K-FLIPS AND VARIATION OF MODULI SCHEME
OF SHEAVES ON A SURFACE, II

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INTRODUCTION

We shall consider some analogy between the wall-crossing problem of moduli schemes of stable sheaves on a surface, and the minimal model program of higher-dimensional varieties. This article is a continuation of [10].

Let $X$ be a non-singular projective surface over $\mathbb{C}$, and $H$ an ample line bundle on $X$. Denote by $M(H)$ (resp. $M^s(H)$) the coarse moduli scheme of rank-two $H$-semistable (resp. $H$-stable) sheaves on $X$ with Chern class $\alpha = (c_1, c_2) \in \text{Pic}(X) \times \mathbb{Z}$.

Let $H_-$ and $H_+$ be $\alpha$-generic polarizations such that just one $\alpha$-wall $W$ separates them. For $a \in [0, 1]$ one can define the $\alpha$-semistability of sheaves on $X$ and the coarse moduli scheme $M(a)$ (resp. $M^s(a)$) of rank-two $\alpha$-semistable (resp. $\alpha$-stable) sheaves with Chern classes $a$ in such a way that $M(e) = M(H_+)$ and $M(1-e) = M(H_-)$ if $a > 0$ is sufficiently small. $M(a)$ is projective over $\mathbb{C}$. Let $a_- < a_+$ be minichambers separated by only one miniwall $a_0$, and denote $M_+ = M(a_+)$, $M_- = M(a_-)$ and $M_0 = M(a_0)$. There are natural morphisms $\phi_- : M_- \to M_0$ and $\phi_+ : M_+ \to M_0$ ([1], [2], [8]). One may say they are morphisms of moduli schemes coming from wall-crossing methods. Let $\phi_- : V_- \to V_0$ be a birational projective morphism such that (1) $V_-$ is normal, (2) $-K_{V_-}$ is $\mathbb{Q}$-Cartier and $\phi_-$-ample, (3) the codimension of the exceptional set $\text{Exc}(\phi_-)$ is more than 1, and (4) the relative Picard number $\rho(V_-/V_0)$ of $\phi_-$ is 1. After the theory of minimal model program, we say a birational projective morphism $\phi_+ : V_+ \to V_0$ is a $K$-flip of $\phi_- : V_- \to V_0$ if (1) $V_+$ is normal, (2) $K_{V_+}$ is $\mathbb{Q}$-Cartier and $\phi_+$-ample, (3) the codimension of the exceptional set $\text{Exc}(\phi_+)$ is more than 1, and (4) the relative Picard number $\rho(V_+/V_0)$ of $\phi_+$ is 1.

**Theorem 0.1.** Fix a closed, finite, rational polyhedral cone $S \subset \text{Am}(X)$ such that $S \cap \partial \text{Am}(X) \subset \mathbb{R}_{\geq 0} \cdot K_X$. If $c_2$ is sufficiently large with respect to $c_1$ and $S$, then for any $\alpha$-generic polarizations $H_-$ and $H_+$ in $S$ separated by just one $\alpha$-wall $W$, and for any adjacent minichambers $a_- < a_+$ separated by a miniwall $a_0$ we have the following.

(i) $M_\pm$ are normal and $\mathbb{Q}$-factorial, $K_{M_\pm}$ are Cartier, $M_\pm$ are l.c.i., and $M_-$ and $M_+$ are isomorphic in codimension 1.

(ii) Suppose $K_X$ does not lie in the $\alpha$-wall, and that $K_X$ and $H_+$ lie in the same connected components of $\text{NS}(X)_{\mathbb{R}} \setminus W$. Then $\rho(M_-/M_0) = 1$ and $\phi_+ : M_+ \to M_0$ is a $K$-flip of $\phi_- : M_- \to M_0$. This morphism $\phi_+$ (resp. $\phi_-$) is the contraction of an extremal ray of $\overline{\text{NE}}(M_+)$ (resp. $\overline{\text{NE}}(M_-)$), which is described in moduli theory.

(iii) Suppose $X$ is minimal and $\kappa(X) > 0$, which means $K_X$ is not numerically equivalent to 0 and contained in $\text{Am}(X)$. Then there is a polarization, say $H_X$.

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contained in $S$ such that no $\alpha$-wall separates $H_X$ and $K_X$, and the canonical divisor of $M(H_X)$ is nef.

