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Homogeneous locally nilpotent derivations of $\mathbb{C}[x,y,z]$ and pencils of rational plane curves

By

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Abstract

In this article, we shall investigate a relation between kernels of weighted homogeneous locally nilpotent derivations (wh-LND, for short) on the polynomial ring $\mathbb{C}[x,y,z]$ in three variables and pencils of rational curves on weighted projective planes. We give a geometric proof to the result about generators of kernels of wh-LND's on $\mathbb{C}[x,y,z]$ due to Daigle [Da98, Da00]. This allows us to reduce the consideration about wh-LND's with respect to a given weight $\mathbf{w}$ to a two-dimensional projective geometry, more precisely, resolutions of certain kinds of pencils $\mathscr{L}$ composed of rational curves on weighted projective planes $\mathbb{P}_{\mathbf{w}}$. Then we shall observe automorphisms on $\mathbb{A}^3$ arising from various $\mathbf{G}_a$-actions according to the number of non-reduced members in $\mathscr{L}$.

§ 1. Introduction

All varieties in this paper are defined over the field of complex numbers $\mathbb{C}$. In affine algebraic geometry, one of the important problems is the comprehension of groups $G_n := \mathrm{Aut}\,(\mathbb{A}^n)$ of automorphisms on the affine spaces $\mathbb{A}^n$ (or equivalently, those on the polynomial rings $\mathbb{C}[x_1, \cdots, x_n]$) for $n \in \mathbb{N}$.[1] In the case of $n = 1$, the consideration is very simple so that there is nothing to do. Meanwhile, the situation becomes drastically complicated for $n \geqq 2$. In order to state adequately, before refering to the case of $n = 2$ in an explicit manner, we shall prepare the notation which is used throughout in this article. Let A_n (resp. J_n) be the subgroup of G_n consisting of affine transformations (resp. de Jonquière transformations) on $\mathbb{A}^n$, that is, an element $\xi \in A_n$ (resp. $\eta \in$

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[1] In this paper, we do not distinguish G_n from $\mathrm{Aut}_{\mathbb{C}}\,(\mathbb{C}[x_1, \cdots, x_n])$.

J_n) is of the form $\xi(x_i) = \sum_{j=1}^{n} a_{ij}x_j + b_i$ $(1 \leqq i \leqq n)$ with $\det(a_{ij}) \neq 0$ (resp. $\eta(x_i) = a_i x_i + f_i(x_{i+1}, \cdots, x_n)$ $(1 \leqq i \leqq n)$, where $a_i \in \mathbb{C}$ with $\prod_{i=1}^{n} a_i \neq 0$ and $f_i \in \mathbb{C}[x_{i+1}, \cdots, x_n]$). An automorphism $\tau \in G_n$ is said to be **tame** if τ is contained in the subgroup $T_n := A_n \vee J_n$ generated by these two subgroups. Conversely, τ is called **non-tame** or **wild** in case of $\tau \in G_n \backslash T_n$.

We come back to state the situation for the case $n = 2$. Then, due to Jung [Ju42], the structure of G_2 is well understood, in fact, $G_2 = T_2$ holds true, i.e., every automorphism on $\mathbb{A}^2$ is tame. In consideration of this fact in dimension two, it seems to be natural to guess that the phenomenon result follows for higher-dimensional case also, namely, $G_n = T_n$ for $n \geqq 3$. However, as far as we are concerned at least with the case of $n = 3$, it is surprisingly known that G_3 is strictly greater than T_3 by the remarkable work of Shestakov and Umirbaev (cf. [SU04$_a$, SU04$_b$]). More precisely to say, for a long time, though the tameness of the Nagata automorphism, say $\sigma_{\mathrm{Nagata}} \in G_3$ (see [Na72] for example for the concrete form of σ_{Nagata}, see Example 1.1 also), which is a complicated looking automorphism on $\mathbb{A}^3$ constructed by Nagata around early 1970's (cf. [Na78]), and the wildness of which remained unknown for the past three decades, they at last showed $\sigma_{\mathrm{Nagata}} \in G_3 \backslash T_3$. Thus σ_{Nagata} is the first example of wild automorphisms on $\mathbb{A}^3$. Nevertheless we know very little about G_3 itself, for instance, we do not know how many generators outside of T_3 are necessary to generate G_3 whole. Certainly σ_{Nagata} must be one of such generators, but it is important to enumerate the other wild automorphisms on $\mathbb{A}^3$ as possible candidates of generators. Here, we note that σ_{Nagata} is obtained from the point of view of $\mathbf{G}_a$-actions on $\mathbb{A}^3 = \mathrm{Spec}\,(\mathbb{C}[x, y, z])$ as follows (see [Mi78, AvdE00, Fr06] or §2 for the interpretation between $\mathbf{G}_a$-actions and locally nilpotent derivations), where we shall use the notation $\mathbb{C}[x, y, z]$ instead of $\mathbb{C}[x_1, x_2, x_3]$:

Example 1.1. Let Δ be a homogeneous locally nilpotent derivation on $\mathbb{C}[x, y, z]$ (cf. Definition 2.1) defined as in the following fashion:

$$\Delta := 2y(xz - y^2)\,\frac{\partial}{\partial x} + z(xz - y^2)^2\,\frac{\partial}{\partial y}$$

Then it is easily seen that $\mathrm{Ker}\,(\Delta) = \mathbb{C}[xz - y^2, z]$. The corresponding co-action $\varphi_\Delta : \mathbb{C}[\,x, y, z\,] \to \mathbb{C}[\,x, y, z\,] \otimes_{\mathbb{C}} \mathbb{C}[\,t\,]$ is written as:

$$\varphi_\Delta : \begin{cases} x \mapsto x + 2y\,(xz - y^2)\,t + z\,(xz - y^2)^2\,t^2 \\ y \mapsto \qquad\qquad y + z\,(xz - y^2)\,t \\ z \mapsto \qquad\qquad\qquad z, \end{cases}$$

where t is the coordinate of $\mathbf{G}_a = \mathrm{Spec}\,(\mathbb{C}[t])$. Putting $t = 1$ in the above equation, we obtain the Nagata automorphism σ_{Nagata}, which is indeed wild by [SU04$_a$, SU04$_b$].

This example indicates a chance to produce the other examples of wild automorphisms on $\mathbb{A}^3$ by making use of homogeneous $\mathbf{G}_a$-actions.

This article is mainly devoted to two affairs. At first, we shall give an alternative proof for the result due to Daigle (cf. [Da98, Da00], see Theorem 2.2 in §2) concerning a characterization of generators of kernels of (weighted) homogeneous locally nilpotent derivations (wh-LND, for short) on $\mathbb{C}[x, y, z]$. Indeed, his original proof [Da98] is rather algebraic.[2] Meanwhile we shall here work with the two dimensional projective geometry admitting quotient singularities and a general theory on $\mathbb{P}^1$-fibrations to obtain a new proof taking the general weights cases into account. In our strategy, it is essential to look at the structure of a certain linear pencil $\mathscr{L}$ on $\mathbb{P}_{\mathbf{w}}$ corresponding to a given wh-LND on $\mathbb{C}[x, y, z]$ with respect to a weight $\mathbf{w}$, which is a slight generalization of the result due to Miyanishi-Sugie [MS81], in particular, we see that $\mathscr{L}$ has at most two multiple members and we are able to restrict the position of singularities on $\mathbb{P}_{\mathbf{w}}$ in terms of multiple members of $\mathscr{L}$ (cf. Theorem 2.7). In the second stage, according to the number of multiple members in $\mathscr{L}$, we shall decide when the corresponding weighted homogeneous $\mathbf{G}_a$-action on $\mathbb{A}^3$ produces a family of wild automorphisms with the aids of techniques due to Kuroda (cf. [Ku11]). Moreover, we propose a question about the wildness of automorphisms on $\mathbb{A}^n$ for $n \geqq 4$ (see Conjecture 4.1). Taking the situations in dimensions $n = 2$ and $n = 3$ simultaneously into account as mentioned in §4, this question seems to be reasonable.

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§ 2. Preliminaries and Main Theorems

In this section, first of all we shall prepare terminologies and general facts which are used consistently in this article. Then we state the main results (Theorems 2.2, 2.7 and 2.10), which are proved in this article. In what follows, we denote by $\mathbb{C}[x, y, z]$ the polynomial ring in three variables instead of $\mathbb{C}[x_1, x_2, x_3]$.

Definition 2.1. Let $\Delta \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x, y, z])$ be a $\mathbb{C}$-derivation.

(1) Δ is said to be a **locally nilpotent derivation** (an **LND**, for short) if for $\forall\, a \in \mathbb{C}[x, y, z]$, there exists $\exists\, N(a) \in \mathbb{N}$ such that $\Delta^{N(a)}(a) = 0$. We denote by $\mathrm{LND}(\mathbb{C}[x, y, z])$ the set of all LND's on $\mathbb{C}[x, y, z]$.

[2]However [Da98] deals additionally with cases where some of (p, q, r) in the statement of Theorem 2.2 is equal to zero. Hence, he treates more general cases.

(2) Let $\mathbf{w} = (p, q, r)$ be a triple of pairwise coprime positive integers and $f(x, y, z) \in \mathbb{C}[x, y, z]$. Let

$$f(x, y, z) = \sum_{i,j,k} a_{i,j,k}\, x^i y^j z^k$$

be a decomposition of f into monomial parts. Then f is said to be **w-homogeneous** if $ip + jq + kr$ is constant whenever $a_{i,j,k} \neq 0$. We denote this constant value by $\deg_{\mathbf{w}}(f)$.

