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<tr>
<td>Author(s)</td>
<td>Matsuo, Tetsuji; Mifune, Takeshi</td>
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<tr>
<td>Citation</td>
<td>IEEE Transactions on Magnetics (2014), 50(2): 177-180</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2014-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/187795">http://hdl.handle.net/2433/187795</a></td>
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</tr>
<tr>
<td>Type</td>
<td>Journal Article</td>
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Reduction of Unphysical Wave Reflection Arising from Space-Time Finite Integration Method

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Several 3D and 4D space-time grids are compared for electromagnetic wave computations using the space-time finite integration (FI) method. To suppress unphysical wave reflections without incurring numerical instability, the constitutive relation is corrected while retaining the symmetric of impedance matrix. The eigenvalue analysis confirms the stability of the 3D space-time FI schemes, where an asymmetric impedance matrix could cause late instability. A 4D space-time FI scheme is also improved by symmetric corrections.

Index Terms—Computational electromagnetics, finite difference methods, numerical stability, time domain analysis.

I. INTRODUCTION

The finite integration (FI) method [1]–[3] provides time-domain electromagnetic wave computation on an unstructured spatial grid. Refs. [4], [5] developed a space-time FI method that realizes non-uniform time-steps on 3D and 4D space-time grids to relax the Courant-Friedrichs-Lewy condition with respect to the smallest spatial grid size, which was applied to an optical device analysis [6]. Refs. [7] and [8] proposed improved 3D and 4D space-time grids to suppress unphysical wave-reflections caused by a nonuniform spatial grid construction [4], [5]. However, the stability of these schemes has not been confirmed for very long computations that might suffer late-time instability [9].

This paper proposes an improved constitutive relation based on a vector correction, where the stability of the resultant scheme is evaluated by an eigenvalue analysis. Computational accuracy resulting from several space-time grids proposed previously are compared.

II. FINITE INTEGRATION METHOD ON A SPACE–TIME GRID

The coordinate system is denoted by \((ct, x, y, z) = (x^0, x^1, x^2, x^3)\) where \(c = 1 / \sqrt{\varepsilon_0 \mu_0}\), and \(\varepsilon_0\) and \(\mu_0\) are the permittivity and permeability of the vacuum. The integral forms of the Maxwell equations without source terms are [5]:

\[
\int F = 0, \quad \int G = 0, \quad (1)
\]

\[
F = -\sum_{i=1}^{3} E_i dx^i dx' + \sum_{j=1}^{3} \mathcal{B}_j dx^i dx', \quad (2)
\]

\[
G = \sum_{j=1}^{3} H_j dx^i dx' + \sum_{l=1}^{3} \mathcal{D}_l dx^i dx'
\]

where \((\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) = c\mathcal{B}\) and \((\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3) = c\mathcal{D}\) \((j, k, l)\) is a cyclic permutation of \((1, 2, 3)\); \(\mathcal{B}_p\) and \(\mathcal{D}_l\) are hypersurfaces in space-time. The electromagnetic variables are defined in the FI method as:

\[
f = \int F, \quad g = \int G \quad (3)
\]

where \(S_p\) and \(S_d\) are the faces of the primal and dual grids that constitute \(\mathcal{B}_p\) and \(\mathcal{D}_l\). The Hodge dual grid [5] is used to express the constitutive equation simply as:

\[
\mathcal{B}_p = \frac{1}{8} \mathcal{D}_l, \quad \mathcal{D}_l = \frac{1}{8} \mathcal{B}_p
\]

where \(\varepsilon_r = 1 / \sqrt{\varepsilon_0 \mu_0}\), \(a\) is a constant determined for each pair of \(S_p\) and \(S_d\); \(\varepsilon_r\) and \(\mu_r\) are the relative permittivity and permeability. Thus, \(f = Zg / a\) is obtained, where \(Z = \sqrt{\mu_r / \varepsilon_r}\) is the impedance.

