Wildness of polynomial automorphisms:
Applications of the Shestakov-Umirbaev theory and its
generalization

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1 Introduction

For each integral domain $R$, we denote by $R[x] = R[x_1, \ldots, x_n]$ the polynomial ring in $n$ variables over $R$, where $n \in \mathbb{N}$, and $x = \{x_1, \ldots, x_n\}$ is a set of variables. For an $R$-subalgebra $A$ of $R[x]$, we consider the automorphism group $\text{Aut}(R[x]/A)$ of the ring $R[x]$ over $A$. We say that $\phi \in \text{Aut}(R[x]/R)$ is affine if $\deg \phi(x_i) = 1$ for $i = 1, \ldots, n$, and elementary if $\phi$ belongs to $\text{Aut}(R[x]/A_i)$ for some $i$, where $A_i := R[x\setminus\{x_i\}]$. Here, $\deg f$ denotes the total degree of $f$ for each $f \in R[x]$. Note that, if $\phi$ is affine, then we have $(\phi(x_1), \ldots, \phi(x_n)) = (x_1, \ldots, x_n)A + (b_1, \ldots, b_n)$ for some $A \in GL_n(R)$ and $b_1, \ldots, b_n \in R$. If $\phi$ is elementary, then there exist $i \in \{1, \ldots, n\}$, $\alpha \in R^\times$ and $f \in A_i$ such that $\phi(x_i) = \alpha x_i + f$ and $\phi(x_j) = x_j$ for $j \neq i$. We denote by $\text{Aff}(R,x)$, $\text{E}(R,x)$, and $\text{T}(R,x)$, the subgroups of $\text{Aut}(R[x]/R)$ generated by all the affine automorphisms, all the elementary automorphisms, and $\text{Aff}(R,x) \cup \text{E}(R,x)$, respectively. An element of $\text{Aut}(R[x]/R)$ is sometimes said to be tame if it belongs to $\text{T}(R,x)$, and wild otherwise.

The following is a fundamental problem in polynomial ring theory.

**Tame Generators Problem.** When is $\text{T}(R,x)$ equal to $\text{Aut}(R[x]/R)$?

The equality holds true if $n = 1$, in which case every element of $\text{Aut}(R[x]/R)$ is affine and elementary.

When $n = 2$, the following result is well-known.

**Theorem 1.1.** Assume that $n = 2$, and $R$ is an integral domain. Then, $\text{T}(R,x)$ is equal to $\text{Aut}(R[x]/R)$ if and only if $R$ is a field.

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Here, the “if” part of the above theorem is due to Jung [8] in the case where \( R \) is of characteristic zero, and to van der Kulk [9] in the general case. The “only if” part of the above theorem is rather easy (cf. [3, Proposition 5.1.9]).

Throughout this report, we denote by \( k \) an arbitrary field of characteristic zero. When \( n = 3 \), Shestakov-Umirbaev [21] gave a criterion to decide whether a given element of \( \text{Aut}(k[x]/k) \) belongs to \( T(k, x) \). As a consequence, they showed the following theorem ([21, Corollary 10]).

**Theorem 1.2** (Shestakov-Umirbaev). \( \text{Aut}(k[x]/k[x_3]) \cap T(k, x) = T(k[x_3], \{x_1, x_2\}) \).

Since some automorphisms, including the famous automorphism of Nagata [16], belong to \( \text{Aut}(k[x]/k[x_3]) \), but do not belong to \( T(k[x_3], \{x_1, x_2\}) \), it was concluded that \( T(k, x) \) is not equal to \( \text{Aut}(k[x]/k) \). At present, the Tame Generators Problem is not solved in the cases where \( n \geq 4 \), and where \( n = 3 \) and the field of fractions of \( R \) is of positive characteristic.

Recently, the author [10], [11] reconstructed and generalized the theory of Shestakov-Umirbaev. This improvement makes it possible to decide more easily and efficiently whether a given element of \( \text{Aut}(k[x]/k) \) belongs to \( T(k, x) \) when \( n = 3 \).

The purposes of this report is to announce some recent results obtained as consequences of the Shestakov-Umirbaev theory and its generalization. For details, we refer to our preprints [12], [13] and [14]. This series of papers (with a total of nearly hundred pages) presents various applications of these theories.

In Sections 2, 4, 5 and 6 of this report, we explain the main results of [12]. These results are derived from Theorem 1.2. To illustrate the usefulness of Theorem 1.2, in Section 3, we show the wildness of some concrete automorphisms by means of a criterion derived from this theorem. Sections 7 and 8, and 9 summarize the main results of [13] and [14]. These papers contain strong results obtained as highly technical applications of the generalized Shestakov-Umirbaev theory.

## 2 Affine reductions and elementary reductions

Let \( \Gamma \) be a finitely generated ordered additive group, and \( w = (w_1, \ldots, w_n) \) an \( n \)-tuple of elements of \( \Gamma \) with \( w \neq (0, \ldots, 0) \) and \( w_i \geq 0 \) for \( i = 1, \ldots, n \). For each nonzero polynomial

\[
f = \sum_{i_1, \ldots, i_n} \lambda_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \in R[x],
\]

we define the \( w \)-degree \( \deg_w f \) of \( f \) to be the maximum among \( \sum_{l=1}^n i_l w_l \) for \( i_1, \ldots, i_n \) with \( \lambda_{i_1, \ldots, i_n} \neq 0 \), where \( \lambda_{i_1, \ldots, i_n} \in R \) for each \( i_1, \ldots, i_l \). We define \( f^w \) to be the sum of
\[
\lambda_{i_{1},\ldots,i_{n}}x_{1}^{i_{1}} \cdots x_{n}^{i_{n}} \text{ for } i_{1}, \ldots, i_{n} \text{ such that } \sum_{i=1}^{n} i_{i}w_{i} = \deg_{w} f. \text{ When } f = 0, \text{ we set } f^{w} = 0 \text{ and } \deg_{w} f = -\infty, \text{ i.e., a symbol which is less than any element of } \Gamma. \text{ Then, for each } \phi \in \text{Aut}(R[x]/R), \text{ it holds that}
\]
\[
\deg_{w} \phi := \sum_{i=1}^{n} \deg_{w} \phi(x_{i}) \geq \sum_{i=1}^{n} w_{i} := |w|. \quad (2.1)
\]

If \( n = 2 \), then \( \deg_{w} \phi = |w| \) implies that \( \phi \) belongs to \( T(R, x) \) (see [12, Section 2] for detail).

Now, we consider two kinds of reductions for \( \phi \in \text{Aut}(R[x]/R) \). We say that \( \phi \) admits an affine reduction for the weight \( w \) if there exists \( \alpha \in \text{Aff}(R, x) \) such that \( \deg_{w} \phi \circ \alpha < \deg_{w} \phi \). We say that \( \phi \) admits an elementary reduction for the weight \( w \) if there exists \( \epsilon \in \text{Aut}(R[x]/A_{i}) \) for some \( i \) such that \( \deg_{w} \phi \circ \epsilon < \deg_{w} \phi \).

Assume that \( n = 2 \) and \( w_{i} \geq 0 \) for \( i = 1, 2 \). Then, we have \( |w| = w_{1} + w_{2} > 0 \) by the assumption that \( (w_{1}, w_{2}) \neq (0, 0) \). Hence, for each \( \phi \in \text{Aut}(R[x]/R) \), it follows from (2.1) that \( \deg_{w} \phi(x_{1}) > 0 \) or \( \deg_{w} \phi(x_{2}) > 0 \). Let \( V(R) \) be the set of \( a/b \in K \) for \( a, b \in R \setminus \{0\} \) such that \( aR + bR = R \), where \( K \) is the field of fractions of \( R \). Note that \( R \setminus \{0\} \) is contained in \( V(R) \), and \( V(R) \) is contained in \( K^{\times} \). If \( R \) is a PID, then we have \( V(R) = K^{\times} \). By definition, \( \phi \) admits an affine reduction if and only if there exist \( a, b, c, d, s, t \in R \) with \( ad - bc \in R^{\times} \) such that
\[
\deg_{w}(a\phi(x_{1}) + b\phi(x_{2}) + s) + \deg_{w}(c\phi(x_{1}) + d\phi(x_{2}) + t) < \deg_{w} \phi(x_{1}) + \deg_{w} \phi(x_{2}).
\]

Since \( \deg_{w} \phi(x_{1}) > 0 \) or \( \deg_{w} \phi(x_{2}) > 0 \), this is equivalent to that \( a\phi(x_{1})^{w} + b\phi(x_{2})^{w} = 0 \) or \( c\phi(x_{1})^{w} + d\phi(x_{2})^{w} = 0 \), and is equivalent to that \( \phi(x_{1})^{w} = u\phi(x_{2})^{w} \) for some \( u \in V(R) \). In particular, we have \( \deg_{w} \phi(x_{1}) = \deg_{w} \phi(x_{2}) \) whenever \( \phi \) admits an affine reduction for the weight \( w \).

