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Descending chain condition for stringy invariants

By

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Abstract

It is known that the degree of the stringy E-function of a log terminal singularity is related to the minimal log discrepancy, and the minimal log discrepancies of certain classes of singularities satisfy the ascending chain condition. We ask if the stringy E-functions of certain classes of varieties satisfy the descending chain condition. As an example, we look at the case of toric varieties. We also comment on the non-standard counting measure, for which the author asked the question of descending chain condition in [T].

§ 1. Introduction

Let (X, B) be a Kawamata log terminal pair over $\mathbb{C}$, where $B = \sum b_i B_i$ is a boundary $\mathbb{R}$-divisor. Batyrev([Ba1], [Ba2]) defined its stringy E-function $E_{st}(X, B; u, v)$ as follows. First of all, the usual E-polynomial of an algebraic variety X is defined as $E(X; u, v) = \sum (-1)^k h^{ij}(H^k_c(X, \mathbb{C}))u^i v^j$. Take a log resolution $\rho : Y \to X$, let $(D_i)_{i \in I}$ denote the exceptional divisors and proper transforms of B_i, and $a(D_i; X, B)$ the discrepancies. For $J \subseteq I$, write $D_J := \bigcap_{j \in J} D_j$ and $D^\circ_J := D_J \setminus \bigcup_{i \notin J} D_i$. The stringy E-function is then defined to be

$$E_{st}(X, B; u, v) := \sum_{J \subseteq I} E(D^\circ_J; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a(D_j;X,B)+1} - 1}.$$

This is independent of the log resolution. Batyrev proved this fact by using motivic integrations. Another method is to use the Weak Factorization Theorem([AKMW]).

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To be rigorous, the "function" $E_{st}(X,B;u,v)$ is an element of a ring $\hat{A}$, which is defined[1] as

$$\left\{ \sum_{\substack{p,q\in\mathbb{R}\\ p-q\in\mathbb{Z}}} a_{pq}u^pv^q \;\middle|\; \begin{array}{l} a_{pq}\in\mathbb{R}, \text{ and there are only finitely many } p,q \\ \text{with } p+q>-N \text{ and } a_{pq}\neq 0, \text{ for any } N \end{array} \right\}.$$

Thus E_{st} is a power series in u and v, in *negative* direction. Note that E_{st} is symmetric in u and v. Let $\hat{A}_{sym}$ denote the subring of $\hat{A}$ consisting of elements symmetric in u and v.

E-polynomials are additive, i.e. $E(X) = E(X\setminus Y) + E(Y)$ for any variety X and its closed subset Y, and multiplicative, i.e. $E(X\times Y) = E(X)E(Y)$ for any varieties X and Y. This shows that the E-polynomial is a kind of measure. The stringy E-function can be considered as a kind of volume that takes the singularities into account. To see how the singularities affect E_{st}, let us look at one summand in the defining formula. It is of the form

$$\begin{aligned} &((uv)^{\dim D_J} + (\text{lower order}))\prod_{j\in J}((uv)^{-a(D_j)} + (\text{lower order})) \\ =\,&(uv)^{\dim X-\sum_{j\in J}(a(D_j)+1)} + (\text{lower order}). \end{aligned}$$

Thus we can roughly say that if the singularities are worse, then $a(D_j)$ is smaller and therefore E_{st} is bigger, assuming that we regard uv as big. To be more specific, it has been known that the degree of E_{st} of a singularity is related to the minimal log discrepancy: For a point $P\in X$, let $\mathrm{mld}(P;X,B)$ denote the minimal log discrepancy over P and let $E_{st}(P;X,B;u,v) = E_{st}(X,B;u,v) - E_{st}(X\setminus P, B|_{X\setminus P};u,v)$. Then the highest total-degree part of $E_{st}(P;X,B;u,v)$ is $(uv)^{\dim X-\mathrm{mld}(P;X,B)}$ times the number of divisors with the minimal log discrepancy. Let us pay attention to the following important properties of the minimal log discrepancies.

(1) The minimal log discrepancy does not increase, and sometimes decreases, in the course of MMP.

(2) The set of minimal log discrepancies is expected to satisfy the ascending chain condition under various settings.

It is natural to ask[2] if something analogous can be said about the stringy E-functions. More precisely, can one define (partial, pre-)order of power series for which uv is big and the stringy E-functions satisfy properties similar to (1) and (2) above?

Let us give a few examples of ordering.

[1] In [Ba2, Definition 2.7], $\hat{A}$ is defined as a certain completion of $\mathbb{Z}[\tau^{\pm}][\theta^{\mathbb{Q}}]$. One may replace $\theta^{\mathbb{Q}}$ by $\theta^{\mathbb{R}}$, and our definition is related to this by $u=\tau\theta^{-1}$ and $v=\tau^{-1}\theta^{-1}$.

[2] The question about DCC is a variant of [T, Problem 2.15]. See §2.

