mathrm{V}}_1(b) \sim (b - b_c)^{\Gamma_-}$ for $b > b_c$. The second component is given by [9]

$$D^{\mathrm{V}}_2(b) = 16\sqrt{\langle M\rangle_0}(I_1 + I_2), \tag{22}$$

where

$$I_1 = -\int_0^1 \left[\frac{2E(k)}{K(k)} - 1\right]^2 kK(k)\frac{F'(\langle M\rangle_0(2k^2-1))}{\langle\langle F(H_0)\rangle\rangle}dk, \tag{23}$$

$$I_2 = -\int_1^\infty \left[\frac{2k^2E(1/k)}{K(1/k)} - 2k^2 + 1\right]^2 K(1/k) \times \frac{F'(\langle M\rangle_0(2k^2-1))}{\langle\langle F(H_0)\rangle\rangle}dk, \tag{24}$$

and K and E are, respectively, the complete elliptic integrals of the first and the second kinds. The integrals I_1 and I_2 converge to nonzero constants in general even in the limit $\langle M\rangle_0 \to 0$. Hence, the second part scales as

$$D^{\mathrm{V}}_2(b) \sim \sqrt{\langle M\rangle_0} \sim (b - b_c)^{\beta/2}. \tag{25}$$

Close to the critical point, the second component D^{V}_2 dominates since it goes to zero slower compared with the first one D^{V}_1. Consequently, the Vlasov critical exponents for Jeans' distributions are

$$\gamma^{\mathrm{V}}_- = \beta/2 = 1/4, \quad \gamma^{\mathrm{V}}_+ = 1, \tag{26}$$

when $\Gamma_\pm = 1$. We stress that the exponent $\gamma^{\mathrm{V}}_- = \beta/2 = \frac{1}{4}$ differs from the classical $\gamma_- = 1$ [23].

We explain that the *strange* exponent $\gamma^{\mathrm{V}}_- = \beta/2$ is due to infinite invariants of the Vlasov equation, called Casimirs. A Casimir is a functional of the distribution function $\int s(f)dq\,dp$, where s is any smooth function. It is an integral of motions of the Vlasov dynamics whenever the distribution solves the Vlasov equation (4) itself. As a consequence, any Casimir introduces a conservation law, and the second component of the stability functional, which gives the *strange* exponent, takes care of all of them.

The variation of the distribution $\delta f = f_h - f_0$ satisfies

$$0 = \iint [s(f_0 + \delta f) - s(f_0)]dq\,dp = \int s'[f_0(J)]\widetilde{\delta f}_0(J)dJ \tag{27}$$

up to the linear order, where $\widetilde{\delta f}_0(J)$ is the Fourier zero mode of δf with respect to the angle θ. This constraint must hold for any smooth functions s, and hence $\widetilde{\delta f}_0(J) = 0$ [22]. Let us derive f_h from the test function with the external field

$$g_h(q,p;b) = \frac{F_h(H_h(q,p;b))}{\langle\langle F_h(H_h(q,p;b))\rangle\rangle}, \tag{28}$$

where F_h is a family of functions of energy and is expanded as

$$F_h = F + hG + O(h^2). \tag{29}$$

By the definition of the susceptibility χ^{V}, the magnetization $\langle M\rangle_h$ is written as $\langle M\rangle_h = \langle M\rangle_0 + h\chi^{\mathrm{V}} + O(h^2)$, and hence

$$H_h = H_0 + h\psi(q) + O(h^2), \tag{30}$$

where

$$\psi(q) = -(\chi^{\mathrm{V}} + 1)\cos q + O(h^2). \tag{31}$$

Substituting the above two expansions into g_h, and ignoring the term of order $O(h^2)$, we have

$$g_h = f_0 + h\left[\frac{F'(H_0)\psi + G(H_0)}{\langle\langle F(H_0)\rangle\rangle} - \frac{\langle\langle F'(H_0)\psi + G(H_0)\rangle\rangle F(H_0)}{\langle\langle F(H_0)\rangle\rangle^2}\right]. \tag{32}$$

Subtracting the Fourier zero mode from $g_h - f_0$, the variation must satisfy $\delta f = g_h - f_0 - \langle g_h - f_0\rangle_J = g_h - \langle g_h\rangle_J$ and hence

$$f_h = f_0 - \frac{h(\chi^{\mathrm{V}} + 1)}{\langle\langle F(H_0)\rangle\rangle}F'(H_0)\left(\cos q - \langle\cos q\rangle_J\right), \tag{33}$$

where we used $\langle F(H_0)\rangle_J = F(H_0)$ and $\langle G(H_0)\rangle_J = G(H_0)$ since H_0 depends on the action J only. Multiplying by $M(q) = \cos q$ and integrating in the μ space, we get the Vlasov susceptibility (9) and the stability functional (10).