(3) Let $A = \mathbb{C}[x, y, z] = \oplus_{n \geqq 0} A_n^{\mathbf{w}}$ be a decomposition of $A = \mathbb{C}[x, y, z]$ into $\mathbf{w}$-homogeneous pieces, i.e., $A_n^{\mathbf{w}}$ is a $\mathbb{C}$-vector space composed of all $\mathbf{w}$-homogeneous polynomials f of $\deg_{\mathbf{w}}(f) = n$. An LND Δ on A is called **w-homogeneous** (**w-h-LND**, for short) in the case of $\Delta(A_n^{\mathbf{w}}) \subseteq A_{n+d}^{\mathbf{w}}$ for $\forall\, n \geqq 0$, where d is an integer that is independent of n. We denote by $\mathrm{LND}^{\mathbf{w}}(A)$ the set of all $\mathbf{w}$-h-LND's on A.

(4) With the notation in (3), we write $\mathbb{P}_{\mathbf{w}} := \mathrm{Proj}\,(\oplus_{n \geqq 0} A_n^{\mathbf{w}})$, which is a weighted projective plane associated with $\mathbf{w}$. For a given $h \in A$, which is $\mathbf{w}$-homogeneous, we denote by $\mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(h)$ the curve in $\mathbb{P}_{\mathbf{w}}$ defined by h.

By the work of Miyanishi (cf. [Mi85]), for any $\Delta \in \mathrm{LND}\,(\mathbb{C}[x, y, z]) \backslash \{0\}$, which is not necessarily $\mathbf{w}$-homogeneous, its kernel $\mathrm{Ker}\,(\Delta)$ is a polynomial ring in two variables, that is, there exist mutually algebraically independent polynomials f, $g \in \mathbb{C}[x, y, z]$ such that $\mathrm{Ker}\,(\Delta) = \mathbb{C}[f, g]$. Note that f and g are not always variables of $\mathbb{C}[x, y, z]$ (cf. Remark 2.5). This result plays an essential role in a lot of places of affine algebraic geometry when we consider quotients of algebraic $\mathbf{G}_a$-actions. On the other hand, this does not yield an information about the candidates for generators of $\mathrm{Ker}\,(\Delta)$. However, whenever we are concerned with $\mathbf{w}$-h-LND's, there exists a nice interpretation in terms of projective geometry as follows:

Theorem 2.2. (cf. [Da98, Da00]) *Let* $\mathbf{w} = (p, q, r)$ *be a triple of pairwise coprime positive integers. For given two irreducible* $\mathbf{w}$*-homogeneous polynomials* f, $g \in \mathbb{C}[x, y, z]$ *such that* $\gcd(\deg_{\mathbf{w}}(f), \deg_{\mathbf{w}}(g)) = 1$*, the following two conditions are equivalent:*

$(\mathbf{A})_{\mathbf{w}}$ $\mathrm{Ker}\,(\Delta) = \mathbb{C}[f, g]$ *for some* $\Delta \in \mathrm{LND}^{\mathbf{w}}\,(\mathbb{C}[x, y, z])$*,*

$(\mathbf{B})_{\mathbf{w}}$ $\mathbb{P}_{\mathbf{w}} \backslash \big(\mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(f) \cup \mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(g)\big) \cong \mathbb{C}^* \times \mathbb{A}^1$.

Remark 2.3. The implication $(\mathrm{A})_{\mathbf{w}} \Rightarrow (\mathrm{B})_{\mathbf{w}}$ can be true even if we do not suppose that p, q and r are not pairwisely coprime (cf. [Da98, Corollary 3.10]). Moreover, under the hypothesis that p, q and r are pairwisely coprime, if f and g satisfy $(\mathrm{A})_{\mathbf{w}}$, then we have $\gcd(\deg_{\mathbf{w}}(f), \deg_{\mathbf{w}}(g)) = 1$ (cf. [Da00, 1.9.]). On the other hand, without the condition that p, q and r are pairwisely coprime, the implication $(\mathrm{B})_{\mathbf{w}} \Rightarrow (\mathrm{A})_{\mathbf{w}}$ does not hold true in general. For instance, let us consider the case $\mathbf{w} = (2,3,6)$ and $f = z$, $g = x^3 + y^2$. Then f and g are $\mathbf{w}$-homogeneous, furthermore, they satisfy $(\mathrm{B})_{\mathbf{w}}$. But it follows in fact that there does not exist a locally nilpotent derivation of $\mathbb{C}[x,y,z]$ whose kernel coincides with $\mathbb{C}[f,g]$. Indeed, assuming to the contrary that we have $\Delta \in \mathrm{LND}^{\mathbf{w}}(\mathbb{C}[x,y,z])$ such that $\mathrm{Ker}\,(\Delta) = \mathbb{C}[f,g] = \mathbb{C}[z, x^3+y^2]$, then Δ is extended to a locally nilpotent derivation on $\mathbb{C}(z)[x,y]$, say $\widetilde{\Delta}$, in a natural way such that:

$$\mathrm{Ker}\,(\widetilde{\Delta}) = \mathrm{Ker}\,(\Delta) \otimes_{\mathbb{C}[z]} \mathbb{C}(z) = \mathbb{C}[z, x^3+y^2] \otimes_{\mathbb{C}[z]} \mathbb{C}(z) = \mathbb{C}(z)[x^3+y^2].$$

On the other hand, note that a kernel of any locally nilpotent derivation on the polynomial ring over the field of characteristic zero is generated by a single variable (cf. [Ren68]). But, $\mathrm{Ker}\,(\widetilde{\Delta})$ is generated by x^3+y^2 over the field $\mathbb{C}(z)$ as seen just above, which is not a variable of $\mathbb{C}(z)[x,y]$. This is a contradiction.

Remark 2.4. From the viewpoint of geometry, it is natural to assume that p, q and r are pairwisely coprime. In fact, letting $c := \gcd\,(p,q,r)$ and $p = cp'$, $q = cq'$, $r = cr'$, we have $\mathbb{P}(p,q,r) \cong \mathbb{P}(p',q',r')$, Furthermore, even in the case $c = 1$, letting $d := \gcd\,(p,q)$ and $p = dp''$, $q = dq''$, it follows that $\mathbb{P}(p,q,r) \cong \mathbb{P}(p'',q'',r)$ (cf. [IF]).

Remark 2.5. A locally nilpotent derivation $\Delta \in \mathrm{LND}^{\mathbf{w}}\,(\mathbb{C}[x,y,z])$ as in $(\mathrm{A})_{\mathbf{w}}$ and $\mathbf{w}$-homogeneous polynomials $f, g \in \mathbb{C}[x,y,z]$ as in $(\mathrm{B})_{\mathbf{w}}$ are related to each other as in the following manner:

$$\Delta\,(*) \;=\; h(f,g)\,\frac{\partial\,(f,g,*)}{\partial\,(x,y,z)}, \tag{2.1}$$

where $h(f,g) \in \mathbb{C}[f,g]\backslash\{0\}$ is $\mathbf{w}$-homogeneous. Indeed, this fact is obtained by Daigle [Da97]. Note that as far as $h(f,g)$ is $\mathbf{w}$-homogeneous and contained in $\mathbb{C}[f,g]\backslash\{0\}$, the resulting Δ obtained as in (2.1) is a $\mathbf{w}$-h-LND with a kernel $\mathbb{C}[f,g]$. But, the choice of such an h becomes often crucial in order to produce a family of wild automorphisms on $\mathbb{A}^3$ (cf. [Ku11]). For instance, let us consider the case of the standard weight $\mathbf{w} = (1,1,1)$, and two homogeneous polynomials $f := z$, $g := xz - y^2$. Then it is not difficult to verify $\mathbb{P}^2\backslash\big(\mathbb{V}_{\mathbb{P}^2}(f) \cup \mathbb{V}_{\mathbb{P}^2}(g)\big) \cong \mathbb{C}^* \times \mathbb{A}^1$. According to (2.1), the corresponding $\Delta \in \mathrm{LND}^{\mathbf{w}}\,(\mathbb{C}[x,y,z])$ is obtained as:

$$\Delta = h\left(2y\,\frac{\partial}{\partial x} + z\,\frac{\partial}{\partial y}\right),$$

where $h \in \mathbb{C}[z, xz-y^2]\backslash\{0\}$ is homogeneous. In the case of $h \in \mathbb{C}[z]\backslash\{0\}$, the resulting co-action produces a family of tame automorphisms on $\mathbb{A}^3$. Meanwhile, if we choose as $h = xz - y^2$ for instance, then Δ coincides with the one in Example 1.1, hence it yields a family of wild automorphisms on $\mathbb{A}^3$.

Remark 2.6. In the assertion of Theorem 2.2, the irreducibility of f and g is necessary. For example, let us consider the case of $f = x$ and $g = xy$ with the ordinary weight $\mathbf{w} = (1,1,1)$. This pair satisfies $(\mathrm{B})_{\mathbf{w}}$ except for the irreducibility. If a suitable $\Delta \in \mathrm{LND}^{\mathbf{w}}(\mathbb{C}[x,y,z])$ has a kernel $\mathrm{Ker}(\Delta) = \mathbb{C}[f,g]$, then y also must be contained in $\mathrm{Ker}(\Delta)$ because it is an inert sub-algebra of $\mathbb{C}[x,y,z]$ (cf. [Mi78]), which is absurd.

Theorem 2.2 is originally due to Daigle (cf. [Da98, Da00]), where his approach for the proof is rather algebraic. The main part of this article is devoted to an alternative proof for Theorem 2.2 by an algebro-geometric one, more precisely, algebraic $\mathbf{G}_a$-actions, resolution of base points of pencils of rational curves on weighted projective planes, a general theory of singular fibers of $\mathbb{P}^1$-fibrations and so on. Especially, the investigation of a certain linear pencil $\mathscr{L}$ occuring in our consideration is indispensable. This should be important not only for an alternative proof but also for formulating the criteria for a family of automorphisms on $\mathbb{A}^3$ arising from $\Delta \in \mathrm{LND}^{\mathbf{w}}(\mathbb{C}[x,y,z])$ (or equivalently, from a pencil $\mathscr{L}$ on $\mathbb{P}_{\mathbf{w}}$) to be wild in terms of the types of $\mathscr{L}$. Indeed, the article due to Kuroda [Ku11], which summarizes results about the criteria of the wildness, seems to show its validity.[3] Besides that, this observation of the pencil $\mathscr{L}$ is a slight generalization of [MS81], hence it is worthwhile to mention it as a theorem as follows.

