The existence of positive solutions to the semilinear elliptic equation involving the Sobolev and the Sobolev-Hardy critical terms

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Abstract

Let $n \geq 3$ and Ω be a C^2-bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$. We consider

$$\begin{cases} \Delta u + \lambda\, u^p + \frac{u^{2^*-1}}{|x|^s} = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $0 < s < 2$, $2^* = \frac{2(n-s)}{n-2}$,

$$\begin{cases} \lambda > 0 \quad \text{if } 1 < p \leq \frac{n+2}{n-2}, \\ 0 < \lambda < \lambda_1(\Omega) \quad \text{if } p = 1, \end{cases}$$

and $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition.

In this paper, we shall prove the equation (0.1) has a positive solution provided that $\frac{n}{n-2} < p < \frac{n+2}{n-2}$. If the mean curvature H at 0 is negative, then for each of the following cases, the equation (0.1) has a positive solution.

1. $1 \leq p < \frac{n+2}{n-2}$.

2. $p = \frac{n+2}{n-2}$ and

$$\begin{cases} 0 < s < 1 \quad \text{if } n = 3, \\ 0 < s < 2 \quad \text{if } n \geq 4. \end{cases}$$

If $H(0) < 0$, then we also prove the existence of a positive solution for the following equation:

$$\begin{cases} \Delta u - \lambda\, u^p + \frac{u^{2^*-1}}{|x|^s} = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where

$$\lambda > 0 \quad \text{and } 1 \leq p < \min\left\{\frac{n}{n-2}, 2^* - 1\right\}.$$

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1 Introduction

In this paper, we consider the existence of positive solutions of the following nonlinear elliptic equations:

$$\begin{cases} \Delta u + \lambda\, u^p + \frac{u^{2^*-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 3$, $0 \in \partial\Omega$, $0 < s < 2$, $2^* = 2^*(s) = \frac{2(n-s)}{n-2}$, $1 \leq p \leq \frac{n+2}{n-2}$ and λ is a real parameter. For the case $p = 1$, this problem was considered by Ghoussoub-Robert [3, 4], and they proved the following result.

Theorem A. *Suppose that Ω is a bounded smooth domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$ and $\lambda < \lambda_1(\Omega)$, where $\lambda_1(\Omega) > 0$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition. Then the equation*

$$\begin{cases} \Delta u + \lambda\, u + \frac{u^{2^*-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a positive solution provided that the mean curvature H of $\partial\Omega$ at 0 is negative.

In an earlier paper by Brézis-Nirenberg [1], among other results, they showed:

Theorem B. *Suppose that Ω is a bounded smooth domain in $\mathbb{R}^n$ with $n \geq 4$. Then for every $\lambda \in (0, \lambda_1(\Omega))$, there exists a positive solution for the equation*

$$\begin{cases} \Delta u + \lambda\, u + u^{\frac{n+2}{n-2}} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this article, we study more general cases. Namely, we obtain the following theorems.

Theorem 1.1. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$. Then the equation*

$$\begin{cases} \Delta u + \lambda\, u^p + \frac{u^{2^*-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has a positive solution if one of the following conditions holds:

(i) $\partial\Omega$ is C^3 at 0, $H(0) < 0$ and

$$\begin{cases} \lambda > 0 & \text{if } 1 < p < \frac{n+2}{n-2}, \\ 0 < \lambda < \lambda_1(\Omega) & \text{if } p = 1. \end{cases}$$

(ii) $\partial\Omega$ is C^3 at 0, $H(0) = 0$, $\lambda > 0$ and $1 < p < \frac{n+2}{n-2}$.
(iii) $\partial\Omega$ is C^2 at 0 (no restriction for the mean curvature), $\lambda > 0$ and $\frac{n}{n-2} < p < \frac{n+2}{n-2}$.
(iv) $\partial\Omega$ is C^3 at 0, $H(0) < 0$, $\lambda > 0$, $p = \frac{n+2}{n-2}$ and

$$\begin{cases} 0 < s < 1 & \text{if } n = 3, \\ 0 < s < 2 & \text{if } n \geq 4. \end{cases}$$

Our proof for Theorem 1.1 is inspired by the idea in Brézis-Nirenberg [1]. However, there is a major difference between our work and [1]. As it is well-known, the Sobolev best constant S_n is actually independent of Ω. This important fact was used in [1] implicitly. For our problem, $\mu_s(\Omega)$ defined in

Lemma 2.1 below does depend on the domain Ω. For the proof of the assertion (iv), besides using the fact that the Sobolev best constant S_n is independent of Ω, we also take advantage of that the energy level c^* in the mountain pass lemma (see the remark of Theorem C) is independent of the choice of v.