Note that \( \phi \) admits an elementary reduction if and only if there exists \( f \in R[\phi(x_{j})] \) such that \( \deg_{w}(\phi(x_{i}) - f) < \deg_{w} \phi(x_{i}) \) for some \( (i, j) \in \{(1, 2), (2, 1)\} \). Since \( \phi(x_{i}) - f \neq 0 \) and \( w_{i} \geq 0 \) for \( l = 1, 2 \), we have \( \deg_{w}(\phi(x_{i}) - f) \geq 0 \), and hence \( \deg_{w} \phi(x_{i}) > 0 \). It follows that \( \deg_{w} f > 0 \), and so \( \deg_{w} \phi(x_{j}) > 0 \). Thus, \( f^{w} \) must be of the form \( c(\phi(x_{j})^{w})^{l} \) or \( d(\phi(x_{j})^{w})^{l} \) for some \( c \in R \setminus \{0\} \) and \( l \in \mathbb{N} \). Therefore, it holds that \( \deg_{w}(\phi(x_{i}) - f) < \deg_{w} \phi(x_{i}) \) for some \( f \in k[\phi(x_{j})] \) if and only if \( \phi(x_{i})^{w} = c(\phi(x_{j})^{w})^{l} \) for some \( c \in R \setminus \{0\} \) and \( l \in \mathbb{N} \).

The following is a basic result on tameness of elements of \( \text{Aut}(R[x]/R) \) for \( n = 2 \). In the case of \( w = (1, 1) \), the result is commonly known (cf. [7, Proposition 1]).

**Proposition 2.1** ([12, Proposition 3.2]). Assume that \( n = 2 \), and \( w := (w_{1}, w_{2}) \in \Gamma^{2} \) is such that \( w \neq (0, 0) \) and \( w_{i} \geq 0 \) for \( i = 1, 2 \). If \( \deg_{w} \phi > |w| \) holds for \( \phi \in T(R, x) \), then \( \phi \) admits an affine reduction or elementary reduction for the weight \( w \).
Next, we recall the notion of coordinate. We call \( f \in R[x] \) a coordinate of \( R[x] \) over \( R \) if \( f \) is equal to \( \phi(x_1) \) for some \( \phi \in \text{Aut}(R[x]/R) \), which is said to be tame if \( \phi \) can be taken from \( T(R, x) \), and wild otherwise. Let \( S \) be an integral domain containing \( R \) as a subring. Then, we may regard \( \text{Aut}(R[x]/R) \) as a subgroup of \( \text{Aut}(S[x]/S) \) by identifying \( \phi \in \text{Aut}(R[x]/R) \) with the automorphism \( \text{id}_S \otimes \phi \) of \( S \otimes_R R[x] \simeq S[x] \) over \( S \). Hence, every coordinate of \( R[x] \) over \( R \) is a coordinate of \( S[x] \) over \( S \). On the other hand, not every coordinate of \( S[x] \) over \( S \) is a coordinate of \( R[x] \) over \( R \). When \( n = 2 \), we say that a coordinate \( f \) of \( S[x] \) over \( S \) is reduced over \( R \) if

\[
\deg_{x_1} \tau(f) + \deg_{x_2} \tau(f) \geq \deg_{x_1} f + \deg_{x_2} f
\]

holds for every \( \tau \in T(R, x) \).

For \( f \in R[x] \), we consider the subgroup

\[
H(f) := \text{Aut}(R[x]/R[\{f\}]) \cap T(R, x)
\]

of \( \text{Aut}(R[x]/R) \).

The following theorem is a consequence of Proposition 2.1.

**Theorem 2.2** ([12, Theorem 4.3]). Assume that \( n = 2 \). Let \( R \subset S \) be an extension of integral domains, and \( f \in R[x] \) a coordinate of \( S[x] \) over \( S \) which is reduced over \( R \).

(i) If \( \deg_{x_1} f = \deg_{x_2} f \), then \( H(f) \) is contained in \( \text{Aff}(R, x) \).

(ii) If \( \deg_{x_1} f < \deg_{x_2} f \) for \( (i, j) \in \{(1, 2), (2, 1)\} \), then \( H(f) \) is contained in \( J(R, x_j, x_i) \).

If \( \deg_{x_1} f = 0 \), then \( H(f) = \text{Aut}(R[x]/R[x_j]) \).

Here, for a permutation \( x_{i_1}, \ldots, x_{i_n} \) of \( x_1, \ldots, x_n \), we denote by \( J(R; x_{i_1}, \ldots, x_{i_n}) \) the set of \( \phi \in \text{Aut}(R[x]/R) \) such that \( \phi(x_{i_l}) \) belongs to \( R[x_{i_1}, \ldots, x_{i_l}] \) for \( l = 1, \ldots, n \). Note that \( J(R; x_{i_1}, \ldots, x_{i_n}) \) forms a subgroup of \( T(R, x) \) consists of the automorphisms \( \phi \) of the form \( \phi(x_{i_l}) = a_l x_{i_l} + h_l \) for \( l = 1, \ldots, n \), where \( a_l \in R^\times \) and \( h_l \in R[x_{i_1}, \ldots, x_{i_{l-1}}] \).

Results explained in Sections 4, 5 and 6 are derived from Theorems 1.2 and 2.2.

## 3 An easy criterion for wildness

The following corollary is an immediate consequence of Theorem 1.2, and Proposition 2.1 applied with \( R = k[x_3] \).

**Corollary 3.1.** Assume that \( n = 3 \). Then, \( \phi \in \text{Aut}(k[x]/k[x_3]) \) does not belong to \( T(k, x) \) if there exist \( w_1, w_2 \in \Gamma \) with \( (w_1, w_2) \neq (0, 0) \) and \( w_i \geq 0 \) for \( i = 1, 2 \) such that the following conditions hold for \( \mathbf{w} := (w_1, w_2, 0) \):

(i) \( \deg_{\mathbf{w}} \phi > |\mathbf{w}| \).