Definition 1.1. Let $f(u,v)$ be an element of $\hat{A}_{sym}$.

- $f(u,v) >_0 0$ if the highest total-degree part is of the form $a_s(uv)^s$ with $a_s > 0$. This defines a partial order.
- Let $\hat{A}'$ denote the field
$$\left\{ \sum_{s\in\mathbb{R}} a_s w^s \,\middle|\, a_s \in \mathbb{R}, \text{there are only finitely many } s > -N \text{ with } a_s \neq 0, \text{ for any } N \right\},$$
and consider the lexicographic order $>$ on $\hat{A}'$. That is, if $f = \sum_{s\le s_0} a_s w^s$ is an element of $\hat{A}'$ with $a_{s_0} \neq 0$, then $f > 0$ if and only if $a_{s_0} > 0$. This order makes $\hat{A}'$ an ordered field. For $t \in \mathbb{R} \cup S^1$, let $f_t(w) := f(tw, t^{-1}w)$, by which we mean $u^p v^q + u^q v^p \mapsto w^{p+q}(t^{p-q} + t^{q-p})$. We obtain a homomorphism $\hat{A}_{sym} \to \hat{A}'$ in this way. (A better choice for the parameter might be $t + t^{-1} \in \mathbb{R}$.) Declare $f(u,v) \gtrsim_t 0$ if $f_t \ge 0$. The relation $\gtrsim_t$ is a total preorder.
- More generally, one can define a partial preorder $\gtrsim_T$ for any $T \subseteq \mathbb{R} \cup S^1$, by "$f \gtrsim_T 0 \Leftrightarrow \forall t \in T, f \gtrsim_t 0$." It is a partial order if T is either infinite or contains a transcendental number. In fact, assume that $f \gtrsim_T 0$ and $0 \gtrsim_T f$ holds for some $f \in \hat{A}_{sym} \setminus \{0\}$. This is equivalent to saying $f_t = 0$ for all $t \in T$. In particular, if we write the highest degree part of f as $(uv)^s g(u,v)$, where g is a homogeneous polynomial, then we have $g(t, t^{-1}) = 0$ for all $t \in T$. Therefore, T must be a finite set of algebraic numbers.

For any of the (pre-)orders above, the stringy E-function strictly decreases after any MMP step, as is easily seen from the defining formula. Similarly, if X is projective, has only terminal singularities, say, and is not uniruled, then X is a minimal model if and only if its stringy E-function is minimal among terminal varieties birational to X. It is tempting to imagine that the set of stringy E-functions satisfies the descending chain condition for certain classes of (X, B) — although the author is not so sure that this point of view is useful in Minimal Model Program.

In the rest of this article, we give further motivating discussions and look at a few examples. In §2, we comment on another stringy invariant called "stringy non-standard point counting"([T]). Since there is a natural ordering for such invariants, the question of DCC will look more natural to the reader than in the case of orderings introduced above, which might have been somewhat artificial. In §3, we look at two toric cases. In the first, we allow only standard coefficients, and the result follows immediately from Ambro's boundedness of index([Am2]). The second is the case of 2-dimensional cyclic quotients, where the coefficients of the boundary are taken from a fixed set of real numbers satisfying DCC. In §4, we ask a few bold questions, and give a couple of remarks.

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§ 2. Non-standard point counting

As explained in Introduction, E-polynomials can be considered as a kind of measure. Over a finite field, counting the number of rational points gives a similar measure, and over a field of characteristic 0, we choose reductions to finite fields and take certain limit to obtain a measure $\mathfrak{N}$. Then we replace E-polynomials by $\mathfrak{N}$ in the definition of stringy E-functions to obtain the stringy version $\mathfrak{N}_{st}$.

We have a natural (partial) ordering for this invariant, so the question of descending chain condition is more natural here. In fact, it was for this invariant that the author originally posed the problem of DCC([T, Problem 2.15]). Below we summarize relevant definitions and results from [T].

Definition 2.1. Let k be a field, $\mathfrak{Var}_k$ the category of varieties over k, $\overline{\mathfrak{Var}_k}$ the set of isomorphism classes in $\mathfrak{Var}_k$. A map $\mu : \overline{\mathfrak{Var}_k} \to R$ to a ring R is called a motivic measure if the following conditions hold:

(i) $\mu(X) = \mu(X \setminus Y) + \mu(Y)$ for any variety X and its closed subvariety Y.

(ii) $\mu(X \times Y) = \mu(X)\mu(Y)$ for any varieties X and Y.

In particular, the assignment $X \mapsto E(X; u, v)$ is a motivic measure.

For a motivic measure μ, we would like to define its stringy version μ_{st}. The following gives a condition for this to be possible.