Theorem 2.7. *Let $\mathbf{w} = (p,q,r)$ be a triple of pairwisely coprime positive integers such that $p \leqq q \leqq r$, and $f,\ g \in \mathbb{C}[x,y,z]$ two irreducible $\mathbf{w}$-homogeneous polynomials of $a := \deg_{\mathbf{w}}(f)$, $b := \deg_{\mathbf{w}}(g)$ satisfying $(\mathrm{A})_{\mathbf{w}}$ as in Theorem 2.2.*[4] *Let $c := \gcd(a,b)$ and let us write $a = ca'$ and $b = cb'$. Let $\mathscr{L}$ be a linear pencil on $\mathbb{P}(p,q,r) = \mathbb{P}_{\mathbf{w}}$ spanned by $f^{b'}$ and $g^{a'}$, namely,*

$$\mathscr{L} := \left\{ B_{[\alpha:\beta]} \;\middle|\; [\alpha:\beta] \in \mathbb{P}(a,b) \right\},$$

where $B_{[\alpha:\beta]}$ is a curve on $\mathbb{P}_{\mathbf{w}}$ defined by $(\alpha f)^{b'} + (\beta g)^{a'} = 0$. Then the following holds true:

(1) *All members of $\mathscr{L}$ except for $B_{[1:0]}$ and $B_{[0:1]}$ are irreducible and reduced.*

(2) *$B_{[1:0]}$ and $B_{[0:1]}$ are irreducible. Furthermore, $B_{[1:0]}$ (resp. $B_{[0:1]}$) is reduced if and only if $b' = 1$ (resp. $a' = 1$).*

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[3]However, the type of $\mathscr{L}$ defined in Remark 2.8 below has some obstacles to formulate a criterion of the wildness. For instance, in the case of $f = y$ and $g = z$, if we take $\mathbf{w}$ as $\mathbf{w} = (1,1,1)$ (resp. $\mathbf{w} = (1,1,2)$, resp. $\mathbf{w} = (1,2,3)$), then the pencil $\mathscr{L}$ spanned by y and z (resp. y^2 and z, resp. y^3 and z^2) are of *Type (0)* (resp. *Type (I)*, resp. *Type (II)*). Thus the pair only (f,g) of polynomials does not determine the type.

[4]In fact $\gcd(a,b) = 1$ holds true (cf. Remark 2.3). However, we shall proceed to see Theorem 2.7 without supposing that $\gcd(a,b) = 1$.

(3) $\mathrm{Bs}\,\mathscr{L}$ *consists of exactly one point, say* P. *For any member* $B_{[\alpha:\beta]} \in \mathscr{L}$ *we have* $(B_{[\alpha:\beta]})_{\mathrm{red}} \backslash \{P\} \cong \mathbb{A}^1$.

(4) *If there is no multiple member in* $\mathscr{L}$, *then* $p = q = 1$, *i.e.*, $\mathbb{P}_{\mathbf{w}} = \mathbb{P}(1, 1, r)$.

(5) *If there exists exactly one multiple member* $B_{[1:0]}$ *(resp.* $B_{[0:1]}$*) in* $\mathscr{L}$, *then* $p = 1$ *and* $\deg_{\mathbf{w}}(f)$ *(resp.* $\deg_{\mathbf{w}}(g)$*) is equal to* 1.

Remark 2.8. The behaviour of the pencil $\mathscr{L}$ changes drastically according to the number of multiple members. Let us say that $\mathscr{L}$ is of *Type* (k) $(0 \leqq k \leqq 2)$ if $\mathscr{L}$ possesses k multiple members. Concerning the explicit description of $\mathscr{L}$ in the sense of defining equations of general members of $\mathscr{L}$, there is almost nothing to consider other than Theorem 2.7, (4) and Theorem 2.10 for *Type (0)*. On the other hand, for the remaining *Type(I)* and *Type(II)*, it is difficult to determine concretely defining equations although there are such examples as in Example 2.9. Meanwhile, even under the condition that $\gcd(\deg_{\mathbf{w}}(f), \deg_{\mathbf{w}}(g)) = 1$, it seems not to be effective to use types just defined in order to formulate a criterion with respect to the wildness as stated before Theorem 2.7 especially for *Type* (I) and *Type* (II). Instead of it, the paper [Ku11] by Kuroda deals with a lot of topics about explicit criteria of wildness, hence we recommend to refer to [Ku11].

Example 2.9. We shall mention examples for *Types (I)* and *(II)* respectively, where we adopt the standard weight $\mathbf{w} = (1, 1, 1)$:

(1) Let us put $f := x$ and $g := x^{b-1}z + a_b y^b + a_{b-1} y^{b-1} x + \cdots + a_1 y x^{b-1} + a_0 x^b$ with $b \geqq 2$ and $a_b \neq 0$. Then (f, g) satisfies $(\mathrm{B})_{\mathbf{w}}$. In the case of $b = 2$ and $a_2 = -1$, the pair (f, g) gives rise to an LND of Nagata type (cf. Remark 2.5). Hence this example is considered as a generalisation of the Nagata automorphism.

(2) Let us put $f := xz - y^2$ and $g := f\,(xz^2 - y^2 z - 2x^2 y) + x^5$ (Yoshihara's quintic, [Yo79]). In this case also, (f, g) satisfies $(\mathrm{B})_{\mathbf{w}}$ (cf. [MS81]).

In any case (1) and (2) above, it follows that $\mathbf{G}_a$-actions arising from Δ in the formula (2.1) in Remark 2.5 yield families of wild automorphisms on $\mathbb{A}^3$ after choosing $h(f, g)$ there suitably (cf. [Ku11]).

As remarked in Remark 2.8, *Type* (0) is not difficult to manipulate to decide the forms of f and g, in addition to this, to see the corresponding $\mathbf{G}_a$-action produces a family of tame automorphisms on $\mathbb{A}^3$ as follows:

Theorem 2.10. *Let* $\mathbf{w} = (p, q, r)$ *be a triple of pairwisely coprime positive integers with* $p \leqq q \leqq r$, *and let* f, $g \in \mathbb{C}[x, y, z]$ *be* $\mathbf{w}$*-homogeneous polynomials of respective* *degrees* $a := \deg_{\mathbf{w}}(f) = ca'$, $b := \deg_{\mathbf{w}}(g) = cb'$ *with* $c = \gcd(a,b)$ *satisfying* $(\mathrm{A})_{\mathbf{w}}$ *as in Theorem 2.2. Let* $\mathscr{L}$ *be the linear pencil on* $\mathbb{P}_{\mathbf{w}} = \mathbb{P}(p,q,r)$ *spanned by* $f^{b'}$ *and* $g^{a'}$*. Suppose that* $\mathscr{L}$ *is of Type* (0)*. Then* f *and* g *are transformed to* x *and* y *by a suitable linear automorphism and any derivation* Δ *obtained as in (2.1) of Remark 2.5 becomes an LND yielding a family of tame automorphisms on* $\mathbb{A}^3$*.*

Remark 2.11. By Theorem 2.10, we know that there is no chance to produce a family of wild automorphisms on $\mathbb{A}^3$ as far as we make use of a pencil $\mathscr{L}$ of *Type* (0). Meanwhile, for pencils of *Type* (I) and *Type* (II), we are often able to construct families of wild automorphisms by choosing the polynomial $h(f,g)$ in the formula (2.1) of Remark 2.5 adequately. For instance, in the case of $\mathbf{w} = (1,2,3)$, let us choose f and g as $f = x \in A_1^{\mathbf{w}}$ and $g = z^2 - x^4 y \in A_6^{\mathbf{w}}$. Then it is easy to verify that the pair (f,g) satisfies $(\mathrm{A})_{\mathbf{w}}$ in Theorem 2.2, and that the linear pencil $\mathscr{L}$ on $\mathbb{P}_{\mathbf{w}} = \mathbb{P}(1,2,3)$ spanned by $f^6 = x^6$ and g is of *Type* (I). As the corresponding $\mathbf{w}$-homogeneous locally nilpotent derivation Δ via the formula (2.1) in Remark 2.5, let us choose for instance:

$$\Delta(*) = -(z^2 - x^4 y)\,\frac{\partial(x, z^2 - x^4 y, *)}{\partial(\ \ x\ ,\ y\ ,\ z\ \ )} = (z^2 - x^4 y)\Big(2z\frac{\partial}{\partial y} + x^4\frac{\partial}{\partial z}\Big).$$

Then the co-action φ_Δ is written as follows:

$$\varphi_\Delta : \begin{cases} x \mapsto x \\ y \mapsto y + 2(z^2 - x^4 y)zt + (z^2 - x^4 y)^2 x^4 t^2 \\ z \mapsto z + (z^2 - x^4 y)x^4 t, \end{cases}$$

and we know in fact that φ_Δ produces a family of wild automorphisms on $\mathbb{A}^3$. Moreover, as far as $h(x,g)$ is $\mathbf{w}$-homogeneous and is taken from $\mathbb{C}[x,g]\backslash\mathbb{C}[x]$, it follows that the corresponding LND via (2.1) of Remark 2.5 yields wild automorphisms. More precisely to say, as seen in [Ku11, Theorem 5.1], for an LND Δ on $\mathbb{C}[x,y,z]$, which is not necessarily homogeneous, under the condition that $\mathrm{Ker}(\Delta)$ contains a tame coordinate of $\mathbb{C}[x,y,z]$, the co-action φ_Δ produces a family of tame automorphisms on $\mathbb{A}^3$ if and only if $\tau^{-1}\circ\Delta\circ\tau$ becomes triangular for a suitable tame automorphism τ of $\mathbb{C}[x,y,z]$. Note that this result due to Kuroda is applicable for *Type (I)* in consideration of Theorem 2.7, (5). We hope that the readers refer to the article [Ku11] by Kuroda, where he summarizes several criteria of wildness of automorphisms on $\mathbb{A}^3$ arising from various LND's on $\mathbb{C}[x,y,z]$ by the different notion. For *Type* (II) also, [Ku11] contains some useful criteria.