(ii) There exists \( (i, j) \in \{(1, 2), (2, 1)\} \) such that \( \deg_{\mathbf{w}} \phi(x_i) < \deg_{\mathbf{w}} \phi(x_j) \), and \( \phi(x_j)^{\mathbf{w}} \) is not equal to \( c(\phi(x_i)^{\mathbf{w}})^l \) for any \( c \in k[x_3] \) and \( l \in \mathbb{N} \).
**Proof.** Suppose to the contrary that \( \phi \) belongs to \( T(k, x) \). Then, \( \phi \) belongs to \( T(k[x_3], \{x_1, x_2\}) \) by Theorem 1.2, since \( \phi \) is an element of \( \text{Aut}(k[x]/k[x_3]) \) by assumption. Regard \( k[x] \) as the polynomial ring in \( x_1 \) and \( x_2 \) over \( k[x_3] \), where we consider the weight \( \mathbf{w}' := (w_1, w_2) \). Then, \( \deg_{\mathbf{w}} m = i_1 w_1 + i_2 w_2 = \deg_{\mathbf{w}} m \) holds for each monomial \( m = x_1^{i_1} x_2^{i_2} x_3^{i_3} \). Hence, we get \( \deg_{\mathbf{w}} f = \deg_{\mathbf{w}} f \) and \( f^{\mathbf{w'}} = f^{\mathbf{w}} \) for each \( f \in k[x] \). It follows that \( \deg_{\mathbf{w'}} \phi = \deg_{\mathbf{w}} \phi \), and is greater than \( |\mathbf{w}| = |\mathbf{w}'| \) by (i). By Proposition 2.1, we know that \( \phi \) admits an affine reduction or elementary reduction for the weight \( \mathbf{w}' \) as an automorphism of the polynomial ring in \( x_1 \) and \( x_2 \) over \( k[x_3] \). On the other hand, since \( \deg_{\mathbf{w'}} \phi(x_l) = \deg_{\mathbf{w}} \phi(x_l) \) and \( \phi(x_l)^{\mathbf{w'}} = \phi(x_l)^{\mathbf{w}} \) for \( l = 1, 2 \), the condition (ii) implies that \( \phi \) does not admit an affine reduction or elementary reduction for the weight \( \mathbf{w}' \). This is a contradiction. Therefore, \( \phi \) does not belong to \( T(k, x) \). \( \square \)

For example, consider Nagata’s automorphism [16] given by

\[
\phi(x_1) = x_1 - 2(x_1 x_3 + x_2^2) x_2 - (x_1 x_3 + x_2^2)^2 x_3, \quad \phi(x_2) = x_2 + (x_1 x_3 + x_2^2) x_3
\]

and \( \phi(x_3) = x_3 \). Let \( \Gamma \) be the additive group \( \mathbb{Z}^2 \) equipped with the lexicographic order with \( \mathbf{e}_1 := (1, 0) \) and \( \mathbf{e}_2 := (0, 1) \). Then, for \( \mathbf{w} = (\mathbf{e}_1, \mathbf{e}_2, 0) \), we have

\[
\deg_{\mathbf{w}} \phi(x_1) = 2 \mathbf{e}_1, \quad \deg_{\mathbf{w}} \phi(x_2) = \mathbf{e}_1, \quad \phi(x_1)^{\mathbf{w}} = -x_1^2 x_3^3, \quad \phi(x_2)^{\mathbf{w}} = x_1 x_3^2.
\]

One easily checks that (i) and (ii) in Corollary 3.1 are satisfied. Thus, \( \phi \) does not belong to \( T(k, x) \).

Ohta [17, Theorem 3] gave two kinds of automorphisms, one of which is defined by

\[
\phi_1(x_1) = (g_1(x_3, f_1 + x_1 x_3^3) - x_2) x_3^{-3}, \quad \phi_1(x_2) = f_1 + x_1 x_3^3, \quad \phi_1(x_3) = x_3,
\]

where \( f_1 \) is a certain element of \( k[x_2, x_3] \), and \( g_1(x, y) \) is a polynomial in \( x \) and \( y \) over \( k \) of the form \( 3x^2 y^5 + (\text{terms of lower degree in } y) \). Here, \( x_1, x_2 \) and \( x_3 \) are denoted by \( z, y \) and \( x \), respectively, in the original text. For the same \( \Gamma \) and \( \mathbf{w} \) as above, we have

\[
\deg_{\mathbf{w}} \phi_1(x_1) = 5 \mathbf{e}_1 + 14 \mathbf{e}_3, \quad \deg_{\mathbf{w}} \phi_1(x_2) = \mathbf{e}_1 + 3 \mathbf{e}_3, \quad \phi_1(x_1)^{\mathbf{w}} = 3 x_1^3 x_3^{14}, \quad \phi_1(x_2)^{\mathbf{w}} = x_1 x_3^3.
\]

It is easy to check that (i) and (ii) in Corollary 3.1 are satisfied. Hence, we conclude that \( \phi_1 \) does not belong to \( T(k, x) \). As this example shows, we can sometimes decide the wildness of \( \phi \in \text{Aut}(k[x]/k) \) from only partial information on \( \phi(x_1), \phi(x_2) \) and \( \phi(x_3) \). Tameness of another automorphism of Ohta is determined at the end of the next section.

### 4 Triangular derivation

Let \( D \) be a locally nilpotent derivation of \( R[x] \) over \( R \), i.e., an element of \( \text{Der}_R R[x] \) such that \( D^l(f) = 0 \) holds for some \( l \in \mathbb{N} \) for each \( f \in R[x] \). When \( R \) contains \( \mathbb{Q} \), an element
exp $D$ of $\text{Aut}(R[x]/R)$ is defined by

$$(\exp D)(f) = \sum_{i \geq 0} \frac{D^i(f)}{i!}$$

for each $f \in R[x]$. We say that $D \in \text{Der}_R R[x]$ is \textit{triangular} if $D(x_i)$ belongs to $R[x_1, \ldots, x_{i-1}]$ for each $i$. If $D$ is triangular, then $D$ is locally nilpotent, and $(\exp D)(x_i) = x_i + f_i$ for each $i$, where $f_i \in R[x_1, \ldots, x_{i-1}]$. Hence, $\exp D$ belongs to $J(R, x_1, \ldots, x_n)$, and so belongs to $T(R, x)$. For $D \in \text{Der}_R R[x]$ and $h \in R[x]$, it is well-known that $hD$ is a locally nilpotent derivation of $R[x]$ if and only if $D$ is a locally nilpotent derivation of $R[x]$, and $h$ belongs to $\ker D$ (cf. [3, Corollary 1.3.34]). Even if $D$ is triangular, $hD$ is not always triangular, and so $\exp hD$ may not belong to $T(k, x)$ for $h \in \ker D \setminus k$. For instance, Nagata’s automorphism is wild, and is of the form $\exp hD$, where

$$h = x_1 x_3 + x_2^2, \quad D = -2x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}.$$ 

This derivation is triangular if $x_1$ and $x_3$ are interchanged.

Thus, the following problem arises.

\textbf{Problem.} Assume that $n = 3$. Let $D$ be a triangular derivation of $k[x]$, and $h$ an element of $\ker D \setminus k$. When does $\exp hD$ belong to $T(k, x)$?

We completely settle this problem as a consequence of a more general result as follows. Let $R$ be a $\mathbb{Q}$-domain, and $D$ a triangular derivation of $R[x_1, x_2]$ such that

$$D(x_1) = a, \quad D(x_2) = \sum_{i=0}^{l} b_i x_1^i,$$

where $l \geq 0$, and $a, b_0, \ldots, b_l \in R$ with $a \neq 0$ and $b_l \neq 0$. Set

$$I = \{i \geq 0 \mid b_i \not\in aR\}, \quad I' = \{1, \ldots, l\} \setminus I,$$

and define $\tau \in \text{Aut}(R[x_1, x_2]/R)$ by

$$\tau(x_1) = x_1, \quad \tau(x_2) = x_2 + \sum_{i \in I'} \frac{b_i}{(i + 1)a} x_1^{i+1}.$$ 

With this notation, we have the following theorem.

\textbf{Theorem 4.1} ([12, Theorem 6.1]). Let $D$ be as above, and $h$ an element of $\ker D \setminus R$. Then, $\exp hD$ belongs to $T(R, \{x_1, x_2\})$ if and only if one of the following conditions holds:

(i) $I = \emptyset$.
(ii) $I = \{0\}$, and $b_0/a$ belongs to $V(R)$ or $\deg \tau(h) = 1$.

In particular, when $V(R) = K^\times$, it follows that $\exp hD$ belongs to $T(R, \{x_1, x_2\})$ if and only if $I = \emptyset$ or $I = \{0\}$, where $K$ is the field of fractions of $R$.  

Applying Theorem 4.1 with \( R = k[x_3] \), we get the following theorem with the aid of Theorem 1.2.