Proposition-Definition 2.2. *Assume that k is of characteristic 0. Let μ be a motivic measure with values in a ring R and $p : \mathbb{R} \to R^\times$ a group homomorphism with $p(1) = \mu(\mathbb{A}^1)$. Write $\mu(\mathbb{A}^1)^s$ for $p(s)$ and assume that $\mu(\mathbb{A}^1)^s - 1$ is invertible for any $s > 0$.*

Let (X, B) be a Kawamata log terminal pair over k. Take a log resolution $\rho : Y \to X$, denote the exceptional divisors and proper transforms of components of B by $(D_i)_{i \in I}$, and the discrepancies by $a(D_i; X, B)$. For $J \subseteq I$, write $D_J := \bigcap_{j \in J} D_j$ and $D_J^\circ := D_J \setminus \bigcup_{i \notin J} D_i$. Then

$$\mu_{st}(X, B) := \sum_{J \subseteq I} \mu(D_J^\circ) \prod_{j \in J} \frac{\mu(\mathbb{A}^1) - 1}{\mu(\mathbb{A}^1)^{a(D_j; X, B)+1} - 1}$$

is independent of the log resolution.

Proof. Proposition 2.3 and Remark 2.8 of [V] can be extended to allow $\mathbb{R}$-divisors as boundaries. Since our assumption implies that μ factors through the ring $\mathcal{R}$ of [V, §1.8(1)], with exponents extended to $\mathbb{R}$, our proposition follows. □

As such a measure, we define the "non-standard point counting measure" $\mathfrak{N}$. Let k be a field and

$$\mathcal{R} := \{A \subseteq k : \text{subring which is finitely generated over the smallest subring}\},$$
$$\mathcal{S} := \{R \subseteq k | R = A_{\mathfrak{m}} \text{ for some } A \in \mathcal{R} \text{ and a maximal ideal } \mathfrak{m}\}.$$

We want to take reductions to finite fields. A variety over k lifts to a scheme over some $A \in \mathcal{R}$ and hence gives rise to schemes over elements of a certain subset of $\mathcal{S}$, which is not necessarily the whole $\mathcal{S}$. Also the schemes obtained in this way depend on the choice of the lifting. To obtain something well-defined, we take the quotient by a filter on $\mathcal{S}$. (See [CK, Ch. 4 and Ch. 6 §2] for generalities on filters, reduced products and ultraproducts.)

A filter $\mathcal{F}$ on $\mathcal{S}$ is a nonempty subset of $2^{\mathcal{S}}$ satisfying

(i) $\emptyset \notin \mathcal{F}$,

(ii) $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$,

(iii) $U \in \mathcal{F}, V \supseteq U \Rightarrow V \in \mathcal{F}$.

A maximal filter is called an ultrafilter.

Let $(A_R)_{R \in \mathcal{S}}$ be a family of sets indexed by $\mathcal{S}$. The reduced product $\prod A_R/\mathcal{F}$ is defined as the quotient of $\prod A_R$ by the equivalence relation $(a_R) \sim (b_R) \Leftrightarrow \{R | a_R = b_R\} \in \mathcal{F}$.

Proposition 2.3. *(1) For any filter $\mathcal{F}$ on $\mathcal{S}$, there exists an ultrafilter $\tilde{\mathcal{F}}$ containing $\mathcal{F}$.*

(2) If each A_R is a ring, then $\prod A_R/\mathcal{F}$ is a ring in a natural way. If each A_R is partially ordered, then $[(a_R)] \geq [(b_R)] \Leftrightarrow \{R | a_R \geq b_R\} \in \mathcal{F}$ gives a partial order.

(3) If each A_R is a field and $\mathcal{F}$ is an ultrafilter, then $\prod A_R/\mathcal{F}$ is a field. If each A_R is totally ordered and $\mathcal{F}$ is an ultrafilter, then $\prod A_R/\mathcal{F}$ is totally ordered.

Proof. (1) [CK, Corollary 4.1.4].

(2), (3) These are consequences of general principles on reduced products. The operators $\{+, \cdot, 0, 1\}$ and the relation $\geq$ are defined in a natural way by [CK, 4.1.6, Proposition 4.1.7]. Since the axioms of fields, ordered sets and ordered fields are given by first-order formulas, the Fundamental Theorem of Ultraproducts([CK, Theorem 4.1.9]) shows (3). Since the axioms of (unital commutative) rings, partially-ordered sets and partially-ordered rings are given by Horn sentences, Proposition 6.2.2 of [CK] shows (2).