Furthermore, we prove the existence theorem of the following:

Theorem 1.2. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$, $\lambda > 0$, $\partial\Omega \in C^3$ at 0, $H(0) < 0$ and $1 \le p < \min\{\frac{n}{n-2}, 2^* - 1\}$. Then the equation*

$$\begin{cases} \Delta u - \lambda\, u^p + \frac{u^{2^*-1}}{|x|^s} = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.3}$$

has a positive solution.

For $\Omega = \mathbb{R}^n_+$, we prove the following theorem by the Pohozaev identity together with the blow-up argument used in the proof of Theorem 1.1.

Theorem 1.3. *For $n \ge 3$ and $\lambda > 0$, there exists a positive solution of the equation*

$$\begin{cases} \Delta u + \lambda\, u^{\frac{n+2}{n-2}} + \frac{u^{2^*-1}}{|x|^s} = 0 \quad \text{in } \mathbb{R}^n_+, \\ u = 0 \quad \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{1.4}$$

provided that

$$\begin{cases} 0 < s < 1 \quad \text{if } n = 3, \\ 0 < s < 2 \quad \text{if } n \ge 4. \end{cases}$$

This paper is organized as follows. Couples of auxiliary lemmas are collected in Section 2. The proof of theorems occupies Section 3-Section 5.

2 Preliminaries

In this section, we prove two lemmas for the proof of main theorems.

Lemma 2.1. *Suppose that Ω is a bounded domain with $0 \in \partial\Omega$, $\lambda > 0$ and assume one of the followings:*
(i) $\partial\Omega$ is C^3 at 0, $H(0) < 0$ and $1 \le p < \frac{n+2}{n-2}$.
(ii) $\partial\Omega$ is C^3 at 0, $H(0) = 0$ and $1 < p < \frac{n+2}{n-2}$.
(iii) $\partial\Omega$ is C^2 at 0 (no restriction for the mean curvature) and $\frac{n}{n-2} < p < \frac{n+2}{n-2}$.
Then there exists a nonnegative function $v_0 \in H^1_0(\Omega) \setminus \{0\}$ such that

$$\max_{t \ge 0} \Phi_{s,p}(tv_0) < \left(\frac{1}{2} - \frac{1}{2^*}\right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}}, \tag{2.1}$$

where

$$\Phi_{s,p}(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{\lambda}{p+1}(u^+)^{p+1} - \frac{1}{2^*}\frac{(u^+)^{2^*}}{|x|^s}\right) dx \quad \text{for } u \in H^1_0(\Omega),$$

and

$$\mu_s(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2 dx \,\middle|\, u \in H^1_0(\Omega) \text{ and } \int_\Omega \frac{|u|^{2^*}}{|x|^s} dx = 1\right\}.$$

For the proof of Lemma 2.1, we apply the following lemma:

Lemma 2.2. *Let $u \in H_0^1(\mathbb{R}^n_+)$ be an entire positive solution of*

$$\begin{cases} \Delta u + \mu_s(\mathbb{R}^n_+)\,\frac{u^{2^*-1}}{|y|^s} = 0 \quad \text{in } \mathbb{R}^n_+, \\ u = 0 \quad \text{on } \partial\mathbb{R}^n_+ \quad \text{with } \int_{\mathbb{R}^n_+} \frac{u^{2^*}}{|y|^s} dy = 1, \end{cases} \tag{2.2}$$

or

$$\begin{cases} \Delta u + \lambda\, u^{\frac{n+2}{n-2}} + \frac{u^{2^*-1}}{|y|^s} = 0 \quad \text{in } \mathbb{R}^n_+, \\ u = 0 \quad \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{2.3}$$

Then the followings hold:

(i)

$$\begin{cases} u \in C^2(\overline{\mathbb{R}^n_+}) \quad \text{if } s < 1 + \frac{2}{n}, \\ u \in C^{1,\beta}(\overline{\mathbb{R}^n_+}) \quad \text{for all } 0 < \beta < 1 \quad \text{if } s = 1 + \frac{2}{n}, \\ u \in C^{1,\beta}(\overline{\mathbb{R}^n_+}) \quad \text{for all } 0 < \beta < \frac{n(2-s)}{n-2} \quad \text{if } s > 1 + \frac{2}{n}. \end{cases}$$

(ii) There is a constant C such that $|u(y)| \le C(1+|y|)^{1-n}$ and $|\nabla u(y)| \le C(1+|y|)^{-n}$.
(iii) $u(y', y_n)$ is axially symmetric with respect to the y_n-axis, i.e., $u(y', y_n) = u(|y'|, y_n)$.