**Theorem 4.2** ([12, Theorem 6.2]). Assume that \( n = 3 \). Let \( D \) be a triangular derivation of \( k[x] \) with \( D(x_1) = 0 \) and \( D(x_2) \neq 0 \), and \( h \) an element of \( \text{ker} \ D \setminus k[x_1] \). Then, \( \exp hD \) belongs to \( T(k, x) \) if and only if \( \partial D(x_3)/\partial x_2 \) belongs to \( D(x_2)k[x_1, x_2] \).

Note that \( \partial D(x_3)/\partial x_2 \) belongs to \( D(x_2)k[x_1, x_2] \) if and only if the coefficient of \( x_2^i \) in \( D(x) \) is divisible by \( D(x) \) in \( k[X] \) for each \( i \geq 1 \), where we regard \( D(x) \) as a polynomial in \( x_2 \) over \( k[x_1] \).

If \( D(x_1) = 0 \) and \( h \) belongs to \( k[x_1] \), then \( hD \) is triangular, and so \( \exp hD \) is tame.

Let \( f \in k[x_1] \setminus \{0\} \) and \( g \in k[x_1, x_2] \), we define a triangular derivation \( T_{f,g} \) of \( k[x] \) by

\[
T_{f,g}(x_1) = 0, \quad T_{f,g}(x_2) = f, \quad T_{f,g}(x_3) = -\frac{\partial g}{\partial x_2}.
\]

Then, we have \( T_{f,g}(fx_3 + g) = 0 \), so \( hT_{f,g} \) is a locally nilpotent derivation of \( k[x] \) for each \( h \in k[x_1, fx_3 + g] \). By Theorem 4.2, it follows that \( \Phi_{f,g}^h := \exp hT_{f,g} \) belongs to \( T(k, x) \) if and only if \( \partial T_{f,g}(x_3)/\partial x_2 = -\partial^2 g/\partial x_2^2 \) belongs to \( T_{f,g}(x_2)k[x_1, x_2] = fk[x_1, x_2] \) for each \( h \in k[x_1, fx_3 + g] \setminus k[x_1] \). Thus, we get a family

\[
\{ \Phi_{f,g}^h \mid (f, g) \in \Lambda, \ h \in k[x_1, fx_3 + g] \setminus k[x_1] \}
\]

of wild automorphisms of \( k[x] \), where \( \Lambda \) is the set of \( (f, g) \in (k[x_1] \setminus \{0\}) \times x_2k[x_1, x_2] \) such that \( \partial^2 g/\partial x_2^2 \) does not belong to \( f k[x_1, x_2] \).

Let us consider the second automorphism of Ohta [17, Theorem 3] defined by

\[
\phi_2(x_1) = x_1, \quad \phi_2(x_2) = x_2 + x_1 f_2, \quad \phi_2(x_3) = x_3 - \sum_{i \geq 0} \sum_{j \geq 1} a_{i,j} ((x_2 + x_1 f_2)^j - x_2^j) x_1^{i-1},
\]

where

\[
f_2 := x_1 x_3 + \sum_{i \geq 0} \sum_{j \geq 1} a_{i,j} x_1^i x_2^j
\]

with \( a_{i,j} \in k \) for each \( i, j \). Here, \( x_1, x_2 \) and \( x_3 \) are denoted by \( x, y \) and \( z \), respectively, in the original text.

**Proposition 4.3.** \( \phi_2 \) belongs to \( T(k, x) \) if and only if \( a_{0,j} = 0 \) for every \( j \geq 2 \).
Proof. Set $D = T_{x_1,g}$ and $\phi = \Phi^2_{x_1,g} = \exp f_2 D$, where $g := f_2 - x_1 x_3$. Then, we claim that $\phi = \phi_2$. In fact, we have $\phi(x_1) = x_1$ and $\phi(x_2) = x_2 + x_1 f_2$, since $D(x_1) = 0$, $f_2 D(x_2) = x_1 f_2$ and $(f_2 D)^2(x_2) = 0$. Since $D(f_2) = 0$, we have $\phi(f_2) = f_2$, and hence

$$x_1 \phi(x_3) + \sum_{i \geq 0} \sum_{j \geq 1} a_{i,j} x_1^i (x_2 + x_1 f_2)^j = \phi(f_2) = f_2 = x_1 x_3 + \sum_{i \geq 0} \sum_{j \geq 1} a_{i,j} x_1^i x_2^j.$$ 

This gives that $\phi(x_3) = \phi_2(x_3)$. Hence, $\phi$ is equal to $\phi_2$. Since $f_2$ is an element of $k[x_1, f_2] \setminus k[x_1]$, it holds that $\phi = \Phi^2_{x_1,g}$ belongs to $T(k, x)$ if and only if $\partial^2 g/\partial x_2^2 = \sum_{i \geq 0} \sum_{j \geq 1} j(j-1) a_{i,j} x_1^i x_2^j - 2$ belongs to $x_1 k[x_1, x_2]$ as mentioned. This condition is equivalent to the condition that $a_{0,j} = 0$ for every $j \geq 2$. \hfill $\square$

5 Tameness and triangularizability

We say that $D \in \text{Der}_k k[x]$ is triangularizable if $\tau^{-1} \circ D \circ \tau$ is triangular for some $\tau \in \text{Aut}(k[x]/k)$. When this is the case, $D$ is locally nilpotent. Moreover, $\exp D$ belongs to $T(k, x)$ if and only if $\tau^{-1} \circ D \circ \tau = \exp D \circ \tau$. If $\tau$ does not belong to $T(k, x)$, however, $\exp D$ does not belong to $T(k, x)$ in general. Actually, as will be remarked after Theorem 8.1, $\exp D$ can be wild even if $\tau^{-1} \circ D \circ \tau = \partial/\partial x_1$ for some $\tau$. On the other hand, it is also not clear whether $D$ is always triangularizable if $\exp D$ belongs to $T(k, x)$. When $n = 2$, every locally nilpotent derivation of $k[x]$ is triangularizable due to Rentschler [19]. When $n \geq 4$, there exists a locally nilpotent derivation $D$ of $k[x]$ which is not triangularizable, but $\exp D$ belongs to $T(k, x)$, by the results of Bass [1], Popov [18] and Smith [20] (see also [6, Sections 3.9]). So Freudenburg [6, Section 5.3] raised the following question.

Question (Freudenburg). Assume that $n = 3$. Is a locally nilpotent derivation $D$ of $k[x]$ always triangularizable if $D$ is tame, i.e., $\exp D$ belongs to $T(k, x)$?

We give a partial affirmative answer to this question as follows.

Theorem 5.1 ([12, Theorem 1.2]). Assume that $n = 3$, and $D$ is a locally nilpotent derivation of $k[x]$ such that $\ker D$ contains a tame coordinate of $k[x]$ over $k$. Then, $\exp D$ belongs to $T(k, x)$ if and only if $\tau^{-1} \circ D \circ \tau$ is triangular for some $\tau \in T(k, x)$.

We note that there exists a locally nilpotent derivation of $k[x]$ for $n = 3$ such that $\ker D$ contains no coordinate of $k[x]$ over $k$ (cf. [4] and [5]). Such a locally nilpotent derivation is never conjugate to a triangular derivation multiplied by an element of $\ker D$. In fact, every triangular derivation of $k[x]$ kills a tame coordinate of $k[x]$ over $k$ if $n \geq 2$. In Section 9, we will discuss tameness of $\exp D$ for such special $D$.

Theorem 5.1 is obtained by Theorem 1.2 and the following theorem.
Theorem 5.2 ([12, Theorem 5.1]). Let $R$ be a Q-domain, and $D$ a locally nilpotent derivation of $R[x_1, x_2]$ over $R$ such that $\exp D$ belongs to $T(R, \{x_1, x_2\})$. Then, there exists $\tau \in T(R, x)$ such that $D' := \tau^{-1} \circ D \circ \tau$ is triangular, or $\deg D'(x_i) \leq 1$ for $i = 1, 2$. If $V(R) = K^\times$, then there exists $\tau \in T(R, x)$ such that $D'$ is triangular, where $K$ is the field of fractions of $R$.

It is interesting to ask the following question.