For the statements about the ring structure, a more down-to-earth explanation can be found in [S, 2.3.4, 2.3.5]. □

For $A \in \mathcal{R}$, write $F_A = \{R \in \mathcal{S} | R \supseteq A\}$. Define

$$\mathcal{F}_0 := \{U \in 2^{\mathcal{S}} | U \supseteq F_A \text{ for some } A \in \mathcal{R}\}.$$

This is a filter. If X is a variety over k, then there exist a ring $A \in \mathcal{R}$ and an algebraic scheme X_A over A such that $X_A \otimes_A k \cong X$. Let us fix them. For each $R \in F_A$, let $\kappa(R)$ be the residue field and count the number of points $n_R := \#(X_A \otimes_A \kappa(R))$. For $R \notin F_A$, set n_R to an arbitrary value, say 0. Then the class of $(n_R)_{R\in\mathcal{S}}$ modulo $\mathcal{F}_0$ is independent of the choices, and this defines a motivic measure $\mathfrak{N} : \overline{\mathfrak{Var}_k} \to \mathbb{Z}^{\mathcal{S}}/\mathcal{F}_0$. If $\mathcal{F}$ is any ultrafilter containing $\mathcal{F}_0$, we have $\mathfrak{N}_{\mathcal{F}} : \overline{\mathfrak{Var}_k} \to \mathbb{Z}^{\mathcal{S}}/\mathcal{F}$.

If k is of characteristic 0, define $p : \mathbb{R} \to \mathbb{R}^{\mathcal{S}}/\mathcal{F}_0$ by $p(s) = (\#\kappa(R)^s)_{R\in\mathcal{S}}$. This function satisfies the assumption of Proposition 2.2, and we can define $\mathfrak{N}_{st}(X, B) \in \mathbb{R}^{\mathcal{S}}/\mathcal{F}_0$ (resp. $\mathfrak{N}_{st,\mathcal{F}}(X, B) \in \mathbb{R}^{\mathcal{S}}/\mathcal{F}$) for any Kawamata log terminal pair (X, B). The target ring is a partially ordered ring (resp. a totally ordered field).

We see the following, as in the case of E_{st}.

- $\mathfrak{N}_{st}$ decreases after each MMP step.
- Minimality can be be expressed in terms of $\mathfrak{N}_{st}$.
- The minimal log discrepancy can be recovered from $\mathfrak{N}_{st}(P; X, B) := \mathfrak{N}_{st}(X, B) - \mathfrak{N}_{st}(X \setminus P, B|_{X\setminus P})$.

Here the value rings are naturally ordered. If one considers the set of the (usual) number of points of *all* varieties over a fixed finite field, they of course satisfy the DCC, since the number of points is a natural number. Thus it is tempting to imagine that DCC holds for $\mathfrak{N}$ and $\mathfrak{N}_{st}$ in a vast generality. We will see certain examples and counterexamples in the remaining sections.

§ 3. Toric cases

Let's look at the case of toric varieties with toric divisors. There is no $H^{p,q}$ for $p \neq q$ in this case, so we don't have to care about which ordering to choose for E_{st}. Let us write $W(P; X, B; w) = E_{st}(P; X, B; w^{1/2}, w^{1/2})$. We have $\mathfrak{N}_{st}(P; X, B) = W(P; X, B; \mathfrak{N}_{st}(\mathbb{A}^1))$, and DCC for $\mathfrak{N}_{st,\mathcal{F}}$ is equivalent to that for E_{st} if $\mathcal{F}$ is a filter containing $\mathcal{F}_0$.

§ 3.1. The case of boundaries with standard coefficients

Let N be a lattice of rank d, M its dual, and $\sigma \subset N \otimes \mathbb{R}$ a rational, strictly convex cone of dimension d. Let v_i be the primitive generators of the 1-dimensional faces of σ. We denote the affine toric variety $\operatorname{Spec}\mathbb{C}[M \cap \sigma^\vee]$ by X, the unique closed torus orbit by 0 and the divisor corresponding to v_i by B_i. For any cone τ, let τ° denote its relative interior, i.e. $\tau \setminus \bigcup(\text{lower dimensional face of } \tau)$.

Proposition 3.1. *(1) For $B = \sum b_i B_i$, $K_X + B$ is $\mathbb{Q}$-Cartier if and only if there exists a linear function $\varphi : N \otimes \mathbb{R} \to \mathbb{R}$ such that $\varphi(v_i) = 1 - b_i$ for any i.*

(2) The series $W(0; X, B; w)$ is calculated as

$$(w-1)^d \sum_{n \in \sigma^\circ \cap N} w^{-\varphi(n)}.$$

Proof. Similar to the proof of Theorem 4.3, [Ba1]. □

Now assume that B has standard coefficients, i.e., $b_i = 1 - 1/n_i$ for some positive integer n_i. We will use the following theorem.

Theorem 3.2 ([Am2], Theorem 1.1). *Let q be the denominator of* $\operatorname{mld}(0; X, B)$*. Then the Cartier index of $K_X + B$ is at most $c_d q^d$, where c_d is a positive constant depending on d only.*

In particular, if q is fixed then the coefficients of B belong to a finite set.

Theorem 3.3. *For d-dimensional affine toric log pairs (X, B) with standard coefficients, the set of $W(0; X, B; w)$ satisfies the descending chain condition.*

Proof. First note that we have only to consider the case of constant minimal log discrepancy m, by the ACC of minimal log discrepancies([Am1]).

The previous theorem tells that there exists a positive integer R such that $R(K_X + B)$ is Cartier whenever $\operatorname{mld}(0; X, B) = m$. This is equivalent to saying $R\varphi(N) \subseteq \mathbb{Z}$.