The greater part of this result has already appeared in [10, Theorem 1.1]. In Section 1, we shall prove the remaining part of this theorem which has not appeared in [10], that largely is the statement about the $\mathbb{Q}$-factoriality of $M_\pm$ and $\rho(M_+/M_0)$. The author was not aware of this part at the time of writing [10]. There is some application; suppose $X$ is minimal and $\kappa(X) > 0$, and fix a polarization $L$ on $X$. If $c_2$ is sufficiently large with respect to $c_1$ and $L$, then one can observe a moduli-theoretic analogue of the minimal model program of $M(L)$. Here “analogue” means that singularities of $M(H_X)$ are not considered. About this analogy, see Introduction in [10] for detail. We remark that a $K$-flip differs from a Thaddeus-type flip in [8].

In Section 2, we give some notes about extremal faces of $\overline{\operatorname{NE}}(M(H)) \subset N_1(M(H))$, where $H$ is an $\alpha$-generic polarization. We shall point out that some extremal faces with dim $\geq 2$ can appear in $\overline{\operatorname{NE}}(M(H))$ when $H$ gets closer to more than one $\alpha$-wall.

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Notation. All schemes are locally of finite type over $\mathbb{C}$ or, more generally, an algebraically closed field of characteristic zero. For a projective scheme $V$ over $\mathbb{C}$, $\operatorname{Num}(V)$ means $\operatorname{Pic}(V)$ modulo numerically equivalence. For any coherent sheaf $E$ on $V$, $\operatorname{Ext}^i_V(E,E)^0$ means the kernel of trace map $\operatorname{Ext}^i_V(E,E) \to H^i(\mathcal{O}_V)$.

1. Proof of Theorem

There is a union of hyperplanes $W \subset \operatorname{Amp}(X)$ called $\alpha$-walls in the ample cone $\operatorname{Amp}(X)$ such that $M(H) = M(H, \alpha)$ changes only when $H$ passes through $\alpha$-walls ([9]). A polarization on $X$ is called $\alpha$-generic if no $\alpha$-wall contains it. Now fix a closed, finite, rational polyhedral cone $S \subset \operatorname{Amp}(X)$ as in Theorem 0.1. Refer to [1, Section 3] about the $\alpha$-stability, minichambers and miniwalls, which appeared in Introduction.

Lemma 1.1. If $c_2$ is sufficiently large with respect to $c_1$ and $S$, then for any $\alpha$-generic polarizations $H_-$ and $H_+$ in $S$ separated by just one $\alpha$-wall $W$, and for any adjacent minichambers $a_-$ and $a_+$ separated by a miniwall $a_0$, (i) $M_\pm$ are normal, (ii) $K_{M_\pm}$ are Cartier, (iii) $M_\pm^s$ are l.c.i., (iv) $M_-$ and $M_+$ are isomorphic in codimension 4, and (v) our natural birational map $M_- \cdots > M_+ \text{ induces } \operatorname{Pic}(M^s) \simeq \operatorname{Pic}(M_+^s)$.

Proof. Fix a polarization $L \in S$. If $c_2$ is sufficiently large w.r.t. $c_1$ and $L$, then $M(L)$ is normal, $M^s(L)$ is of expected dimension, and the codimension of $\operatorname{Sing}'(M(L))$ in $M(L)$ is greater than 4 by [5] and [11], where $\operatorname{Sing}'(M(L)) \subset M(L)$ is the closed subset consisting of sheaves $E$ such that $E$ is not $L$-stable or that $\operatorname{Ext}^2_X(E,E)^0 \neq 0$. One can check (iv) in a similar way to [10, Lemma 2.4]. Now we compare $M(L)$ with $M_+$. By (iv) and the deformation theory of simple sheaves, $M_+^s$ is of expected dimension so it is l.c.i., and

\[
\operatorname{codim}(\operatorname{Sing}'(M_+^s), M_+) > 4.
\]

Thereby $M_+^s$ is normal. Since $H_\pm$ are $\alpha$-generic and $a_\pm$ are minichambers, if a rank-two sheaf $E$ with Chern classes $\alpha$ is $a_-$-semistable and not $a_+$-semistable, then $E$
is $H$-semistable for any polarization $H$, and so our birational map $M_+ \cdots > M_-$ is isomorphic near $M_+ \setminus M_+^s$. Thus $M_+$ is normal near $M_+ \setminus M_+^s$, and accordingly $M_+$ itself is normal. Item (v) follows item (iv) and (1) because of Fact 1.3 below. Last, $M_+$ is the GIT quotient of an open subset $R_+$ of some Quot-scheme on $X$. Let $E$ be a universal family of $R_+$ on $X \times R_+$. Since $a_+$ is not a miniwall, one can check that the line bundle $R\text{Hom}_{p_2}(E, E)$ on $R_+$ descends to a line bundle on $M_+$, that equals $K_{M_+}$.