Following these results, we propose a scenario of relaxation as follows [15,19]: When the external field is switched on, the system gets trapped in a QSS to keep Casimir invariants and this trapping gives the *strange* critical exponent $\gamma^{\mathrm{V}}_- = \beta/2$. However, the Vlasov dynamics is not the true dynamics for finite systems, thus Casimirs are not exactly conserved but evolve on a time scale which diverges with N. Consequently, the system goes to the BG equilibrium recovering the classical exponent after the equilibration.

IV. NUMERICAL TESTS

The Vlasov exponent is verified by N-body simulations, which are performed by the fourth-order symplectic integrator [24] with the time step $\Delta t = 0.1$. We compute susceptibility for two families of Jeans' class states. One is the thermal equilibrium

$$F(E) = e^{-E/T}, \tag{34}$$

whose control parameter is $b = 1/T$ and the critical point is $T_c = \frac{1}{2}$. The other is Fermi-Dirac type

$$F(E) = \frac{1}{e^{(E-\mu)/T} + 1} \tag{35}$$

with fixed $T = \frac{1}{5}$, whose control parameter is $b = -\mu$ and the critical point is $\mu_c \simeq 0.239\,346$. The latter is an example of a family of out-of-equilibrium QSSs, and the critical exponent β is confirmed as $\frac{1}{2}$ by solving the self-consistent equation (6). The Fermi-Dirac-type families are obtained, approximately at least, by starting from waterbag initial states. The values of parameters μ and T are controlled by suitably choosing the waterbag initial states with the aid of a nonequilibrium statistical mechanics [20]. For both cases, the Vlasov predictions are in good agreement with the N-body simulations for time scales shorter than the equilibration one, as shown in Fig. 1.

The scenario of relaxation proposed in the last of Sec. III is examined by direct N-body simulations, shown in Fig. 2. For $t < 0$, the system is at equilibrium with a temperature $T = 0.499 < 1/2 = T_c$. The external field with a small magnitude $h = 0.01$ is switched on at $t = 0$, and the system jumps to the QSS predicted by the linear response theory based on the Vlasov equation. In the long time regime, Casimirs are no longer invariants due to the presence of rare collisions [6], and the system goes towards equilibrium. Simulations indicate that the time scale of relaxation from the QSS to equilibrium grows linearly with N, as found for isolated inhomogeneous QSSs in Ref. [25].

V. GENERALIZATION OF SYSTEMS

For simplicity, we have concentrated in the HMF model, but the present theory can be applied to generalized systems. Let us consider the Hamiltonian

$$H_N = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2}\sum_{i,j=1}^{N} K_N(r_i - r_j)[1 - \cos(q_i - q_j)] - \sum_{i=1}^{N} h_{r_i}(t)\Theta(t)\cos q_i, \tag{36}$$

where r_i is the ith lattice point on the one-dimensional lattice $r_{i+1} - r_i = 1$, the lattice has the periodic boundary condition by identifying r_0 with r_N, and the factor $K(r)$ is even, non-negative and satisfies [26]

$$\sum_{i=1}^{N} K_N(r_i) = 1. \tag{37}$$

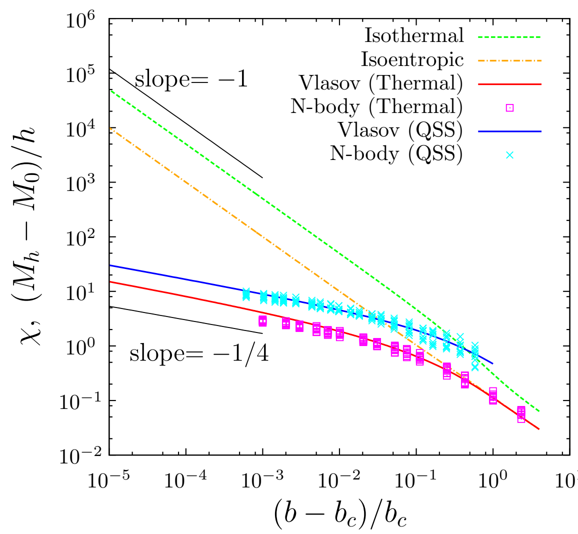

FIG. 1. (Color online) Susceptibilities as functions of the normalized parameter $(b_c - b)/b_c$ in log-log plot. Lines report theoretical predictions of the isothermal χ^T (green broken line), the isoentropic χ^S (orange dashed line), and the Vlasov χ^V (red lower solid line) susceptibilities for the thermal equilibrium family. We remark that χ^S is computed explicitly by using the exact solution in the microcanonical statistics [11] and by taking the invariance of the entropy during the quasistatic adiabatic process into account. The Vlasov susceptibility for a QSS family of the Fermi-Dirac type is also reported (blue upper solid line). Points are computed in N-body simulations and represent $(M_h - M_0)/h$, where M_h is the time average in the period of $t \in [0,500]$. $N = 10^6$ and $h = 10^{-2}$ for the thermal equilibrium family (purple square), and $N = 10^7$ and $h = 10^{-3}$ for the QSS family (light blue cross). For each b, 10 points are plotted corresponding to 10 realizations.