Concerning the proof of the existence of a solution, assertions (i) and (ii) for the equation (2.2), see Egnell [2] and Ghoussoub-Robert [3, 4]. The existence of a solution for the equation (2.3) is obtained in Theorem 1.3. For the rest part of the proof for Lemma 2.2, we refer to Lin-Wadade [5].

Proof of Lemma 2.1. We first show (i). Ghoussoub-Robert [3, 4] proved that if $H(0) < 0$, then $\mu_s(\Omega)$ is attained by a positive function v_0 with

$$\int_\Omega |\nabla v_0|^2 dx = \mu_s(\Omega) \quad \text{and} \quad \int_\Omega \frac{v_0^{2^*}}{|x|^s} dx = 1.$$

Then since $\lambda > 0$, we have

$$\begin{aligned} &\max_{t\ge 0} \Phi_{s,p}(tv_0) = \max_{t\ge 0} \left(\frac{t^2}{2} \int_\Omega |\nabla v_0|^2 dx - \frac{\lambda\, t^{p+1}}{p+1} \int_\Omega v_0^{p+1} dx - \frac{t^{2^*}}{2^*} \int_\Omega \frac{v_0^{2^*}}{|x|^s} dx \right) \\ &< \max_{t\ge 0} \left(\frac{t^2}{2} \int_\Omega |\nabla v_0|^2 dx - \frac{t^{2^*}}{2^*} \int_\Omega \frac{v_0^{2^*}}{|x|^s} dx \right) = \max_{t\ge 0} \left(\frac{t^2}{2} \mu_s(\Omega) - \frac{t^{2^*}}{2^*} \right) = \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}}. \end{aligned}$$

Thus (i) is proved.

Next, we shall prove assertions (ii) and (iii). Since $\mu_s(\Omega) \le \mu_s(\mathbb{R}^n_+)$, we consider two cases. First, if $\mu_s(\Omega) < \mu_s(\mathbb{R}^n_+)$, then $\mu_s(\Omega)$ can be achieved by some function in $H_0^1(\Omega)$, see [3, 4] and [5]. In this case, (1.2) can be proved by the same fashion used in the proof of the assertion (i). Hereafter, we only need to deal with the remaining case that $\mu_s(\Omega) = \mu_s(\mathbb{R}^n_+)$.

Without loss of generality, we may assume that in a neighborhood of 0, $\partial\Omega$ can be represented by $x_n = \varphi(x')$, $x' = (x_1, \cdots, x_{n-1})$, where $\varphi(0) = 0$, $\nabla'\varphi(0) = 0$, $\nabla' = (\partial_1, \cdots, \partial_{n-1})$, and the outer normal of $\partial\Omega$ is $-e_n$.

Let $u \in H_0^1(\mathbb{R}^n_+)$ be an entire positive solution of (2.2), and take U and $\tilde{U}$ to be neighborhoods of 0 such that $\Psi(U) = B_{r_0}(0)$ and $\Psi(\tilde{U}) = B_{\frac{r_0}{2}}(0)$, respectively. We define for $\varepsilon > 0$,

$$v_\varepsilon(x) := \varepsilon^{-\frac{n-2}{2}} u\left(\frac{\Psi(x)}{\varepsilon} \right) \quad \text{for } x \in \Omega \cap U, \quad \text{and} \quad \hat{v}_\varepsilon := \eta v_\varepsilon \quad \text{in } \Omega, \tag{2.4}$$

where $\eta \in C_c^\infty(U)$ is a positive cut-off function with $\eta \equiv 1$ in $\tilde{U}$, and

$$\Psi(x) := (x', x_n - \varphi(x')) \quad \text{for } x \in \overline{\Omega} \cap \overline{U}.$$

For $t \geq 0$, we have

$$\Phi_{s,p}(t\hat{v}_\varepsilon) \leq \frac{t^2}{2}\int_\Omega |\nabla \hat{v}_\varepsilon|^2 dx - \frac{\lambda\, t^{p+1}}{p+1}\int_\Omega \hat{v}_\varepsilon^{\,p+1} dx - \frac{t^{2^*}}{2^*}\int_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*}}{|x|^s} dx. \tag{2.5}$$

