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Takagi's Class Field Theory
– From where? and to where? –

By

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§ 1. Introduction

After the publication of his doctoral thesis [T-1903] in 1903, Teiji Takagi (1875–1960) had not published any academic papers until 1914 when World War I started. In the year he began his own investigation on class field theory. The reason was to stay in the front line of mathematics still "after the cessation of scientific exchange between Japan and Europe owing to World War I"; see [T-1942, Appendix I Reminiscences and Perspectives, pp.195–196] or the quotation below from the English translation [T-1973, p.360] by Iyanaga.

The last important scientific message he received at the time from Europe should be Fueter's paper [Fu-1914] which contained a remarkable result on *Kronecker's Jugendtraum* (Kronecker's dream in his young days):

Kronecker's Jugendtraum The roots of an Abelian equation over an imaginary quadratic field k are contained in an extension field of k generated by the singular moduli of elliptic functions with complex multiplication in k and values of such elliptic functions at division points of their periods.

(See Subsections 3.2 and 3.3 in Section 3.)

K. Fueter treated Abelian extensions of k of odd degrees.

Theorem 1.1 (Fueter). *Every abelian extension of an imaginary quadratic number field k with an odd degree is contained in an extension of k generated by suitable roots of unity and the singular moduli of elliptic functions with complex multiplication in k.*

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(See Subsection 6.2 in Section 6.)

It should be noted that Takagi proved Kronecker's Jugendtraum for the Gaussian field $\mathbb{Q}(\sqrt{-1})$ in his doctoral thesis. (See the next section.)

From 1897 to 98, H. Weber published a series of papers [Wb-1897] to introduce his concepts of congruence ideal class groups and congruence class fields, and proved the following result:

Theorem 1.2 (Weber). *Let k be an imaginary quadratic field, and φ an elliptic function with complex multiplication in k. Then the singular modulus of φ and the values of φ at division points of its periods by an integral ideal $\mathfrak{m}$ of k generate an Abelian extension of k whose Galois group is isomorphic to the congruence ideal class group modulo $\mathfrak{m}$ of k.*

(See Subsection 5.1 in Section 5.)

Hence to examine Kronecker's Jugendtraum for k well farther, one should seek some good criteria for an Abelian extension of k to be a congruence class field of k.

In 1897, apart from Weber's works, D. Hilbert published a big scale report [Hi-1897] (frequently quoted as 'the Bericht') on the theory of algebraic numbers. It contained a systematic treatment of Kummer extensions. In its §58, we see the first idea of his class fields. The title is

§58. Der Fundamentalsatz von den relativ-zyklischen Körpern mit der Relativdifferente 1. Die Bezeichnung dieser Körper als Klassenkörper. (The fundamental theorem of relative-cyclic extensions with relative difference 1. The characterization of these fields as class fields.)

Then he developed his theory of relative quadratic number fields in [Hi-1898, Hi-1899b], and announced a refined concept of his class fields by [Hi-1899a] in 1899. (See Subsection 5.2 below.) These works of Hilbert should have been very influential at least over Takagi who wrote the following comment on his class field theory in the preface of his textbook [T-1948]:

> The theory of Kummer extensions of the Bericht was, as Hilbert states, the most advanced achievement at the time, although we have moved far ahead now. It was Hilbert himself who set up in flames the general theory of Abelian extensions with his penetrating insight into the theory of Kummer extensions. His special class field theory was the first step of his program. Far beyond any perspectives at the time, however, the general class field theory is established and states that every Abelian extension is a class field. The theory of Abelian extensions has now come to the end of its first stage. (English translation from

Miyake [Mi-2007, p.162–163])

From the words of Takagi, "Far beyond any perspectives at the time, however," it seems probable that Takagi started his own investigation without any full scale overview on his class field theory. Instead, he would have tried to find a good criterion for a cyclic extension of odd prime degree over an algebraic number field to be a congruence class field. In [T-1942, Appendix I Reminiscences and Perspectives, pp.195–196] he stated the following comment:

> Hilbert considered only unramified class fields. From the standpoint of the theory of algebraic functions which are defined by Riemann surfaces, it is natural to limit considerations to unramified cases. I do not know precisely whether Hilbert himself stuck to this constraint, but anyway what he had written induced me to think so. However, after the cessation of scientific exchange between Japan and Europe owing to World War I, [I started my own investigation,] ... was freed from that idea and [soon found, ...] ..., that every Abelian extension might be a class field, if the latter is not limited to the unramified case. I thought at first this could not be true. Were this to be false, the idea should contain some error. I tried my best to find this error. At that period, I almost suffered from a nervous breakdown. I dreamt often that I had resolved the question. I woke up and tried to recover the reasoning, but in vain. I made my utmost effort to find a counterexamp le .... (English translation is from [T-1973, p.360] with [supplemented translation] from the original Japanese text added by the author.)

In the next section, we review Iwasawa's article [T-1973, Appendix 1] or [Iw-1990] on Takagi's main works [T-1903, T-1920a] and [T-1922].

In Section 3, we pick up two keywords 'reciprocity law' and 'Kronecker's Jugendtraum'; each of them will be discussed in Subsections 3.1 and 3.2; in the former, we also review the process, how the theory of algebraic numbers has been formulated; in the latter, Kronecker's Jugendtraum will be discussed in connection with complex multiplication of elliptic functions; in the last Susection 3.3, we prepared a mathematical review on complex multiplication, especially to show its effect on arithmetic of imaginary quadratic number field, for the convenience of readers who are not much familiar with it.

Then a brief note on analytic method created by Dirichlet will be commented in Section 4.

In Section 5, the independent contributions of Weber and Hilbert on concepts of class fields are investigated in two subsections. Then the works of Ph. Furtwängler and K. Fueter will be reviewed in the succeeding Section 6.

Section 7 is a short comment on Takagi's works in the historical context.

The main theme of Section 8 is the general reciprocity law of E. Artin. Subsection 8.1 is for Tschebotareff's Density Theorem; it is very important in its content, and further the method of the proof gave an essential effect on the proof of Artin's general reciprocity law. Then follows Subsection 8.2 where Artin's general reciprocity law is discussed. It reduced the Principal Ideal Theorem to a proposition on the transfer of a metabelian group to its commutator subgroup; and Furtwängler proved it in [Fw-1930]. These are discussed in the last Subsection 8.3 of this section.

In the last Section 9 of this article, some relevant topics will be discussed. Subsection 9.1 contains comments on local class field theory and algebraic proof of class field theory. The second and the third subsections are devoted to 'idele' and 'idele version' of the main theorem of class field theory.

The final Subsection 9.4 is a glimpse of non-Abelian world, especially on p-extensions of algebraic number fields, the gate of which was widely opened by Shafarevich and Iwasawa. To close this paper, we add the most recent and brilliant results of M. Ozaki in [Oz-2009] on the Galois groups of the maximal unramified p-extensions of algebraic number fields. He showed that an arbitrary finite p-group is realized as the Galois group of the maximal *unramified* p-extension of an algebraic number field, and the similar proposition of pro-p-groups with countably many generators (Theorems 9.3 and 9.4 in Subsection 9.4).

§ 2. From Iwasawa's article on Takagi's major works

Kenkichi Iwasawa reviewed Takagi's works [T-1903, T-1920a, T-1922] in a clear way in his article, 'On papers of Takagi in number theory' [Iw-1990] (or [T-1973, 2nd edit., Appendix I, pp.342–351]).

In its Section 1, Iwasawa gives a brief description of Takagi's thesis [T-1903], Über die im Bereiche der rationalen komplexen Zahlen Abelscher Zahlkörper. (On Abelian number fields over the rational complex number field.)

> In a paper [22] [= [Kr-1853] in our bibliography] of 1853, Kronecker announced that every abelian extension of the rational field $\mathbb{Q}$ is a subfield of a cyclotomic field. He also stated that all abelian extensions of the quadratic field $\mathbb{Q}(\sqrt{-1})$ can be obtained similarly by dividing the lemniscate instead of the circle. This is the origin of what is now called Kronecker's Jugendtraum, namely,

his conjecture that all abelian extensions of an imaginary quadratic field k can be generated by the singular values of the elliptic modular function $j(u)$ and the division values of elliptic functions which have complex multiplication in k. In this thesis **6** [= [T-1903] in our bibliography], Takagi proved Kronecker's statement on $\mathbb{Q}(\sqrt{-1})$.

And Iwasawa explains the method of Takagi. Here we extract the technical essence to cast a glance at the idea.

Since the class number of this specific quadratic number field is equal to 1, Takagi could use the method of Hilbert to prove Kronecker-Weber Theorem (p.135 in Subsection 3.2). Takagi constructed two extensions $L(p^h, \mu)$ and $M(p^h)$ of the quadratic field for every power of a prime number p by evaluating Jacobi's elliptic function $sn(u)$ and Weierstrass' function $\wp(u) = su(u)^{-2}$; $L(p^h, \mu)$ is cyclic of degree p^h and ramifies only at the prime element μ with $\mathrm{N}(\mu) \equiv 1 \mod p^h$ or $\mod 2^{h+2}$ if $p = 2$; $M(p^h)$ is an abelian extension of type $p^h \times p^h$ and ramifies only at the prime factors of p. Then Takagi generalized the method by which Hilbert proved Kronecker-Weber Theorem in [Hi-1896] utilizing so-called Hilbert Theory of [Hi-1894]. (Here Hilbert used values of exponential function at division points of the period $2\pi\sqrt{-1}$: $\zeta_{p^h} = e^{2\pi\sqrt{-1}/p^h}$.)