Taking the limit $N \to \infty$ so that

$$K(r) = \lim_{N\to\infty} N K_N(Nr) \tag{38}$$

and

$$\int_{-1/2}^{1/2} K(r)dr = 1, \tag{39}$$

we get the effective one-particle Hamiltonian

$$H_h[f] = \frac{p^2}{2} + V_r[f](q,t) - h_r(t)\cos q, \tag{40}$$

where

$$V_r[f](q,t) = -\int_{-1/2}^{1/2} dr' K(r - r') \times \iint \cos(q - q')f(q',p',r',t)dq'\,dp'. \tag{41}$$

The single body distribution $f(q,p,r,t)$ evolves as the Vlasov equation [27]

$$\frac{\partial f}{\partial t} + \{H_h[f], f\} = 0. \tag{42}$$

We consider the linear response for the uniform stable stationary configuration $f_0(q,p)$, which does not depend on

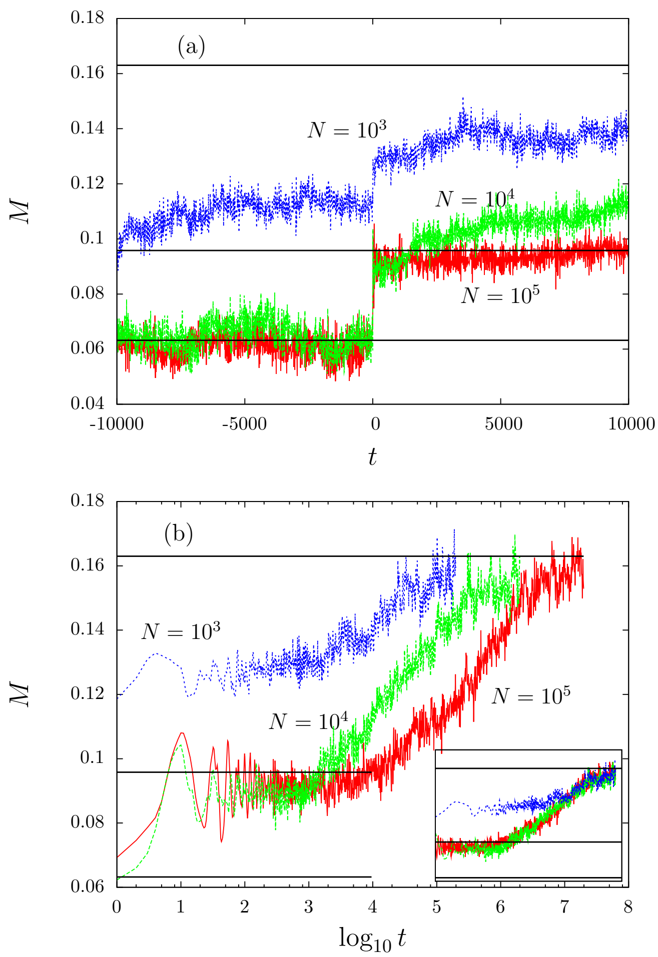

FIG. 2. (Color online) N-body simulations in the HMF model with external field. (a) Short time evolutions of magnetization. (b) Long time evolutions. The horizontal axis is in the logarithmic scale, which is scaled as $\log_{10}(t/N)$ in the inset. $N = 10^3(100)$, $10^4(10)$, and $10^5(1)$, where the inside of braces is the number of realizations over which the orbits are averaged. The system is in thermal equilibrium with $T = 0.499$ in $t < 0$, and the external field turns on with $h = 0.01$ at $t = 0$. The direction of magnetization vector (M_x, M_y) is reset to the x direction at $t = 0$. In each panel, three horizontal lines represent equilibrium level with the quasistatic adiabatic susceptibility (upper), the QSS level predicted by the Vlasov linear response theory (middle), and thermal equilibrium level without external field (lower).

