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$K$-FLIPS AND VARIATION OF MODULI SCHEME OF SHEAVES ON A SURFACE, II (Higher Dimensional Algebraic Geometry)

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K-FLIPS AND VARIATION OF MODULI SCHEME
OF SHEAVES ON A SURFACE, II

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INTRODUCTION

We shall consider some analogy between the wall-crossing problem of moduli schemes of stable sheaves on a surface, and the minimal model program of higher-dimensional varieties. This article is a continuation of [10].

Let $X$ be a non-singular projective surface over $\mathbb{C}$, and $H$ an ample line bundle on $X$. Denote by $M(H)$ (resp. $M^s(H)$) the coarse moduli scheme of rank-two $H$-semistable (resp. $H$-stable) sheaves on $X$ with Chern class $\alpha = (c_1, c_2) \in \text{Pic}(X) \times \mathbb{Z}$.

Let $H_-$ and $H_+$ be $\alpha$-generic polarizations such that just one $\alpha$-wall $W$ separates them. For $a \in [0, 1]$ one can define the $\alpha$-semistability of sheaves on $X$ and the coarse moduli scheme $M(a)$ (resp. $M^s(a)$) of rank-two $\alpha$-semistable (resp. $\alpha$-stable) sheaves with Chern classes $\alpha$ in such a way that $M(e) = M(H_+)$ and $M(1-e) = M(H_-)$ if $\epsilon > 0$ is sufficiently small. $M(a)$ is projective over $\mathbb{C}$. Let $a_- < a_+$ be minichambers separated by only one miniwall $a_0$, and denote $M_+ = M(a_+)$, $M_- = M(a_-)$ and $M_0 = M(a_0)$. There are natural morphisms $\phi_- : M_- \to M_0$ and $\phi_+ : M_+ \to M_0$ ([1], [2], [8]). One may say they are morphisms of moduli schemes coming from wall-crossing methods. Let $\phi_- : V_- \to V_0$ be a birational projective morphism such that (1) $V_-$ is normal, (2) $-K_{V_-}$ is $\mathbb{Q}$-Cartier and $\phi_-$-ample, (3) the codimension of the exceptional set $\text{Exc}(\phi_-)$ is more than 1, and (4) the relative Picard number $\rho(V_-/V_0)$ of $\phi_-$ is 1. After the theory of minimal model program, we say a birational projective morphism $\phi_+ : V_+ \to V_0$ is a \textit{K-flip} of $\phi_- : V_- \to V_0$ if (1) $V_+$ is normal, (2) $K_{V_+}$ is $\mathbb{Q}$-Cartier and $\phi_+$-ample, (3) the codimension of the exceptional set $\text{Exc}(\phi_+)$ is more than 1, and (4) the relative Picard number $\rho(V_+/V_0)$ of $\phi_+$ is 1.

\textbf{Theorem 0.1.} Fix a closed, finite, rational polyhedral cone $S \subset \overline{\text{Amp}}(X)$ such that $S \cap \partial \overline{\text{Amp}}(X) \subset \mathbb{R}_{\geq 0} \cdot K_X$. If $c_2$ is sufficiently large with respect to $c_1$ and $S$, then for any $\alpha$-generic polarizations $H_-$ and $H_+$ in $S$ separated by just one $\alpha$-wall $W$, and for any adjacent minichambers $a_- < a_+$ separated by a min iw all $a_0$ we have the following.

(i) $M_\pm$ are normal and $\mathbb{Q}$-factorial, $K_{M_\pm}$ are Cartier, $M_\pm^s$ are l.c.i., and $M_-$ and $M_+$ are isomorphic in codimension 1.

(ii) Suppose $K_X$ does not lie in the $\alpha$-wall, and that $K_X$ and $H_+$ lie in the same connected components of $\text{NS}(X) \mathbb{R} \setminus W$. Then $\rho(M_-/M_0) = 1$ and $\phi_+ : M_+ \to M_0$ is a $K$-flip of $\phi_- : M_- \to M_0$. This morphism $\phi_+$ (resp. $\phi_-$) is the contraction of an extremal ray of $\overline{\text{NE}}(M_+)$ (resp. $\overline{\text{NE}}(M_-)$), which is described in moduli theory.