In Section 2, Iwasawa reviewed Weber's works [Wb-1897] on congruence ideal groups and class fields.

Let k be an arbitrary algebraic number field k of finite degree, $\mathfrak{m}$ an integral ideal in k and $I_\mathfrak{m}$ the group of those ideals of k which are relatively prime to $\mathfrak{m}$. Weber defined a congruence ideal group $H_\mathfrak{m}$ mod $\mathfrak{m}$ as a subgroup of $I_\mathfrak{m}$ which contains the Strahl subgroup $S_\mathfrak{m}$ mod $\mathfrak{m}$,

$$S_\mathfrak{m} := \{(\xi) \mid \xi \in k, \xi \equiv 1 \mod \mathfrak{m}\}$$

where (ξ) is the principal ideal generated by $\xi \in K$; hence we have a series of abelian groups

$$S_\mathfrak{m} \subseteq H_\mathfrak{m} \subseteq I_\mathfrak{m};$$

the indices of the subgroups of $I_\mathfrak{m}$ are finite; the quotient group $I_\mathfrak{m}/H_\mathfrak{m}$ is called a congruence ideal class group. Suppose that such a group $I_\mathfrak{m}/H_\mathfrak{m}$ is given. Weber called a finite extension K of k a class field over k for $I_\mathfrak{m}/H_\mathfrak{m}$ if a prime ideal $\mathfrak{p}$ of absolute degree 1 in $I_\mathfrak{m}$ is completely decomposed in K exactly when $\mathfrak{p}$ belongs to the subgroup $H_\mathfrak{m}$. Following the idea of Dirichlet, Weber then proved by using analytic properties of L-series that

$$[I_\mathfrak{m} : H_\mathfrak{m}] \leq [K : k]$$

for such a class field K.

Iwasawa further states, "Although Weber did not prove the existence of a class field K over k for a given congruence ideal class group $I_{\mathfrak{m}}/H_{\mathfrak{m}}$, he showed that if such K exists, it is unique for $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ and that the existence of K implies the existence of infinitely many prime ideals of absolute degree 1 in each coset of the factor group $I_{\mathfrak{m}}/H_{\mathfrak{m}}$, a generalization of the classical theorem of Dirichlet on prime numbers in an arithmetic progression." (On Weber's works, also see Subsection 5.1 below.)

In Section 3, Takagi's paper [T-1920a], Über eine Theorie des relativ Abel'schen Zahlkörpers (On a theory of relative Abelian number fields), on his class field theory is discussed.

> In this paper, Takagi started with a new definition of class fields as follows. Let K be a finite Galois extension of a number field k with degree $[K : k] = n$. For each integral divisor $\mathfrak{m}$ of k, the extension K/k defines a congruence ideal class group $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ in k, where $H_{\mathfrak{m}}$ is the subgroup of $I_{\mathfrak{m}}$ generated by $S_{\mathfrak{m}}$ and by all norms $N_{K/k}(\mathfrak{A})$ of ideals $\mathfrak{A}$ of K, prime to $\mathfrak{m}$. $\cdots$ In fact, there exists an integral divisor $\mathfrak{f}$ such that $\cdots$ [if $\mathfrak{m}$ is a multiple of $\mathfrak{f}$, then $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ is cannonically identified with $I_{\mathfrak{f}}/H_{\mathfrak{f}}$]. $\cdots$ The group $C_{K,k}$ $(= C\ [= I_{\mathfrak{f}}/H_{\mathfrak{f}}])$ is called the ideal class group of k, associated with the Galois extension K/k, and the integral divisor $\mathfrak{f}$ is called the conductor of K/k. Let h be the order of $C_{K,k}$: $h = [C : 1]$. Then we obtain the so-called second fundamental inequality of class field theory:
>
> $$h \leq n = [K : k].$$
>
> Now, Takagi called the Galois extension K of k a class field over k when the equality
>
> $$h = n$$
>
> holds for the h and n above. If $\mathfrak{m}$ is an integral divisor of k such that $C_{K,k} = I_{\mathfrak{m}}/H_{\mathfrak{m}}$, the above equality means $[I_{\mathfrak{m}} : H_{\mathfrak{m}}] = [K : k]$, and K is then called a class field over k for the ideal class group $I_{\mathfrak{m}}/H_{\mathfrak{m}}$.

Then Iwasawa gave a list of the fundamental results of Takagi [T-1920a].

Theorem 2.1. *A finite Galois extension K of a number field k is a class field over k if and only if K/k is an Abelian extension.*

Theorem 2.2 (Existence Theorem). *For every congruence ideal group $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ of k, there exists a class field K over k for $I_{\mathfrak{m}}/H_{\mathfrak{m}}$.*

As a consequence of the existence, analytic method implies that each coset of $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ of k contains infinitely many prime ideals of k with absolute degree 1. This is a generalization of Dirichlet's Prime Number Theorem for arithmetic progressions.

Theorem 2.3 (Uniqueness Theorem). *Let K and K' be class fields over k and let $C_{K,k} = I_{\mathfrak{m}}/H_{\mathfrak{m}}$ and $C_{K',k} = I_{\mathfrak{m}}/H'_{\mathfrak{m}}$ for a divisor $\mathfrak{m}$ which is divisible by both of the conductors of K/k and K'/k. Then $k \subseteq K' \subseteq K$ if and only if $H_{\mathfrak{m}} \subseteq H'_{\mathfrak{m}} \subseteq I_{\mathfrak{m}}$. In particular, the class field K is unique for the given ideal class group $I_{\mathfrak{m}}/H_{\mathfrak{m}}$.*

Theorem 2.4 (Isomorphism Theorem). *The Galois group of a class field K/k is isomorphic to the ideal class group $C_{K,k}$ associated with K/k.*

Note that this Isomorphism Theorem follows from the first three theorems and the Fundamental Theorem of Abelian Groups; indeed, if K/k is a cyclic extension then the ideal class group $C_{K,k}$ has to be cyclic because K/k can not contain any (l,l)-type subfields over k for prime l. Hence the equality $h = n$ implies Isomorphism Theorem for the cyclic case. The canonical isomorphism of the two groups of the theorem will be given by Artin's General Reciprocity Law (cf. Subsection 8.2).

Theorem 2.5 (Conductor Theorem). *A prime divisor of k is ramified in a class field K/k if and only if it divides the conductor $\mathfrak{f}$ of K/k.*

Theorem 2.6 (Decomposition Theorem). *In a class field K/k, the relative degree of an unramified prime ideal $\mathfrak{p}$ of k is equal to the order of the class of $\mathfrak{p}$ in the ideal class group $C_{K,k} = I_{\mathfrak{f}}/H_{\mathfrak{f}}$.*

Iwasawa states further,

> In Takagi's proof of those theorems in **13** [= [T-1920a] in our bibliography] the key steps were the proof of the following two statements:
>
> a) Let l be an odd prime and let K be a cyclic extension of degree l over k with discriminant $\mathfrak{d} = \mathfrak{f}^{l-1}$. Then K is a class field over k and its conductor is a factor of the ideal $\mathfrak{f}$ of k.
>
> b) Suppose that the ground field k contains a primitive l-th root of unity, l being an odd prime as above. Then, for each congruence ideal class group $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ in k with order l, there exists a cyclic extension K of degree l over k such that K is a class field for the given $I_{\mathfrak{m}}/H_{\mathfrak{m}}$.
>
> The proof of a) was carried out by computing (in modern terminology) the orders of the cohomology groups of the cyclic Galois group of K/k, acting on various abelian groups such as the unit group and the ideal class group of K. The computation gave the first fundamental inequality of class field theory,
>
> $$h \geq n,$$

for the extension K/k, and hence the equality $h = n$. In proving $h \geq n$, Takagi also obtained the Norm Theorem for K/k which states that an element α of k is the norm of an element of K if and only if α is a norm for every local extension associated with K/k. For the proof of b), Takagi fixed an integral divisor $\mathfrak{m}$ of k and counted the number N of congruence ideal class group $I_{\mathfrak{m}}/H_{\mathfrak{m}}$ in k with order $[I_{\mathfrak{m}} : H_{\mathfrak{m}}] = l$. On the other hand, using a) and the theory of Kummer extensions of Hilbert [15] [= [Hi-1897] in our bibliography], he showed that there exist at least N class fields K of degree l over k with conductor dividing $\mathfrak{m}$. This of course proved b). At the same time, the argument also yielded that the ideal $\mathfrak{f}$ is actually the conductor of the extension K/k in a).