III. 3D-SPACE-TIME GRIDS WITH 2D SPACE

A. 3D-Space-Time Grids

Fig. 1 illustrates the space-time grids proposed in this paper, where domains (I) and (II) have uniform time-steps \(\Delta t^0\) and \(\Delta t^0/2\), respectively; domain (III) is the connecting domain. For simplicity, the spatial cell size is set to 1 by normalization. Fig. 2(a) and (b) shows the grid structure of corner parts of domain (III) examined in [7] and [8] to suppress the unphysical wave reflection due to spatial irregularity. In this article, the grids shown in Figs. 1, 2(a), and 2(b) are called types A, B, and C, respectively. Fig. 3 illustrates the three types of corner for domain (III).

Similar to a type-B grid, a type-A grid has dual edges that are not orthogonal to the corresponding primal faces (Fig. 1), where (4) is not exactly applicable. This study examines two types of constitutive relations described by (5) and (6):

\[
\int_{S_p} c \mathcal{B}_p dx^i dx' = -\int_{S_d} c \mathcal{D}_l dx^i dx' = a \quad (4)
\]
Stability Analysis

The grid in Fig. 1 has a period $\Delta x^0_0$ along the $x^0$-direction. Consequently, the variables consisting of $f$ and $g$ given by (3) are periodically allocated along the $x^0$-direction on the spacet ime grid. The variables are accordingly denoted $\mathbf{V}^t$, $\mathbf{V}^d$, ... where $\mathbf{V}^{n+1}$ is assigned after $\mathbf{V}^n$ by the time interval $\Delta t^0$. The variable vector $\mathbf{V}^n$ is divided into $\mathbf{v}^n$ and $\mathbf{u}^n$ where the components of $\mathbf{v}^n$ are linearly independent and the components of $\mathbf{u}^n$ are given by the linear combination of the components of $\mathbf{v}^n$. The numerical stability of the time-marching scheme is evaluated using the eigenvalues of $\partial / \partial t^{n+1} / \partial \mathbf{v}^n$.

A small spacet ime grid having spatial domain size of $40 \times 40$ containing domain (II) with $18 \times 18$ spatial cells is used for the numerical eigenvalue analysis imposing spatially periodic boundary conditions. If $\Delta x^0_0$ is small, all the eigenvalues of $\partial / \partial t^{n+1} / \partial \mathbf{v}^n$ given by the type-A grid are on the unit circle. If $\Delta x^0_0$ is large, some of the eigenvalues move outside the unit circle, causing numerical instability. Setting $\Delta t = 5/6$, Fig. 5 contrasts the maximum $\Delta x^0_0 / c$ that does not induce instability for the different grid types and corrections. Note that a type-B grid with correction yields eigenvalues for which the absolute values slightly exceed 1 by about $10^{-4}$, even with smaller $\Delta x^0_0 / c$ than that shown in Fig. 5, that will cause late-time instability [9, 10]. Consequently, a type-B grid should not be used for lengthy computations with correction, even though a late-time instability was not found in [8]. In contrast, a type-A grid with symmetric or asymmetric correction increases the maximum time-step without inducing a late-time instability. As is shown in Appendix B, the scheme for a type-A grid with symmetric or asymmetric correction is equivalent to that for type-C, which is unaffected by the choice between (8) and (9).

**Fig. 2.** Corner of domain (III): (a) type-B grid and (b) type-C grid.

**Fig. 3.** Comparison of three types of corner for domain (III).

**Fig. 4.** Corner of domain (III) of a type-A grid: (a) variables on a primal grid, (b) variables on a dual grid, (c) directions of variables, and (d) vectorial correction.

**Fig. 5.** Maximum $\Delta x^0_0 / c$ for stable computation with respect to grid type and correction.
boundary conditions are imposed. The time-step $\Delta t = 0.5$ in domain (I) and normalization. The normalized initial condition is given by

$$f_0 = 3 \exp\{-3(x^2+y^2)/25\}.$$  

Fig. 8. Discrepancy of $\delta_3$ between the space-time F1 method on a type-A grid and the FDTD method: (a) without and (b) with symmetric correction.

Fig. 7. Discrepancy in $\delta_3$ between the space-time F1 method on a type-A grid and the FDTD method: (a) without and (b) with symmetric correction.

Fig. 9. 4D Space-time grid of type A: (a) primal grid, and (b) dual grid.