Question. Let $R$ be a Q-domain, and $D$ a locally nilpotent derivation of $R[x]$ over $R$. Does $\exp D$ belong to $T(R, x)$ whenever $\exp hD$ belongs to $T(R, x)$ for some $h \in \ker D \setminus \{0\}$?

As a corollary to Theorem 5.2, we get the following result.

Corollary 5.3 ([12, Corollary 5.2]). Let $R$ be a Q-domain, and $D$ a locally nilpotent derivation of $R[x]$ over $R$ for $n = 2$. If $\exp hD$ belongs to $T(R, x)$ for some $h \in \ker D \setminus \{0\}$, then $\exp D$ belongs to $T(R, x)$.

6 Invariant coordinates

In Theorem 2.2, we described a rough structure of the subgroup $H(f)$ of $\text{Aut}(R[x]/R)$ for a coordinate $f \in R[x]$ of $S[x]$ over $S$ which is reduced over $R$. In this section, we assume that the field $K$ of fractions of $R$ is of characteristic zero, and determine the precise structure of $H(f)$ and classify $f$ such that $H(f)$ has at least two elements.

By the following lemma, we may assume that $S = K$.

Lemma 6.1 ([12, Lemma 7.1]). Assume that $K$ is of characteristic zero. If $f \in R[x_1, x_2]$ is a coordinate of $S[x_1, x_2]$ over $S$, then $f$ is a coordinate of $K[x_1, x_2]$ over $K$.

Now, we define four types of elements of $R[x]$ which are coordinates of $K[x]$ over $K$ as follows: For $a \in R \setminus \{0\}$, $g \in R[x_1]$ with $\deg g \geq 2$, and $u(z) \in K[z^l] \setminus K$ and $1 \neq \zeta \in R^\times$ with $\zeta^l = 1$ for some $l \geq 2$, we define

$$f_1 = ax_2 + g, \quad f_2 = ax_1 + u((\zeta - 1)x_2 + g),$$

where we assume that $g$, $\zeta$ and $u(z)$ are such that $u((\zeta - 1)x_2 + g)$ belongs to $R[x]$. For $\tau \in \text{Aff}(K, x)$ such that $\tau(x_1) = ax_1 + \beta x_2 + \gamma$ for $\alpha, \beta, \gamma \in K$ with $\alpha, \beta \neq 0$, and for $v \in K[x_1]$ with $\deg v = 2$, or $v \in K[x_1^l] \setminus K$ for some $l \geq 2$, we define

$$f_3 = \tau(x_1), \quad f_4 = \tau(x_2 + v),$$

where we assume that $\tau$ and $v$ are such that $\tau(x_2 + v)$ belongs to $R[x]$. 9
For $f_1, \ldots, f_4$ as above, we define subsets $H_1, \ldots, H_4$ of $\text{Aut}(R[x]/R)$ as follows:

- $H_1 = J(R; x_1, x_2) \cap \text{Aut}(R[x]/R[f_1])$.
- $H_2$ is the set of $\phi \in \text{Aut}(R[x]/R[x_1])$ such that $\phi(x_2) = \xi x_2 + (\xi - 1)\zeta^{-1}g$. Here, $\xi \in R$ is such that $(\xi - 1)\zeta^{-1}g$ belongs to $R[x_1]$, and $\xi^m = 1$, where $m$ is the maximal integer for which $u(z)$ belongs to $R[z^m]$.
- $H_3 = \text{Aff}(R, x) \cap \tau \circ \text{Aut}(K[x]/K[x_1]) \circ \tau^{-1}$.
- $H_4 = \text{Aff}(R, x) \cap \tau \circ \text{Aut}(K[x]/K[x_2 + v]) \circ \tau^{-1}$.

In the notation above, we have the following result.

**Theorem 6.2** ([12, Theorem 7.2]). Assume that $n = 2$ and $K$ is of characteristic zero.

(i) Let $f \in R[x]$ be a coordinate of $K[x]$ over $K$ which is reduced over $R$. If $\deg_{x_1} f \geq \deg_{x_2} f \geq 1$ and $H(f) \neq \{\text{id}_{R[x]}\}$, then $f$ has the form of $f_i$ for some $i \in \{1, 2, 3, 4\}$.

(ii) If $f_i$ is reduced over $R$ for $i \in \{1, 2, 3, 4\}$, then we have $H(f_i) = H_i$.

In the case where $R = k[x_3]$, the above theorem and Theorem 1.2 imply the following corollary.

**Corollary 6.3** ([12, Corollary 7.5]). Assume that $n = 3$. Let $f \in k[x]$ be a coordinate of $k(x_3)[x_1, x_2]$ over $k(x_3)$. If $H := \text{Aut}(k[x]/k[x_3, f]) \cap T(k, x)$ has at least two elements, then one of the following holds for some $\tau \in T(k[x_3], \{x_1, x_2\})$:

(i) $\tau(f) = ax_1 + b$ for some $a, b \in k[x_3]$ with $a \neq 0$, and $H = \text{Aut}(k[x]/k[x_1, x_3])$.

(ii) $\tau(f) = ax_2 + g$ for some $a \in k[x_3]$ and $g \in k[x_1, x_3]$ with $\deg_{x_1} g \geq 2$ for which the leading coefficient of $g$, as a polynomial in $x_1$ over $k[x_3]$, does not belong to $ak[x_3]$. Moreover, we have $H = J(k[x_3]; x_1, x_2) \cap \text{Aut}(k[x]/k[ax_2 + g, x_3])$.

If $K$ is of positive characteristic, the statements of Lemma 6.1 and Theorem 6.2 do not hold in general (cf. [12, Section 7]).

### 7 Generalized Shestakov-Umirbaev theory

In the following sections, we explain the main results of [13] and [14]. These papers are devoted to applications of the generalized Shestakov-Umirbaev theory [10], [11]. In this section, we mention some consequences of this theory used in [13] and [14]. In what follows, we assume that $n = 3$ unless otherwise stated, and a *wild automorphism* always means an element of $\text{Aut}(k[x]/k)$ not belonging to $T(k, x)$.

For $w = (w_1, w_2, w_3) \in \Gamma^3$, we define rank $w$ to be the rank of the $\mathbb{Z}$-submodule of $\Gamma$ generated by $w_1$, $w_2$ and $w_3$. For $\phi \in \text{Aut}(k[x]/k)$, consider the following conditions:

1. $\phi(x_1)^w$, $\phi(x_2)^w$ and $\phi(x_3)^w$ are algebraically dependent over $k$, and are pairwise algebraically independent over $k$.
(2) $\phi(x_i)^w$ does not belong to $k[\{\phi(x_j)^w \mid j \neq i\}]$ for $i = 1, 2, 3$.

The generalized Shestakov-Umirbaev theory implies the following sufficient condition for wildness, where $\Gamma_{>0} := \{\alpha \in \Gamma \mid \alpha > 0\}$.

**Proposition 7.1** ([13, Section 1]). If $\phi \in \text{Aut}(k[x]/k)$ is such that (1) and (2) hold for some $w \in (\Gamma_{>0})^3$ with rank $w = 3$, then $\phi$ is wild.

We call $P \in k[x]$ a W-test polynomial if, for each $\phi \in \text{Aut}(k[x]/k)$, it holds that $\phi$ is wild whenever there exist a totally ordered additive group $\Gamma$ and $w \in (\Gamma_{>0})^3$ with rank $w = 3$ as follows:

(a) $\deg_w \phi(P) < \deg_w \phi(x_{i_1})$ for some $i_1 \in \{1, 2, 3\}$;
(b) $\deg_w \phi(x_{i_2})$ and $\deg_w \phi(x_{i_3})$ are linearly independent over $\mathbb{Z}$ for some $i_2, i_3 \in \{1, 2, 3\}$.

It is sometimes useful to use a W-test polynomial for showing that an automorphism is wild. The following proposition follows from Proposition 7.1.