In Section 4, Iwasawa explains the last chapter of [T-1920a] which is devoted to the proof of Kronecker's Jugendtraum.

In Section 5, he gave a comment of Takagi's another major paper [T-1922], Über das Reciprocitätsgesetz in einem beliebigen algebraischen Zahlkörper. (On the reciprocity law in an arbitrary algebraic number field.) In this paper, Takagi discussed the reciprocity laws of the power residue symbol and the norm residue symbol following Hilbert and Furtwängler; (see Subsections 5.2 and 6.1); he could much simplified Furtwängler's arguments by using his class field theory, although he handled only the case of a prime exponent l.

§ 3. Two key words

As we saw in the previous section, there were two key words behind the tide towards Takagi's class field theory; namely,

(1) reciprocity law, and

(2) Kronecker's Jugendtraum.

The latter more directly affected class field theory than the former. The former, however, produced the main stream which created the theory of algebraic numbers and grew it up to Takagi-Artin class field theory.

§ 3.1. Reciprocity Law and Theory of Algebraic Numbers

The quadratic reciprocity law seems to be independently found by three mathematicians: L. Euler [Eu-1744], A.-M. Legendre [Le-1785] and C. F. Gauss [Ga-1801].

It was, however, Gauss who was most influential. The work [Eu-1744] of Euler was published in 1744, but was not well known until it was pointed out by Kronecker in [Kr-1875] much later in 1875.

The main theme of the book *Disquisitiones Arithmeticae* [Ga-1801] of Gauss is actually the reciprocity law. In [F-1994] Günther Frei presented his opinion that even the cyclotomy of Chapter 7 was prepared as an introduction to the planned but unpublished Chapter [Ga-1801*] for the theory of finite fields which should have been prepared for investigation into higher power residue reciprocity law. (This article [Ga-1801*] was posthumously published in 1863; hence E. Galois did not see it when he wrote his article [Gl-1846] on finite fields in 1846.) Twenty seven years had passed when Gauss wrote his first paper [Ga-1828] on the theory of biquadric residue in 1828; the second paper [Ga-1832] appeared four more years later. In these papers, he introduced a new concept of integers of form $m + n\sqrt{-1}, (m, n \in \mathbb{Z})$, showed the Fundamental Theorem of Arithmetic for them, and his formulation of the biquadratic reciproci ty law. He did not, however, published any proofs to it.

Frei stated in [F-1994], "Jacobi was the first to give a statement of the Cubic Reciprocity Law in print, $\cdots$ (see [[Ja-1827] in our bibliography])."

Proofs of cubic and biquadratic reciprocity laws were published in 1844 by Ferdinand Gotthold Eisenstein in [Ei-1844a], [Ei-1844c] and in [Ei-1844d]. He also showed the Fundamental Theorem of Arithmetic for the integers generated by a cubic root of unity $(-1 + \sqrt{-3})/2$ in [Ei-1844b].

Then Ernst Eduard Kummer followed the line with a long series of papers [Ku-1846, Ku-1845, Ku-1847, Ku-1852, Ku-1857, Ku-1856, Ku-1858, Ku-1859a, Ku-1859b] published in the period 1845–59, and could finally show the l-th power residue reciprocity law in the l-th cyclotomic field for odd prime l in 1859. For the purpose, first he had to establish the Fundamental Theorem of Arithmetic in the ring $\mathbb{Z}[\zeta_l]$ of cyclotomic integers generated by the l-th root of unity ζ_l. There do not exist, however, sufficiently many 'prime elements' in the ring if $l \geq 5$. Hence he had to introduce 'ideal divisors', and developed his full scale theory on them by [Ku-1847] in 1847. (A gap in this paper was filled in the paper [Ku-1856] of himself nine years later in 1856.)

The next stage is the establishment of general theory of algebraic numbers. It may be worth noting that only one of two quadratic number fields $\mathbb{Q}(\sqrt{l})$ and $\mathbb{Q}(\sqrt{-l})$ is contained in the cyclotomic field $\mathbb{Q}(\zeta_l)$ for an odd prime number l; indeed, we have $\mathbb{Q}(\sqrt{l^*}) \subset \mathbb{Q}(\zeta_l)$ where l^* is one of $\pm l$ which satisfies $l^* \equiv 1 \mod 4$. It was well understood at the time, furthermore, that arithmetic in both of these quadratic number fields were closely related to that of binary quadratic forms with discriminants l^* and $-l^*$.

P. G. Lejeune Dirichlet presented his unit theorem of algebraic number fields in his paper [Di-1846] in 1846 when Kummer was ready to publish his theory of ideal divisors for cyclotomic integers by [Ku-1847]. Dirichlet actually published his results on units earlier in 1841 as [Di-1841]; however, the contents appeared in the garment of arithmetic of norm forms.

It was Richard Dedekind and Leopold Kronecker who followed Kummer to develop general theories of algebraic numbers. Dedekind was very successful with his concepts of number fields (Körper), modules (Module), rings of algebraic integers (Ordnung), and ideals (Ideale) which were developed in the series of appendices [De-1871, De-1879, De-1893] to the 2nd, 3rd and 4th editions of Dirichlet's textbook *Vorlesungen über Zahlentheorie* (Lectures on Number Theory). Kronecker formulated his divisor theory around 1860, but published it much later in 1882 ([Kr-1882]). His method utilized forms of many variables which seems much influenced by Gauss' second proof of the Fundamental Theorem of Algebra in [Ga-1815]. (Cf. A. N. Kolmogorov and A. P. Yushkevich (edit.) [KY-1992, Ch. II].)

Note that E. I. Zolotareff also established a divisor theory in a general algebraic number fields in his paper [Zo-1880] under a quite different motivation. (Cf. *ibid.*)

In 1897, Hilbert wrote his report [Hi-1897] on algebraic numbers. This became a standard text for the theory of algebraic numbers. Then in the next year, he reported his results on relative quadratic fields by [Hi-1898] though its title was 'Über die Theorie der relativ-Abelschen Zahlkörper' (On the theory of relative Abelian number fields), and gave, in 1899, the formulation of his unramified class field theory in [Hi-1899a] whose title was 'Über die Theorie der relativquadratischen Zahlkörper' (On the theory of relative quadratic number field), however it may oddly sound. His theory of relative quadratic number fields was fully demonstrated in [Hi-1899b]; here he introduced *quadratic norm residue symbol* and showed its reciprocity law, proved quadratic residue reciprocity law, and showed the existence of his class field when the class number of the base field was 2 and 4.

Philipp Furtwängler followed Hilbert's track to show the existence of the class fields of Hilbert and proved higher power residue reciprocity law for powers of a prime. His works will be reviewed in Subsection 6.1 below.

§ 3.2. Complex multiplication of elliptic functions and Kronecker's Jugendtraum

Here we start again with Gauss' *Disquisitiones Arithmeticae* [Ga-1801]. At the beginning of Chapter 7 on cyclotomy, he added the following phrase:

> The principles of the theory which we are going to explain actually extend much farther than we will indicate. For they can be applied not only to circular functions but just as well to other transcendental functions, e.g. to those which depend on the integral $\int [1/\sqrt{(1-x^4)}]\,dx$ and also to various types of congruences. Since, however, we are preparing a substantial work on transcendental functions and since we will treat congruences at length as we continue our discussion of arithmetic, we have decided to consider only circular functions here. And although we could discuss them in all their generality, we reduce them to the simplest case in the following article, both for the sake of brevity and in order that the new principles of this theory may be more easily understood. (From English translation of [Ga-1801] by Arthur A. Clarke, S.J.)

Then in 1827, Niels Henrik Abel established his theory of elliptic functions by the paper [Ab-1827], Recherches sur les fonctions elliptiques (Researchs on elliptic functions).

In the last Section X, Abel showed a basic result on 'complex multiplication' (named by Kronecker [Kr-1857a]) with explicit examples. Also see his succeeding paper [Ab-1828] on the transformation of elliptic functions.

Abel was also interested in algebraic equations which were algebraically solvable; he analyzed the solvability by radicals and found his criterion ([Ab-1826, Ab-1829]) in the essential case. This is the reason why we call a commutative group an Abelian group.

Abel used the term 'fonctions elliptiques' for elliptic integrals after Legendre; the latter investigated them in the world of real numbers $\mathbb{R}$; but Abel did in the wide world of complex numbers $\mathbb{C}$ from the beginning, and considered their inverse functions to find their double periodicity. It was Carl Gustav Jacobi who introduced the term 'elliptic integrals' and called the inverse functions of the complex elliptic integrals 'elliptic functions'; see [Ja-1829a, Ja-1829b]. He dared propose the change of the term 'elliptic function' which had been used since Legendre's paper [Le-1793] of 1793.[1]

Kronecker was attracted by Abel's works, and in the article [Kr-1853], he formulated

Theorem 3.1 (Kronecker-Weber Theorem). *Every abelian extension of the rational field $\mathbb{Q}$ is a subfield of a cyclotomic field.*

He also stated that all abelian extensions of the quadratic field $\mathbb{Q}(\sqrt{-1})$ can be obtained similarly by dividing the lemniscate instead of circle, and mathematically formulated

[1]M. Takase noticed this change of terminology and pointed out the correspondence between Jacobi and Legendre in 1829 on the matter; cf. Takase's article in Japanese in *Sûgaku Seminar*, Oct., (2002), 37–43, Nihon Hyôronsha, Tokyo.

his Jugendtraum (dream in his young days) in [Kr-1857a, Kr-1857b, Kr-1862].