Next we recall a fact concerning $\text{Pic}(M^s_+)$ from [6]. For a moment we assume $M^s_+$ has a universal family $E$ on $X \times M^s_+$. Let $K$ be the Grothendieck group of $X \times X$ and let $\tilde{K}$ be the kernel of $\xi : K \to \mathbb{Z}$, that is defined by $\xi(C) = \chi(C \boxtimes \pi^*_1E \boxtimes \pi^*_2E)$. Here $\boxtimes$ denotes the tensor product of complexes. Let $\sigma : X \times X \to X \times X$ be the map exchanging factor and let $\text{Pic}(X \times X)^\sigma$ be the subgroup consisting of line bundles invariant under $\sigma$. The map $\psi : \tilde{K} \to \text{Pic}(M_+)$ defined by

$$
\psi(C) = \det \left( (p_1)_! \left( p_{23}^*(C) \boxtimes p_{12}^*(E) \boxtimes p_{13}^*(E) \right) \right) \quad (C \in \tilde{K})
$$

induces a homomorphism

$$
\Phi_\pm : \text{Pic}(X \times X)^\sigma \otimes \mathbb{Z} \to \text{Pic}(M^s_+) \otimes \mathbb{Z}[1/12],
$$

as explained in [6, p. 132]. One can define $\Phi$ also when $M^s_+$ do not necessarily admit a universal family.

**Proposition 1.2.** Let $a_\pm$ be a minichamber satisfying assumptions in Lemma 1.1. If $c_2$ is sufficiently large with respect to $c_1$ and $\mathcal{S}$, then

$$
\Phi_\pm \otimes \mathbb{Q} : \text{Pic}(X \times X)^\sigma \otimes \mathbb{Q} \oplus \mathbb{Q} \to \text{Pic}(M^s_+) \otimes \mathbb{Q}
$$

is isomorphic.

**Proof.** One can verify this from Lemma 1.1 (v) and by reading [6] (especially Lemma 3.10.) carefully. \hfill \Box

Before the proof of Theorem 0.1, recall a useful fact at [SGA2, p.132].

**Fact 1.3.** Let $W$ be any quasi-projective and l.c.i. scheme with codim(Sing($W$), $W$) $\geq 4$. Then for any closed subset $\Lambda \subset W$ of codimension at least two, the restriction map $\text{Pic}(W) \to \text{Pic}(W \setminus \Lambda)$ is an isomorphism.

Now we shall prove two propositions; those and [10] end the proof of Theorem 0.1.

**Proposition 1.4.** Let $a_+$ be a minichamber satisfying assumptions in Lemma 1.1. Suppose $c_2$ is so large with respect to $c_1$ and $\mathcal{S}$ that $M_\pm$ are normal, $M^s_\pm$ are l.c.i., codim(Sing($M^s_\pm$), $M^s_\pm$) $\geq 4$, codim($M^s_\pm \setminus M^s_\pm, M_\pm$) $\geq 2$, and the homomorphisms at (4) are isomorphic. Then $M_\pm$ are $\mathbb{Q}$-factorial.

**Proof.** First remark that assumptions in this proposition holds for $c_2 \gg 0$ from Lemma 1.1, Proposition 1.2, [11], and [3, Theorem 9.1.2.]. We shall verify this only for $M_+$. Let $U$ be the open set $M_+ \setminus \text{Sing}(M_+)$ in $M_+$. If $\text{Cl}(M_+)$ means its divisor class group generated by Weil divisors, then we have

$$
\text{Cl}(M_+) \to \text{Cl}(U) \simeq \text{Pic}(U) \to \text{Pic}(M^s_+),
$$
where the first map is restriction, the second map is isomorphism since $U$ is smooth, and the third map is an extension map, which is assured by Fact 1.3. Next, we have the following diagram.