(iii) Suppose $X$ is minimal and $\kappa(X) > 0$, which means $K_X$ is not numerically equivalent to 0 and contained in $\overline{\text{Amp}}(X)$. Then there is a polarization, say $H_X$,
contained in $\mathcal{S}$ such that no $\alpha$-wall separates $H_X$ and $K_X$, and the canonical divisor of $M(H_X)$ is nef.

The greater part of this result has already appeared in [10, Theorem 1.1.]. In Section 1, we shall prove the remaining part of this theorem which has not appeared in [10], that largely is the statement about the $\mathbb{Q}$-factoriality of $M_{\pm}$ and $\rho(M_{\pm}/M_0)$. The author was not aware of this part at the time of writing [10]. There is some application; suppose $X$ is minimal and $\kappa(X) > 0$, and fix a polarization $L$ on $X$. If $c_2$ is sufficiently large with respect to $c_1$ and $L$, then one can observe a moduli-theoretic analogue of the minimal model program of $M(L)$. Here “analogue” means that singularities of $M(H_X)$ are not considered. About this analogy, see Introduction in [10] for detail. We remark that a $K$-flip differs from a Thaddeus-type flip in [8].

In Section 2, we give some notes about extremal faces of $\overline{\text{NE}}(M(H)) \subset N(M(H))$, where $H$ is an $\alpha$-generic polarization. We shall point out that some extremal faces with $\dim \geq 2$ can appear in $\overline{\text{NE}}(M(H))$ when $H$ gets closer to more than one $\alpha$-wall.

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Notation. All schemes are locally of finite type over $\mathbb{C}$ or, more generally, an algebraically closed field of characteristic zero. For a projective scheme $V$ over $\mathbb{C}$, $\text{Num}(V)$ means Pic$(V)$ modulo numerically equivalence. For any coherent sheaf $E$ on $V$, $\text{Ext}_{V}^{1}(E, E)^{0}$ means the kernel of trace map $\text{Ext}_{V}^{1}(E, E) \rightarrow H^{0}(\mathcal{O}_{V})$.

1. Proof of Theorem

There is a union of hyperplanes $W \subset \text{Amp}(X)$ called $\alpha$-walls in the ample cone $\text{Amp}(X)$ such that $M(H) = M(H, \alpha)$ changes only when $H$ passes through $\alpha$-walls ([9]). A polarization on $X$ is called $\alpha$-generic if no $\alpha$-wall contains it. Now fix a closed, finite, rational polyhedral cone $\mathcal{S} \subset \text{Amp}(X)$ as in Theorem 0.1. Refer to [1, Section 3] about the $\alpha$-stability, minichambers and miniwalls, which appeared in Introduction.

Lemma 1.1. If $c_2$ is sufficiently large with respect to $c_1$ and $\mathcal{S}$, then for any $\alpha$-generic polarizations $H_{-}$ and $H_{+}$ in $\mathcal{S}$ separated by just one $\alpha$-wall $W$, and for any adjacent minichambers $a_{-}$ and $a_{+}$ separated by a miniwall $a_{0}$, (i) $M_{\pm}$ are normal, (ii) $K_{M_{\pm}}$ are Cartier, (iii) $M_{\pm}^{s}$ are l.c.i., (iv) $M_{-}$ and $M_{+}$ are isomorphic in codimension 4, and (v) our natural birational map $M_{-} \cdots \rightarrow M_{+}$ induces $\text{Pic}(M_{+}) \cong \text{Pic}(M_{-})$.