Kronecker's Jugendtraum. All abelian extensions of an imaginary quadratic field k can be generated by the singular values of the elliptic modular function $j(\tau)$ and the division values of elliptic functions which have complex multiplication in k.

He used the term 'Jugendtraum' much later in 1880 in his letter [Kr-1880b] to Dedekind. (He originally put 'singular moduli' in place of the singular values of the elliptic modular function $j(\tau)$ of the statement. Here we give this mathematically refined form.)

It should be noted that Kronecker also suggested

Theorem 3.2 (Principal Ideal Theorem for imaginary quadratic number fields). *Every ideal of an imaginary quadratic field k is represented by a number in the field generated by the singular values of the elliptic modular function $j(\tau)$ corresponding to elliptic functions which have complex multiplication in k.*

§ 3.3. Mathematical Review on Complex Multiplication

(a) Elliptic Functions

An elliptic function $\varphi(z)$ is a meromorphic function on the whole complex plane $\mathbb{C}$ which has a pair of independent periods ω_1 and ω_2; here 'independent' means that the two lines $\mathbb{R}\omega_1$ and $\mathbb{R}\omega_2$ are different. We choose the indices so that the imaginary part of the quotient ω_1/ω_2 is positive. Put $\Omega := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$; this is a free $\mathbb{Z}$-module of rank 2; let us call it the module of periods of φ. Then for $\omega \in \Omega$, we have $\varphi(z+\omega) = \varphi(z)$. Hence φ may be considered as a meromorphic function on the complex torus $\mathbb{C}/\Omega$; this is a quotient Abelian group of the additive group $\mathbb{C}$ by its discrete subgroup Ω.

All of those elliptic functions (including constant functions) whose module of periods contains Ω form an algebraic function field K_Ω of one variable over $\mathbb{C}$; we call it the elliptic function field with the module of periods Ω. Suppose that a $\mathbb{Z}$-module Ω of rank 2 in $\mathbb{C}$ is given; then the Weierstrass $\wp$-function $\wp(z) = \wp(\Omega; z)$ and its derivative $\wp'(z) = \wp'(\Omega; z)$ are given by the series

$$\wp(\Omega; z) = \frac{1}{z^2} + \sum_{\omega\in\Omega\setminus\{0\}} \Big\{\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big\}, \qquad \wp'(\Omega; z) = -2\sum_{\omega\in\Omega} \frac{1}{(z-\omega)^3};$$

these two series define meromorphic functions on $\mathbb{C}$ which generate the elliptic function field $K_\Omega = \mathbb{C}(\wp(z), \wp'(z))$ with a relation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \quad g_2, g_3 \in \mathbb{C}, \quad \Delta := g_2^3 - 27g_3^2 \neq 0.$$

The coefficients $g_2 = g_2(\Omega)$, $g_3 = g_3(\Omega)$ and hence $\Delta = \Delta(\Omega)$ are determined by Ω as Eisenstein series

$$g_2(\Omega) = 60 \sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{\omega^4}, \qquad g_3(\Omega) = 140 \sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{\omega^6}.$$

Note that we have $g_2(\lambda\Omega) = \lambda^{-4} g_2(\Omega)$ and $g_3(\lambda\Omega) = \lambda^{-6} g_3(\Omega)$ for $\lambda \in \mathbb{C} \setminus \{0\}$.

Theorem 3.3. *The following three statements are equivalent:*
(1) Two elliptic function fields K_Ω and $K_{\Omega'}$ are isomorphic over $\mathbb{C}$ as abstract fields;
(2) $g_2(\Omega)^3/\Delta(\Omega) = g_2(\Omega')^3/\Delta(\Omega')$;
(3) There exists $\xi \in \mathbb{C} \setminus \{0\}$ such that $\Omega' = \xi\Omega := \{\xi\omega \mid \omega \in \Omega\}$.

Because of the theorem, the quantity

$$j(\Omega) := \frac{g_2(\Omega)^3}{\Delta(\Omega)} = \frac{g_2(\Omega)^3}{g_2(\Omega)^3 - 27 g_3(\Omega)^2}$$

is called the modulus of the elliptic function field K_Ω.

(b) The Elliptic Modular Function

As it is easily seen, we have $j(\lambda\Omega) = j(\Omega)$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Put $\tau_\Omega := \omega_1/\omega_2$; then $\tau_\Omega \in \mathbb{C} \setminus \mathbb{R}$.

We suppose that the imaginary part $\mathrm{Im}(\tau_\Omega)$ is positive by changing the indices if necessary; that is, τ_Ω is a point on the upper half complex plane. Conversely, for a given point τ on the half plane, we have a $\mathbb{Z}$-module $\Omega_\tau := \mathbb{Z} \cdot \tau + \mathbb{Z} \cdot 1$ of rank 2, and an elliptic function $\wp(\Omega_\tau; z)$ whose module of periods is Ω_τ. It is clear that $\tau = \tau_{\Omega_\tau}$. Thus we have a holomorphic function $j(\tau) := j(\Omega_\tau)$ on the upper half plane. Because of the theorem in the preceding **(a)**, the function $j(\tau)$ is called the *elliptic modular function.* For a given module of periods Ω, the value $j(\tau_\Omega)$ is independent from the choice of its basis ω_1, ω_2. If we take another basis ω_1', ω_2' and $\tau' := \omega_1'/\omega_2'$, then we have

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}, ad - bc = 1$, and hence $\tau' = (a\tau + b)/(c\tau + d)$. Therefore we have

Proposition 3.4. *For τ on the upper half plane and for $A \in \mathrm{SL}_2(\mathbb{Z})$, let $A(\tau)$ be the linear fractional transformation defined as above. Then we have $j(A(\tau)) = j(\tau)$.*

(c) Complex Multiplication

Let $\varphi(z)$ be an elliptic function with the module Ω of its periods. For $\mu \in \mathbb{C} \setminus \{0\}$, $\psi(z) := \varphi(\mu z)$ is an elliptic function with the module of periods $\Omega' = \mu^{-1}\Omega$.

Proposition 3.5. *The notation being as above, $\varphi(z)$ and $\psi(z)$ are algebraically dependent over $\mathbb{C}$ if and only if $\Omega \cap \Omega'$ is of rank 2. Hence, in the case, $\Omega \cap \Omega'$ is of finite index both in Ω and in Ω'.*

Definition 3.6. An elliptic function $\varphi(z)$ has complex multiplication by $\mu \in \mathbb{C} \setminus \mathbb{R}$ if $\varphi(z)$ and $\varphi(\mu z)$ are algebraically dependent over $\mathbb{C}$.

Now suppose that $\varphi(z)$ has complex multiplication by μ. Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be the module of periods of $\varphi(z)$. Then there is an integer $N \neq 0$ such that $N\Omega \subset \mu^{-1}\Omega$; hence we have $N\mu\Omega \subset \Omega$. Writing this relation in terms of the basis ω_1 and ω_2, we have

$$N\mu \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}, ad - bc \neq 0$. Therefore, $N\mu$ is an eigenvalue of the 2×2 matrix in $\mathrm{GL}_2(\mathbb{Q})$ and a root of a quadratic equation with coefficients in $\mathbb{Q}$.

The notation being as above, we have $N\mu = c\tau + d, c, d \in \mathbb{Z}$ by the equation multipled by ω_2^{-1}. By the choice of μ and τ, we see $c \neq 0$; hence μ and τ generate the same imaginary quadratic number field $k := \mathbb{Q}(\tau)$.

From now on, by changing the complex variable z to $\omega_2 z$ or passing to the isomorphic elliptic function field induced by multiplying ω_2^{-1}, we assume $\Omega = \mathbb{Z} \cdot \tau + \mathbb{Z} \cdot 1$. Then it is contained in the imaginary quadratic field $k = \mathbb{Q}(\tau)$. Put

$$\mathfrak{o} := \{\alpha \in k \mid \alpha\Omega \subset \Omega\};$$

this is an order of k, that is, a subring of the ring of integers $\mathfrak{o}_k$ of rank 2 as $\mathbb{Z}$-module, and Ω is a fractional ideal of $\mathfrak{o}$. For simplicity, here we consider only the case of the maximal order, $\mathfrak{o} = \mathfrak{o}_k$.

We now take a fractional ideal $\mathfrak{a}$ of k as a module of periods. Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals of k. Then two complex tori $\mathbb{C}/\mathfrak{a}$ and $\mathbb{C}/\mathfrak{b}$ are isomorphic if and only if there exists $\alpha \in \mathbb{C}$ such that $\mathfrak{b} = \alpha\mathfrak{a}$. This equality shows that $\alpha \in k$.