C. Unphysical Wave Reflection

Wave propagation is simulated to examine the three types of grid on the computational domain (Fig. 6(a)). For simplicity, the permittivity and permeability are set uniformly to unity by normalization. The normalized initial condition is given by $E_1 = E_2 = 0$ and $\delta_3 = \exp\{-3(x^2+y^2)/25\}$. Spatially periodic boundary conditions are imposed. The time-step $\Delta t$ is set to 0.5 in domain (I) and $\Delta t$ is set to 5/6. Fig. 6(b) depicts the distribution of $\delta_3$ at $x^0 = 40$.

Fig. 7 depicts the distributions of discrepancy $\Delta \delta_3$ between $\delta_3$ at $x^0 = 40$ obtained by the FDTD method and the space-time F1 method with a type-A grid with and without symmetric correction. The FDTD method is executed with the same uniform spatial grid and time-step as in domain (I). The discrepancy seen in $x^1 \geq 15$ and $x^2 \geq 20$ is mainly caused by numerical dispersion whereas that in $x^1 \leq 15$ or $x^2 \leq 20$ is caused by an unphysical wave reflection at domain (III). Fig. 8 compares the latter discrepancy in domain (I). It shows that both the asymmetric and symmetric corrections effectively reduce the unphysical wave reflection equally to the type-C grid. The type-B grid also achieves accurate results under correction, but suffers a late-time instability before $10^6\Delta t$.

IV. 4D-SPACE-TIME GRID WITH 3D SPACE

The computational domain shown in Fig. 6(a) is extended to the $x^2$-direction for a 4D simulation. Fig. 9 illustrates a 4D space-time grid of type A at the corner of domain (III). For the 4D computation, two types of constitutive relations given by (10) and (11) are examined as well:

$$\mathbf{h}_i^e = \frac{c_4 \Delta x^0}{Z} \mathbf{b}_i^e, \quad \mathbf{h}_i^m = \frac{c_4 \Delta x^0}{2Z} \mathbf{b}_i^m,$$

$$g_{ij}^{s1/4} = \frac{c_4 \Delta x^0}{4Z} f_{ij}^{s1/4}, \quad (i, j = 1, 3) \quad (10)$$

$$h_{ii}^e = \frac{\mu c_4 \Delta x^0}{Z} [b_{ii}^e + (-1)\{(1 - \Delta l)b_{ii}^e\}],$$

$$h_{ii}^m = \frac{c_4 \Delta x^0}{2Z} [\alpha b_{ii}^m + (-1)\{(\alpha - \Delta l)b_{ii}^m\}],$$

$$g_{ij}^{s1/4} = \frac{c_4 \Delta x^0}{4Z} \left[\alpha f_{ij}^{s1/4} + (-1)\{(\alpha - \Delta l)f_{ij}^{s1/4}\}\right],$$

$$\quad (i, j = 1, 3) \quad (11)$$

where $b$ and $h$ are the magnetic flux and the integration of magnetic field given by (3); respectively; $f$ and $g$ allocated as in Fig. 9 are also given by (3); $\Delta l^e = \Delta l - 3(c_4 \Delta x^0)^2/32 \approx \Delta l$, $\eta = \Delta l^e / \Delta l \approx 1$ and $\alpha^e = (1 + \alpha) / 2 \approx 1$. Based on a vector correction, (11) gives a more accurate approximation of the constitutive equation than (10) that gives a diagonal matrix without correction.

The constitutive relations for $h_{12}$ and $g_{12}$ are given by

$$h_{12}^e = \frac{c_4 \Delta x^0}{2Z} (2\alpha - 2\Delta l)b_{12}^e,$$

$$g_{12}^{s1/4} = \frac{c_4 \Delta x^0}{4Z} (2\alpha - 2\Delta l)f_{12}^{s1/4}, \quad (j = 1, 3) \quad (12)$$

To avoid asymmetric impedance matrix, (12) is replaced by

$$h_{12}^e = \frac{c_4 \Delta x^0}{2Z} [\zeta h_{12}^e - 2\eta(1 - \Delta l)(b_{12}^e - b_{12}^m) - (\alpha - \Delta l)(b_{11}^m - b_{22}^m)],$$

$$g_{12}^{s1/4} = \frac{c_4 \Delta x^0}{4Z} (\alpha - \Delta l)(3f_{12}^{s1/4} - f_{11}^{s1/4} + f_{22}^{s1/4})$$

$$\quad (j = 1, 3) \quad (13)$$

where $\zeta = 3(\alpha - \Delta l) + 2\eta(1 - \Delta l) \approx 5(1 - \Delta l)$. The correction with (6), (9), (11), and (13) is called the symmetric correction whereas the correction with (6), (8), (11), and (12) is called the asymmetric correction.