One can decompose σ into simplicial cones which are generated by subsets of $\{v_i\}$: There is a fan $\Sigma = \{\sigma_k\}_{k \in K}$ with $|\Sigma| = \sigma$, $\dim \sigma_k = d_k$ and $\sigma_k = \sum_{l=1}^{d_k} \mathbb{R}_{\geq 0} v_{i_{kl}}$ for some sequence $i_{k1}, \ldots, i_{kd_k}$. Let $P_k = \{\sum a_l v_{i_{kl}} | 0 < a_l \leq 1\}$. Write K° for the set of indices k such that $\sigma_k^\circ \subset \sigma^\circ$. Then $W(0; X, B; w)$ can be written as

$$(w-1)^d \sum_{k \in K^\circ} \left(\prod_{l=1}^{d_k} \frac{1}{1 - w^{-(1-b_{i_{kl}})}} \sum_{n \in P_k \cap N} w^{-\varphi(n)} \right).$$

Let $\{\beta_i | i \in I\}$ be the set of possible values of coefficients in Theorem 3.2. Then each summand in this expression belongs to

$$S_f := \left\{ \prod_{j=1}^{d'} \frac{1}{1 - w^{-(1-\beta_{f(j)})}} \sum_{d \in \frac{1}{R}\mathbb{Z}, 0 < d \leq \sum_{j=1}^{d'}(1-\beta_{f(j)})} c_d w^{-d} \,\middle|\, c_d \in \mathbb{Z}_{\geq 0} \right\}$$

for some $d' \leq d$ and $f : \{1, 2, \ldots, d'\} \to I$. There might be multiple (or no) $k \in K^\circ$ for a given f, but they add up to an element of S_f. For each f, the set S_f satisfies DCC, since its elements are $\prod_{j=1}^{d'}(1-w^{-(1-\beta_{f(j)})})^{-1}$ times polynomials of $w^{-1/R}$ whose degrees are bounded and whose coefficients are nonnegative integers. It is easy to see that if S and T satisfy DCC then so does $\{s+t|s \in S, t \in T\}$, and we are done. $\square$

§ 3.2. 2-dimensional case

Let $(0 \in S_{n,q}) = (0 \in \mathbb{C}^2)/\langle \frac{1}{n}(q,1)\rangle$ be a cyclic quotient singularity of dimension 2, and E and F the images of $0 \times \mathbb{C}$ and $\mathbb{C} \times 0$. This is the toric surface associated to $N = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{n}(q,1)$ and $\sigma = \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(0,1)$, and E and F are the toric divisors corresponding to $(1,0)$ and $(0,1)$.

Theorem 3.4. *Assume that $\mathcal{B} \subset (0,1)$ satisfies the descending chain condition. Then the set*

$$\{W(0; S_{n,q}, bE + cF; w)|n \geq q > 0, \gcd(n,q) = 1, b, c \in \mathcal{B}\}$$

satisfies the descending chain condition.

Again the question is whether there can be an infinite descending sequence with the same degree, since ACC for minimal log discrepancies is already known([Al]). In the case of toric varieties with standard coefficients, the result followed from the boundedness of Cartier indices for a fixed value of minimal log discrepancy. We have the following analogue for toric surfaces with $\mathbb{R}$-coefficients, from which we deduce the above theorem.

Theorem 3.5. *Let the notations be as above, and let Div^+ denote the semigroup of effective Cartier divisors. For any real number m, there exists a finite set C of positive real numbers satisfying the following: For any (n,q,b,c) with $b, c \in \mathcal{B}$ and $\mathrm{mld}(0; S_{n,q}, bE+cF) = m$, the log anti-canonical divisor $(1-b)E+(1-c)F$ is contained in $C \cdot \mathrm{Div}^+ := \{\sum c_i D_i | c_i \in C, D_i \in \mathrm{Div}^+\}$.*

Proof of "Theorem 3.5 ⇒ Theorem 3.4". We have only to show that the set $\{W(0; S_{n,q}, bE+cF; w)|\mathrm{mld}(0; S_{n,q}, bE+cF) = m\}$ satisfies DCC. Take C as in Theorem 3.5.