Theorem 3.7. *For two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of an imaginary quadratic number field k, the elliptic function fields $K_{\mathfrak{a}}$ and $K_{\mathfrak{b}}$ are isomorphic over $\mathbb{C}$ as abstract fields if and only if $\mathfrak{a}$ and $\mathfrak{b}$ belong to the same ideal class of k.*

(d) Division Points of Elliptic Curves with Complex Multiplication and Congruence ideal groups

The elliptic function field K_Ω for a module of periods Ω is generated by the Weierstrass $\wp$-function $\wp(\Omega; z)$ and its derivative $\wp'(\Omega; z)$ over $\mathbb{C}$ as explained in (a); these functions define an embedding of the complex torus $\mathbb{C}/\Omega$ into the projective plane $\mathbb{P}^2$ with homogeneous coordinate $(X : Y : Z)$ whose image is the elliptic curve C_Ω defined by

$$\begin{aligned} &Y^2 Z = 4X^3 - g_2(\Omega) X Z^2 - g_3(\Omega) Z^3, \\ &\quad X = \wp(\Omega; z), \quad Y = \wp'(\Omega; z), \quad Z = 1, \quad \text{for } z \notin \Omega, \\ &\quad X = 0, \quad Y = 1, \quad Z = 0, \quad \text{for } z \in \Omega. \end{aligned}$$

We may be much familiar with its affine form

$$y^2 = 4x^3 - g_2(\Omega)x - g_3(\Omega),$$

with $x = X/Z, y = Y/Z$. (Note that $z = 0$ is a pole of $\wp(\Omega; z)$ of order 2 and a pole of $\wp'(\Omega; z)$ of order 3.) Hence division points of the elliptic curve C_Ω just correspond to division points of the periods in the complex torus $\mathbb{C}/\Omega$.

Now, let $\mathfrak{a}$ be a fractional ideal of an imaginary quadratic number field k, and consider the complex torus $\mathbb{C}/\mathfrak{a}$. Then for a natural number m, the set of m-th division points corresponds to the subgroup $m^{-1}\mathfrak{a}/\mathfrak{a}$ of the complex torus.

Let α be an integer of k. Then $\alpha\mathfrak{a} \subset \mathfrak{a}$. Therefore, we have an endomorphism of $\mathbb{C}/\mathfrak{a}$ by multiplication of α. Suppose that it fixes each of the m-th division points on $\mathbb{C}/\mathfrak{a}$. Then we have $(\alpha - 1)m^{-1}\mathfrak{a} \subset \mathfrak{a}$. Hence we have $\alpha - 1 \in (m) = m\mathfrak{o}_k$ by multipying both sides by $m\mathfrak{a}^{-1}$, and

$$\alpha \equiv 1 \mod (m).$$

We may replace m with an integral ideal $\mathfrak{m}$ of k, and obtain $\mathfrak{m}$-division points on $\mathbb{C}/\mathfrak{a}$. This is the idea behind congruence ideal class groups which Weber introduced. To define them, however, we have to introduce 'multiplicative congruence'.

If two integers α and β in k are relatively prime to an integral ideal $\mathfrak{m}$ and $\alpha \equiv \beta \mod (\mathfrak{m})$, then they induce the same permutation on the quotient group $\mathfrak{m}^{-1}\mathfrak{a}/\mathfrak{a}$ by multiplication; in the case, we simply denote $\alpha/\beta \equiv 1 \mod (\mathfrak{m})$ even when $\alpha/\beta - 1$ might not belong to the set $\mathfrak{m}$; note that an element γ of k is expressed as $\gamma = \alpha/\beta$ with two integers relatively prime to $\mathfrak{m}$ if the principal ideal (γ) is relatively prime to $\mathfrak{m}$.

§ 4. Analytic method

In 1837, P. G. Lejeune Dirichlet introduced ingenious analytic method into number theory with his L-series to prove the so-called Dirichlet prime number theorem on arithmetic progressions; it states that there are infinitely many prime numbers in an arithmetic progression whenever its common difference is relatively prime to the initial term. Then he refined the analytic method to show his 'class number formula' of binary quadratic forms ([Di-1837, Di-1838, Di-1839]; today, it is known as his class number formula of quadratic fields; see Subsection 3.1.) Here he created modern analytic number theory.

Later, Dedekind studied pure cubic fields ([De-1900]), introduced his zeta-function of an algebraic number field ([De-1877]), and encouraged F. G. Frobenius to develop the theory of group characters ([Fr-1896b]). (The paper [De-1900] was a reproduction based on a manuscript prepared in 1871 or 1872 and published in 1900.) The original motivation of Dedekind should have been to establish a 'class number formula' for a pure cubic field $\mathbb{Q}(\sqrt[3]{a}), a \in \mathbb{Q} \setminus \mathbb{Q}^3$ which would be analogous to that of Dirichlet for quadratic fields. Note that a pure cubic field is a non-abelian extension of the rational number field. (Cf. eg. Hawkins [Hk-1970, Hk-1974] and Miyake [Mi-1989b].)

§ 5. Two concepts of class fields: Weber and Hilbert

§ 5.1. Weber

In his two papers [Wb-1886], Heinrich Martin Weber (1842 – 1913) presented a proof of Kronecker-Weber Theorem, and then started to challenge Kronecker's Jugendtraum and published *Elliptische Functionen und algebraische Zahlen* ([Wb-1891]). (The mathematical propositions of them were given in Subsection 3.2.) There was a gap in his proof of Kronecker-Weber Theorem. Hence, the paper [Hi-1896] of Hilbert gave the first complete proof to it. Weber could show in [Wb-1891] that, for an imaginary quadratic field k, each of the singular values of the elliptic modular function $j(\tau)$ for elliptic functions with fractional ideals of k as period modules generate the same unramified abelian extension $\tilde{k}$ of k. He could also show that the values of the division points of the periods of the elliptic functions generate abelian extensions of $\tilde{k}$; but at the time, he could not see that they were abelian extensions over the base quadratic field k.

His book [Wb-1891] was much enlarged and published as the third volume [Wb-1908] of his textbook *Lehrbuch der Algebra* [Wb-1894]. He also showed Principal Ideal Theorem for imaginary quadratic fields by utilizing special values of Dedekind's η function;

(see Subsection 3.2). His article 'Komplexe Multiplikation' [Wb-1900] in *Encyklopädie der Mathematischen Wissenschaften* is worth mentioning as a good introduction to complex multiplication of elliptic functions.

Then as we saw in Section 2, he introduced his Strahl (ray) ideal class group $I_{\mathfrak{m}}/S_{\mathfrak{m}}$ for an integral ideal $\mathfrak{m}$ of an arbitrary algebraic number field k in a series of papers [Wb-1897], and showed for an imaginary quadratic number field k that the field generated by the values of the elliptic functions at $\mathfrak{m}$-division points over $\tilde{k}$ is abelian extensions over the quadratic field k. With the result, he called such extensions 'class fields' (Classenkörper) of k.

§ 5.2. Hilbert

As was mentioned in the preceding subsection, David Hilbert (1862 – 1943), much younger than Weber, started to investigate algebraic number fields with his ramification theory in a Galois extension ([Hi-1894]) in 1894, and gave an amazingly simple proof to Kronecker-Weber Theorem in his paper [Hi-1896]. Then by the request of Deutschen Mathematiker-Vereinigung (German Mathematician's Union), he wrote up the famous gigantic report [Hi-1897] (usually called Hilbert-Bericht, Zahlbericht or simply the Bericht here in this article) on the theory of algebraic numbers. This contains a systematic treatment of Kummer extensions $k(\sqrt[m]{a}), a \in k$, of an algebraic number field k which contains m-th roots of unity.

Theorem 94 of the Bericht states that, in a cyclic unramified extension K/k of algebraic number fields of prime degree l, ideals in at least l ideal classes of k become principal ideals in K. Because of the generality of the theorem, Hilbert called a cyclic unramified extension a 'class field' (Klassenkörper) associated to the ideal classes which become principal ideal classes in the extension. The section containing the theorem is

§58. Der Fundamentalsatz von den relativ-zyklischen Körpern mit der Relativdifferente 1. Die Bezeichnung dieser Körper als Klassenkörper. (The fundamental theorem of relative-cyclic extensions with relative difference 1. The characterization of these fields as class fields.)

Then he gave much more sophisticated definition of his class fields in the paper [Hi-1899a] showing his results on relative quadratic extensions. This class field $\tilde{k}$ of an algebraic number field k would be an unramified abelian extension whose Galois group over k is isomorphic to the absolute ideal class group of k. He required a certain decomposition law of ideals in $\tilde{k}/k$. Furthermore, he stated the following two properties:

(1) $\tilde{k}$ would be the maximal unramified abelian extension of k;

(2) Principal Ideal Theorem: all ideals of k become principal ideals in $\tilde{k}$.