In the next §3, we shall prove at first Theorem 2.2 in terms of homogeneous $\mathbf{G}_a$-actions, and we see simultaneously Theorem 2.7 (1) and (2). According to the number of non-reduced members in $\mathscr{L}$ (cf. Theorem 2.7), we shall divide the property on $\mathscr{L}$ into three types *Type* (k) $(0 \leqq k \leqq 2)$ (see Remark 2.8). Certainly, this division is appropriate and natural from the viewpoint of geometry, however this is not so effective for the purpose to obtain a criterion of wildness. Indeed, for *Type* (0) only, we are able to state in an explicit manner as in Theorem 2.10 and we shall prove it in §3. For the remaining *Type* (*I*) and *Type* (*II*), the corresponding LND's can produce very often families of wild automorphisms on $\mathbb{A}^3$ (cf. [Ku11]). In consideration of this fact, we shall mention a problem concerning wild automorphisms on the affine space $\mathbb{A}^n$ for $n \geqq 4$ arising from homogeneous locally nilpotent derivations.

§ 3. Proofs of Theorems 2.2, 2.7 and 2.10

In this section, we shall prove Theorems 2.2 and 2.7 whose proofs proceed algebro-geometrically. Indeed, in our argument to confirm the implication $(\mathrm{A})_{\mathbf{w}} \Rightarrow (\mathrm{B})_{\mathbf{w}}$, the observation of the linear pencil $\mathscr{L}$ as in Theorem 2.7 is indispensable. We consider then the cases *Type* (*0*) (see Remark 2.8) especially and show Theorem 2.10.

§ 3.1.

We shall prepare the notation used in what follows of this section. Let $\mathbf{w} = (p,q,r)$ be a triple of pairwisely coprime positive integers such that $p \leqq q \leqq r$. Let $A = \mathbb{C}[x,y,z]$ be a polynomial ring in three variables x, y and z, and let us assign weights p, q, r to variables x, y, z, respectively. Let $A = \oplus_{n \geqq 0} A_n^{\mathbf{w}}$ be a decomposition of A into $\mathbf{w}$-homogeneous pieces as in Definition 2.1. According to this grading on A, we have a $\mathbf{G}_m$-action on $X := \mathrm{Spec}\,(\mathbb{C}[x,y,z])$:

$$t \cdot (x,y,z) = (t^p x, t^q y, t^r z), \quad t \in \mathbf{G}_m$$

with the quotient $\mathbb{P}_{\mathbf{w}} := \mathbb{P}(p,q,r)$. Since the closures of $\mathbf{G}_m$-orbits intersect each other at only the origin $o := (0,0,0) \in X$, we have the quotient morphism:

$$\pi : X \backslash \{o\} \longrightarrow \mathbb{P}_{\mathbf{w}}.$$

Though π can not be defined at o, after the weighted blow-up at o with respect to $\mathbf{w} = (p,q,r)$, say:

$$\sigma : \widetilde{X} \supseteq \widetilde{E} \longrightarrow o \in X,$$

the composite $\widetilde{\pi} := \pi \circ \sigma$ yields an $\mathbb{A}^1$-bundle over $\mathbb{P}_{\mathbf{w}}$, where $\widetilde{E} \cong \mathbb{P}_{\mathbf{w}}$ is the exceptional divisor of σ.

§ 3.2.

At first, we shall prove an implication $(\mathrm{B})_{\mathbf{w}} \Rightarrow (\mathrm{A})_{\mathbf{w}}$. Hence, let us suppose that $f, g \in A$ are $\mathbf{w}$-homogeneous such that the affine surface $U := \mathbb{P}_{\mathbf{w}} \backslash (\mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(f) \cup \mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(g))$ is isomorphic to $\mathbb{C}^* \times \mathbb{A}^1$, and we denote by $\rho : U \to Z := \mathbb{C}^*$ an $\mathbb{A}^1$-bundle, which is just the projection to the first factor. We often use the same notation Z to denote a section of ρ even if we do not mention it explicitly. Let us set $\widetilde{U} := \widetilde{\pi}^{-1}(U)$, and $\widetilde{Z} := \widetilde{\pi}^{-1}(Z)$.

Lemma 3.1. *$\widetilde{U} \cong U \times \mathbb{A}^1$ and $\widetilde{Z} \cong Z \times \mathbb{A}^1$, where $\mathbb{A}^1$ corresponds to the direction of fibers of $\widetilde{\pi}$. Furthermore, $\widetilde{U} \cap \widetilde{E} \cong U \times \{0\}$ and $\widetilde{Z} \cap \widetilde{E} \cong Z \times \{0\}$ under these isomorphisms.*

Proof. Note that $\mathrm{Pic}\,(U) = (0)$, which implies that $\widetilde{\pi}|_{\widetilde{U}} : \widetilde{U} \to U$ gives rise to a trivial $\mathbb{A}^1$-bundle. Thus the first assertion are obtained. Then the second assertion is obvious. $\square$

By Lemma 3.1, the image $\widehat{U} := \sigma(\widetilde{U})\backslash\{o\}$ of $\widetilde{U}$ with o deleted off has an $\mathbb{A}^1$-bundle structure over $\widehat{Z} := \sigma(\widetilde{Z})\backslash\{o\}$ arising from ρ, say:

$$\widehat{\rho} : \widehat{U} \longrightarrow \widehat{Z} \cong \mathbb{C}^* \times \mathbb{C}^*,$$

which is, in fact, trivial, i.e., $\widehat{U} \cong \widehat{Z} \times \mathbb{A}^1$ as $\mathrm{Pic}\,(\widehat{Z}) = (0)$. Let δ be a locally nilpotent derivation corresponding to $\widehat{\rho}$ on the coordinate ring $\mathbb{C}[\widehat{U}]$ of $\widehat{U}$.[5]

Lemma 3.2. *By multiplying δ with $(fg)^N$ for a suitable $N \geqq 0$, the resulting $(fg)^N\delta$ becomes a derivation on A.*

Proof. Since $U = \mathbb{P}_{\mathbf{w}}\backslash(\mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(f) \cup \mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(g))$, we have $\widehat{U} = X\backslash(\mathbb{V}_X(f) \cup \mathbb{V}_X(g))$, i.e., $\mathbb{C}[\widehat{U}] = A[f^{-1}, g^{-1}]$. By noting that A is finitely generated, it follows that $(fg)^N\delta \in \mathrm{Der}_{\mathbb{C}}\,(A)$ for a suitable $N \geqq 0$ (cf. [Mi84, Lemma, p.1471], [KPZ]). $\square$

Now we decompose $(fg)^N\delta$ into homogeneous parts with respect to $\mathbf{w}$ as follows:

$$(fg)^N\delta = \delta_d + \delta_{d+1} + \cdots + \delta_e, \quad \delta_j(A_n^{\mathbf{w}}) \subseteq A_{n+j}^{\mathbf{w}} \quad (\forall\, n \geqq 0,\, d \leqq j \leqq e), \quad \delta_d, \delta_e \neq 0.$$

Lemma 3.3. *δ_d and δ_e are locally nilpotent derivations on A such that* $\mathrm{Ker}\,(\delta_d) = \mathrm{Ker}\,(\delta_e) = \mathbb{C}[f, g]$.

Proof. Since f and g are units in $\mathbb{C}[\widehat{U}] = A[f^{-1}, g^{-1}]$, $f|_{\widehat{U}}$ and $g|_{\widehat{U}}$ do not vanish anywhere, in particular, they are constants along general orbits of a $\mathbf{G}_a$-action induced by δ. Thus f and g are contained in $\mathrm{Ker}\,(\delta) = \mathbb{C}[\widehat{Z}]$, thence the derivation $(fg)^N\delta$ as in Lemma 3.2 is in fact locally nilpotent. Then it is easy to confirm that δ_d and δ_e are also locally nilpotent derivations. On the other hand, as $f, g \in \mathrm{Ker}\,(\delta)$ and they are $\mathbf{w}$-homogeneous, we see that $\delta_d(f) = \delta_e(f) = \delta_d(g) = \delta_e(g) = 0$, so that $\mathbb{C}[f, g]$ is contained

[5]Note that such a δ is unique up to the multiplication of elements from the sub-algebra $\mathbb{C}[\widehat{Z}]$ of $\mathbb{C}[\widehat{U}]$, which is the coordinate ring of the base variety $\widehat{Z}$.

in $\mathrm{Ker}\,(\delta_d)$ and $\mathrm{Ker}\,(\delta_e)$. The remaining is to verify $\mathrm{Ker}\,(\delta_d) = \mathbb{C}[f,g] = \mathrm{Ker}\,(\delta_e)$. Since δ_d is a $\mathbf{w}$-homogeneous locally nilpotent derivation on $A = \mathbb{C}[x,y,z]$, its kernel $\mathrm{Ker}\,(\delta_d)$ is a polynomial ring in two variables generated by two $\mathbf{w}$-homogeneous polynomials, say F and G by [Mi85], [Zur]. If $\mathbb{C}[F,G]$ is strictly larger than $\mathbb{C}[f,g]$, then at least one of f and g is not linear with respect to F and G, say:

$$f = \sum_{\mu_i \deg_{\mathbf{w}}(F)+\nu_i \deg_{\mathbf{w}}(G)=\deg_{\mathbf{w}}(f)} a_i F^{\mu_i} G^{\nu_i}, \tag{3.1}$$

for some $a_i \in \mathbb{C}$. Since (3.1) is not linear and $\mathbf{w}$-homogeneous, it can be decomposed into several $\mathbf{w}$-homogeneous factors, in particular, f can not be irreducible. This is a contradiction. Hence, it follows that $\mathbb{C}[F,G] = \mathbb{C}[f,g]$ as desired. As for δ_e also, the argument is same. Thus we complete the proof. $\square$

Thus we obtain the direction $(\mathrm{B})_{\mathbf{w}} \Rightarrow (\mathrm{A})_{\mathbf{w}}$.