The function φ in Proposition 3.1 is given by $\varphi(n_1, n_2) = (1-b)n_1 + (1-c)n_2$, and our function $W(0; S_{n,q}, bE + cF; w)$ is

$$\begin{aligned}
&(w-1)^2 \sum_{(n_1,n_2)\in\sigma^\circ\cap N} w^{-((1-b)n_1+(1-c)n_2)} \\
&= (w-1)^2 \frac{1}{(1-w^{-(1-b)})(1-w^{-(1-c)})} \sum_{(n_1,n_2)\in(0,1]^2\cap N} w^{-((1-b)n_1+(1-c)n_2)}.
\end{aligned}$$

From $(1-b)E+(1-c)F \in C\cdot\mathrm{Div}^+$ it follows that $(1-b)n_1+(1-c)n_2 \in C\cdot\mathbb{Z}_{\geq 0}$. We also have $0 < (1-b)n_1+(1-c)n_2 < 2$. Thus there are only finitely many possibilities for $(1-b)n_1+(1-c)n_2$, say $e_1,\dots,e_n$. The sum $\sum_{(n_1,n_2)\in(0,1]^2\cap N} w^{-((1-b)n_1+(1-c)n_2)}$ belongs to $\{k_1w^{-e_1}+\cdots+k_nw^{-e_n} | k_1,\dots,k_n \in \mathbb{Z}_{\geq 0}\}$, and we see that for a fixed (b,c) the set of $W(0;S_{n,q},bE+cF;w)$ satisfies DCC. Since $1-b$ and $1-c$ belong to $C\cdot\mathbb{Z}_{\geq 0}\cap(0,1)$, there are only finitely many possibilities for b and c. □

Proof of Theorem 3.5. The proof presented here depends on [Al].

First of all, one can always discard finite possibilities for (n,q). In fact, for a fixed (n,q), the minimal log discrepancy at $0\in(S_{n,q},bE+cF)$ is a piecewise linear function of (b,c). Since $\mathcal{B}$ is assumed to satisfy DCC, there are only finitely many possibilities for (b,c).

To use the calculation of [Al], let us give another characterization of $S_{n,q}$: A normal surface singularity is isomorphic to $S_{n,q}$ for some (n,q) if and only if the exceptional curve of its minimal resolution is a chain of smooth rational curves. To be more precise, let r and $f_1,\dots,f_r$ be the integers determined by $f_i>1$ and

$$n/q = f_1 - \cfrac{1}{f_2 - \frac{1}{f_3 - \cdots - \frac{1}{f_r}}}.$$

This is called the Hirzebruch-Jung continued fraction([BPV, Ch. III, §5]). Let C_0 and C_{r+1} be the strict transforms of E and F. Then the exceptional curves of the minimal resolution of $S_{n,q}$ can be labelled $C_1,\dots,C_r$ so that $C_0,C_1,\dots,C_r,C_{r+1}$ form a chain in this order, and the self intersection of C_i is $-f_i$. By the following lemma, n and q can written as $(-1)^r\det(C_i.C_j)_{i,j=1}^r$ and $(-1)^{r-1}\det(C_i.C_j)_{i,j=2}^r$.

Lemma 3.6. *Let $f_1,f_2,\dots,f_r$ be positive integers and assume that $f_i>1$ for either $i=1,\dots,r-1$ or $i=2,\dots,r$. Let n_i and q_i be the coprime positive integers such that*

$$n_i/q_i = f_i - \cfrac{1}{f_{i+1} - \frac{1}{f_{i+2} - \cdots - \frac{1}{f_r}}}.$$

(1) One has recursive relations

$$n_i = f_in_{i+1} - q_{i+1},$$
$$q_i = n_{i+1}.$$

(2)

$$n_i = \det\begin{pmatrix} f_i & -1 \dots & 0 & 0 \\ -1 & f_{i+1} \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \dots & f_{r-1} & -1 \\ 0 & 0 \dots & -1 & f_r \end{pmatrix}.$$

Proof. (1) is straightforward. For (2), expand the determinant along the first column. □

Lemma 3.7. *Let ε be a positive real number.*

(1) There exist finite sets $\{(n_k, q_k)\}$ and $\{(f_{k1}, \dots, f_{kr_k}, f'_{k1}, \dots, f'_{kr'_k})\}$ such that the following holds: One has $\mathrm{mld}(S_{n,q}) > \varepsilon$ if and only if

(i) $(n, q) = (n_k, q_k)$ for some k, or

(ii) The integers f_i determined as above are given by

$$\begin{aligned} f_1 = f_{k1}, \dots, f_{r_k} &= f_{kr_k}, \\ f_{r_k+1} = \dots = f_{r_k+A} &= 2, \\ f_{r_k+A+1} = f'_{k1}, \dots, f_{r_k+A+r'_k} &= f'_{kr'_k} \end{aligned}$$

for some k and $A = r - r_k - r'_k$.