He also showed in [Hi-1899b] the quadratic reciprocity law in k via the reciprocity law for the quadratic norm residue symbols, and the existence of his class field when the class number of k was 2 or 4. The results of this paper is previously announced in [Hi-1898]. Although he claimed the properties (1) and (2) for his class fields, his main target would have been to show reciprocity laws of power residues and norm residues in an algebraic number field with appropriate roots of unity.

He left this subject, algebraic number theory, after the above-mentioned works. Then Furtwängler pursued Hilbert's project for unramified class field theory and higher power residue reciprocity law as it will be seen in the succeeding section.

§ 6. Further Development

§ 6.1. Furtwängler

Philipp Furtwängler (1869 – 1940) went farther along the way where Hilbert handled the quadratic case; he aimed to generalize Kummer's results on reciprocity law in cyclotomic fields and showed the power residue reciprocity law for an odd prime degree l in a general algebraic number field which contains the l-th roots of unity by his paper [Fw-1904] in 1904. Then, in 1907, he constructed Hilbert's class field for a general algebraic number field in [Fw-1907]. He could also show the power residue reciprocity law for an arbitrary power l^e of an odd prime l in the series of papers [Fw-1909] by utilizing Hilbert's class fields he constructed.

It was in 1911 when he was able to prove the required decomposition law of ideals in the Hilbert class field ([Fw-1911]).

He did not seem to be seriously interested in Weber's class fields nor Kronecker's Jugendtraum.

The assertion (1) of Hilbert, which were pointed out in the preceding subsection, was established in 1920 when Takagi showed his class field theory in which the ramifications in an Abelian extension became clear as it was characterized as a congruence class field.

In 1930, Furtwängler came back to show Principal Ideal Theorem (the assertion (2) above) with laborious calculation of the transfer of a metabelian group to its commutator subgroup. This approach was opened by Artin with his general reciprocity law as will be seen in one of the following sections.

§ 6.2. Fueter

There was also a big advance for Kronecker's Jugendtraum early in the twentieth century. In 1914, Swiss mathematician Karl Rudolf Fueter (1880 – 1950) succeeded in proving a theorem on Abelian extensions of imaginary quadratic number fields in the paper 'Abel'sche Gleichungen in quadratisch-imaginären Zahlkörpern' [Fu-1914].

Theorem 6.1 (Fueter). *Every abelian extension of an imaginary quadratic number field k with an odd degree is contained in an extension of k generated by suitable roots of unity and the singular moduli of elliptic functions with complex multiplication in k.*

It was known at the time, however, that the extensions of k given in the theorem are not large enough to cover all abelian extensions of k.

This result apparently stimulated Takagi.

§ 7. Takagi

Teiji Takagi (1875 – 1960) gave the following list of seven references in the foreword of his major paper [T-1920a] on his class field theory as basis and stimulation for his work: Weber's paper [Wb-1897] on his congruence class fields and his book [Wb-1908] which is a much enlarged version of [Wb-1891], Hilbert's papers [Hi-1898, Hi-1899b] on relative quadratic extensions and [Hi-1899a] of his unramified class fields, Furtwängler's paper [Fw-1907] on construction of Hilbert's class fields, and Fueter's paper [Fu-1914].

At the end of the foreword of his paper [T-1922] on reciprocity law, Takagi gave a list of six references, Hilbert's papers [Hi-1898, Hi-1899a, Hi-1899b], Furtwängler's two papers [Fw-1907, Fw-1909], and his own paper [T-1920a] of his class field theory.

As a result of Takagi's determination of the conductors of Abelian extensions, the assertion (1) of Hilbert was proved as was stated in Subsection 6.1:

Theorem 7.1 (Maximality of Hilbert's Class Field). *Hilbert's class filed $\tilde{k}$ of an algebraic number field k is the maximal unramified abelian extension of k.*

At the end of the abstract [T-1920b] for International Mathematicians Congress held at Strasbourg in 1920, Takagi proposed a problem whether it would be possible or not to define 'ideal classes' of an algebraic number field to determine non-abelian normal extensions of the base field. Indeed, we do not see any reason to define the congruence ideal group $H_{\mathfrak{m}}$ in association with a Galois extensions K/k as the subgroup of $I_{\mathfrak{m}}$ which is generated by $S_{\mathfrak{m}}$ and by the norm group $\{N_{K/k}(\mathfrak{A}) \mid \mathfrak{A}$ is an ideal of K prime to $\mathfrak{m}\}$. In case where k is an imaginary quadratic field, the Strahl (ray) ideal group $S_{\mathfrak{m}}$ may be reasonable on account of $\mathfrak{m}$-division points of elliptic curves with complex multiplication in k. If we see it, however, from the side of an Abelian extension K of a general algebraic number field k, we do not find any explana tion to enlarge the norm group by the Strahl group. Here we recall the words of Takagi, "Far beyond any perspectives at the time, however, $\cdots$."

It should also be noted that Takagi refined multiplicative congruence by introducing 'signatures at infinite places'. Weber owed his idea of class fields to arithmetic of imaginary quadratic number fields and did not seem to be aware of necessity of the signatures. It was Hilbert who introduced them in his investigation of relative quadratic extensions in general in [Hi-1899b]. Today, we say an integral divisor of an algebraic number field k to mean a combination of an integral ideal and some infinite places to indicate distribution of signatures there.

§ 8. After Takagi

§ 8.1. Tschebotareff's Density Theorem

In 1926, Nikolai Tschebotareff published his paper [Ts-1926] in which he proved so called his density theorem. The proposition was formulated for a Galois extension by Frobenius [Fr-1896a] in 1896. The origin of the idea of it goes back to the paper [Kr-1880a] of Kronecker. Takagi gave a comment on the theorem in a footnote of his book [T-1948, p.261] as follows:

> It was a kind of daydream of Kronecker to control algebraic extensions Ω/k with the density $\Delta(\Omega)$. He only conjectured the existence of Δ_a; he was lucky enough to hit the target here again. The existence was established by Tschebotareff after a partial success of Frobenius. We named it as 'Kronecker's density' after the prophet (cf. Hilbert's Bericht [Hi-1897, §50]).

The theorem determines Kronecker's density of those prime ideals in a Galois extension K over k whose Frobenius automorphisms belong to a prescribed conjugacy class of the group. More precisely, the density of a conjugacy class in a finite group is the number of elements in the class divided by the order of the group; the density $\Delta(M)$ of a set M of prime ideals of K is give by

$$\Delta(M) = \lim_{s \to 1+0} \frac{\sum_{\mathfrak{p} \in M} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{p})^s}}{\log \frac{1}{s-1}}$$

if the limit exists. (The density $\Delta(\Omega)$ for a Galois extension Ω/k in Takagi's comment is $\Delta(M)$ for the set M of those prime ideals of Ω whose Frobenius automorphisms are the identity of the group; this M represents the set of those prime ideals of k which split completely in Ω/k.)

Theorem 8.1 (Tschebotareff's Density Theorem). *For a Galois extension K/k, the density of those prime ideals in K whose Frobenius automorphisms belong to a given conjugacy class of the Galois group exists and coincides with the density of the conjugacy class in the Galois group.*

This is a natural generalization of the Dirichlet prime number theorem on arithmetic progressions showed in Dirichlet [Di-1837] if we 'naturally' identify the multiplicative group $(\mathbb{Z}/m\mathbb{Z})^\times$ with the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ of the mth cyclotomic field $\mathbb{Q}(\zeta_m)$. The original conjecture of Kronecker was on the density of the set of those prime numbers p for which a given polynomial with integer coefficients has the prescribed number (a in the above quoted comment of Takagi) of solutions mod p. Because of the contribution of Frobenius, Helmut Hasse later chose the term 'Frobenius automorphism' in his Bericht [Ha-1926] on class field theory.

The proof of Tschebotareff in [Ts-1926] was well analyzed by Otto Schreier in his paper [Srr-1927] and applied by Artin to show his general reciprocity law which we shall see in the next subsection.

§ 8.2. Artin's general reciprocity law

Much influenced by Takagi's class field theory, Emil Artin (1898 – 1962) formulated his general reciprocity law and showed that reciprocity laws for power residue and norm residue are easily derived from the one conjecturally proposed by him in [Ar-1924]. Then he could publish a proof to it in [Ar-1927] with the help of Schreier's analysis of the proof of Tschebotareff as was mentioned at the end of the preceding subsection.

Theorem 8.2 (Artin's General Reciprocity Law). *Let K be the class field of k corresponding to a congruence ideal group $H_\mathfrak{f}$ with the conductor $\mathfrak{f}$. Then the canonical isomorphism from $I_\mathfrak{f}/H_\mathfrak{f}$ onto the Galois group $\mathrm{Gal}(K/k)$ is given by the Artin map which assigns the Frobenius automorphism to each prime ideal of $I_\mathfrak{f}$.*

This work of Artin clarified the arithmetic nature of the isomorphism of an congruence ideal class group onto the Galois group of the Takagi's class field associated to it. Takagi highly praised this work of Artin ([T-1927]).

