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Non-existence of certain Galois representations with a uniform tame inertia weight: A resume

By

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Abstract

In this paper, we announce some results on the non-existence of certain semistable Galois representations. We apply them to a conjecture of Rasmussen and Tamagawa.

§1. Main results

Our main concern in this paper is the non-existence of certain semistable Galois representations of a number field. Let \( \ell \) be a prime number and \( K \) a number field of degree \( d \) and discriminant \( d_K \). Choose an algebraic closure \( \bar{K} \) of \( K \). Fix non-negative integers \( n, r \) and \( w \), and a prime number \( \ell_0 \neq \ell \). Put \( \bullet := (n, \ell_0, r, w) \). Let \( \text{Rep}_{\mathbb{Q}_\ell}(G_K) \) be the set of isomorphism classes of \( n \)-dimensional \( \ell \)-adic representations \( V \) of the absolute Galois group \( G_K = \text{Gal}(\bar{K}/K) \) of \( K \) which satisfy the following four conditions:

(A) For any place \( \lambda \) of \( K \) above \( \ell \), the restriction of \( V \) to the decomposition group of (an extension to \( \bar{K} \) of) \( \lambda \) is semistable and has Hodge-Tate weights in \([0, r] \).

(B) For some place \( \lambda_0 \) of \( K \) above \( \ell_0 \), the representation \( V \) is unramified at \( \lambda_0 \) and the characteristic polynomial \( \det(T - \text{Fr}_{\lambda_0}|V) \) has rational integer coefficients. Furthermore, the roots of the above characteristic polynomial have complex absolute value \( q_\lambda^{w/2} \) for every embedding \( \mathbb{Q}_\ell \) into \( \mathbb{C} \). Here \( \text{Fr}_{\lambda_0} \) and \( q_{\lambda_0} \) are the arithmetic Frobenius and the order of the residue field of \( \lambda_0 \), respectively.

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(C) For any finite place $v$ of $K$ not above $\ell$, the action of the inertia group at $v$ on $\bar{V}$ is unipotent. Here $\bar{V}$ is a residual representation of $V^1$.

(D) The representation $\bar{V}$ has a filtration of $G_K$-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{n-1} \subset \bar{V}_n = \bar{V}$$

such that $\bar{V}_k$ has dimension $k$ for each $0 \leq k \leq n$.

Furthermore, we denote by $\text{Rep}_{\mathbb{Q}_\ell}(G_K)^{\text{cyc}1}$ the subset of $\text{Rep}_{\mathbb{Q}_\ell}(G_K)$ whose elements $V$ satisfy the additional property (E) below:

(E) For each $1 \leq k \leq n$, the $G_K$-action on the quotient $\bar{V}_k/\bar{V}_{k-1}$ is given by a power of the mod $\ell$ cyclotomic character $\chi_\ell$.

Example 1.1. Let $X$ be a proper smooth scheme over $K$ which has semistable reduction everywhere and has good reduction at some place of $K$ above $\ell_0$. Let $n$ be the $w$-th Betti number of $X(\mathbb{C})$ and $w \leq r$. Then the dual of $H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ satisfies the conditions (A), (B) and (C).

Our main results are the following:

Theorem 1.2 ([O], Theorem 3.10). Suppose that $w$ is odd or $w > 2r$. Then there exists an explicit constant $C$ depending only on $d_K, n, \ell_0, r$ and $w$ such that $\text{Rep}_{\mathbb{Q}_\ell}(G_K)^{\text{cyc}1}$ is empty for any prime number $\ell > C$.

Theorem 1.3 ([O], Theorem 3.11). Suppose that $w$ is odd or $w > 2r$. Then there exists an explicit constant $C'$ depending only on $K, n, \ell_0, r$ and $w$ such that $\text{Rep}_{\mathbb{Q}_\ell}(G_K)^*$ is empty for any prime number $\ell > C'$ which does not split in $K$.

The key of the proofs of the above Theorems is a relation between tame inertia weights and Frobenius weights. This relation is obtained by a result of Caruso [Ca] which gives an upper bound of tame inertia weights of semistable Galois representations.