(2) In (ii), fix k. Let m_1, q_1, m_2 and q_2 be determined by

$$m_1/q_1 = f_{kr_k} - \cfrac{1}{f_{k,r_k-1} - \cfrac{1}{f_{k,r_k-2} - \dots - \cfrac{1}{f_{k1}}}}, m_2/q_2 = f'_{k1} - \cfrac{1}{f'_{k2} - \cfrac{1}{f'_{k3} - \dots - \cfrac{1}{f'_{kr'_k}}}}.$$

If $b, c > \varepsilon$ and A is sufficiently large, then the minimal log discrepancy of $(S_{n,q}, bE + cF)$ is the minimum of

$$a_1 = \frac{\frac{1-b}{m_1-q_1}(A + \frac{m_2}{m_2-q_2}) + \frac{1-c}{m_2-q_2}\frac{q_1}{m_1-q_1}}{A + \frac{m_2}{m_2-q_2} + \frac{q_1}{m_1-q_1}} \tag{3.1}$$

and

$$a_2 = \frac{\frac{1-c}{m_2-q_2}(A + \frac{m_1}{m_1-q_1}) + \frac{1-b}{m_1-q_1}\frac{q_2}{m_2-q_2}}{A + \frac{m_1}{m_1-q_1} + \frac{q_2}{m_2-q_2}}. \tag{3.2}$$

Proof. This is in [Al], Lemma 3.3 and its proof. □

Since we are assuming that $\mathrm{mld}(S_{n,q}, bE + cF) = m$, the minimal log discrepancies of $S_{n,q}$ are bounded below by a positive number. Thus (1) of the previous lemma can be applied. Finite possibilities of n, q can be discarded by the remark at the beginning of the proof, so we only consider the case (ii). The coefficients b and c are also bounded below by a positive number, since $\mathcal{B} \subset (0, 1)$ satisfies DCC. Again discarding finite possibilities, we may assume that the minimal log discrepancy is a_1 or a_2 in (2).

By symmetry we may assume $\frac{1-b}{m_1-q_1} \leq \frac{1-c}{m_2-q_2}$. Let us compare a_1 and a_2. Note that the denominators in (3.1) and (3.2) are both equal to $A+m_1/(m_1-q_1)+m_2/(m_2-q_2)-1$, and we have

$$\begin{aligned}
&\left(A+\frac{m_1}{m_1-q_1}+\frac{m_2}{m_2-q_2}-1\right)(a_2-a_1)\\
=&\left[\frac{1-c}{m_2-q_2}\left(A+\frac{m_1}{m_1-q_1}\right)+\frac{1-b}{m_1-q_1}\frac{q_2}{m_2-q_2}\right]\\
&-\left[\frac{1-b}{m_1-q_1}\left(A+\frac{m_2}{m_2-q_2}\right)+\frac{1-c}{m_2-q_2}\frac{q_1}{m_1-q_1}\right]\\
=&\left(\frac{1-c}{m_2-q_2}-\frac{1-b}{m_1-q_1}\right)(A+1)\\
\geq& 0.
\end{aligned}$$

Thus we have $a_1 \leq a_2$ and $\mathrm{mld}(S_{n,q}, bE+cF) = a_1 = m$. We claim that $\frac{1-b}{m_1-q_1} = m$ if A is sufficiently large. In fact, since $\{\frac{1-b}{m_1-q_1} | b \in \mathcal{B}\}$ satisfies ACC, one may take $\varepsilon > 0$ such that $\{\frac{1-b}{m_1-q_1} | b \in \mathcal{B}\} \cap (m-\varepsilon, m]$ is empty or consists of m. Think of a_1 as a function of b, c and A. As A goes to infinity, it converges to $\frac{1-b}{m_1-q_1}$ uniformly for $b, c \in [0,1]^2$. Also, $a_1 \geq \frac{1-b}{m_1-q_1}$ if $\frac{1-b}{m_1-q_1} \leq \frac{1-c}{m_2-q_2}$. Thus, if A is sufficiently large, then a_1 belongs to $[\frac{1-b}{m_1-q_1}, \frac{1-b}{m_1-q_1}+\varepsilon)$. In order to have $a_1 = m$, one must have $\frac{1-b}{m_1-q_1} \in (m-\varepsilon, m]$. If $b \in \mathcal{B}$, then $\frac{1-b}{m_1-q_1} = m$ holds by our choice of ε. From $a_1 = m$, it also follows that $\frac{1-c}{m_2-q_2} = m$, and therefore that $(1-b)E + (1-c)F = m((m_1-q_1)E + (m_2-q_2)F)$.

It suffices to see that $(m_1-q_1)E + (m_2-q_2)F$ is Cartier. By the toric description, this is equivalent to saying that $(m_1-q_1)q + (m_2-q_2)$ is divisible by n. Let n_i and s_i be the coprime positive integers such that

$$n_i/s_i = f_i - \cfrac{1}{f_{i+1} - \cfrac{1}{f_{i+2} - \cdots - \cfrac{1}{f_r}}}.$$

Claim 3.8. *(1) If $r_k + A + 1 \geq i > r_k$, then $n_i = s_i + (m_2 - q_2)$.*

(2) If $r_k \geq i \geq 1$, then $s'_i n_i = n'_i s_i + (m_2 - q_2)$, where n'_i and s'_i are defined as the coprime positive integers such that

$$n'_i/s'_i = f_i - \cfrac{1}{f_{i+1} - \cfrac{1}{f_{i+2} - \cdots - \cfrac{1}{f_{r_k} - 1}}}.$$

Proof. We use descending inductions.