**Proposition 7.2** ([14, Proposition 6.1]). Let $P$ be an element of $k[x]$ not belonging to $k[x \setminus \{x_i\}]$ for $i = 1, 2, 3$. Then, $P$ is a W-test polynomial if the following conditions hold for every totally ordered additive group $\Gamma$ and $w \in (\Gamma_{>0})^3$ such that $P^w$ is not a monomial:

(i) $P^w$ is not divisible by $x_i - g$ for any $g \in k[x \setminus \{x_i\}] \setminus k$ for $i = 1, 2, 3$;
(ii) $P^w$ is not divisible by $x_i^{s_i} - cx_j^{s_j}$ for any $c \in k^\times$, $s_i, s_j \in \mathbb{N}$ and $i, j \in \{1, 2, 3\}$ with $i \neq j$.

By this proposition, we can check that $P = x_1x_3 - \sum_{i=1}^{t} \alpha_i x_2^{i-1}$ and $x_2 - Px_3$ are W-test polynomials if $t \geq 2$, where $\alpha_1, \ldots, \alpha_{t-1} \in k$ and $\alpha_t \in k^\times$. This result is used to prove Theorems 9.1 and 9.3.

### 8 Absolutely wild and totally wild coordinates

We say that a coordinate $f$ of $k[x]$ over $k$ is absolutely wild if $D(f) = 0$ implies that $\exp D$ is wild for every nonzero locally nilpotent $D$ of $k[x]$, and totally wild if $\phi(f) = f$ implies that $\phi$ is wild for every idempotent $\phi \neq \phi \in \text{Aut}(k[x]/k)$. Since $D(f) = 0$ implies $(\exp D)(f) = f$, “totally wild” implies “absolutely wild”. We claim that “absolutely wild” implies “wild”. In fact, if $f$ is a tame coordinate, then there exists $\sigma \in T(k, x)$ such that $\sigma(x_1) = f$, for which we have $D(f) = 0$, and $\exp D$ belongs to $T(k, x)$, where $D := \sigma \circ (\partial/\partial x_2) \circ \sigma^{-1}$. In [13], we construct totally wild coordinates, and absolutely wild coordinates which are not totally wild as follows.

For $\theta(z) \in k[z] \setminus k$, we define a locally nilpotent derivation $D_\theta$ of $k[x]$ by

$$D_\theta(x_1) = -\theta'(x_2), \quad D_\theta(x_2) = x_3, \quad D_\theta(x_3) = 0,$$
where \( \theta'(z) \) is the derivative of \( \theta(z) \). Then, \( f_\theta := x_1x_3 + \theta(x_2) \) belongs to \( \ker D_\theta \). Hence, \( f_\theta D_\theta \) is a locally nilpotent derivation of \( k[x] \). Set \( \sigma_\theta = \exp f_\theta D_\theta \), and \( y_1 = \sigma_\theta(x_1) \). We consider the subgroup

\[
G_\theta := \text{Aut}(k[x]/k[y_1]) \cap T(k, x)
\]

of \( \text{Aut}(k[x]/k) \). Note that \( G_\theta = \{ \text{id}_{k[x]} \} \) if and only if \( y_1 \) is a totally wild coordinate. If \( G_\theta \) is a finite group, then \( y_1 \) is an absolutely wild coordinate. Actually, \( \exp D \) has an infinite order for every locally nilpotent derivation \( D \neq 0 \), since \( (\exp D)^l = \exp lD \neq \text{id}_{k[x]} \).

Let \( a \) and \( b \) be the coefficients of \( z^d \) and \( z^{d-1} \) in \( \theta(z) \), respectively, where \( d := \deg \theta(z) \). We set \( c = -b/(ad) \) and write \( \theta(z) = \sum_{i=0}^d u_i(z-c)^i \), where \( u_i \in k \) for each \( i \). Then, we have \( u_d = a, \ u_{d-1} = 0 \) and \( u_0 = \theta(c) \). Let \( e \in \mathbb{N} \) be the the positive generator of the ideal of \( \mathbb{Z} \) generated by \( 2i - 1 \) for \( 1 \leq i \leq d \) with \( u_i \neq 0 \), and define

\[
T_\theta = \{ \zeta \in k^\times \mid \zeta^e = 1 \}.
\]

For each \( \zeta \in T_\theta \), we define an element \( \phi_\zeta \) of \( \text{J}(k; x_3, x_2, x_1) \) by \( \phi_\zeta(x_3) = \zeta x_3 \),

\[
\phi_\zeta(x_2 - c) = \zeta^2(x_2 - c) + \zeta(\zeta - 1)\theta(c)x_3, \quad \phi_\zeta(x_1) = x_1 + g_\zeta,
\]

where

\[
g_\zeta := (\zeta\theta(x_2) - \theta(\phi_\zeta(x_2)) + (1 - \zeta)\theta(c))(\zeta x_3)^{-1}.
\]

Here, we note that \( g_\zeta \) always belongs to \( k[x] \) for \( \zeta \in T_\theta \).

In the notation above, we have the following theorem.

**Theorem 8.1** ([13, Theorem 6.1]). For each \( \zeta \in T_\theta \), the automorphism \( \phi_\zeta \) belongs to \( G_\theta \). The map \( \iota : T_\theta \ni \zeta \mapsto \phi_\zeta \in G_\theta \) is an injective homomorphism of groups. If \( d \geq 9 \) and \( d \neq 10, 12 \), then \( \iota \) is an isomorphism.

By this theorem, we know that there exist a number of totally wild coordinates, and absolutely wild coordinates which are not totally wild as follows. If \( d \geq 9 \) and \( d \neq 10, 12 \), then \( G_\theta \) is isomorphic to \( T_\theta \). Since \( T_\theta \) is a finite group, it follows that \( G_\theta \) is a finite group. Hence, \( y_1 \) is an absolutely wild coordinate as mentioned. Furthermore, \( y_1 \) is a totally wild coordinate if and only if \( T_\theta = \{ 1 \} \). Since some \( \theta(z) \)’s satisfy \( T_\theta = \{ 1 \} \) and others do not, it follows that there exist various totally wild coordinates, and absolutely wild coordinates which are not totally wild.

Note that

\[
\text{Aut}(k[x]/k[y_1]) = \sigma_\theta \circ \text{Aut}(k[x]/k[x_1]) \circ \sigma_\theta^{-1},
\]

so every element of \( \sigma_\theta \circ \text{Aut}(k[x]/k[x_1]) \circ \sigma_\theta^{-1} \) not belonging to \( G_\theta \) is wild. Hence, if \( d \geq 9 \) and \( d \neq 10, 12 \), then \( \exp D \) is wild even for the locally nilpotent derivation

\[
D := \sigma_\theta \circ \left( \frac{\partial}{\partial x_3} \right) \circ \sigma_\theta^{-1},
\]
since \( \exp D = \sigma_0 \circ (\exp \partial / \partial x_3) \circ \sigma_0^{-1} \), and \( \exp D \) has an infinite order. If \( y_1 \) is a totally wild coordinate, then \( \sigma_0 \circ \tau \circ \sigma_0^{-1} \) is wild even for \( \tau \in \text{Aut}(k[x]/k) \) defined by

\[
\tau(x_1) = x_1, \quad \tau(x_2) = x_2, \quad \tau(x_3) = -x_3.
\]

As these examples show, the existence of absolutely wild or totally wild coordinates means the existence of a very large class of wild automorphisms of \( k[x] \).

9 Local slice constructions

The rank \( r \) of \( D \in \text{Der}_k k[x] \) is by definition the minimal number \( r \geq 0 \) for which \( D(\sigma(x_i)) \neq 0 \) holds for \( i = 1, \ldots, r \) for some \( \sigma \in \text{Aut}(k[x]/k) \) (cf. [5]). As mentioned after Theorem 5.1, every triangular derivation of \( k[x] \) is of rank less than \( n \) when \( n \geq 2 \). If \( n = 2 \), every locally nilpotent derivation of \( k[x] \) is of rank at most one by Rentschler [19]. Freudenburg [4], [5] first gave locally nilpotent derivations of \( k[x] \) of rank \( n \) for \( n \geq 3 \) using his method of local slice constructions. It is not easy to construct such a locally nilpotent derivation \( D \), for which it is previously not known whether \( \exp D \) is tame.

