Ramification of local fields and Fontaine's property (P$_m$) : a resume (Algebraic Number Theory and Related Topics 2009)

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Ramification of local fields and Fontaine’s property (\(P_m\)): a résumé

By

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Abstract

This is a résumé of our results ([7], [9]) on Fontaine’s property (\(P\)) which is an effective tool for estimating the ramification of torsion Galois representations.

\section{Fontaine’s property (\(P_m\))}

Let \(K\) be a complete discrete valuation field with perfect residue field \(k\) of characteristic \(p > 0\), \(K^{\text{alg}}\) a fixed algebraic closure of \(K\), \(K\) the separable closure of \(K\) in \(K^{\text{alg}}\) and \(v_K\) the valuation on \(K^{\text{alg}}\) normalized by \(v_K(K^\times) = \mathbb{Z}\). We denote by \(G_K\) the absolute Galois group of \(K\). Let \(G^{(m)}_K\) be the \(m\)th upper numbering ramification group in the sense of [3]. Namely, we put \(G^{(m)}_K = G^{m-1}_K\), where the latter is the upper numbering ramification group defined in [6].

One of the classical problems in ramification theory is to obtain a ramification bound of torsion geometric Galois representations. Assume \(\text{char}(K) = 0\) for the moment. Let \(e\) be the absolute ramification index of \(K\). Consider a proper smooth variety \(X_K\) over \(K\) and put \(X_{\bar{K}} = X_K \times_K \bar{K}\). In [3], Fontaine conjectured the upper numbering ramification group \(G^{(m)}_K\) acts trivially on the \(r\)th étale cohomology group \(V = H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})\) for \(m > e(n + r/(p-1))\) if \(X_K\) has good reduction. This is equivalent to the inequality \(u_{L/K} \leq e(n + r/(p-1))\), where \(L\) is defined by \(G_L = \text{Ker}(G_K \to \text{Aut}(V))\) and \(u_{L/K}\) is the infimum of the real numbers \(m\) such that \(G^{(m)}_K \subset G_L\). For \(e = 1\) and \(r < p - 1\), this conjecture was proved independently by himself ([4], for \(n = 1\)) and Abrashkin ([1], for any \(n\)). There are also similar ramification bounds if \(X_K\) has semi-stable reduction by

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Caruso-Liu ([2]) and Hattori ([5]). All of them used Fontaine’s property \( (P_m) \) studied in [3] to obtain each ramification bound.

Now, let us go back to the general case where \( K \) is of either characteristic. For an algebraic extension \( E/K \), we denote by \( \mathcal{O}_E \) the integral closure of \( \mathcal{O}_K \) in \( E \). Fontaine’s property \( (P_m) \) is the following condition for a finite Galois extension \( L \) of \( K \) and a real number \( m \):

\[
(P_m) \quad \text{For any algebraic extension } E/K, \text{ if there exists an } \mathcal{O}_K\text{-algebra homomorphism } \mathcal{O}_L \to \mathcal{O}_E/a_{E/K}^m, \text{ then there exists a } K\text{-embedding } L \hookrightarrow E,
\]

where \( a_{E/K}^m = \{ x \in \mathcal{O}_E \mid v_K(x) \geq m \} \). We define the greatest upper numbering ramification break \( u_{L/K} \) as above. Fontaine proved the following:

**Proposition 1.1** ([3], Prop. 1.5). Let \( L \) be a finite Galois extension of \( K \), \( m \) a real number and \( e_{L/K} \) the ramification index of \( L/K \). Then there are following relations:

(i) If we have \( m > u_{L/K} \), then \( (P_m) \) is true.

(ii) If \( (P_m) \) is true, then we have \( m > u_{L/K} - e_{L/K}^{-1} \).

**Remark.** The above (i) can be generalized to the imperfect residue field case by Abbes-Saito’s ramification theory ([9], Prop. 4.3).