In what follows, we investigate each integral in (2.5) precisely. By a change of the variable $\frac{\Psi(x)}{\varepsilon} = y$ and Lemma 2.2, we get

$$\begin{aligned}
&\int_\Omega |\nabla \hat{v}_\varepsilon|^2 dx = \int_{\Omega\cap U} \eta^2 |\nabla v_\varepsilon|^2 dx - \int_{\Omega\cap U} \eta(\Delta\eta) v_\varepsilon^2 dx \\
&\leq \int_{\mathbb{R}^n_+} |\nabla u(y)|^2 dy - 2\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)^2 \partial_n u(y)\, \nabla' u(y)\cdot(\nabla'\varphi)(\varepsilon y')dy \\
&+ \int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)^2 (\partial_n u(y))^2 |(\nabla'\varphi)(\varepsilon y')|^2 dy - \varepsilon^2 \int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)(\Delta\eta)\left(\Psi^{-1}(\varepsilon y)\right) u(y)^2 dy \\
&= \int_{\mathbb{R}^n_+} |\nabla u(y)|^2 dy - 2\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)^2 \partial_n u(y)\, \nabla' u(y)\cdot(\nabla'\varphi)(\varepsilon y')dy + O(\varepsilon^2).
\end{aligned}$$

Using integration by parts and $\nabla' u \equiv 0$ on $\partial\mathbb{R}^n_+$, we obtain

$$\begin{aligned}
I :=& -2\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)^2 \partial_n u(y)\, \nabla' u(y)\cdot(\nabla'\varphi)(\varepsilon y')dy \\
=& -\frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)^2 \partial_n u(y)\, \nabla' u(y)\cdot\nabla'\left[\varphi(\varepsilon y')\right] dy \\
=& \frac{4}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)\nabla'\left[\eta\left(\Psi^{-1}(\varepsilon y)\right)\right]\cdot \partial_n u(y)\, \nabla' u(y)\, \varphi(\varepsilon y')dy \\
&+ \frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)^2 \nabla'\partial_n u(y)\cdot\nabla' u(y)\, \varphi(\varepsilon y')dy \\
&+ \frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)^2 \partial_n u(y)\sum_{i=1}^{n-1}\partial_{ii}u(y)\, \varphi(\varepsilon y')dy \\
=& \frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)^2 \partial_n u(y)\sum_{i=1}^{n-1}\partial_{ii}u(y)\, \varphi(\varepsilon y')dy + O(\varepsilon^n).
\end{aligned}$$

Applying the equation (2.2) and integration by parts, we have

$$\begin{aligned}
I' :=& \frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)^2 \partial_n u(y)\sum_{i=1}^{n-1}\partial_{ii}u(y)\, \varphi(\varepsilon y')dy \\
=& -\frac{2\mu_s(\mathbb{R}^n_+)}{2^*\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\left(\Psi^{-1}(\varepsilon y)\right)^2 \frac{\partial_n\left[u(y)^{2^*}\right]}{|y|^s}\varphi(\varepsilon y')dy
\end{aligned}$$

$$-\frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}}\eta\left(\Psi^{-1}(\varepsilon y)\right)^2\partial_n\left[(\partial_n u(y))^2\right]\varphi(\varepsilon y')dy$$
$$=-\frac{2s\mu_s(\mathbb{R}^n_+)}{2^*\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}}\eta\left(\Psi^{-1}(\varepsilon y)\right)^2\frac{u(y)^{2^*}y_n}{|y|^{s+2}}\varphi(\varepsilon y')dy$$
$$+\frac{1}{\varepsilon}\int_{\overline{B^+_{\frac{r_0}{\varepsilon}}}\cap\partial\mathbb{R}^n_+}\eta\left(\Psi^{-1}(\varepsilon y)\right)^2(\partial_n u(y))^2\varphi(\varepsilon y')dS_y+O(\varepsilon^n)=:J_1+J_2+O(\varepsilon^n).$$

If $\partial\Omega$ is C^3 at 0, φ can be expanded by

$$\varphi(y')=\sum_{i=1}^{n-1}\alpha_i y_i^2+O(|y'|^3). \tag{2.6}$$

Thus we see that

$$J_1=-\frac{2s\mu_s(\mathbb{R}^n_+)}{2^*\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}}\eta\left(\Psi^{-1}(\varepsilon y)\right)^2\frac{u(y)^{2^*}y_n}{|y|^{s+2}}\varphi(\varepsilon y')dy$$
$$=-\frac{2s\mu_s(\mathbb{R}^n_+)}{2^*\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}\setminus B^+_{\frac{r_0/2}{\varepsilon}}}\eta\left(\Psi^{-1}(\varepsilon y)\right)^2\frac{u(y)^{2^*}y_n}{|y|^{s+2}}\varphi(\varepsilon y')dy$$
$$-\frac{2s\mu_s(\mathbb{R}^n_+)}{2^*\varepsilon}\int_{B^+_{\frac{r_0/2}{\varepsilon}}}\frac{u(y)^{2^*}y_n}{|y|^{s+2}}\varphi(\varepsilon y')dy=:J_{1,1}+J_{1,2},\quad\text{and}$$

$$|J_{1,1}|\le C\varepsilon\int_{\{\frac{r_0}{2}\le|\varepsilon y|<r_0\}}|y|^{2^*(1-n)+1-s}dy=O(\varepsilon^{\frac{n(n-s)}{n-2}}).$$