As an application of his reciprocity law, Artin formulated, in the paper [Ar-1930], Principal Ideal Theorem in terms of the metabelian group $\mathrm{Gal}(\tilde{\tilde{k}}/k)$ where $\tilde{\tilde{k}}$ is the Hilbert class field of the Hilbert class field $\tilde{k}$ of an algebraic number field k, that is, the second class field of k; namely, the transfer homomorphism from $\mathrm{Gal}(\tilde{\tilde{k}}/k)$ to its commutator subgroup $\mathrm{Gal}(\tilde{\tilde{k}}/\tilde{k})$ is trivial.

Then by the paper [Ar-1931] published in 1931, Artin introduced his L-functions with group characters of the Galois group of a general Galois extension K/k to decompose Dedekind's zeta-function $\zeta_K(s)$ into a product of these L-functions and Dedekind's zeta $\zeta_k(s)$ of the base field k. This is a far-reaching generalization of what Dedekind did for the Galois closure of a pure cubic extension of the rational number field. The theory of group characters has been well developed by Frobenius [Fr-1896b] who were encouraged by Dedekind. This paper of Frobenius was published in 1896; however, he himself nor Dedekind did not give any L-series with group characters of the group of a general non-Abelian Galois extension. Frobenius seemed much engaged in the theory of representation of finite groups.

Here we give the reciprocity law for complex multiplication to close this subsection (cf. [Shm-1971, Theorem 5.7, p.123]).

Theorem 8.3 (Reciprocity Law of Elliptic Modular Function). *Let k be an imaginary quadratic number field and $\mathfrak{a}$ a fractional ideal of k. Then the extension $k(j(\mathfrak{a}))/k$ generated by the singular value $j(\mathfrak{a})$ of the elliptic modular function is the Hibert class field of k, whose Galois group is isomorphic to the ideal class group of k. For a prime ideal $\mathfrak{p}$ of k, moreover, let $\alpha(\mathfrak{p})$ be the Frobenius automorphism of $\mathfrak{p}$ in* $\mathrm{Gal}(k(j(\mathfrak{a}))/k)$. *Then we have $j(\mathfrak{a})^{\alpha(\mathfrak{p})} = j(\mathfrak{p}^{-1}\mathfrak{a})$.*

§ 8.3. Principal Ideal Theorem

Furtwängler, informed of the result of Artin in the paper [Ar-1930] in advance, carried out a laborious calculation and succeeded in proving Principal Ideal Theorem in [Fw-1930]. He could show his powerful ability even at about 60 years old, which makes us think of his young days when classical invariant theory still flourished in Germany. Their papers [Ar-1930, Fw-1930] were published in the same volume of the journal, *Abhandl. Math. Sem. Univ. Hamburg* in 1930.

It should be pointed out here again that Weber showed the principal ideal theorem for imaginary quadratic fields in his book [Wb-1908] using special values of Dedekind

η-function.

Later in 1949, F. Terada showed Tannaka's generalization of Principal Ideal Theorem in genus fields in his paper [Te-1949] by following the line of the method of Furtwängler.

In 1997, Hiroshi Suzuki finally proved a beautiful 'Principal Ideal Theorem' in a unified form which includes all of Hilbert's Theorem 94, the original Principal Ideal Theorem and Terada's Theorem in his paper 'On the Capitulation Problem' [Su-1997]:

Theorem 8.4 (Suzuki)**.** *Let k be a finite cyclic extension of an algebraic number field k_0 of finite degree and K be an unramified extension of k which is abelian over k_0. Then the number of those $\mathrm{Gal}(k/k_0)$-invariant ideal classes of k which become principal in K is divisible by the degree $[K:k]$ of the extension K/k.*

As is easily seen, Hilbert's Theorem 94 is the case of this theorem where $k = k_0$ and K/k is cyclic; the original Principal Ideal Theorem is the case where $k = k_0$ and K is the Hilbert class field $\tilde{k}$ of k; and Terada's Theorem is the case where K is the maximal Abelian extension of k_0 contained in the Hilbert class field $\tilde{k}$.

§ 9. Some Topics

§ 9.1. Local class field theory and algebraic proof

In 1930, F. K. Schmidt showed local class field theory in his paper [Smd-1930]. He put a stress in the introduction that he used global class field theory and hence needed transcendental (analytic) method behind the second fundamental inequality.

On the result, E. Noether gave a comment that the way of the proof should be converted; that is, the simpler local class field theory should be proved first independently, and then the complicated global class field theory should be shown by the 'local-global relation' of norm residue symbol. (Cf. the testimony of Hasse in his lecture notes *Class Field Theory* [Ha-1973, p.68].) Thus the problem of finding an algebraic proof of class field theory was asked by Noether.

It was the paper [Ch-1940] of Claude Chevalley (1909 – 1984) which first showed an algebraic proof of global class field theory.

The theory of class formation of Y. Kawada (a part of it is a joint work with I. Satake) may be another answer to the problem of algebraic proof ([Ka-1955, KS-1955, Ka-1956]).

We may study local class field theory directly and independently from the global one in the books of Serre [Se-1962] and of Iwasawa [Iw-1986], for example; the article [Hz-1975] of Hazewinkel may also be helpful.

There has not yet been presented any construction of global class field theory with local ones and the local-global relation of norm residue symbol.

§ 9.2. Idele

One of the motivations of Chevalley to introduce the concept of 'élément idéal' in his paper [Ch-1940] was to avoid somewhat clumsy Strahl ideal classes, and tried to go far up to a kind of limit by taking all integral divisors instead of setting conductors. It naturally led him to consider the maximal abelian extension k^{ab} of the ground field k in a fixed algebraic closure of k; k^{ab}/k is an infinite extension. Hence Krull's works [Krl-1928] and [Krl-1932] gave a foundation to him.

Then Iwasawa [Iw-1950] and Tate [Ta-1950] independently furnished the group of ideles with the good topology and developed functional analysis on the group to describe zeta- and L-functions of the base field.

Finally, Weil introduced adeles (additive ideles) in the lecture notes [We-1959]. And these became a kind of natural language for (algebraic) number theory.

The idele class group of the ground field k is the quotient of the whole idele group by its discrete subgroup of diagonally embedded 'global numbers' $k^{\times}$. It describes the Galois group $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ of the maximal abelian extension of the base field by Artin's general reciprocity law. Then we see the local-global principle most naturally as in the next subsection.

See also Iyanaga's article [Iy-1975].

§ 9.3. Class Field Theory — Idele Version

Let k be an algebraic number field and k^{ab} the maximal Abelian extension of k in the algebraic closure $\bar{k}$ of k in the complex number field $\mathbb{C}$.

For a prime ideal $\mathfrak{p}$ of k, let $k_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion of k, and $\mathfrak{o}_{\mathfrak{p}}$ the ring of integers of $k_{\mathfrak{p}}$. Then the unit group $\mathfrak{o}_{\mathfrak{p}}{}^{\times}$ of $k_{\mathfrak{p}}$ is the set of invertible elements in $\mathfrak{o}_{\mathfrak{p}}$. Fix a local parameter π so that the unique maximal ideal of $\mathfrak{o}_{\mathfrak{p}}$ is $\pi\mathfrak{o}_{\mathfrak{p}}$. Then we have $k_{\mathfrak{p}}^{\times} \simeq \langle\pi\rangle \times \mathfrak{o}_{\mathfrak{p}}{}^{\times}$. Furthermore, let $k_{\mathfrak{p}}^{\mathrm{ab}}$ be the maximal Abelian extension of $k_{\mathfrak{p}}$ in its fixed algebraic closure $\bar{k_{\mathfrak{p}}}$, and $k_{\mathfrak{p}}^{\mathrm{urab}}$ the maximal unramified Abelian extension of $k_{\mathfrak{p}}$ in $k_{\mathfrak{p}}^{\mathrm{ab}}$.

Theorem 9.1 (Local Artin Map). *Let the notation be as above. Then there exists an injective homomorphism* $\alpha_{\mathfrak{p}} : k_{\mathfrak{p}}^{\times} \to \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}})$ *which satisfies the three conditions (i), (ii) and (iii):*
(i) the image of $\alpha_{\mathfrak{p}}$ *is dense in* $\mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}})$*;*
(ii) $\alpha_{\mathfrak{p}}(\mathfrak{o}_{\mathfrak{p}}{}^{\times}) = \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}^{\mathrm{urab}})$*;*
(iii) $\alpha_{\mathfrak{p}}(\pi) \bmod \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}^{\mathrm{urab}})$ *is the Frobenius automorphism of* $k_{\mathfrak{p}}^{\mathrm{urab}}/k_{\mathfrak{p}}$*;* $\mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{urab}}/k_{\mathfrak{p}})$ *is an Abelian free pro-finite group generated by* $\alpha_{\mathfrak{p}}(\pi) \bmod \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}^{\mathrm{urab}})$*.*

Note that $\alpha_{\mathfrak{p}}(\pi) \bmod \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}^{\mathrm{urab}})$ in (iii) of the theorem is independent from the choice of the local parameter π.