In this section, we summarize the main results of [14], where we give a large family of locally nilpotent derivations of \( k[x] \) by means of local slice construction, and determine tameness of \( \exp hD \) for each \( D \) and \( h \in \ker D \setminus \{0\} \). The family includes the locally nilpotent derivations of Freudenburg, and many other locally nilpotent derivations of rank three. The result is that \( \exp hD \) is always wild unless \( hD \) is triangularizable by a tame automorphism, i.e., \( \tau^{-1} \circ (hD) \circ \tau \) is triangular for some \( \tau \in \text{Aut}(k, x) \). This gives a partial affirmative answer to the question of Freudenburg (Section 5).

Now, for \( i = 0, 1 \), let \( t_i \) be a positive integer, and \( \alpha_j^i \) an element of \( k \) for \( j = 1, \ldots, t_i \) with \( \alpha_{t_i}^i = 1 \). We define a sequence \( (b_i)_{i=0}^\infty \) of integers by

\[
b_0 = b_1 = 0 \quad \text{and} \quad b_{i+1} = t_i b_i - b_{i-1} + \xi_i \quad \text{for} \quad i \geq 1,
\]

where \( t_i := t_0 \) if \( i \) is an even number, and \( t_i := t_1 \) otherwise, and where \( \xi_i := 1 \) if \( i \equiv 0, 1 \) (mod 4), and \( \xi_i := -1 \) otherwise. For each \( i \geq 1 \), we define \( \eta_i(y, z) \in k[y, z] \) by

\[
\eta_i(y, z) = z^{t_i b_i + 1} + \sum_{j=1}^{t_i} \alpha_j^i y^j z^{(t_i - j)b_i} \quad \text{if} \quad i \equiv 0, 1 \quad \text{(mod 4)}
\]

\[
\eta_i(y, z) = y^{t_i} + \sum_{j=1}^{t_i} \alpha_j^i z^{j b_i - 1} y^{t_i - j} \quad \text{otherwise},
\]

where \( \alpha_j^0 := \alpha_j^0 \) if \( i \) is an even number, and \( \alpha_j^1 := \alpha_j^1 \) otherwise for each \( j \). Set

\[
r = x_1 x_2 x_3 - \sum_{i=1}^{t_0} \alpha_i^0 x_2^i - \sum_{j=1}^{t_1} \alpha_j^1 x_1^j,
\]

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and define a sequence $(f_i)_{i=0}^\infty$ of rational functions by $f_0 = x_2$, $f_1 = x_1$ and $f_{i+1} = f_{i-1}^{-1}q_i$ for each $i \geq 1$ by induction on $i$, where $q_i = \eta_i(f_i, r)$. Note that
\[
q_1 = r + \sum_{j=1}^{t_1} \alpha_j^1 x_1^j = x_1x_2x_3 - \sum_{i=1}^{t_0} \alpha_i^0 x_2^i,
\]
and $r = x_2f_2 - \sum_{j=1}^{t_1} \alpha_j^1 x_1^j$. If $t_0 = 2$, then we have $f_2 = x_1x_3 - \sum_{i=1}^{t_0} \alpha_i^0 x_2^i$.

If $t_1 = 2$, then we can define the sequence $(f_i)_{i=0}^\infty$ from the same data $t_0$, $t_1$, $(\alpha_j^0)_{j=1}^{t_0-1}$ and $(\alpha_j^1)_{j=1}^{t_1-1}$. To distinguish it from the original one, we denote it by $(f_i')_{i=0}^\infty$.

When $f_i$ and $f_{i+1}$ belong to $k[\mathbf{x}]$, we consider a derivation $D_i := \Delta(f_i, f_{i+1})$ of $k[\mathbf{x}]$. Here, for $g_1, g_2 \in k[\mathbf{x}]$, we define a derivation $\Delta(g_1, g_2)$ of $k[\mathbf{x}]$ by $\Delta(g_1, g_2)(g_3) = \det(\partial g_i/\partial x_j)_{i,j}$ for each $g_3 \in k[\mathbf{x}]$. For example, we have
\[
D_1(x_1) = 0, \quad D_1(x_2) = -x_1 \quad \text{and} \quad D_1(x_3) = - \sum_{i=2}^{t_0} (i-1)\alpha_i^0 x_2^{i-2}
\]
by (9.1). Hence, $D_1$ is triangular. When $f_i'$ and $f_{i+1}'$ belong to $k[\mathbf{x}]$, we define $D_i' = \Delta(f_i', f_{i+1}')$ similarly.

Set $a_i = t_ib_i + \xi_i$ for each $i \geq 0$, and let $I$ be the set of $i \in \mathbb{N}$ such that $a_j > 0$ for $j = 1, \ldots, i$. Then, we have $a_0 = 0$, $a_1 = 1$ and $a_{i+1} = t_{i+1}a_i - a_{i-1}$ for $i \geq 1$. From this, we get
\[
I = \begin{cases} 
\{1\} & \text{if } t_0 = 1 \\
\{1, 2\} & \text{if } (t_0, t_1) = (2, 1) \\
\{1, 2, 3, 4\} & \text{if } (t_0, t_1) = (3, 1) \\
\mathbb{N} & \text{otherwise.}
\end{cases}
\]

In the notation above, we have the following result.

**Theorem 9.1** ([13, Theorem 2.1]). The following assertions hold for each $i \in I$:

(i) $f_i$ and $f_{i+1}$ belong to $k[\mathbf{x}]$, and $D_i$ is a locally nilpotent derivation of $k[\mathbf{x}]$ such that $D_i(r) = -f_if_{i+1}$. Furthermore, we have the following:

(1) If $i$ is the maximum of $I$, then $D_i$ is not irreducible and $\ker D_i \neq k[f_i, f_{i+1}]$.

(2) If $i$ is not the maximum of $I$, then $D_i$ is irreducible and $\ker D_i = k[f_i, f_{i+1}]$.

(ii) Assume that $t_0 = 2$. Then, we have $\tau_2^{-1} \circ D_i \circ \tau_2 = D_i'_{i-1}$. Hence, $D_2$ is triangularizable by a tame automorphism. Moreover, the following assertions hold:
(a) If $t_1 = 2$, then we have $\tau^{-1} \circ D_i \circ \tau = D_0$, where $\tau := (\tau_2 \circ \tau_1)^{i/2}$ if $i$ is an even number, and $\tau := (\tau_2 \circ \tau_1)^{(i-1)/2} \circ \tau_2$ otherwise.

(b) If $t_1 \geq 3$ and $i \geq 3$, then $\exp hD_i$ is wild for every $h \in \ker D_i \setminus \{0\}$.

(iii) If $t_0 \geq 3$ and $i \geq 2$, then $\exp hD_i$ is wild for every $h \in \ker D_i \setminus \{0\}$.

Here, $D \in \Der_k k[x]$ is said to be irreducible if $D(k[x])$ is contained in no proper principal ideal of $k[x]$.

Recall that $\pl D := D(k[x]) \cap \ker D$ forms an ideal of $\ker D$ for each $D \in \Der_k k[x]$, and is called the plinth ideal of $D$. Assume that $D$ is locally nilpotent. Then, we have $\pl D \neq \{0\}$ unless $D = 0$. Owing to Miyanishi [15], it holds that $\pl D = \ker D$ if and only if $D$ is irreducible and of rank one when $n = 3$. By Daigle-Kaliman [2, Theorem 1], $\pl D$ is always a principal ideal of $\ker D$ when $n = 3$.

We use the following lemma to determine the rank of a locally nilpotent derivation.

**Lemma 9.2** ([14, Lemma 2.5]). Let $D \neq 0$ be an irreducible locally nilpotent derivation of $k[x]$. If $\ker D$ contains a coordinate $p$ of $k[x]$ over $k$, then there exists $s \in k[x]$ such that $D(s)$ belongs to $k[p] \setminus \{0\}$.