§ 3.3.

We shall prove the other direction $(\mathrm{A})_{\mathbf{w}} \Rightarrow (\mathrm{B})_{\mathbf{w}}$. Let $f \in A_a^{\mathbf{w}}$ and $g \in A_b^{\mathbf{w}}$ be $\mathbf{w}$-homogeneous irreducible polynomials of $A = \mathbb{C}[x,y,z]$ satisfying $(\mathrm{A})_{\mathbf{w}}$, i.e., $\mathrm{Ker}\,(\Delta) = \mathbb{C}[f,g]$ for a suitable $\Delta \in \mathrm{LND}^{\mathbf{w}}\,(A)$. We denote by

$$\nu : X \longrightarrow Z := \mathrm{Spec}\,(\mathbb{C}[f,g]) \cong \mathbb{A}^2$$

the morphism induced by the inclusion $A \supseteq \mathbb{C}[f,g]$. Let us put $c := \gcd\,(a,b)$ and write $a = ca'$, $b = cb'$.[6] Then we investigate the linear pencil $\mathscr{L}$ on $\mathbb{P}_{\mathbf{w}}$ generated by $f^{b'}$ and $g^{a'}$, namely:

$$\mathscr{L} := \left\{ B_{[\alpha:\beta]} \;\middle|\; [\alpha:\beta] \in \mathbb{P}(a,b) \right\} \subseteq \left| \mathscr{O}_{\mathbb{P}_{\mathbf{w}}}(a'b'c) \right|, \tag{3.2}$$

where $B_{[\alpha:\beta]}$ is the projective curve on $\mathbb{P}_{\mathbf{w}}$ defined by

$$(\alpha f)^{b'} + (\beta g)^{a'} = 0. \tag{3.3}$$

On the other hand, let $C_{[\alpha:\beta]}$ be the affine curve on Z defined by the same equation as (3.3), and let us denote by Λ the linear pencil on Z consisting of $C_{[\alpha:\beta]}$'s, where $[\alpha:\beta]$ ranges over $\mathbb{P}(a,b)$. Then we have the following:

Lemma 3.4.

(1) *Any member of* $\Lambda \backslash \{ C_{[1:0]}, C_{[0:1]} \}$ *is irreducible and reduced.*

[6] As remarked in Remark 2.3, it follows in fact that $c = 1$. But, in the argument of **3.3**, we do not assume that $c = 1$.

(2) $C_{[1:0]}$ *and* $C_{[0:1]}$ *are irreducible. (Even so, they are not necessarily reduced.)*

(3) *Any member of* $\nu^*\Lambda\backslash\{\,\nu^*C_{[1:0]}, \nu^*C_{[0:1]}\,\}$ *is irreducible and reduced.*

(4) $\nu^*C_{[1:0]}$ *and* $\nu^*C_{[0:1]}$ *are irreducible. (Even so, they are not necessarily reduced.)*

Proof. Note that Ker$\,(\Delta)$ is an inert sub-algebra of A (cf. [Mi78]), thence we have only to confirm (1) in order to obtain (3) also. Assume to the contrary that some member $C_{[\alpha:\beta]}$ except for $C_{[1:0]}$ and $C_{[0:1]}$ is not scheme-theoretically irreducible, which means that $(\alpha f)^{b'} + (\beta g)^{a'}$ can be decomposed in Ker$\,(\Delta) = \mathbb{C}[f, g]$ as follows:

$$(\alpha f)^{b'} + (\beta g)^{a'} = \prod_{k=1}^{N} P_k^{\mu_k}, \quad P_k \in \mathbb{C}\,[f, g],\, \mu_k \in \mathbb{N} \quad (1 \leqq k \leqq N),$$

where $N \geqq 2$ or $N = 1$ with $\mu_1 \geqq 2$ holds for some $[\alpha : \beta] \in \mathbb{P}(a, b)\backslash\{[1 : 0], [0 : 1]\}$. Note that in this decomposition each irreducible factor P_k, which is different from f and g, in the right hand side is $\mathbf{w}$-homogeneous, thus P_k is actually contained in $\mathbb{C}\,[f^{b'}, g^{a'}]$. Then we can see the contradiction by taking $\mathbf{w}$-degrees of the both sides into account. Thus we obtain (1). Since $f, g \in A$ are irreducible, the remaing assertions (2) and (4) are obvious. □

Before looking at Bs $\mathscr{L}$, we shall observe the linear pencil $\nu^*\Lambda$ on X. By the construction, we see:

$$\text{Bs}\ (\nu^*\Lambda) = \mathbb{V}_X(f) \cap \mathbb{V}_X(g) = \nu^{-1}(\overline{o}), \tag{3.4}$$

where $\overline{o}$ is the origin of the affine plane Z with a system of coordinates (f, g).

Lemma 3.5. dim Bs$\,(\nu^*\Lambda) = 1$ *and it is composed of the image by* $\sigma : \widetilde{X} \to X$ *of some fibers of* $\widetilde{\pi} : \widetilde{X} \to \mathbb{P}_{\mathbf{w}}$.

Proof. At first, we shall verify dim Bs$\,(\nu^*\Lambda) = 1$. Since A is factorial, it is easy to see that dim Bs$\,(\nu^*\Lambda) \leqq 1$. Furthermore, by noting that Bs$\,(\nu^*\Lambda) = \nu^{-1}(\overline{o})$ is defined by the ideal $(f, g) \subseteq A$ generated by two elements in A, it follows that dim Bs$\,(\nu^*\Lambda) = 1$ unless $\nu^{-1}(\overline{o})$ is empty. The fact that $\nu^{-1}(\overline{o})$ is non-empty follows from [Bo02], but we are able to see it by the elementary argument as follows. Indeed, assuming to the contrary that $\mathbb{V}_X(f) \cap \mathbb{V}_X(g) = \emptyset$, this implies that $\mathbb{V}_X(g)$ is contained in a fiber of the polynomial map defined by f:

$$f :\, X\, \ni (s, t, u) \mapsto f(s, t, u)\, \in\, \mathbb{A}^1 = \text{Spec}\,(\mathbb{C}\,[f]).$$

Thus $f = gh + \gamma$ for suitable $h \in A$ and $\gamma \in \mathbb{C}$. If necessary, by interchanging the role of f and g, we may and shall assume that deg$\,(f) \leqq$ deg$\,(g)$, where deg$\,(\cdot)$ means the ordinary degree. The comparison of degree says that h must be a constant. Then $\mathbb{C}[f,g]$ can not be a polynomial ring of dimension 2, which is a contradiction. Thus $\dim \nu^{-1}(\overline{o}) = 1$ as desired. Since $\nu^{-1}(\overline{o})$ is invariant under the $\mathbf{G}_m$-action on X, the second assertion is then obtained. □

By Lemma 3.5, we have $\mathrm{Bs}\,(\nu^*\Lambda) = \cup_{j=1}^{s} l_j$, where $l_j = \sigma_*(\widetilde{\pi}^{-1}(P_j))$ and P_j is a point on $\mathbb{P}_{\mathbf{w}}$ $(1 \leqq j \leqq s)$. Then it is not difficult to see that $\mathrm{Bs}\,\mathscr{L} = \{ P_1, \cdots, P_s \}$. In fact, we have furthermore the following result:

Lemma 3.6.

(1) $s = 1$, *i.e.,* $\mathrm{Bs}\,\mathscr{L}$ *consists of only one point, say* $\mathrm{Bs}\,\mathscr{L} = \{ P \}$.

(2) $B_{[\alpha:\beta]} \backslash \{ P \} \cong \mathbb{A}^1$ *for general* $[\alpha : \beta] \in \mathbb{P}(a,b)$.

Proof. Let us put $S_{[\alpha:\beta]} := \nu^* C_{[\alpha:\beta]} \backslash (\cup_{j=1}^{s} l_j)$. As Δ is a $\mathbf{w}$-homogeneous locally nilpotent derivation, there exists $q \in \mathrm{Ker}\,(\Delta) = \mathbb{C}[f,g]$, which is $\mathbf{w}$-homogeneous, such that $A[q^{-1}] = \mathbb{C}[f,g,q^{-1}][u]$, where $u \in \mathrm{Frac}\,(A)$ is algebraically independent over $\mathbb{C}(f,g)$ (cf. [Mi78]). Since $q \in \mathbb{C}[f,g]$ is $\mathbf{w}$-homogeneous, we know that $\mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(q)$ is composed of several members of Λ. Hence $\nu|_{S_{[\alpha:\beta]}} : S_{[\alpha:\beta]} \to C_{[\alpha:\beta]} \backslash \{\overline{o}\}$ yields an $\mathbb{A}^1$-bundle structure for a general $[\alpha : \beta] \in \mathbb{P}(a,b)$. Meanwhile $S_{[\alpha:\beta]}$ is invariant under the $\mathbf{G}_m$-action on X and its quotient coincides with $B_{[\alpha:\beta]} \backslash \{ P_1, \cdots, P_s \}$. Thus $S_{[\alpha:\beta]}$ has two morphisms:

$$B_{[\alpha:\beta]} \backslash \{ P_1, \cdots, P_s \} \xleftarrow{\pi} S_{[\alpha:\beta]} \xrightarrow{\nu} C_{[\alpha:\beta]} \backslash \{\overline{o}\} \tag{3.5}$$

depending on a $\mathbf{G}_m$-action and a $\mathbf{G}_a$-action on it. The fiber of ν in (3.5) is isomorphic to $\mathbb{A}^1$ which is mapped to a curve in $B_{[\alpha:\beta]} \backslash \{ P_1, \cdots, P_s \}$. This is possible only if $B_{[\alpha:\beta]} \backslash \{ P_1, \cdots, P_s \}$ is a rational curve with only one-place at infinity, therefore it follows that $s = 1$, i.e., $\mathrm{Bs}\,\mathscr{L}$ is composed of only one point. Thus we confirm the first assertion. The second one is then easy to see by Bertini's Theorem. □

In particular, $\mathscr{L}$ is a linear pencil on $\mathbb{P}_{\mathbf{w}}$ consisting of rational curves with only one base point $\mathrm{Bs}\,\mathscr{L} = \{P\}$. Moreover, we have the following:

Lemma 3.7.