Proof. Fix a polarization $L \in \mathcal{S}$. If $c_2$ is sufficiently large w.r.t. $c_1$ and $L$, then $M(L)$ is normal, $M(L)$ is of expected dimension, and the codimension of $\text{Sing}'(M(L))$ in $M(L)$ is greater than 4 by [5] and [11], where $\text{Sing}'(M(L)) \subset M(L)$ is the closed subset consisting of sheaves $E$ such that $E$ is not $L$-stable or that $\text{Ext}_{X}^{2}(E, E)^{0} \neq 0$. One can check (iv) in a similar way to [10, Lemma 2.4.]. Now we compare $M(L)$ with $M_{+}$. By (iv) and the deformation theory of simple sheaves, $M_{+}^{s}$ is of expected dimension so it is l.c.i., and

\begin{equation}
\text{codim}(\text{Sing}'(M_{+}), M_{+}) > 4.
\end{equation}

Thereby $M_{+}^{s}$ is normal. Since $H_{\pm}$ are $\alpha$-generic and $a_{\pm}$ are minichambers, if a rank-two sheaf $E$ with Chern classes $\alpha$ is $a_{-}$-semistable and not $a_{+}$-semistable, then $E$
is $H$-semistable for any polarization $H$, and so our birational map $M_+ \cdots > M_-$ is isomorphic near $M_+ \setminus M_+^s$. Thus $M_+$ is normal near $M_+ \setminus M_+^s$, and accordingly $M_+$ itself is normal. Item (v) follows item (iv) and (1) because of Fact 1.3 below. Last, $M_+$ is the GIT quotient of an open subset $R_+$ of some Quot-scheme on $X$. Let $\mathcal{E}$ be a universal family of $R_+$ on $X \times R_+$. Since $a_+$ is not a minichamber, one can check that the line bundle det $R\text{Hom}_{p_2}(\mathcal{E}, \mathcal{E})$ on $R_+$ descends to a line bundle on $M_+$, that equals $K_{M_+}$.

Next we recall a fact concerning $\text{Pic}(M_+^s)$ from [6]. For a moment we assume $M_+^s$ has a universal family $\mathcal{E}$ on $X \times M_+^s$. Let $K$ be the Grothendieck group of $X \times X$ and let $\tilde{K}$ be the kernel of $\xi : K \to \mathbb{Z}$, that is defined by $\xi(C) = \chi(C \otimes \pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{E})$. Here $\otimes$ denotes the tensor product of complexes. Let $\sigma : X \times X \to X \times X$ be the map exchanging factor and let $\text{Pic}(X \times X)^\sigma$ be the subgroup consisting of line bundles invariant under $\sigma$. The map $\psi : \tilde{K} \to \text{Pic}(M_+)$ defined by

$$\psi(C) = \text{det}((p_1)_! (p_{23}^*(C) \otimes p_{12}^* \mathcal{E} \otimes p_{13}^* \mathcal{E})) \quad (C \in \tilde{K})$$

induces a homomorphism

$$\Phi_\pm : \text{Pic}(X \times X)^\sigma \otimes \mathbb{Z} \to \text{Pic}(M_+^s) \otimes \mathbb{Z} \left[ \frac{1}{12} \right],$$

as explained in [6, p. 132]. One can define $\Phi$ also when $M_+^s$ do not necessarily admit a universal family.

**Proposition 1.2.** Let $a_\pm$ be a minichamber satisfying assumptions in Lemma 1.1. If $c_2$ is sufficiently large with respect to $c_1$ and $S$, then

$$\Phi_\pm \otimes \mathbb{Q} : \text{Pic}(X \times X)^\sigma \otimes \mathbb{Q} \oplus \mathbb{Q} \to \text{Pic}(M_+^s) \otimes \mathbb{Q}$$

is isomorphic.

**Proof.** One can verify this from Lemma 1.1 (v) and by reading [6] (especially Lemma 3.10.) carefully. $\square$

Before the proof of Theorem 0.1, recall a useful fact at [SGA2, p.132].

**Fact 1.3.** Let $W$ be any quasi-projective and l.c.i. scheme with codim($\text{Sing}(W), W$) $\geq 4$. Then for any closed subset $\Lambda \subset W$ of codimension at least two, the restriction map $\text{Pic}(W) \to \text{Pic}(W \setminus \Lambda)$ is an isomorphism.

Now we shall prove two propositions; those and [10] end the proof of Theorem 0.1.