Wave propagation is simulated with $c = 1$, $\Delta l = 5/6$, and normalized initial conditions given by $E_1 = E_2 = E_3 = 0$, $\delta_3 = \exp\{-3(x^2+y^2)/25\}$, $\delta_5 = \exp\{-3(x^2+y^2)/25\}$, and $\delta_5 = \exp\{-3(x^2+y^2)/25\}$. For the different grid types and corrections, Fig. 10 contrasts the maximum $\Delta x^0$ that does not
induce an apparent instability before $10^4 \Delta t^0$. A type-A grid with symmetric and asymmetric corrections increases the maximum time-step. The type-A grid with the symmetric correction and the type-C grid do not develop numerical instability even after $10^2 \Delta t^0$ if $\Delta x^0 \leq 0.57$. However type-A and type-B grids with asymmetric correction suffer late-time instability, even with smaller $\Delta x^0$ than that shown in Fig. 10. Fig. 11 depicts the distribution for $|B|$ at $x^0 = 40$ given by a type-A grid with symmetric correction and the discrepancy $|\Delta B|$ when compared with that from the FDTD method. Fig. 12 contrasts the discrepancy in domain (I) when $\Delta x^0 = 0.5$ over grid type and correction. Fig. 12 shows that both the asymmetric and symmetric corrections effectively reduce the unphysical wave reflection.

**APPENDIX**

A. Correction for type B grid

The correction given by (9) and (10) in [8] is unfortunately not correct, where (9) and (10a) should have similar form to (5) and the first and second equations of (6) in this paper; $\varepsilon_0$ in (10b) in [8] should be replaced by $Z_c\Delta x^0/4$.

B. Equivalence to type C grid

Variables $d_{12}^{xy/j} (j = 1, 2, 3)$ are updated as

$$
d_{12}^{1+1/4} = d_{12}^{-1+1/4} + h_x^{1/14} - h_x^0,
$$

$$
d_{12}^{-1+1/4} = d_{12}^{1+1/4} + g_x^{1+1/4} - g_x^{1+1/4},
$$

$$
d_{12}^{1+3/4} = d_{12}^{-1+3/4} + g_x^{1+3/4} - g_x^{-1+3/4},\tag{14}
$$

By setting, $b_0^{n} = h_x^{n} + b_0^{n}$, $b_0^{n} = (h_x^{n} + h_x^0) / 2 = (c_4 \Delta x^0/2Z) b_0$, $f_0^{n} = f_n^{1+3/4} + b_0^{n}$, and $b_0$ are updated as

$$
d_{12}^{+1/1} = d_{12}^{+1/1} + (g_x^{n+1/4} + h_x^0 - h_x^{n+1/4}),
$$

$$
d_{12}^{-1/1} = d_{12}^{-1/1} + (g_x^{n+1/4} - h_x^{n+1/4}),
$$

$$
d_{12}^{+3/4} = d_{12}^{+3/4} + (g_x^{n+3/4} - h_x^{n+3/4}),
$$

$$
d_{12}^{+3/4} = d_{12}^{+3/4} + (h_x^{n+3/4} - h_x^0),\tag{15}
$$

$$
f_0^{+1/1} = f_0^{-1/1} - e_{11}^{+1/1} + e_{11}^{-1/1},
$$

$$
f_0^{-1/1} = f_0^{+1/1} - e_{11}^{+1/1} + e_{11}^{-1/1} + e_{11}^{+1/2} - e_{11}^{-1/2},
$$

$$
b_0^{+1} = f_0^{-3/4} - e_{11}^{+3/4} + e_{11}^{-3/4} \tag{16}
$$

where subscripts N and S indicate the positions shown in Fig. 2(b); $d_2^{xy/j} (K=I, II)$ are similarly updated using $h_x^0$, $h_x^0$, and $g_x$. The scheme above is equivalent to that for a type-C grid.

**REFERENCES**