(1) *Any member of* $\mathscr{L} \backslash \{B_{[1:0]}, B_{[0:1]}\}$ *is irreducible and reduced.*

(2) *If a member of* $\mathscr{L}$ *is irreducible and reduced, then it does not contain the point from* $\mathrm{Sing}\,(\mathbb{P}_{\mathbf{w}}) \backslash \{P\}$.

(3) $B_{[\alpha:\beta]} \backslash \{P\} \cong \mathbb{A}^1$ *if* $B_{[\alpha:\beta]}$ *is irreducible and reduced.*

(4) *If* $B_{[\alpha:\beta]}$ *is not reduced,*[7] *then it contains at most one point from* $\mathrm{Sing}\,(\mathbb{P}_{\mathbf{w}})\backslash\{P\}$.

Proof. By taking Lemma 3.4 into account, any member $B_{[\alpha:\beta]}$ of $\mathscr{L}$ except for $B_{[1:0]}$ and $B_{[0:1]}$ is irreducible and reduced, moreover, $B_{[1:0]}$ and $B_{[0:1]}$ are irreducible (but not necessarily reduced). Thus we see (1). Now let $\mu : V \to \mathbb{P}_{\mathbf{w}}$ be the composition of the resolution of $\mathrm{Sing}\,(\mathbb{P}_{\mathbf{w}})$ and the shortest succession of blow-ups at $\mathrm{Bs}\,\mathscr{L} = \{P\}$ including infinitely near points such that the proper transform $\widetilde{\mathscr{L}}$ on V of $\mathscr{L}$ is free of base points. Then $\widetilde{\mathscr{L}}$ yields a $\mathbb{P}^1$-fibration $\Phi_{\widetilde{\mathscr{L}}} : V \to \mathbb{P}(a,b) = \mathbb{P}(\widetilde{\mathscr{L}}) = \mathbb{P}(\mathscr{L}) \cong \mathbb{P}^1$ and there exists exactly one component, say E, contained in $\mu^{-1}(P) \subseteq V$ which is projected onto $\mathbb{P}^1$ via $\Phi_{\widetilde{\mathscr{L}}}$. Furthermore, E is a section of $\Phi_{\widetilde{\mathscr{L}}}$ by virtue of Lemma 3.6, (2). Note that any exceptional component in $\mathrm{Exc}(\mu)$ other than E has a self-intersection number less than or equal to -2. Let

$$\widetilde{B_{[\alpha:\beta]}} + \sum a_i E_i, \quad E_i \subseteq \mathrm{Exc}(\mu) \tag{3.6}$$

be the member of $\widetilde{\mathscr{L}}$ corresponding to $B_{[\alpha:\beta]}$, where $\widetilde{B_{[\alpha:\beta]}}$ is the proper transform on V of $B_{[\alpha:\beta]}$. If $B_{[\alpha:\beta]}$ is irreducible and reduced (this is the case if $[\alpha:\beta] \neq [1:0], [0:1]$), then the multiplicity in the fiber (3.6) of the component $(\widetilde{B_{[\alpha:\beta]}})_{\mathrm{red}}$ is equal to 1. Hence by supposing that the fiber (3.6) is not irreducible, there must exist a component from $\{E_i\}$ whose self-intersection number is -1 (cf. [Mi78, 2.2. Lemma, p.115]). This is absurd. Thus $\{E_i\} = \emptyset$, which implies that $B_{[\alpha:\beta]}$ does not pass through any point in $\mathrm{Sing}\,(\mathbb{P}_{\mathbf{w}})\backslash\{P\}$ and $B_{[\alpha:\beta]}\backslash\{P\} \cong \widetilde{B_{[\alpha:\beta]}}\backslash(\widetilde{B_{[\alpha:\beta]}} \cap E) \cong \mathbb{A}^1$ as desired in (2) and (3). As for (4), assume to the contrary that for example $B_{[1:0]}$ passes at least two points in $\mathrm{Sing}\,(\mathbb{P}_{\mathbf{w}})\backslash\{P\}$, say P_1, P_2, the corresponding member of $\widetilde{\mathscr{L}}$ is written as follows:

$$\widetilde{B_{[1:0]}} + \sum a_i E_i^{(1)} + \sum b_j E_j^{(2)} + \sum c_k F_k, \tag{3.7}$$

$$\cup E_i^{(1)} \subseteq \mu^{-1}(P_1), \quad \cup E_j^{(2)} \subseteq \mu^{-1}(P_2), \quad \cup F_k \subseteq \mu^{-1}\Big(\big(\mathrm{Sing}\,(\mathbb{P}_{\mathbf{w}}) \cup \{P\}\big)\backslash\{P_1, P_2\}\Big).$$

By assumption, $\{E_i^{(1)}\} \neq \emptyset$ and $\{E_j^{(2)}\} \neq \emptyset$. Furthermore, note that $\{F_k\} \neq \emptyset$. In fact, otherwise $\widetilde{B_{[1:0]}}$ meets the section E of $\Phi_{\widetilde{\mathscr{L}}}$, hence the multiplicity of this component in the fiber (3.7) must be equal to 1. Then there must exist a (-1)-curve in the fiber (3.7) except for $\widetilde{B_{[1:0]}}$, which is absurd (cf. [Mi78, 2.2. Lemma]). Thus $\{F_k\} \neq \emptyset$. As $\widetilde{B_{[1:0]}}$ must be a unique (-1)-curve in the fiber (3.7), the contraction of it yields anew a fiber of a $\mathbb{P}^1$-fibration in which the proper transforms of components $E_i^{(1)}$, $E_j^{(2)}$ and F_k meeting $\widetilde{B_{[1:0]}}$ intersect each other in a common point, which is a contradiction bacause of the fact that the different three components can not meet in a common point in a fiber of a $\mathbb{P}^1$-fibration (cf. [Mi78, 2.2. Lemma]). Thus we complete the proof. □

[7] Note that this can occur only for $[\alpha:\beta] = [1:0]$ or $[0:1]$ by (1).

When we restrict the rational map $\Phi_{\mathscr{L}} : \mathbb{P}_{\mathbf{w}} \dashrightarrow \mathbb{P}^1$ defined by $\mathscr{L}$ to the affine surface $S := \mathbb{P}_{\mathbf{w}} \backslash (B_{[1:0]} \cup B_{[0:1]})$, this gives rise to an $\mathbb{A}^1$-bundle:

$$\varphi := \Phi_{\mathscr{L}}|_S : S \longrightarrow \mathbb{P}(a, b) \backslash \{ [1:0], [0:1] \} \cong \mathbb{C}^*,$$

by Lemma 3.7. Because of $\mathrm{Pic}(\mathbb{C}^*) = 0$, it follows that φ is in fact trivial to deduce $S \cong \mathbb{C}^* \times \mathbb{A}^1$. Thus we see the direction $(\mathrm{A})_{\mathbf{w}} \Rightarrow (\mathrm{B})_{\mathbf{w}}$. Hence we complete the proof for Theorem 2.2. Moreover, we obtain the assertions (1), (2) and (3) in Theorem 2.7 also.

§ 3.4.

As verified in Lemma 3.7, all members contained in $\mathscr{L} \backslash \{ B_{[1:0]}, B_{[0:1]} \}$ are irreducible and reduced, in other words, $B_{[1:0]}$ and $B_{[0:1]}$ are the only possibilities as members of $\mathscr{L}$ to be non-reduced. As already defined in Remark 2.8, we shall divide $\mathscr{L}$ into three types *Type* (k) $(0 \leqq k \leqq 2)$ according to the number k of non-reduced members in $\mathscr{L}$.

3.4.1. *Type (0)* is not difficult to analyse. Thus let us suppose for a while that all members of $\mathscr{L}$ are irreducible and reduced. Then, in fact, the possibility of the weight is fairly restricted as follows:

Lemma 3.8. *The weight $\mathbf{w} = (p, q, r)$ satisfies $p = q = 1$.*

Proof. By the same reason as in the proof of Lemma 3.7, any member of $\mathscr{L}$ does not contain points from $\mathrm{Sing}(\mathbb{P}_{\mathbf{w}}) \backslash \{P\}$, where we recall that P is the base point of $\mathscr{L}$ (see Lemma 3.7). This implies that $\mathrm{Sing}(\mathbb{P}_{\mathbf{w}}) \subseteq \{P\}$. Therefore we may and shall assume that the weight $\mathbf{w} = (p, q, r)$ satisfies $p = q = 1$ as desired. □

Lemma 3.9. *After an application of a suitable linear automorphism on $\mathbb{A}^3$, f and g are transformed to coordinates x and y.*

Proof. In the case of $r = 1$, namely, the case of $\mathbb{P}_{\mathbf{w}} = \mathbb{P}^2$, it is easy to see that $\mathscr{L}$ is composed of lines passing through P, hence the assertion is obvious to see. In what follows, we consider the case of $r \geqq 2$. Since every member of the pencil $\mathscr{L}$ spanned by $f^{b'}$ and $g^{a'}$ is irreducible and reduced, it must be that $a' = b' = 1$, i.e., $a = b$, where $a = \deg_{\mathbf{w}}(f)$ and $b = \deg_{\mathbf{w}}(g)$. We shall show in fact that $a = b = 1$. Taking the argument to prove Lemma 3.7 into account, the minimal resolution $\mu : \Sigma_r \cong V \to \mathbb{P}_{\mathbf{w}}$ of the singularity on $\mathbb{P}_{\mathbf{w}} = \mathbb{P}(1, 1, r)$, which is just the weighted blow-up at $P = [0:0:1]$ with respect to the weight $\frac{1}{r}(1, 1)$, coincides with the resolution of $\mathrm{Bs}\,\mathscr{L} = \{P\}$. Thus the proper transform $\mu_*^{-1}\mathscr{L}$ consists of rulings on $V \cong \Sigma_r$. Therefore we see that $\mathscr{L}$ is spanned by x and y to deduce that $a = b = 1$. Hence f and g are transformed to x and y by making use of a linear automorphism on $\mathbb{A}^3$ as desired. $\square$

Let us consider a locally nilpotent derivation Δ corresponding to $\mathscr{L}$ via the formula (2.1) in Remark 2.5. In order to show the fact that φ_Δ yields a family of tame automorphisms on $\mathbb{A}^3$, we may assume that $f = x$ and $g = y$ since f and g are brought to x and y by an appropriate *linear* automorphism by virtue of Lemma 3.9. Then Δ is of the form $\Delta = h(x, y)\frac{\partial}{\partial z}$, where $h(x, y)$ is a homogeneous polynomial of x and y in a usual sense. Thus it is easy to confirm that φ_Δ gives rise to tame automorphisms on $\mathbb{A}^3$ actually. Summarizing arguments performed above, we obtain Theorem 2.7, (4) and Theorem 2.10.