the lattice point r. The external field modifies the state from $f_0(q,p)$ to $f_0(q,p) + f_1(q,p,r,t)$. We define the modification of magnetization depending on r as

$$M_r^1(t) = \iint \cos q f_1(q,p,r,t) dq\, dp. \tag{43}$$

Using the periodicity of the lattice, we expand the modification $M_r^1(t)$, the factor $K(r)$, and the external field $h_r(t)$ as

$$M_r^1(t) = \sum_{n\in\mathbb{Z}} \tilde{M}_n^1(t) e^{2\pi i n r}, \quad K(r) = \sum_{n\in\mathbb{Z}} \tilde{K}_n e^{2\pi i n r}, \tag{44}$$

and

$$h_r(t) = \sum_{n\in\mathbb{Z}} \tilde{h}_n(t) e^{2\pi i n r}. \tag{45}$$

The Laplace transform with respect to time t gives

$$M_r^1(t) = \sum_{n\in\mathbb{Z}} \frac{e^{2\pi i n r}}{2\pi} \int_\Gamma \frac{F(\omega)}{1 - \tilde{K}_n F(\omega)} \hat{h}_n(\omega) e^{-i\omega t}, \tag{46}$$

where Γ is the Bromwich contour, $\hat{h}_n(\omega)$ is the Laplace transform of $\tilde{h}_n(t)$,

$$F(\omega) = \int_0^\infty dt\, e^{i\omega t} \iint \cos q_t \{\cos q, f_0\} dq\, dp, \tag{47}$$

and q_t is the solution to the canonical equation associated with the Hamiltonian $H_0[f_0]$, which has zero external field.

Setting the external field as $h_r(t) \to h(t \to \infty)$, we have $\tilde{h}_0 \to h$ and $\tilde{h}_n \to 0$ $(n \neq 0)$. As discussed in [9], the surviving response is provided by the pole of $\hat{h}_0(\omega)$ at $\omega = 0$, and other poles give dampings by the stability assumption of f_0. The linear response is hence written by the same stability functional $D^{\rm V}$ with the HMF model [9] as

$$M_r^1(t) \to \frac{F(0)}{1 - F(0)} h = \frac{1 - D^{\rm V}}{D^{\rm V}} h, \tag{48}$$

where we used the fact $\tilde{K}_0 = 1$ from Eq. (39). From the above expression, we conclude that the Vlasov susceptibility and the critical exponents $\gamma_\pm^{\rm V}$ in the system (36) are the same with those in the HMF model, for uniform stable stationary configurations.

VI. SUMMARY AND DISCUSSIONS

We investigated the critical exponent of susceptibility in the HMF model, which is a mean-field ferromagnetic model and is approximately described by the Vlasov dynamics. The classical mean-field theory gives the critical exponent 1 both in the high- and low-energy phases, but the linear response theory for the Vlasov systems reveals that the exponent is half that of magnetization in the low-energy phase, which is typically $\frac{1}{4}$. This scaling is obtained not only in thermal equilibrium states, but also in one-parameter families of quasistationary states of the Jeans type, when the families have continuous phase transitions. Apart from the HMF model, the present theory can be applied to uniform stable stationary configurations of generalized systems, whose interaction depends on the distance between two lattice points on which particles are.

Some remarks are discussed in the following. The first remark is about the validity of the linear response theory close to critical points. The theory assumes that δf is vanishing when $h \to 0$ with satisfying the condition $|h\chi^{\rm V}| \ll \langle M \rangle_0$. The Vlasov susceptibility can be, therefore, computed by use of the Vlasov linear theory even for a large $\chi^{\rm V}$ since it is computed in the limit of $h \to 0$.

The second remark is on the spectrum analysis used to compute susceptibilities [28] in the inhomogeneous phase. This method does not consider all of the integrals of motion but can be used to describe approximations of the linear theory for nonintegrable systems.

The third remark is for the critical exponents in the homogeneous phase. In the homogeneous equilibrium, the two susceptibilities satisfy $\chi^T = \chi^{\rm V} = T_{\rm c}/(T - T_{\rm c})$ for $T > T_{\rm c}$. Then, the isolated system shows the classical exponent, although the dynamics keeps an infinite number of Casimir invariants. Thus, Casimir constraints do not always bring about the *strange* critical exponent, and it depends on the initial equilibrium state.

Another remark is that the existence of invariants may break some thermodynamic laws. Indeed, local temperature in isolated crystalline clusters is not uniform by conservation of angular and translational momenta [29].

We remark on other studies of the critical exponents in the Vlasov framework. Based on the theory on unstable manifolds of the Vlasov-Poisson equation [30], Ivanov *et al.* [31] found numerically that scaling laws are different from those predicted by the classical theory. However, they start from unstable spatially homogeneous Maxwell distributions, and no critical exponents are discussed in literature for stable states and QSSs.

We end this article remarking on observations in experiments. Dynamical systems could get trapped in QSSs, therefore, measures on experimental setups will show the Vlasov prediction for systems with large enough number of particles.

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