An Abelian free pro-finite group generated by a single element is isomorphic to the pro-finite completion $\tilde{\mathbb{Z}}$ of the additive group of integers $\mathbb{Z}$ and is a compact topological group. It is naturally isomorphic to the direct product of all of the p-adic completion $\mathbb{Z}_p$ where p runs over all prime numbers because of the Chinese Remainder Theorem.

The adele ring $\mathbb{A}$ of the rational number field $\mathbb{Q}$ may be defined as follows:

$$\mathbb{A} := \tilde{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \mathbb{R}.$$

The p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$ is considered as $\mathbb{Q}_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Q}$. The field of real numbers $\mathbb{R}$ of the last term in the above expression of $\mathbb{A}$ stands for the Archimedean completion of $\mathbb{Q}$. The gobal numbers $\mathbb{Q}$ are diagonally embedded in $\mathbb{A}$ and form a discrete subring.

The adele ring $k_{\mathbb{A}}$ of an algebraic number field k is defined as $k_{\mathbb{A}} := k \otimes_{\mathbb{Q}} \mathbb{A}$.

The idele group $k_{\mathbb{A}}^{\times}$ is the set of all invertible adeles; its group structure is multiplicative, and furnished with the topology to act naturally on the adele ring $k_{\mathbb{A}}$.

Let p be a prime number and $\mathfrak{p}$ a prime ideal of k which divides p. Let $k_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completeion of k. Since $k \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is the direct sum of $k_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ dividing p, the local field $k_{\mathfrak{p}}$ is naturally embedded in the adele ring $k_{\mathbb{A}}$; hence we have a natural embedding $\iota_{\mathfrak{p}} : k_{\mathfrak{p}}^{\times} \to k_{\mathbb{A}}^{\times}$ of the multiplicative group of the local field into the idele group.

On the other hand, we have embeddings of k into $k_{\mathfrak{p}}$, and so embeddings k^{ab} into $k_{\mathfrak{p}}^{\mathrm{ab}}$ for every $\mathfrak{p}$ dividing p which are consistent with the embeddings $\iota_{\mathfrak{p}}$. Hence we have the restriction homomorphism

$$\rho_{\mathfrak{p}} : \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}) \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$$

which is injective because $k_{\mathfrak{p}}^{\mathrm{ab}} = k^{\mathrm{ab}} \cdot k_{\mathfrak{p}}$.

The Archimedean (or infinite) part $k_{\infty}^{\times}$ of the idele group $k_{\mathbb{A}}^{\times}$ is naturally identified with the multiplicative group $(k \otimes_{\mathbb{Q}} \mathbb{R})^{\times}$. If k has a real conjugate, then the Archimedean completion at the corresponding infinite place is $\mathbb{R}$, and hence its multiplicative group has two connected components. Let $k_{\infty,0}^{\times}$ be the connected component of the unity in $(k \otimes_{\mathbb{Q}} \mathbb{R})^{\times}$.

With these preparation, we are ready to state the idele version of class field theory.

Theorem 9.2 (Global Artin Map). *Let the notation be as above. Then there exists a surjective homomorphism $\alpha_k : k_{\mathbb{A}}^{\times} \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$ which satisfies the two conditions (i) and (ii):*
(i) the kernel of α is the topological closure of $k^{\times} \cdot k_{\infty,0}^{\times}$ in $k_{\mathbb{A}}^{\times}$;
(ii) For each prime divisor $\mathfrak{p}$ of k, we have $\alpha_k \circ \iota_{\mathfrak{p}} = \rho_{\mathfrak{p}} \circ \alpha_{\mathfrak{p}}$.

We now see by these two theorems on the Artin maps that the local-global relation in class field theory is supplied by the diagonally embedded global numbers of k.

On account of (ii), $\alpha_k \circ \iota_{\mathfrak{p}}(\pi_{\mathfrak{p}})$ represents the Frobenius automorphism of $\mathfrak{p}$ on each Abelian extension K/k where $\mathfrak{p}$ is unramified. In general, however, it is only determined as a coset of $\alpha_k \circ \iota_{\mathfrak{p}}(\mathfrak{o}_{\mathfrak{p}}{}^{\times})$.

§ 9.4. Shafarevich and Iwasawa — Non-Abelian World, especially, p-Extensions —

After the establishment of Takagi-Artin class field theory in 1927, Arnold Scholz (1904 – 1942) and Olga Taussky (1906 – 1995) took a step toward non-Abelian world in 1934. Their main theme in the paper [ST-1934] was the 'capitulation problem' which asked more detailed process for a prime ideal of the base field to become a principal ideal in the Hilbert class field. They dealt the case of $p = 3$ and, in particular, determined the Galois group of the maximal unramified p-extension of $\mathbb{Q}(\sqrt{-4027})$; it is a non-Abelian group of order 3^5.

Then in 1937 after his studies [Slz-1929a, Slz-1929b, Slz-1929c, Slz-1934, Slz-1936], Scholz opened the gate toward p-extensions, p being a prime number, with construction problem of p-extensions in his paper [Slz-1937]. It is sorry to say, however, that he could not continue his investigation on the subject because of his short life.

In 1947, I.R. Shafarevich published a paper 'On p-extensions' (Russian) [Shf-1947]; in 1954, he did two elaborated papers 'On the construction of fields with a given Galois group of order l^{α}' (Russian) [Shf-1954a] and 'On the construction of fields with given solvable Galois groups' (Russian) [Shf-1954b]. Then in 1964, he presented the paper 'On the tower of class fields' (Russian) [GS-1964] with co-author E. Golod which showed the existence of infinite towers of unramified p-extensions of quadratic fields. We may see his intension toward non-Abelian world in his article 'Abelian and Nonabelian Mathematics'

[Shf-1991].

In 1958, K. Iwasawa published a paper 'On solvable extensions of algebraic number fields' [Iw-1958]; here he proved that the Galois group of the maximal solvable extension k^{sol} over the maximal Abelian extension k^{ab} of an algebraic number field k of finite degree is a free pro-finite solvable group with countable number of generators. This is remarkable because the structure of $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ is very complicated as we see it by its idelic description given in the preceding subsection. Actualy Iwasawa also reached a similar result for the Galois group of the maximal nilpotent extension k^{nil} over the maximal Abelian extension k^{ab} of k; cf. also Miyake [Mi-1990]. Note that a pro-finite-nilpotent-group is a direct product of pro-p-groups where prime numbers p runs over some set of prime numbers. After he looked into the Galois groups of local fields, he started his investigation in $mathbbZ_p$-extensions in the series of papers [Iw-1959a, Iw-1959b] and [Iw-1959c]. (He first called $\mathbb{Z}_p$-extensions Γ-extensions.) His class number formula for $\mathbb{Z}_p$-extensions was given in [Iw-1959a] with a proof.

Here we may recall a paper [De-1901] of Dedekind in which he showed the structure of the Galois group of the infinite extension $\mathbb{Q}(\mu(p^\infty))$ obtained by all of p^n-th roots of unity for $n = 1, 2, 3, \ldots$, which is isomorphic to the multiplicative group $\mathbb{Z}_p^\times$ of the units of the p-adic completion $\mathbb{Z}_p$.

In 1962, Iwasawa studied the class numbers of cyclotomic fields in [Iw-1962], and, finally in 1973, presented his full scale theory of $\mathbb{Z}_p$-extensions in his celebrated paper 'On $\mathbf{Z}_\ell$-extensions of algebraic number fields' [Iw-1973].

We should here mention Helmut Koch's influential book *Galoissche Theorie der p-Erweiterungen* (Galois Theory of p-Extensions) [Ko-1970]. Its English translation [Ko-2002] was published in 2002 with a brief historical description as Postscript with additional references.

To close our article, let me point out a few recent results. One of them is the paper 'Galois Theoretic Local-Global Relations in Nilpotent Extensions of Algebraic Number Fields' [Mi-1989c] of the author which presents somewhat analogous local-global relations to the idelic description of the Galois group $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ in the previous subsection.

Others are most recent remarkable results of Manabu Ozaki. In 2007, he successfully presented 'Non-abelian Iwasawa theory of $\mathbb{Z}_p$-extensions' [Oz-2007]. Then in 2009, he showed the existence of algebraic number fields with prescribed finite p-groups or finitely generated pro-p-groups as the Galois groups of the *maximal unramified p-extensions*.

To be more precise, let p be a prime number. For a number field F (not necessary of finite degree), let $L_p(F)$ denote the maximal unramified p-extension over F, and put $\tilde{G}_F(p) := \mathrm{Gal}(L_p(F)/F)$.

Theorem 9.3 (Ozaki). *For an arbitrarily given finite p-group G, there exists a number field F of finite degree such that $\tilde{G}_F(p)$ is isomorphic to G.*

In the case where we take account of number fields of infinite degree, Ozaki obtained a similar result:

Theorem 9.4 (Ozaki). *Let $\mathcal{C}_p$ be the set of all isomorphism classes of the pro-p-groups $\tilde{G}_F(p)$ where F runs over all algebraic extensions of the rational number field $\mathbb{Q}$. Then $\mathcal{C}_p$ is exactly equal to the set of all isomorphism classes of pro-p-groups with countably many generators.*

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