3.4.2. We shall here consider the case of *Type* (I) for the assertion Theorem 2.7, (5), namely, $\mathscr{L}$ contains only one non-reduced member $B_{[1:0]} = b'\mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(f)$ with $b' \geqq 2$, and all other members of $\mathscr{L}$ are scheme-theoretically irreducible. Indeed, we have the following:

Lemma 3.10.

(1) $\#\mathrm{Sing}\,(\mathbb{P}_{\mathbf{w}}) \leqq 2$.

(2) $\mathrm{Sing}\,(\mathbb{P}_{\mathbf{w}}) \subseteq B_{[1:0]}$.

(3) *In case of* $\#\mathrm{Sing}\,(\mathbb{P}_{\mathbf{w}}) = 2$, *one of singularities must coincide with* P *(recall that* $\mathrm{Bs}\,\mathscr{L} = \{P\}$*), and the other is located on* $B_{[1:0]}$.

Proof. All the assertions can be confirmed by virtue of Lemma 3.7. $\square$

In particular, by Lemma 3.10, we may assume that $p = 1$. Since any member $B_{[\alpha:\beta]} \in \mathscr{L}$ except for $B_{[1:0]} = b'\mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(f)$ satisfies $B_{[\alpha:\beta]}\backslash\{P\} \cong \mathbb{A}^1$, it follows that $\mathbb{P}_{\mathbf{w}}\backslash B_{[1:0]} \cong \mathbb{A}^2$. Concerning the $\mathbf{w}$-degree $a = \deg_{\mathbf{w}}(f)$ of $f \in A_a^{\mathbf{w}}$, we can see in fact the following:

Lemma 3.11. $a = 1$.

Proof. Let us consider the divisor $D := \mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(x) \subseteq \mathbb{P}_{\mathbf{w}}$. If $\deg_{\mathbf{w}}(f) = a \geqq 2$, then $D \neq \mathrm{Supp}\,(B_{[1:0]})$. In consideration of the fact $\mathbb{P}_{\mathbf{w}}\backslash\mathrm{Supp}\,(B_{[1:0]}) \cong \mathbb{A}^2$, the restriction $D\backslash(D \cap B_{[1:0]})$ is a principal divisor. Hence there exists a suitable rational function $h \in \mathbb{C}(\mathbb{P}_{\mathbf{w}})$ such that $\mathrm{div}_{\mathbb{P}_{\mathbf{w}}}(h) = D - n(B_{[1:0]})_{\mathrm{red}}$ for some $n \geqq 1$. This is absurd to see $a = 1$, which completes the proof. $\square$

Thus we finish the proof of Theorem 2.7.

§ 4. Problems for polynomial rings of dimension $\geqq 4$

In this section, we shall review the situation in dimension 2 and that in dimension 3 just investigated in §3 about generators of kernels of weighted homogeneous locally nilpotent derivations. This observation seems to be something in order that we would be able to list the candidates for wild automorphisms on the affine spaces $\mathbb{A}^n$ with $n \geqq 4$.

§ 4.1.

Let us begin with the review in the case of dimension 2, so let Δ be an LND on $\mathbb{C}[x,y]$, which is homogeneous with respect to $\mathbf{w} = (p,q)$ with $\gcd(p,q) = 1$. Then $\mathrm{Ker}(\Delta)$ is generated by a single $\mathbf{w}$-homogeneous polynomial (cf. [Ren68]), say $\mathrm{Ker}(\Delta) = \mathbb{C}[f]$ such that the inclusion $\mathrm{Ker}(\Delta) \subseteq \mathbb{C}[x,y]$ gives rise to the morphism:

$$\rho : \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[x,y]) \longrightarrow \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[f]), \quad (a,b) \mapsto f(a,b),$$

which is in fact an $\mathbb{A}^1$-bundle (cf. [Mi78]). Therefore this generator f of $\mathrm{Ker}(\Delta)$ must be a coordinate of $\mathbb{C}[x,y]$. Furthermore, Δ is described as

$$\Delta = cf^N\Big(f_x\frac{\partial}{\partial y} - f_y\frac{\partial}{\partial x}\Big),$$

for some $N \geqq 0$ and $c \in \mathbb{C}^*$, and it yields a family of tame automorphisms on $\mathbb{A}^2$. Moreover it follows that $\mathbb{P}_{\mathbf{w}} \backslash \mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(f) \cong \mathbb{P}^1 \backslash \{1\,pt\} \cong \mathbb{A}^1$.

§ 4.2.

Again let us look at the case of dimension 3 (see §3). Assuming that Δ is a $\mathbf{w}$-homogeneous locally nilpotent derivation on $A = \mathbb{C}[x,y,z]$ with respect to a given weight $\mathbf{w} = (p,q,r)$, the kernel $\mathrm{Ker}(\Delta)$ is generated by two $\mathbf{w}$-homogeneous polynomials $f, g \in A$ satisfying $\mathbb{P}_{\mathbf{w}} \backslash (\mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(f) \cup \mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(g)) \cong \mathbb{C}^* \times \mathbb{A}^1$ (cf. Theorem 2.2). The $\mathbb{A}^1$-fibration corresponding to the inclusion $\mathrm{Ker}(\Delta) \subseteq A$:

$$\rho : \mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x,y,z]) \longrightarrow \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[f,g]), \quad (a,b,c) \mapsto (f(a,b,c), g(a,b,c))$$

is not necessarily an $\mathbb{A}^1$-bundle. With the notation in §3, in the case of *Type* (*0*), the polynomials f and g are transformed to x and y by an application of a suitable linear automorphism on $\mathbb{A}^3$, in particular, they are coordinates on $\mathbb{A}^3$ to see that ρ is an $\mathbb{A}^1$-bundle (see Lemma 3.9). As a consequence, we see that Δ in this case gives rise to a family of tame automorphisms (cf. Theorem 2.10). Meanwhile, in the case of *Type* (*I*) or *Type* (*II*), it can be that f or g is not a coordinate of A, which implies that ρ is no longer an $\mathbb{A}^1$-bundle. For instance, the linear pencil $\mathscr{L}$ on $\mathbb{P}^2$ generated by $f^2 = x^2$ and $g = xz - y^2$ satisfies $(\mathrm{B})_{\mathbf{w}}$ with $\mathbf{w} = (1,1,1)$ and it is of *Type* (*I*). Then the fiber

$\rho^{-1}\big((\alpha,\beta)\big)$ of ρ is isomorphic to $\mathbb{A}^1$ scheme-theoretically unless $\alpha = 0$. On the other hand, $\rho^{-1}\big((0,\beta)\big)$ is composed of two disjoint $\mathbb{A}^1$'s unless $\beta = 0$, and $\rho^{-1}\big((0,0)\big)$ is a double $\mathbb{A}^1$. Then we are able to construct, in fact, a family of wild automorphisms on $\mathbb{A}^3$ by making use of a suitable LND corresponding to $\mathscr{L}$ (cf. [Ku11]).

§ 4.3.

Taking the observation just above into account, we shall think of a reasonable way to construct the candidate of a family of wild automorphisms on $\mathbb{A}^n$ for $n \geqq 4$. For instance, let us look at the action of $\mathbf{G}_a$ on $\mathbb{A}^4 = \operatorname{Spec}(\mathbb{C}[x,y,z,w])$ via the following $\delta \in \mathrm{LND}^{\mathbf{w}}(\mathbb{C}[x,y,z,w])$, where $\mathbf{w} = (1,1,1,a)$ ($a \in \mathbb{N}$) is a weight attached to a system of coordinates (x,y,z,w) on $\mathbb{A}^4$:

$$\Delta = (xz - y^2)\Big(x\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z}\Big).$$

Then the co-action is described as follows:

$$\varphi_\Delta : \begin{cases} x \mapsto & x \\ y \mapsto & y + x\,(xz - y^2)\,t \\ z \mapsto & z + 2y\,(xz-y^2)\,t + x\,(xz-y^2)^2\,t^2 \\ w \mapsto & w, \end{cases}$$

which is a straightforward extension of the family of Nagata automorphisms in dimension 4. Note that $\operatorname{Ker}(\Delta) \subseteq \mathbb{C}[x,y,z,w]$ is generated by x, $xz - y^2$ and w, in particular, two of three generators of $\operatorname{Ker}(\Delta)$ are variables of $\mathbb{C}[x,y,z,w]$. Moreover, it follows that $\mathbb{P}_{\mathbf{w}} \backslash (\mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(x) \cup \mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(xz - y^2) \cup \mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(w)) \cong \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{A}^1$. It is known in fact that φ_Δ induces a family of tame automorphisms on $\mathbb{A}^4$. In consideration of this example and the situation of dimension $\leqq 3$ explained just above, it seems to be reasonable to propose the following problem:

Problem 4.1. Let $\mathbf{w} = (w_1, \cdots, w_n)$ be a weight attached to the variables of the polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$ such that w_i's are pairwise coprime. Suppose that there exist $(n-1)$-polynomials $f_1, \cdots, f_{n-1} \in \mathbb{C}[x_1, \cdots, x_n]$, which are $\mathbf{w}$-homogeneous, satisfying the following conditions:

(i) $\mathbb{P}_{\mathbf{w}} \backslash \big(\mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(f_1) \cup \cdots \cup \mathbb{V}_{\mathbb{P}_{\mathbf{w}}}(f_{n-1})\big) \cong (\mathbb{C}^*)^{n-2} \times \mathbb{A}^1$, and

(ii) At least $(n-2)$'s of $f_1, \cdots, f_{n-1}$ are not variables of $\mathbb{C}[x_1, \cdots, x_n]$.