(1) The case $i = r_k + A + 1$ is clear, since we have $m_2 = n_{r_k+A+1}$ and $q_2 = s_{r_k+A+1}$ by definition. For i with $r_k + A + 1 > i > r_k$, assume that $n_{i+1} = s_{i+1} + (m_2 - q_2)$ holds. Since $f_i = 2$, one has $n_i = 2n_{i+1} - s_{i+1}$ and $s_i = n_{i+1}$, and therefore $n_i - s_i = n_{i+1} - s_{i+1} = m_2 - q_2$.

(2) Similarly, one has $n_i = f_i n_{i+1} - s_{i+1}$ and $s_i = n_{i+1}$. For $i = r_k$, we have $n'_i = f_{r_k} - 1$, $s'_i = 1$ and

$$s'_i n_i - n'_i s_i = (f_{r_k} n_{r_k+1} - s_{r_k+1}) - (f_{r_k} - 1) n_{r_k+1} = m_2 - q_2$$

by (1). For $1 \le i < r_k$, we have $n'_i = f_i n'_{i+1} - s'_{i+1}$, $s'_i = n'_{i+1}$ and

$$\begin{aligned} & s'_i n_i - n'_i s_i \\ =& n'_{i+1}(f_i n_{i+1} - s_{i+1}) - (f_i n'_{i+1} - s'_{i+1}) n_{i+1} \\ =& s'_{i+1} n_{i+1} - n'_{i+1} s_{i+1} \\ =& m_2 - q_2, \end{aligned}$$

assuming the equality for $i + 1$. □

We have $n_1 = n$ and $s_1 = q$ by definition. By Lemma 3.6, n'_1 is equal to the numerator of the reversed continued fraction

$$1 - \cfrac{1}{f_{r_k} - \frac{1}{f_{r_k-1} - \cdots - \frac{1}{f_1}}},$$

which is $m_1 - q_1$. Therefore

$$(m_1 - q_1) q + (m_2 - q_2) = n'_1 s_1 + (m_2 - q_2) = s'_1 n_1 = s'_1 n,$$

concluding the proof of the theorem. □

Remark 3.9. The proof actually shows that $m^{-1}((1-b)E + (1-c)F)$ is Cartier except for finitely many (n, q, b, c).

§ 4. Questions

In what generality does the descending chain condition hold? I will mention several cases that occur to mind.

Question 4.1. Let μ be either E or $\mathfrak{N}$. In the case of E, choose an ordering from Definition 1.1. Does the descending chain condition hold for the following sets?

(1) For a positive integer d, the set

$$\{\mu_{st}(X) - \mu(X \setminus P) | P \in X \text{ is a } d\text{-dimensional isolated log terminal singularity}\}.$$

(2) For a terminal projective variety X, the set

$$\left\{ \mu_{st}(Y) \,\middle|\, \begin{array}{l} Y \text{ is terminal projective, } X \text{ and } Y \text{ are birational and} \\ g^* K_X \ge h^* K_Y \text{ for a common resolution } g : W \to X,\ h : W \to Y \end{array} \right\}.$$

(3) $\{\mu(X) | X$ is a smooth proper variety$\}$.

(4) $\{\mu_{st}(X) | X$ is a log terminal proper variety$\}$.

(5) For a fixed proper variety Y and a fixed real number $b \in (0,1)$, the set

$$\{\mu_{st}(X, bB) | X \text{ is proper, } B \cong Y \text{ and } (X, bB) \text{ is (Kawamata) log terminal}\}.$$

Let us consider the set (3).

For the measure E and the order $>_0$ or the preorder $\gtrsim_1$, DCC is trivial since the Hodge numbers are nonnegative integers for a smooth proper variety. On the other hand, DCC fails for the preorder $\gtrsim_{-1}$, since a smooth projective curve C gives $E(C; -w, -w) = w^2 - 2g(C) \cdot w + 1$. When T is sufficiently large, e.g. $T = S^1$, the question of DCC with respect to $\gtrsim_T$ seems interesting.

Over a field of positive characteristic, one can consider the set (3) with $\mu = \mathfrak{N}$. The following example shows that DCC does not hold here.

Example 4.2. Let p be a prime number, and k a field containing $\bar{\mathbb{F}}_p$. By [GV], there exists an infinite sequence C_i of supersingular curves with $g(C_i)$ strictly increasing. The supersingularness means that the jacobian of C_i is isogenous to a product of supersingular elliptic curves. Therefore, for a sufficiently divisible n_0, C_i has a model over $\mathbb{F}_{p^{n_0}}$ and its zeta function is $(1-p^{n_0/2}t)^{2g(C_i)}/(1-t)(1-p^{n_0}t)$. Thus $\#(C_i(\mathbb{F}_{p^n})) = p^n + 1 - 2g(C_i)p^{n/2}$ when n is a multiple of n_0, and $\mathfrak{N}(C_1) > \mathfrak{N}(C_2) > \mathfrak{N}(C_3) > \dots$ holds.

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