§ 2. Rasmussen-Tamagawa Conjecture

We describe an application, which is a special case of the Rasmussen-Tamagawa conjecture ([RT]) related with the finiteness of the set of isomorphism classes of abelian varieties with constrained prime power torsion. Our work is motivated by this conjecture. We denote by $\overline{K}_\ell$ the maximal pro-$\ell$ extension of $K(\mu_\ell)$ which is unramified away from $\ell$.

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1A residual representation $\bar{V}$ is not uniquely defined (it depends on the choice of a $G_K$-stable lattice), but the validity of conditions (C), (D) or (E) does not depend on the choice of $V$. 
Definition 2.1. Let \( g \geq 0 \) be an integer. We denote by \( \mathcal{A}(K, g, \ell) \) the set of \( K \)-isomorphism classes of abelian varieties \( A \) over \( K \), of dimension \( g \), which satisfy the following equivalent conditions:

1. \( K(A[\ell^\infty]) \subset \bar{K}_\ell \);
2. The abelian variety \( A \) has good reduction outside \( \ell \) and \( A[\ell] \) admits a filtration

\[
\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{2g-1} \subset \bar{V}_{2g} = A[\ell]
\]
such that \( \bar{V}_k \) has dimension \( k \) for each \( 0 \leq k \leq 2g \). Furthermore, for each \( 1 \leq k \leq 2g \), the \( G_K \)-action on the space \( \bar{V}_k/\bar{V}_{k-1} \) is given by a power of the mod \( \ell \) cyclotomic character \( \chi_\ell \).

The equivalence of (1) and (2) follows from the criterion of Néron-Ogg-Shafarevich and Lemma 3 of [RT]². The set \( \mathcal{A}(K, g, \ell) \) is a finite set because of the Shafarevich conjecture proved by Faltings. Rasmussen and Tamagawa conjectured that this set is in fact empty for any \( \ell \) large enough:

Conjecture 2.2 ([RT], Conjecture 1). The set \( \mathcal{A}(K, g, \ell) \) is empty for any prime \( \ell \) large enough.

It is known that this conjecture holds under the following conditions:

(i) \( K = \mathbb{Q} \) and \( g = 1 \) ([RT], Theorem 2);
(ii) \( K \) is a quadratic number field other than the imaginary quadratic fields of class number one and \( g = 1 \) ([RT], Theorem 4).

We consider the semistable reduction case of Conjecture 2.2.

Definition 2.3. (1) We denote by \( \mathcal{A}(K, g, \ell)_{st} \) the set of \( K \)-isomorphism classes of abelian varieties in \( \mathcal{A}(K, g, \ell) \) with everywhere semistable reduction.

(2) We denote by \( \mathcal{A}(K, g, \ell_0, \ell)_{st} \) the set of \( K \)-isomorphism classes of abelian varieties \( A \) over \( K \) with everywhere semistable reduction, of dimension \( g \), which satisfy the following condition: The abelian variety \( A \) has good reduction at some place of \( K \) above \( \ell_0 \) and \( A[\ell] \) admits a filtration of \( G_K \)-modules

\[
\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{2g-1} \subset \bar{V}_{2g} = A[\ell]
\]
such that \( \bar{V}_k \) has dimension \( k \) for each \( 0 \leq k \leq 2g \).

We have \( \mathcal{A}(K, g, \ell)_{st} \subset \mathcal{A}(K, g, \ell_0, \ell)_{st} \). We can show the following easily as corollaries of Theorems 1 and 2³:

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²Lemma 3 of [RT] is stated in the setting \( K = \mathbb{Q} \). However, an easy argument allows us to extend this setting to any number field \( K \).

³Rasmussen and Tamagawa have shown the emptiness of the set \( \mathcal{A}(K, g, \ell)_{st} \) for \( \ell \) large enough by using the result of [Ra] instead of [Ca] (unpublished).
Corollary 2.4 ([O], Corollary 4.5). There exists an explicit constant $D$ depending only on $d_K$ and $g$ such that the set $\mathcal{A}(K, g, \ell)_{st}$ is empty for any prime number $\ell > D$.

Corollary 2.5 ([O], Corollary 4.6). There exists an explicit constant $D'$ depending only on $K$, $g$ and $\ell_0$ such that the set $\mathcal{A}(K, g, \ell_0, \ell)_{st}$ is empty for any prime number $\ell > D'$ which does not split in $K$.

References