\[
\begin{array}{c}
\Phi_{+} \otimes \mathbb{Q} : \text{Pic}(X \times X) \otimes \mathbb{Q} \oplus \mathbb{Q} \longrightarrow \text{Pic}(M_{+}) \otimes \mathbb{Q} \\
\Phi_{+} \otimes \mathbb{Q} : \text{Pic}(X \times X) \otimes \mathbb{Q} \oplus \mathbb{Q} \longrightarrow \text{Pic}(M_{+}^{s}) \otimes \mathbb{Q},
\end{array}
\]

where $\Phi_{+}$ is defined at the equation (1.13) in [6] since $H_{\pm}$ are $\alpha$-generic and $a_{+}$ is not a miniwall, and the second column is a restriction map. Proposition 1.2 implies that the second column is surjective. On the other hand, the assumptions in this proposition implies that the second column is injective. As a result we get a homomorphism $\text{Cl}(M_{+}) \to \text{Pic}(M_{+}) \otimes \mathbb{Q}$. Thus we end the proof.

For a projective morphism $f$, we define $N_{1}(f)$ and $\overline{\text{NE}}(f)$ according to [4, Example 2.16], an extremal ray or extremal face of $\overline{\text{NE}}(f)$ according to [4, Definition 1.15], and the contraction of an extremal ray or face according to [4, Definition 1.25].

**Proposition 1.5.** Let $a_{\pm}$ be minichambers as in **Theorem 0.1**. Suppose $c_{2}$ is sufficiently large with respect to $c_{1}$ and $S$ that conclusions in **Lemma 1.1** and **Proposition 1.2** hold good. Then we have the following. Let $t$ be any point in $\phi_{+}(\text{Exec}^{\alpha} \phi_{+}) \subset M_{0}$, and let $l \simeq \mathbb{P}^{1}$ be any line in $\phi_{+}^{-1}(t) \simeq \mathbb{P}^{N}$. Then $\mathbb{R}_{\geq 0} \cdot l$ is an extremal ray of $\overline{\text{NE}}(M_{+})$, and $\phi_{+}$ is the contraction of this extremal ray. In particular $\rho(M_{+}/M_{0}) = 1$. The similar statement holds also for $\phi_{-} : M_{-} \to M_{0}$.

**Proof.** We check it for $a_{+}$; the proof is the same for $a_{-}$. For simplicity suppose that $M_{+}^{s}$ has a universal family $\mathcal{E}$ on $X \times M_{+}^{s}$, but the proof goes in a similar way for general case. The set

\[(5) \quad M_{+} \supset P_{+} = \{[E] \mid E \text{ is not } a_{-}\text{-semistable}\}\]

is contained in $M_{+}^{s}$ since we consider rank-two case. Take a point $t \in \phi_{+}(P_{+})$. By **Proposition 2.1.** in [10], it holds that $\phi_{+}^{-1}(t) \simeq \mathbb{P}^{N}$, and there is a nontrivial exact sequence on $X \times \mathbb{P}^{N}$

\[(6) \quad 0 \longrightarrow \pi_{1}^{*}F \otimes \mathcal{O}_{\mathbb{P}^{N}}(1) \longrightarrow \mathcal{E}|_{\phi_{+}^{-1}(t)} \otimes \pi_{2}^{*}L \longrightarrow \pi_{1}^{*}G \longrightarrow 0,\]

where $F$ and $G$ are coherent sheaves on $X$, which depends on the choice of $t$, and $L$ is a line bundle on $\phi_{+}^{-1}(t)$. Let $l \simeq \mathbb{P}^{1}$ be a line in $\phi_{+}^{-1}(t)$. Then (6) implies that $\text{ch}(\mathcal{E}|_{l}) = \text{ch}(E) + \mathcal{O}(1) \cdot \text{ch}(F)$ in $A(X \times l)$, where $E$ is a rank-two sheaf with Chern classes $\alpha$. Let $C$ be a class in $\overline{K}$. Because of the definition of $\overline{K}$ and the G.R.R. theorem, we have