Moreover, note that

$$\begin{cases}\varepsilon\int_{\mathbb{R}^n_+\setminus B^+_{\frac{r_0/2}{\varepsilon}}}u(y)^{2^*}|y|^{1-s}dy=O(\varepsilon^{\frac{n(n-s)}{n-2}}),\\ \varepsilon^2\int_{B^+_{\frac{r_0/2}{\varepsilon}}}u(y)^{2^*}|y|^{2-s}dy=O(\varepsilon^2),\end{cases} \tag{2.7}$$

which is integrable because $2^*(1-n)+2-s+n<0$, i.e., $n^2-(2+s)n+4>0$. Thus by using (2.6) and (2.7), we get

$$J_{1,2}=-\frac{2s\varepsilon\mu_s(\mathbb{R}^n_+)}{2^*}\sum_{i=1}^{n-1}\alpha_i\int_{\mathbb{R}^n_+}\frac{u(y)^{2^*}y_i^2y_n}{|y|^{2+s}}dy+O(\varepsilon^2)$$
$$=-\frac{2s\varepsilon\mu_s(\mathbb{R}^n_+)}{2^*(n-1)}\int_{\mathbb{R}^n_+}\frac{u(y)^{2^*}|y'|^2y_n}{|y|^{2+s}}dy\sum_{i=1}^{n-1}\alpha_i+O(\varepsilon^2)=-K_1H(0)\varepsilon+O(\varepsilon^2),$$

where

$$H(0):=\frac{1}{n-1}\sum_{i=1}^{n-1}\alpha_i\quad\text{and}\quad K_1:=\frac{2s\mu_s(\mathbb{R}^n_+)}{2^*}\int_{\mathbb{R}^n_+}\frac{u(y)^{2^*}|y'|^2y_n}{|y|^{2+s}}dy. \tag{2.8}$$

In the above estimate, we used the fact $u(y',y_n)=u(|y'|,y_n)$. Next, we see that

$$J_2=\frac{1}{\varepsilon}\int_{\overline{B^+_{\frac{r_0}{\varepsilon}}}\cap\partial\mathbb{R}^n_+}\eta\left(\Psi^{-1}(\varepsilon y)\right)^2(\partial_n u(y))^2\varphi(\varepsilon y')dS_y$$

$$= \frac{1}{\varepsilon}\int_{(\overline{B^+_{\frac{r_0}{\varepsilon}}}\setminus\overline{B^+_{\frac{r_0/2}{\varepsilon}}})\cap\partial\mathbb{R}^n_+} \eta\left(\Psi^{-1}(\varepsilon y)\right)^2 \left(\partial_n u(y)\right)^2 \varphi(\varepsilon y')dS_y$$

$$+ \frac{1}{\varepsilon}\int_{\overline{B^+_{\frac{r_0/2}{\varepsilon}}}\cap\partial\mathbb{R}^n_+} \left(\partial_n u(y)\right)^2 \varphi(\varepsilon y')dS_y =: J_{2,1} + J_{2,2}, \quad \text{and}$$

$$|J_{2,1}| \le \frac{C}{\varepsilon}\int_{\{\frac{r_0}{2}<|\varepsilon y'|\le r_0\}} |(\partial_n u)(y',0)|^2|\varphi(\varepsilon y')|dy'$$
$$\le C\varepsilon\int_{\{\frac{r_0}{2}<|\varepsilon y'|\le r_0\}} |y'|^{-2n+2}dy' = O(\varepsilon^n).$$

Moreover, note that $|(\partial_n u)(y',0)|^2|y'|^3 = O(|y'|^{-2n+3})$ for large $|y'|$ and $2n-3 > n-1$ for $n \ge 3$. Hence, it is integrable and

$$\begin{cases} \varepsilon\int_{\{|\varepsilon y'|>\frac{r_0}{2}\}} |(\partial_n u)(y',0)|^2|y'|^2dy' = O(\varepsilon^n), \\ \varepsilon^2\int_{\{|\varepsilon y'|\le\frac{r_0}{2}\}} |(\partial_n u)(y',0)|^2|y'|^3dy' = O(\varepsilon^2). \end{cases} \tag{2.9}$$