**Proposition 1.4.** Let $a_+$ be a minichamber satisfying assumptions in Lemma 1.1. Suppose $c_2$ is so large with respect to $c_1$ and $S$ that $M_\pm$ are normal, $M_+^s$ are l.c.i., codim($\text{Sing}(M_\pm), M_\pm$) $\geq 4$, codim($M_\pm \setminus M_\pm^s, M_\pm$) $\geq 2$, and the homomorphisms at (4) are isomorphic. Then $M_\pm$ are $\mathbb{Q}$-factorial.

**Proof.** First remark that assumptions in this proposition holds for $c_2 \gg 0$ from Lemma 1.1, Proposition 1.2, [11], and [3, Theorem 9.1.2.]. We shall verify this only for $M_+$. Let $U$ be the open set $M_+ \setminus \text{Sing}(M_+)$ in $M_+$. If CD($M_+$) means its divisor class group generated by Weil divisors, then we have

$$\text{Cl}(M_+) \to \text{Cl}(U) \simeq \text{Pic}(U) \to \text{Pic}(M_+^s),$$
where the first map is restriction, the second map is isomorphism since \( U \) is smooth, and the third map is an extension map, which is assured by Fact 1.3. Next, we have the following diagram:

\[
\begin{array}{ccc}
\Phi_+ \otimes \mathbb{Q} : \text{Pic}(X \times X)^e \otimes \mathbb{Q} \oplus \mathbb{Q} & \longrightarrow & \text{Pic}(M_+) \otimes \mathbb{Q} \\
\Phi_+ \otimes \mathbb{Q} : \text{Pic}(X \times X)^e \otimes \mathbb{Q} \oplus \mathbb{Q} & \longrightarrow & \text{Pic}(M_+^s) \otimes \mathbb{Q},
\end{array}
\]

where \( \Phi_+ \) is defined at the equation (1.13) in [6] since \( H_\pm \) are \( \alpha \)-generic and \( a_+ \) is not a miniwall, and the second column is a restriction map. Proposition 1.2 implies that the second column is surjective. On the other hand, the assumptions in this proposition implies that the second column is injective. As a result we get a homomorphism \( \text{Cl}(M_+) \to \text{Pic}(M_+) \otimes \mathbb{Q} \). Thus we end the proof. \( \square \)

For a projective morphism \( f \), we define \( N_1(f) \) and \( \overline{\text{NE}}(f) \) according to [4, Example 2.16], an extremal ray or extremal face of \( \overline{\text{NE}}(f) \) according to [4, Definition 1.15], and the contraction of an extremal ray or face according to [4, Definition 1.25].

**Proposition 1.5.** Let \( a_\pm \) be minichambers as in Theorem 0.1. Suppose \( c_2 \) is sufficiently large with respect to \( c_1 \) and \( S \) that conclusions in Lemma 1.1 and Proposition 1.2 hold good. Then we have the following. Let \( t \) be any point in \( \phi_+(\text{Exec}(\phi_+)) \subset M_0 \), and let \( l \simeq \mathbb{P}^1 \) be any line in \( \phi_+(t) \simeq \mathbb{P}^{N_{t}} \). Then \( \mathbb{R}_{\geq 0} \cdot l \) is an extremal ray of \( \overline{\text{NE}}(M_+) \), and \( \phi_+ \) is the contraction of this extremal ray. In particular \( \rho(M_+/M_0) = 1 \). The similar statement holds also for \( \phi_- : M_- \to M_0 \).

**Proof.** We check it for \( a_+ \); the proof is the same for \( a_- \). For simplicity suppose that \( M_+^s \) has a universal family \( \mathcal{E} \) on \( X \times M_+ \), but the proof goes in a similar way for general case. The set

\[
M_+ \supset P_+ = \{ [E] \mid E \text{ is not } a_- \text{-semistable} \}
\]

is contained in \( M_+^s \) since we consider rank-two case. Take a point \( t \in \phi_+(P_+) \). By Proposition 2.1. in [10], it holds that \( \phi_+^{-1}(t) \simeq \mathbb{P}^N \), and there is a nontrivial exact sequence on \( X \times \mathbb{P}^N \)