\[
\text{deg}(\psi(C) \cdot l) = \left[\text{ch}(p_{23}^{*}C \otimes p_{13}^{*}E|_{t} \otimes p_{13}^{*}E|_{l}) \cdot p_{23}^{*}td(X \times X)|_{1 \times X \times X} \cdot \mathcal{O}(1)\right]_{1 \times X \times X} = \chi(X \times X, C \otimes (\pi_{1}^{*}F \otimes \pi_{2}^{*}E + \pi_{2}^{*}F \otimes \pi_{1}^{*}E)).
\]
By the projection formula and again by the definition of $\tilde{K}$, the last term equals
\[
\chi(X \times X, C \otimes \{\pi_1^*(F + G + F - G) \boxtimes \pi_2^*(E) + \pi_2^*(F + G + F - G) \boxtimes \pi_1^*(E)\}) / 2
\]
\[
- \chi(X \times X, C \boxtimes \{\pi_1^*(F - G) \boxtimes \pi_2^*(E) + \pi_2^*(F - G) \boxtimes \pi_1^*(E)\}) / 2
\]
\[
= [\pi_1^*td(X) \cdot \pi_2^*td(X) \cdot ch(C) \cdot \{\pi_1^*ch(F - G) \cdot \pi_2^*ch(E) + \pi_2^*ch(F - G) \cdot \pi_1^*ch(E)\}]_0 / 2
\]
\[
= [\{\pi_1^*(ch(C) \cdot \pi_2^*(td(X)ch(E))) + \pi_2^*(ch(C) \cdot \pi_1^*(td(X)ch(E)))\} \cdot td(X)ch(F - G)]_0 / 2.
\]
From [1, Section 3], if we denote $\xi = c_1(F) - c_1(G) \in NS(X)$, $n = c_2(F)$ and $m = c_2(G)$, then $W^\xi = \{H \in \text{Amp}(X) | H \cdot \xi = 0\}$ equals $W$ and one can check that
\[
\text{td}(X) \cdot ch(F - G) = (0, \xi, (a_0 - 1)(H_+ - H_-) \cdot \xi).
\]
Then there are exact sequences on $X \times l_i$
\[
0 \rightarrow \pi_i^*F \otimes \mathcal{O}_{\mathbb{P}^N}(1) \rightarrow \mathcal{E}|_{l_i} \otimes \pi_2^*L \rightarrow \pi_i^*G \rightarrow 0,
\]
where $F_i$ and $G_i$ are coherent sheaves on $X$, and $L_i$ is a line bundle on $l_i$, for $i = 1, 2$. Since the wall defined by $\xi_i = c_1(F_i) - c_1(G_i)$ equals $W$ for $i = 1, 2$, there is a rational number $r$ such that $\xi_1 = r\xi_2$ in Num($X$). Then (4) and (7) imply that $l_1 \equiv r \cdot l_2$ in $N_1(M+/M_0)$. As a result, we have $\overline{\text{NE}}(\phi_+) = \mathbb{R}_{\geq 0} \cdot l$.

Now $\mathbb{R}_{\geq 0} \cdot l$ is an extremal ray of $\overline{\text{NE}}(M_+)$. Indeed, let $u_i \in \overline{\text{NE}}(M_+)$ $(i = 1, 2)$ satisfy that $u_1 + u_2 \in \mathbb{R}_{\geq 0} \cdot l$. Then, for any $H \in \text{Amp}(M_0)$, $0 = (u_1 + u_2) \cdot \phi_+^*(H) = u_1 \cdot \phi_+^*(H) + u_2 \cdot \phi_+^*(H)$. Since $u_i \in \overline{\text{NE}}(M_+)$, we have $u_i \cdot \phi_+^*(H) \geq 0$, and hence $u_i \cdot \phi_+^*(H) = 0$ for $i = 1, 2$. Recall that, by Example-Exercise 3-5-1 in [7], a natural inclusion $N_1(\phi_+) \subset N_1(M_+)$ identifies $\overline{\text{NE}}(\phi_+)$ with
\[
\{z \in \overline{\text{NE}}(M_+) \mid z \cdot \phi_+^*(H) = 0 \text{ for any } H \in \text{Amp}(M_0)\}.
\]
Thereby $u_i \in \overline{\text{NE}}(\phi_+) = \mathbb{R}_{\geq 0} \cdot l$.

Last, $\phi_+$ is the contraction of $\mathbb{R}_{+} \cdot l$. Indeed, for any irreducible curve $C \subset M_+$, one can verify that $\phi_+(C)$ is a point if and only if $C \subset \mathbb{R}_{+} \cdot l$ by using arguments above. Also it holds that $\phi_+(\mathcal{O}_{M_+}) \simeq \mathcal{O}_{M_0}$, since one can show that $M_0$ is normal from conclusions in Lemma 1.1 and Serre’s criterion of normality, and so we conclude the proof of this proposition.