Given a torsion Galois representation \( V \) of \( G_K \), we can use the above proposition to bound its ramification as follows: Let \( L/K \) be the finite Galois extension defined by \( G_L = \text{Ker}(G_K \to \text{Aut}(V)) \). If \( V \) is of some geometric origin, it is often possible to verify \( (P_m) \) for \( L/K \) and a suitable \( m \). Then the above inequality of (ii) gives the upper bound \( u_{L/K} < m + e_{L/K}^{-1} \). Thus it will be useful to sharpen the inequality. Indeed, our first main theorem below shows that we can improve the bound to \( u_{L/K} \leq m \). This result is actually used in [5], Section 5, Proposition 5.6.

For a finite Galois extension \( L \) of \( K \), we put

\[
m_{L/K} = \inf \{ m \in \mathbb{R} \mid (P_m) \text{ is true for } L/K \}
\]

By the above proposition, we have the inequalities

\[
u_{L/K} - e_{L/K}^{-1} \leq m_{L/K} \leq u_{L/K}.
\]

More precisely, we have the following equality:

**Theorem 1.2** ([9], Prop. 3.3). We have \( u_{L/K} = m_{L/K} \).

An outline of the proof: We can prove easily that \( (P_m) \) is not true if \( m = u_{L/K} \) (hence \( u_{L/K} = m_{L/K} \)) in the case where \( L/K \) is at most tamely ramified. Hence we may assume \( L/K \) is wildly ramified. It suffices to show the inequality \( u_{L/K} - (e')^{-1} \leq m_{L/K} \) with an
arbitrarily large integer $e'$. Take an arbitrary finite tamely ramified Galois extension $K'$ of $K$ and put $L' = LK'$. Then we have $u_{L'/K} = u_{L/K}$ since $L/K$ is wildly ramified. If we apply (ii) of Proposition 1.1 to $L'/K$, and (i) of Proposition 1.1 to $K'/K$ respectively, then we can prove the inequality $u_{L/K} - e_{L/K}^{-1} \leq m_{L/K}$.

§ 2. ($P_m$) at the ramification break

Let $L$ be a finite Galois extension of $K$ and $e$ its ramification index. Assume $L/K$ to be totally and wildly ramified for simplicity. In this section, we completely determine the truth of ($P_m$)\(^1\). The equality $u_{L/K} = m_{L/K}$ in Theorem 1.2 gives no information about the truth of ($P_m$) at the break $m = u_{L/K}$. The behavior of ($P_m$) at the break depends on the residue field:

**Theorem 2.1** ([7], Thm. 1.1). The property ($P_m$) is true for $m = u_{L/K}$ if and only if the residue field $k$ has no Galois extension whose degree is divisible by $p$.

This theorem can be proved by using the local class field theory of Serre and Hazewinkel. On the other hand, we consider a weaker property ($P_m^e$) as follows:

($P_m^e$) For any totally ramified extension $E/K$ of degree $e$, if there exists an $O_K$-algebra homomorphism $O_L \to O_E/a_{E/K}^m$, then there exists a $K$-embedding $L \hookrightarrow E$.

Then we have the following theorem, which is a similar result as Theorem 2.1. The proof employs the notion of a non-Archimedean metric on the set of all Eisenstein polynomials over $K$.

**Theorem 2.2** ([8], Thm. A, Prop. 5.1). The property ($P_m^e$) is true for $m = u_{L/K}$ if and only if the residue field $k$ has no Galois extension of degree $p$.

Remark. Both (i) and (ii) of Proposition 1.1 remain true for a finite totally and wildly ramified Galois extension $L$ of $K$ if we consider ($P_m^e$) instead of ($P_m$). However, ($P_m^e$) does not satisfy the equality in Proposition 1.2 in general. In fact, we can check that ($P_m^e$) is true for $m = u_{L/K}$ if and only if ($P_m^e$) is true for $m > u_{L/K} - e_{L/K}^{-1}$.

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\(^1\)The results in this section were obtained by joint work with Takashi Suzuki after the talk in “Algebraic Number Theory and Related Topics 2009”.

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