\[
0 \longrightarrow \pi_1^*F \otimes \mathcal{O}_{\mathbb{P}^N}(1) \longrightarrow \mathcal{E}|_{\phi_+^{-1}(t)} \oplus \pi_2^*L \longrightarrow \pi_1^*G \longrightarrow 0,
\]

where \( F \) and \( G \) are coherent sheaves on \( X \), which depends on the choice of \( t \), and \( L \) is a line bundle on \( \phi_+^{-1}(t) \). Let \( l \simeq \mathbb{P}^1 \) be a line in \( \phi_+^{-1}(t) \). Then (6) implies that \( ch(\mathcal{E}|_l) = ch(E) + \mathcal{O}_l(1) \cdot ch(F) \) in \( A(X \times l) \), where \( E \) is a rank-two sheaf with Chern classes \( \alpha \). Let \( C \) be a class in \( \tilde{K} \). Because of the definition of \( \tilde{K} \) and the G.R.R. theorem, we have

\[
\deg(\psi(C) \cdot l) = [p_1 \cdot (ch(p_{23}^*C \otimes p_{12}^*\mathcal{E}|_l \otimes p_{13}^*\mathcal{E}|_l) \cdot p_{23}^*\text{td}(X \times X))]_{1,1} \cdot \mathcal{O}_l(1)
\]

\[
= [p_1 \cdot \{ch(p_{23}^*C) \cdot \{p_{2}^*\mathcal{O}_l(1) \cdot p_{2}^*ch(F)\} \cdot p_{23}^*\text{td}(X \times X)\}]_{1,1} \cdot \mathcal{O}_l(1)
\]

\[
= [ch(C) \cdot \text{td}(X \times X) \cdot \{\pi_1^*ch(F)\pi_1^*ch(E) + \pi_2^*ch(F)\pi_2^*ch(E)\}]_{0,0,0} \cdot \mathcal{O}_l(1)
\]

\[
= \chi(X \times X, C \otimes (\pi_1^*F \otimes \pi_2^*E + \pi_2^*F \otimes \pi_2^*E)).
\]
By the projection formula and again by the definition of $K$, the last term equals
\[
\chi(X \times X, C \boxtimes \{ \pi_1^*(F + G + F - G) \boxtimes \pi_2^*(E) + \pi_2^*(F + G + F - G) \boxtimes \pi_1^*(E) \}) / 2
\]
\[
= \chi(X \times X, C \boxtimes \{ \pi_1^*(F - G) \boxtimes \pi_2^*(E) + \pi_2^*(F - G) \boxtimes \pi_1^*(E) \}) / 2
\]
\[
= [\pi_1^*td(X) \cdot \pi_2^*td(X) \cdot ch(C) \cdot \{ \pi_1^*ch(F - G) \cdot \pi_2^*ch(E) + \pi_2^*ch(F - G) \cdot \pi_1^*ch(E) \}]_{0} / 2
\]
\[
= [\{ \pi_1^*(ch(C) \cdot \pi_2^*(td(X)ch(E))) + \pi_2^*(ch(C) \cdot \pi_1^*(td(X)ch(E))) \} \cdot td(X)ch(F - G)]_{0} / 2.
\]
From [1, Section 3], if we denote $\xi = c_1(F) - c_1(G) \in \text{NS}(X)$, $n = c_2(F)$ and $m = c_2(G)$, then $W^K = \{ H \in \text{Amp}(X) | H \cdot \xi = 0 \}$ equals $W$ and one can check that $td(X) \cdot ch(F - G) = (0, \xi, (a_0 - 1)(H_+ - H_-) \cdot \xi)$. Thereby one can verify that
\[
\deg(\psi(C) \cdot l) = \left[ \left\{ \pi_1^*(C \cdot \pi_2^*td(X)) + \pi_2^*(C \cdot \pi_1^*td(X)) \right\} \right]^{0} \cdot (H_+ - H_-) \cdot \xi / 2.
\]
Now we shall show that $\text{rk}\, N_1(M_+/M_0) = 1$. If we pick two points $t_1$ and $t_2$ in $\phi_+(P_+)$, then $\phi_+^{-1}(t_i) \simeq \mathbb{P}^N$ for $i = 1, 2$. Fix lines $l_i \subset \phi_+^{-1}(t_i)$. Then there are exact sequences on $X \times l_i$
\[
0 \to \pi_1^*F_i \otimes \mathcal{O}_{\mathbb{P}^N}(1) \to \mathcal{E}|_{l_i} \otimes \pi_2^*L_i \to \pi_1^*G_i \to 0,
\]
where $F_i$ and $G_i$ are coherent sheaves on $X$, and $L_i$ is a line bundle on $l_i$, for $i = 1, 2$. Since the wall defined by $\xi_i = c_1(F_i) - c_1(G_i)$ equals $W$ for $i = 1, 2$, there is a rational number $r$ such that $\xi_1 = r \xi_2$ in $\text{Num}(X)$. Then (4) and (7) imply that $l_1 \equiv r \cdot l_2$ in $N_1(M_+/M_0)$. As a result, we have $\overline{\text{NE}}(\phi_+) = \mathbb{R}_{\geq 0} \cdot l$.