Thus by using (2.6) and (2.9), we get

$$J_{2,2} = \varepsilon\sum_{i=1}^{n-1}\alpha_i\int_{\mathbb{R}^{n-1}} \left((\partial_n u)(y',0)\right)^2 y_i^2dy' + O(\varepsilon^2)$$
$$= \frac{\varepsilon}{n-1}\int_{\mathbb{R}^{n-1}} |(\nabla u)(y',0)|^2|y'|^2dy'\sum_{i=1}^{n-1}\alpha_i + O(\varepsilon^2) = K_2H(0)\varepsilon + O(\varepsilon^2),$$

where

$$K_2 := \int_{\mathbb{R}^{n-1}} |(\nabla u)(y',0)|^2|y'|^2dy'. \tag{2.10}$$

After all, we get

$$\int_\Omega |\nabla\hat{v}_\varepsilon|^2dx \le \mu_s(\mathbb{R}^n_+) - K_1H(0)\varepsilon + K_2H(0)\varepsilon + O(\varepsilon^2). \tag{2.11}$$

Next, by changing the variable $\frac{\Psi(x)}{\varepsilon} = y$, we have

$$\int_\Omega \hat{v}_\varepsilon^{p+1}dx = \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}}\int_{\mathbb{R}^n_+} u^{p+1}dy + O(\varepsilon^{\frac{n(p+1)}{2}}). \tag{2.12}$$

Furthermore, the integral $\int_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*}}{|x|^s}dx$ can be estimated as follows. By a change of the variable $\frac{\Psi(x)}{\varepsilon} = y$, we have

$$\int_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*}}{|x|^s}dx = \int_{B^+_{\frac{r_0/2}{\varepsilon}}} \frac{u^{2^*}}{\left|\frac{\Psi^{-1}(\varepsilon y)}{\varepsilon}\right|^s}dy. \tag{2.13}$$

Since $\Psi^{-1}(y) = (y', y_n + \varphi(y'))$, it holds $|\Psi^{-1}(y)|^2 = |y|^2 + 2y_n\varphi(y') + (\varphi(y'))^2$, and then

$$\frac{1}{\left|\frac{\Psi^{-1}(\varepsilon y)}{\varepsilon}\right|^s} = \frac{1}{|y|^s}\cdot\frac{1}{\left(1+\frac{2y_n\varphi(\varepsilon y')}{\varepsilon|y|^2}+\frac{(\varphi(\varepsilon y'))^2}{\varepsilon^2|y|^2}\right)^{\frac{s}{2}}}$$

$$= \frac{1}{|y|^s}\left(1 - \frac{sy_n\varphi(\varepsilon y')}{\varepsilon|y|^2} - \frac{s\left(\varphi(\varepsilon y')\right)^2}{2\varepsilon^2|y|^2}\right) + C\frac{1}{|y|^s}\left(\frac{2y_n\varphi(\varepsilon y')}{\varepsilon|y|^2} + \frac{\left(\varphi(\varepsilon y')\right)^2}{\varepsilon^2|y|^2}\right)^2. \tag{2.14}$$

Thus from (2.13) and (2.14), we obtain

$$\begin{aligned}
\int_{\Omega\cap\tilde U}\frac{v_\varepsilon^{2^*}}{|x|^s}dx &= \int_{\mathbb{R}^n_+}\frac{u^{2^*}}{|y|^s}dy - \frac{s}{\varepsilon}\int_{B^+_{\frac{r_0/2}{\varepsilon}}}\frac{u(y)^{2^*}\,y_n\varphi(\varepsilon y')}{|y|^{2+s}}dy + O(\varepsilon^2)\\
&= 1 - s\varepsilon\sum_{i=1}^{n-1}\alpha_i\int_{\mathbb{R}^n_+}\frac{u(y)^{2^*}\,y_i^2y_n}{|y|^{2+s}}dy + O(\varepsilon^2)\\
&= 1 - \frac{s\varepsilon}{n-1}\int_{\mathbb{R}^n_+}\frac{u(y)^{2^*}|y'|^2y_n}{|y|^{2+s}}dy\sum_{i=1}^{n-1}\alpha_i + O(\varepsilon^2) = 1 - \frac{2^*K_1}{2\mu_s(\mathbb{R}^n_+)}H(0)\varepsilon + O(\varepsilon^2),
\end{aligned}$$

where K_1 is the same positive constant as in (2.8).