Then is $\Delta \in \operatorname{Der}(\mathbb{C}[x_1, \cdots, x_n])$ of the form:

$$\Delta(*) = h\,\frac{\partial(f_1, \cdots, f_{n-1}, *)}{\partial(x_1, \cdots, x_{n-1}, x_n)}$$

locally nilpotent giving rise to a family of wild automorphisms on $\mathbb{A}^n$ by choosing $h \in \mathbb{C}[f_1, \cdots, f_{n-1}]$, which is $\mathbf{w}$-homogeneous, adequately ?[8]

Example 4.2. In the case of $n = 4$, let us for example consider the following polynomials f_i's ($1 \leqq i \leqq 3$) in $\mathbb{C}[x, y, z, w]$:

$$f_1 := xw^4 + 2yw^2(xz - y^2) + z(xz - y^2)^2, \; f_2 := yw^2 + z(xz - y^2), \; f_3 := w.$$

It is then verified that f_1, f_2 and f_3 satisfy the conditions (i) and (ii) in Problem 4.1 with $\mathbf{w} = (1, 1, 1, 1)$. The corresponding $\mathbf{w}$-homogeneous derivation Δ is of the form:

$$\Delta\,(*) := h\,\frac{\partial\,(f_1, f_2, f_3, *)}{\partial\,(\,x,\; y,\; z,\; w)}, \tag{4.1}$$

where h is $\mathbf{w}$-homogeneous contained in $\mathbb{C}[f_1, f_2, f_3]\backslash\{0\}$. Note that the morphism corresponding to the inclusion $\mathbb{C}[x, y, z, w] \supseteq \mathbb{C}[f_1, f_2, f_3]$, say

$$\nu : X := \operatorname{Spec}(\mathbb{C}[x, y, z, w]) \longrightarrow Z := \operatorname{Spec}(\mathbb{C}[f_1, f_2, f_3]) \cong \mathbb{A}^3, \tag{4.2}$$

$$(a, b, c, d) \mapsto (f_1(a, b, c, d), f_2(a, b, c, d), f_3(a, b, c, d))$$

yields us an $\mathbb{A}^1$-fibration. Indeed, the scheme-theoretic fiber $\nu^{-1}\big((\alpha, \beta, \gamma)\big)$ over the point $(\alpha, \beta, \gamma) \in Z$ with $\gamma \neq 0$ is isomorphic to the curve in $\mathbb{A}^3\,(x, y, z)$ defined by:

$$x + 2y\Big(\frac{xz - y^2}{\gamma^4}\Big) + z\Big(\frac{xz - y^2}{\gamma^4}\Big)^2 - \alpha = y + z\Big(\frac{xz - y^2}{\gamma^4}\Big) - \beta = 0,$$

which is isomorphic to $\mathbb{A}^1$ as we can construct an automorphism on $\mathbb{A}^3$ transforming x and y to $x + 2y\Big(\frac{xz-y^2}{\gamma^4}\Big) + z\Big(\frac{xz-y^2}{\gamma^4}\Big)^2$ and $y + z\Big(\frac{xz-y^2}{\gamma^4}\Big)$, respectively, by imitating the construction of the Nagata automorphism as in Remark 2.5. Thus the restriction of ν onto $U := X\backslash\mathbb{V}_X(w)$ gives rise to an $\mathbb{A}^1$-bundle over $U_0 := Z\backslash\mathbb{V}_Z(w) \cong \mathbb{C}^* \times \mathbb{A}^1$, which is actually trivial, i.e., $U \cong U_0 \times \mathbb{A}^1$ as $\operatorname{Pic}(U_0) = 0$. By the same fashion as in Lemma 3.2, an LND on $\mathbb{C}[U] = \mathbb{C}[x, y, z, w, w^{-1}]$ corresponding to this $\mathbb{A}^1$-bundle $\nu|_U$ can be extended to that on $\mathbb{C}[x, y, z, w]$, say Δ', after multiplying w^N by choosing $N \geqq 0$ appropriately. Since the difference between Δ and Δ' as derivations on $\mathbb{C}[U]$ is then a multiplication of an element from $\mathbb{C}[f_1, f_2, f_3 = w, w^{-1}]$, it is not difficult to see that Δ is, in fact, locally nilpotent. By choosing $h \in \mathbb{C}[f_1, f_2, f_3]\backslash\{0\}$ in the formula (4.1) suitably, for instance $h = f_1$, we expect furthermore that the co-action φ_Δ produces a family of wild automorphisms on $\mathbb{A}^4$.

[8] If we ignore the condition that w_i's are pairwise coprime, then we have an example where the derivation of the form as in Problem 4.1 is never locally nilpotent even satisfying (i) and (ii). For instance, the pair $(z, x^3 + y^2)$ in Remark 2.3 in the case of $\mathbf{w} = (2, 3, 6)$ becomes such an example.

References

[AM75] S. Abhyankar and T.T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. 276 (1975), 148–166.

[AvdE00] A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics Vol.190, Birkhäuser, Basel, Boston, Berlin, 2000.

[Ba84] H. Bass, *A nontriangular action of* $\mathbf{G}_a$ *on* $\mathbb{A}^3$, J. Pure Appl. Algebra 33 (1984), 1–5.

[Bo02] P. Bonnet, *Surjectivity of quotient maps for algebraic* $(\mathbb{C},+)$*-actions and polynomial maps with contractible fibers*, Transformation Groups 7 (2002), 3–14.

[Da97] D. Daigle, *On some properties of locally nilpotent derivations*, J. Pure Appl. Algebra 114 (1997), 221–230.

[Da98] D. Daigle, *Homogeneous locally nilpotent derivations of* $k[x,y,z]$, J. Pure Appl. Algebra 128 (1998), 109–132.

[Da00] D. Daigle, *On kernels of homogeneous locally nilpotent derivations of* $k[x,y,z]$, Osaka J. Math. 37 (2000), 689–699.

[Fr98] G. Freudenburg, *Actions of* $\mathbf{G}_a$ *on* $\mathbb{A}^3$ *defined by homogeneous derivations*, J. Pure Appl. Algebra 126 (1998), 169–181.

[Fr06] G. Freudenburg, *Algebraic theory of locally nilpotent derivations*, Encyclopaedia Math. Sci. Vol.136, Springer, Berlin, 2006.

[IF] A.R. Iano-Fletcher, *Working with weighted complete intersections*, In: Explicit Birational Geometry of 3-folds (edited by A. Corti and M. Reid), Cambridge University Press, 2000, pp. 101–173.

[Ju42] H. Jung, *Über ganze birationale transformationen der Ebene*, J. Reine Angew. Math. 184 (1942), 161–174.

[KM78] T. Kambayashi and M. Miyanishi, *On flat fibrations by the affine line*, Illinois J. Math. 22 (1978), 662–671.

[KPZ] T. Kishimoto, Yu. Prokhorov, M. Zaidenberg, *Group actions on affine cones*, arXiv:0905.4647 (to appear).

[Ku11] S. Kuroda, *Wildness of polynomial automorphisms: Applications of the Shestakov-Umirbaev theory and its generalization*, preprint. (to appear)

[Mi78] M. Miyanishi, *Lectures on curves on rational and unirational surfaces*, Springer-Verlag (Berlin, Heidelberg, New York), 1978, Published for Tata Inst. Fund. Res., Bombay.

[Mi84] M. Miyanishi, *An algebro-topological characterization of the affine space of dimension three*, Amer. J. Math. 106 (1984), 1469–1486.

[Mi85] M. Miyanishi, *Normal affine subalgebras of a polynomial ring*, Algebraic and Topological Theories – To the memory of Dr. Takehiko MIYATA (Tokyo), Kinokuniya, 1985, pp. 37–51.

[MS81] M. Miyanishi and T. Sugie, *On a projective plane curve whose complement has logarithmic Kodaira dimension* $-\infty$, Osaka J. Math. 18 (1981), 1–11.

[Na72] M. Nagata, *On Automorphism Group of* $k[x,y]$, Lectures in Mathematics, Department of Mathematics, Kyoto University, Vol. 5, Kinokuniya Book-Store Co. Ltd., Tokyo, 1972.

[Na78] M. Nagata, *Polynomial rings and affine spaces*, Regional Conference Series in Math. Vol.37, AMS, Providence, RI, 1978.

[Ren68] R. Rentschler, *Opérations du groupe additif sur le plan affine*, C. R. Acad. Sci. Paris Sér. A-B, 267 (1968), 384–387.

[SU04$_a$] I.P. Shestakov and U.U. Umirbaev, *Poisson brackets and two-generated subalgebras of rings of polynomials*, J. Amer. Math. Soc. 17 (2004), 181–196.

[SU04$_b$] I.P. Shestakov and U.U. Umirbaev, *The tame and the wild automorphisms of polynomial rings in three variables*, J. Amer. Math. Soc. 17 (2004), 197–227.

[Su74] M. Suzuki, *Propriétés topologiques des polynômes de deux variables complexes et automorphismes algébriques de l'espace* $\mathbb{C}^2$, J. Math. Soc. Japan 26 (1974), 241–257.

[Yo79] H. Yoshihara, *On plane rational curves*, Proc. Japan Acad. (Ser. A) 55 (1979), 152–155.

[Zur] V.D. Zurkowski, *Locally finite derivations*, preprint.