Now $\mathbb{R}_{\geq 0} \cdot l$ is an extremal ray of $\overline{\text{NE}}(M_+)$. Indeed, let $u_i \in \overline{\text{NE}}(M_+)$ ($i = 1, 2$) satisfy that $u_1 + u_2 \in \mathbb{R}_{\geq 0} \cdot l$. Then, for any $H \in \text{Amp}(M_0)$, $0 = (u_1 + u_2) \cdot \phi_+^*(H) = u_1 \cdot \phi_+^*(H) + u_2 \cdot \phi_+^*(H)$. Since $u_i \in \overline{\text{NE}}(M_+)$, we have $u_i \cdot \phi_+^*(H) \geq 0$, and hence $u_i \cdot \phi_+^*(H) = 0$ for $i = 1, 2$. Recall that, by Example-Exercise 3-5-1 in [7], a natural inclusion $N_1(\phi_+) \subset N_1(M_+)$ identifies $\overline{\text{NE}}(\phi_+)$ with
\[
\left\{ z \in \overline{\text{NE}}(M_+) \mid z \cdot \phi_+^*(H) = 0 \text{ for any } H \in \text{Amp}(M_0) \right\}.
\]

Thereby $u_i \in \overline{\text{NE}}(\phi_+) = \mathbb{R}_{\geq 0} \cdot l$.

Last, $\phi_+$ is the contraction of $\mathbb{R}_+ \cdot l$. Indeed, for any irreducible curve $C \subset M_+$, one can verify that $\phi_+(C)$ is a point if and only if $C \subset \mathbb{R}_+ \cdot l$ by using arguments above. Also it holds that $\phi_+(\mathcal{O}_{M_+}) \simeq \mathcal{O}_M$, since one can show that $M_0$ is normal from conclusions in Lemma 1.1 and Serre’s criterion of normality, and so we conclude the proof of this proposition.}

\[\square\]

2. Some extremal faces of $M(H)$

Now we suppose that a polarization $H_+$ is $\alpha$-generic and contained in an $\alpha$-chamber $C$, with which two different $\alpha$-walls $W_1$ and $W_2$ contact, that a polarization $H_0$ is contained in $W_1 \cap W_2 \cap \overline{C}$, and that no $\alpha$-wall except $W_1$ and $W_2$ contains $H_0$. Similarly to [1, Section 3], for $\alpha \in [0, 1]$ one can define the $\alpha$-stability of a coherent sheaf on $X$ and the moduli scheme $M(\alpha)$ of $\alpha$-semistable rank-two sheaves on $X$ with fixed Chern classes in such a way that $M(1) = M(H_0)$ and $M(\epsilon) = M(H_\pm)$ if $\epsilon \geq 0$ is sufficiently small. Let $a_\pm$ be minichambers separated by just one miniwall $a_0$. Then Proposition 2.1 below says that $\rho(M_+/M_0)$ can be greater than $1$, $\overline{\text{NE}}(M_+)$ can have an extremal face with dim $\geq 2$, and so $\overline{\text{NE}}(M_+)$ can admit a “polyhedral-like part.”
Let \( P_+ \subset M_+^s \) be the set defined at (5). Every member \( E \in P_+ \) has a Harder-Narasimhan filtration with respect to \( a_- \), that is given by a nontrivial exact sequence