After all, each integral can be estimated by

$$\begin{cases}
\int_\Omega|\nabla\hat v_\varepsilon|^2dx \le \mu_s(\mathbb{R}^n_+) - K_1H(0)\varepsilon + K_2H(0)\varepsilon + O(\varepsilon^2),\\
\int_\Omega \hat v_\varepsilon^{\,p+1}dx = \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}}\int_{\mathbb{R}^n_+}u^{p+1}dy + O(\varepsilon^{\frac{n(p+1)}{2}}),\\
\int_{\Omega\cap\tilde U}\frac{v_\varepsilon^{2^*}}{|x|^s}dx = 1 - \frac{2^*K_1}{2\mu_s(\mathbb{R}^n_+)}H(0)\varepsilon + O(\varepsilon^2).
\end{cases} \tag{2.15}$$

By (2.5) and (2.15), we have for $t \ge 0$,

$$\begin{aligned}
\Phi_{s,p}(t\hat v_\varepsilon) &\le \frac{t^2}{2}\left(\mu_s(\mathbb{R}^n_+) - K_1H(0)\varepsilon + K_2H(0)\varepsilon + O(\varepsilon^2)\right)\\
&- \frac{\lambda\,t^{p+1}}{p+1}\left(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}}\int_{\mathbb{R}^n_+}u^{p+1}dy + O(\varepsilon^{\frac{n(p+1)}{2}})\right) - \frac{t^{2^*}}{2^*}\left(1 - \frac{2^*K_1}{2\mu_s(\mathbb{R}^n_+)}H(0)\varepsilon + O(\varepsilon^2)\right).
\end{aligned} \tag{2.16}$$

We claim that the right-hand side has the maximum point t_m expressed by

$$\begin{aligned}
t_m &= \mu_s(\mathbb{R}^n_+)^{\frac{1}{2^*-2}} + \frac{1}{2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}}K_1H(0)\varepsilon + \frac{1}{2^*-2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}}K_2H(0)\varepsilon\\
&- \frac{\lambda}{2^*-2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-2-p}{2^*-2}}\int_{\mathbb{R}^n_+}u^{p+1}dy\,\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} + O(\varepsilon^2) + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}+1}) + O(\varepsilon^{n+2-(n-2)p}).
\end{aligned} \tag{2.17}$$

Indeed, set

$$t_m = \mu_s(\mathbb{R}^n_+)^{\frac{1}{2^*-2}} + A(\varepsilon) \quad \text{with } A(\varepsilon) \to 0 \tag{2.18}$$

as $\varepsilon \to 0$. Since t_m is the maximum point, we have

$$\begin{aligned}
&\mu_s(\mathbb{R}^n_+) - K_1H(0)\varepsilon + K_2H(0)\varepsilon + O(\varepsilon^2) - \lambda t_m^{p-1}\left(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}}\int_{\mathbb{R}^n_+}u^{p+1}dy + O(\varepsilon^{\frac{n(p+1)}{2}})\right)\\
&- t_m^{2^*-2}\left(1 - \frac{2^*K_1}{2\mu_s(\mathbb{R}^n_+)}H(0)\varepsilon + O(\varepsilon^2)\right) = 0.
\end{aligned} \tag{2.19}$$

By substituting (2.18) into (2.19), we see that

$$\mu_s(\mathbb{R}^n_+) - K_1H(0)\varepsilon + K_2H(0)\varepsilon + O(\varepsilon^2)$$

$$
\begin{aligned}
&- \lambda \Big(\mu_s(\mathbb{R}^n_+)^{\frac{1}{2^*-2}} + A(\varepsilon)\Big)^{p-1} \left(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} \int_{\mathbb{R}^n_+} u^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}})\right) \\
&- \Big(\mu_s(\mathbb{R}^n_+)^{\frac{1}{2^*-2}} + A(\varepsilon)\Big)^{2^*-2} \left(1 - \frac{2^* K_1}{2\mu_s(\mathbb{R}^n_+)} H(0)\varepsilon + O(\varepsilon^2)\right) \\
&= \mu_s(\mathbb{R}^n_+) - K_1 H(0)\varepsilon + K_2 H(0)\varepsilon + O(\varepsilon^2) \\
&- \lambda\Big(\mu_s(\mathbb{R}^n_+)^{\frac{p-1}{2^*-2}} + (p-1)\mu_s(\mathbb{R}^n_+)^{\frac{p-2}{2^*-2}} A(\varepsilon) + O(A(\varepsilon)^2)\Big)\left(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} \int_{\mathbb{R}^n_+} u^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}})\right) \\
&- \Big(\mu_s(\mathbb{R}^n_+) + (2^*-2)\mu_s(\mathbb{R}^n_+)^{\frac{2^*-3}{2^*-2}} A(\varepsilon) + O(A(\varepsilon)^2)\Big) \left(1 - \frac{2^* K_1}{2\mu_s(\mathbb{R}^n_+)} H(0)\varepsilon + O(\varepsilon^2)\right) \\
&= \frac{2^*-2}{2} K_1 H(0)\varepsilon + K_2 H(0)\varepsilon - \lambda\, \mu_s(\mathbb{R}^n_+)^{\frac{p-1}{2^*-2}} \int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} + O(\varepsilon^2) + O(A(\varepsilon)^2) \\
&- A(\varepsilon) \left((2^*-2)\mu_s(\mathbb{R}^n_+)^{\frac{2^*-3}{2^*-2}} - \frac{2^*(2^*-2)}{2} \mu_s(\mathbb{R}^n_+)^{-\frac{1}{2^*-2}} K_1 H(0)\,\varepsilon + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}})\right) = 0,
\end{aligned}
$$