Since $\pl D$ is a principal ideal of $\ker D$, Lemma 9.2 implies that $\pl D$ is generated by an element of $k[p] \setminus \{0\}$ if $D$ is irreducible, and $\ker D$ contains a coordinate $p$ of $k[x]$ over $k$. On the other hand, if $t_0 = 2$, $t_1 \geq 3$ and $i \geq 3$, or if $t_0 \geq 3$, $(t_0, t_1) \neq (3, 1)$ and $i \geq 2$, then we have $\pl D_i = f_i f_{i+1} \ker D_i$ (cf. [14, Proposition 1.2]). Since $f_i$ and $f_{i+1}$ are algebraically independent over $k$, we see that $f_i f_{i+1}$ does not belong to $k[p]$ for any coordinate $p$ of $k[x]$ over $k$. Thus, we conclude that $D_i$ is of rank three. In [14], we also determined the rank of $D_i$ for the other cases.

Next, take $i \in \mathbb{N}$ with $i \geq 2$, and assume that $t_0 \geq 3$ if $i = 2$, and $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$ if $i \geq 3$. Let $\lambda(y) \in k[y] \setminus \{0\}$ and $\mu(y, z) = \sum_{j \geq 1} \mu_j(y)z^j \in zk[y, z] \setminus \{0\}$ be such that $\gcd(\lambda(y), \mu_j(y)) = 1$ for some $j \geq 1$, where $\mu_j(y) \in k[y]$ for each $j \geq 1$. We set $$ r_i = \lambda(f_i)\bar{r} - \mu(f_i, f_{i-1}), \quad \text{where} \quad \bar{r} := \begin{cases} x_2 & \text{if } i = 2 \\ r & \text{if } i \geq 3. \end{cases} $$

Then, we define $$ \tilde{f}_{i+1} = \tilde{\eta}_i(f_i, r_i \lambda(f_i)^{-1}) \lambda(f_i)^{a_i} f_{i-1}^{-1}, $$
where $$ \tilde{\eta}_2(y, z) := y + \sum_{j=1}^{t_0} \alpha_j^0 z^{j-1} $$
and $\tilde{\eta}_i(y, z) := \eta_i(y, z)$ for $i \geq 3$.

With this notation and assumptions, we have the following result.
Theorem 9.3 ([13, Theorem 3.1]). (i) \( \tilde{f}_{i+1} \) belongs to \( k[x] \), and \( \tilde{D}_{i} := \Delta_{(f_{i}, \tilde{f}_{i+1})} \) is an irreducible locally nilpotent derivation of \( k[x] \) such that \( \ker \tilde{D}_{i} = k[f_{i}, \tilde{f}_{i+1}] \). Moreover, we have \( \tilde{D}_{i}(r_{i}) = -\lambda(f_{i})\tilde{f}_{i+1} \) if \( i = 2 \), and \( \tilde{D}_{i}(r_{i}) = -\lambda(f_{i})f_{i}\tilde{f}_{i+1} \) if \( i \geq 3 \).

(ii) Assume that \( \lambda(y) \) belongs to \( k^{\times} \), \( \mu(y, z) \) belongs to \( zk[z] \), and \( i = 2 \). Then, \( \exp h\tilde{D}_{2} \) is tame if and only if \( h \) belongs to \( k[\tilde{f}_{3}] \) for \( h \in \ker \tilde{D}_{2} \). In the other case, \( \exp h\tilde{D}_{i} \) is wild for each \( h \in \ker \tilde{D}_{i} \setminus \{0\} \).

In the same situation, the following proposition holds.

Proposition 9.4 ([14, Proposition 1.5]). (i) If \( i \geq 3 \), then we have \( \operatorname{rank} \tilde{D}_{i} = 3 \).

(ii) If \( \lambda(y) \) belongs to \( k^{\times} \), then \( \operatorname{rank} \tilde{D}_{2} = 2 \), and \( \tilde{f}_{3} \) is a coordinate of \( k[x] \) over \( k \).

(iii) Assume that \( \lambda(y) \) does not belong to \( k^{\times} \). If \( t_{0} \geq 4 \), or \( \mu_{j}(y) \) does not belong to \( \sqrt{\lambda(y)k[y]} \) for some \( j \geq 2 \), then we have \( \operatorname{rank} \tilde{D}_{2} = 3 \). If \( t_{0} = 3 \), and \( \mu_{j}(y) \) belongs to \( \sqrt{\lambda(y)k[y]} \) for every \( j \geq 2 \), then we have \( \operatorname{pl} \tilde{D}_{2} = \tilde{f}_{3} \ker \tilde{D}_{2} \).

The locally nilpotent derivations of Freudenburg [4] are obtained as follows. Assume that \( t_{j} = 3 \) and \( \alpha_{1}^{j} = \alpha_{2}^{j} = 0 \) for \( j = 0, 1 \). Then, we have \( I = \mathbb{N} \) by (9.3). By Theorem 9.1 (i), it follows that \( f_{i} \) and \( f_{i+1} \) belong to \( k[x] \), and \( D_{i} \) is an irreducible locally nilpotent derivation of \( k[x] \) with \( \ker D_{i} = k[f_{i}, f_{i+1}] \) for each \( i \geq 1 \). It is easy to check that \( a_{1} = 1 \), \( a_{2} = 2 \), \( f_{2} = x_{1}x_{3} - x_{2}^{2} \) and \( a_{i+1} = 3a_{i} - a_{i-1} \) for every \( i \geq 2 \). Moreover, we have \( r = x_{2}f_{2} - x_{1}^{2} \), \( f_{0} = x_{2} \), \( f_{1} = x_{1} \) and \( f_{i+1} = f_{i-1}^{-1}(r^{a_{i}} + f_{i}^{3}) \) for every \( i \geq 2 \). From this, we see that \( (\iota^{-1} \circ D_{i} \circ \iota)_{i=1}^{\infty} \) is the same as the sequence of locally nilpotent derivations of “Fibonacci type” given by Freudenburg [4], where \( \iota \in \text{Aut}(k[x]/k) \) is such that \( \iota(x_{2}) = -x_{2} \) and \( \iota(x_{i}) = x_{i} \) for \( i = 1, 3 \). According to Theorem 9.1 (ii), \( \exp hD_{i} \) is wild for each \( h \in \ker D_{i} \setminus \{0\} \) for every \( i \geq 2 \). Next, for \( l, m \in \mathbb{N} \), set \( \lambda(y) = y^{l} \) and \( \mu(y, z) = -z^{m} \). Then, it follows from Theorem 9.3 that \( \tilde{f}_{3} \) belongs to \( k[x] \), \( \tilde{D}_{2} \) is an irreducible locally nilpotent derivation of \( k[x] \) such that \( \ker \tilde{D}_{2} = k[f_{2}, \tilde{f}_{3}] \), and \( \exp h\tilde{D}_{2} \) is wild for each \( h \in \ker \tilde{D}_{2} \setminus \{0\} \). In this case, we have \( r_{2} = f_{2}^{l}x_{2} + x_{1}^{m} \). Since \( \tilde{\eta}_{2}(y, z) = y + z^{2} \), and \( a_{2} = t_{0} - 1 = 2 \), we get

\[
\tilde{\eta}_{2} \left( f_{2}, \frac{r_{2}}{\lambda(f_{2})} \right) (f_{2}^{l})^{2} = \left( f_{2} + \frac{r_{2}}{f_{2}^{l}} \right) f_{2}^{2l} = f_{2}^{2l+1} + r_{2}^{2} = x_{1}(f_{2}^{2l}x_{3}^{2} - 2f_{2}^{l}x_{1}^{m-1}x_{2} + x_{1}^{2m-1}),
\]

and so \( \tilde{f}_{3} = f_{2}^{2l}x_{3} - 2f_{2}^{l}x_{1}^{m-1}x_{2} + x_{1}^{2m-1} \). We note that, if \( m = 2l+1 \), then \( \tilde{D}_{2} \) is the same as the homogeneous locally nilpotent derivation of “type (2, 4l+1)” given by Freudenburg [4].

References


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