\[
0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0,
\]

and then one can check that the wall defined by \( \xi(E) := c_1(F) - c_1(G) \in \text{NS}(X) \) equals \( W_1 \) or \( W_2 \) because of the way to derive \( a_\pm \) from \( H_\pm \). For \( j = 1, 2 \), we define a set

\[
P_+ \supset P_+^{(j)} = \{ [E] \in P_+ \mid \text{the wall defined by } \xi(E) \text{ equals } W_j \}.
\]

Then, from the uniqueness of \( a_-\)-HNF, \( P_+^{(j)} \) is a union of some connected components of \( P_+ \), and it holds that \( P_+^{(1)} \cap P_+^{(2)} = \emptyset \).

**Proposition 2.1.** Suppose that both \( P_+^{(1)} \) and \( P_+^{(2)} \) are non-empty. Then \( \overline{\text{NE}}(M_+) \) has a two-dimensional extremal face spanned by \( \mathbb{R}_{\geq 0} \cdot l_1 \) and \( \mathbb{R}_{\geq 0} \cdot l_2 \), where \( l_j \simeq \mathbb{P}^1 \) is a line contained in \( \phi_+^{-1}(t_j) \simeq \mathbb{P}^{N_j} \) with some \( t_j \in \phi_+(P_+^{(j)}) \), for \( j = 1, 2 \). The morphism \( \phi_+ \) is the contraction of this extremal face.

**Proof.** If a sheaf \( E_j \in M_+^s \) is a member of \( l_j \subset P_+ \), then one can check that \( \mathbb{R} \cdot \xi(E_1) \) does not contain \( \xi(E_2) \) in \( \text{Num}(X) \) since \( W_1 \neq W_2 \). Thus it follows from (7) that the ray \( \mathbb{R}_{\geq 0} \cdot l_1 \) does not contain \( l_2 \) in \( N_1(M_+) \). In a similar way to the proof of Proposition 1.5, we can check that (i) \( \overline{\text{NE}}(\phi_+) = \mathbb{R}_{\geq 0} \cdot l_1 + \mathbb{R}_{\geq 0} \cdot l_2 \), (ii) this is a two-dimensional extremal face of \( \overline{\text{NE}}(M_+) \), and (iii) \( \phi_+ \) is the contraction of this extremal face.

Similarly, suppose that different \( \alpha \)-walls \( W_j \) \((1 \leq j \leq N)\) contact with an \( \alpha \)-chamber \( C \) containing \( H_+ \) and satisfy that \( \cap_{j=1}^{N} W_j \cap C \) is non-empty. Then \( \rho(M_+/M_0) \) can be \( N \) or more, and \( \overline{\text{NE}}(M_+) \) can have an extremal face with \( \dim \geq N \).

**Remark 2.2.** There does exist an example of a surface \( X \), a class \( \alpha \) with \( 4c_2 - c_1^2 \gg 0 \), an \( \alpha \)-chamber \( C \), two \( \alpha \)-walls \( W_1 \) and \( W_2 \), an \( \alpha \)-generic polarization \( H_+ \), a polarization \( H_0 \), a minichamber \( a_+ \) and a miniwall \( a_0 \) such that both \( P_+^{(1)} \) and \( P_+^{(2)} \) are non-empty. We leave it to the reader to find such examples. In rank-two case, the definition of \( \alpha \)-walls is rather numerical. Hence if one grasps the structure of \( \text{Amp}(X) \), then it may be just a calculating exercise to find such an example. Remark that, when \( X \) is an Abelian surface, \( \text{Amp}(X) \) is just a connected component of the big cone of \( X \).

**References**


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