2. Some extremal faces of $M(H)$

Now we suppose that a polarization $H_+$ is $\alpha$-generic and contained in an $\alpha$- chamber $\mathcal{C}$, with which two different $\alpha$-walls $W_1$ and $W_2$ contact, that a polarization $H_0$ is contained in $W_1 \cap W_2 \cap \mathcal{C}$, and that no $\alpha$-wall except $W_1$ and $W_2$ contains $H_0$. Similarly to [1, Section 3], for $\alpha \in [0, 1]$ one can define the $\alpha$-stability of a coherent sheaf on $X$ and the moduli scheme $M(\alpha)$ of $\alpha$-semistable rank-two sheaves on $X$ with fixed Chern classes in such a way that $M(1) = M(H_0)$ and $M(\epsilon) = M(H_+)$ if $\epsilon \geq 0$ is sufficiently small. Let $a_\pm$ be minichambers separated by just one miniface $a_0$. Then Proposition 2.1 below says that $\rho(M+/M_0)$ can be greater than 1, $\overline{\text{NE}}(M_+)$ can have an extremal face with dim $\geq 2$, and so $\overline{\text{NE}}(M_+)$ can admit a “polyhedral-like part.”
Let $P_+ \subset M_+^a$ be the set defined at (5). Every member $E \in P_+$ has a Harder-Narasimhan filtration with respect to $a_-$, that is given by a nontrivial exact sequence
\[ 0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0, \]
and then one can check that the wall defined by $\xi(E) := c_1(F) - c_1(G) \in \text{NS}(X)$ equals $W_1$ or $W_2$ because of the way to derive $a_\pm$ from $H_\pm$. For $j = 1, 2$, we define a set
\[ P_+ \supset P^{(j)}_+ = \{ [E] \in P_+ \mid \text{the wall defined by } \xi(E) \text{ equals } W_j \}. \]

Then, from the uniqueness of $a_-$-HNF, $P^{(j)}_+$ is a union of some connected components of $P_+$, and it holds that $P^{(1)}_+ \cap P^{(2)}_+ = \emptyset$.

**Proposition 2.1.** Suppose that both $P^{(1)}_+$ and $P^{(2)}_+$ are non-empty. Then $\overline{\text{NE}}(M_+)$ has a two-dimensional extremal face spanned by $\mathbb{R}_{\geq 0} \cdot l_1$ and $\mathbb{R}_{\geq 0} \cdot l_2$, where $l_j \simeq \mathbb{P}^1$ is a line contained in $\phi^{-1}_+(t_j) \simeq \mathbb{P}^{N_j}$ with some $t_j \in \phi_+(P^{(j)}_+)$, for $j = 1, 2$. The morphism $\phi_+$ is the contraction of this extremal face.

**Proof.** If a sheaf $E_j \in M_+^a$ is a member of $l_j \subset P_+$, then one can check that $\mathbb{R} \cdot \xi(E_1)$ does not contain $\xi(E_2)$ in $\text{Num}(X)$ since $W_1 \neq W_2$. Thus it follows from (7) that the ray $\mathbb{R}_{\geq 0} \cdot l_1$ does not contain $l_2$ in $N_1$. In a similar way to the proof of Proposition 1.5, we can check that (i) $\overline{\text{NE}}(\phi_+) = \mathbb{R}_{\geq 0} \cdot l_1 + \mathbb{R}_{\geq 0} \cdot l_2$, (ii) this is a two-dimensional extremal face of $\overline{\text{NE}}(M_+)$, and (iii) $\phi_+$ is the contraction of this extremal face. 

Similarly, suppose that different $a$-walls $W_j$ $(1 \leq j \leq N)$ contact with an $a$-chamber $C$ containing $H_+$ and satisfy that $\cap_{j=1}^N W_j \cap C$ is non-empty. Then $\rho(M_+/M_0)$ can be $N$ or more, and $\overline{\text{NE}}(M_+)$ can have an extremal face with dim $\geq N$.

**Remark 2.2.** There does exist an example of a surface $X$, a class $\alpha$ with $4c_2 - c_1^2 \gg 0$, an $a$-chamber $C$, two $a$-walls $W_1$ and $W_2$, an $\alpha$-generic polarization $H_+$, a polarization $H_0$, a minichamber $a_+$ and a miniwall $a_0$ such that both $P^{(1)}_+$ and $P^{(2)}_+$ are non-empty. We leave it to the reader to find such examples. In rank-two case, the definition of $a$-walls is rather numerical. Hence if one grasps the structure of $\text{Amp}(X)$, then it may be just a calculating exercise to find such an example. Remark that, when $X$ is an Abelian surface, $\text{Amp}(X)$ is just a connected component of the big cone of $X$.

**References**


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