where note that

$$
\frac{n+2}{2} - \frac{(n-2)p}{2} \le 2, \text{ i.e., } p \ge 1, \quad \text{and} \quad \frac{n+2}{2} - \frac{(n-2)p}{2} \le \frac{n(p+1)}{2}, \text{ i.e., } p \ge \frac{1}{n-1}.
$$

Then we have

$$
\begin{aligned}
A(\varepsilon) &= \left((2^*-2)\mu_s(\mathbb{R}^n_+)^{\frac{2^*-3}{2^*-2}} - \frac{2^*(2^*-2)}{2} \mu_s(\mathbb{R}^n_+)^{-\frac{1}{2^*-2}} K_1 H(0)\,\varepsilon + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}})\right)^{-1} \\
&\times \left(\frac{2^*-2}{2} K_1 H(0)\varepsilon + K_2 H(0)\varepsilon - \lambda\, \mu_s(\mathbb{R}^n_+)^{\frac{p-1}{2^*-2}} \int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} + O(\varepsilon^2)\right) + O(A(\varepsilon)^2) \\
&=: B(\varepsilon) + O(A(\varepsilon)^2).
\end{aligned}
$$

Here, note that $\lim_{\varepsilon \to 0} \dfrac{B(\varepsilon)}{A(\varepsilon)} = 1$, which implies $A(\varepsilon) = B(\varepsilon) + O(B(\varepsilon)^2)$. Moreover, we have

$$
\begin{aligned}
B(\varepsilon) &= \left(\frac{1}{2^*-2} \mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} + \frac{2^*}{2(2^*-2)} \mu_s(\mathbb{R}^n_+)^{-\frac{2\cdot 2^*-5}{2^*-2}} K_1 H(0)\varepsilon + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}})\right) \\
&\times \left(\frac{2^*-2}{2} K_1 H(0)\varepsilon + K_2 H(0)\varepsilon - \lambda\, \mu_s(\mathbb{R}^n_+)^{\frac{p-1}{2^*-2}} \int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} + O(\varepsilon^2)\right) \\
&= \frac{1}{2} \mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_1 H(0)\varepsilon + \frac{1}{2^*-2} \mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_2 H(0)\varepsilon + O(\varepsilon^2) \\
&- \frac{\lambda}{2^*-2} \mu_s(\mathbb{R}^n_+)^{-\frac{2^*-2-p}{2^*-2}} \int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}+1}) + O(\varepsilon^{n+2-(n-2)p}). \qquad (2.20)
\end{aligned}
$$

As a consequence, we obtain (2.17). By (2.16) and (2.17), we see that

$$
\begin{aligned}
&\max_{t \ge 0} \Phi_{s,p}(t\hat{v}_\varepsilon) \\
&\le \frac{1}{2}\Bigg[\mu_s(\mathbb{R}^n_+)^{\frac{2}{2^*-2}} + 2\mu_s(\mathbb{R}^n_+)^{\frac{1}{2^*-2}} \Bigg(\frac{1}{2} \mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_1 H(0)\varepsilon + \frac{1}{2^*-2} \mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_2 H(0)\varepsilon
\end{aligned}
$$

$$
- \frac{\lambda}{2^*-2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-2-p}{2^*-2}} \int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}}\Bigg) + O(\varepsilon^2) + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}+1}) + O(\varepsilon^{n+2-(n-2)p})\Bigg]
$$

$$
\times \left(\mu_s(\mathbb{R}^n_+) - K_1 H(0)\varepsilon + K_2 H(0)\varepsilon + O(\varepsilon^2)\right) - \frac{\lambda}{p+1}\Bigg[\mu_s(\mathbb{R}^n_+)^{\frac{p+1}{2^*-2}} + (p+1)\mu_s(\mathbb{R}^n_+)^{\frac{p}{2^*-2}}
$$

$$
\times \Bigg(\frac{1}{2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_1 H(0)\varepsilon + \frac{1}{2^*-2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_2 H(0)\varepsilon
$$

$$
- \frac{\lambda}{2^*-2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-2-p}{2^*-2}} \int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}}\Bigg) + O(\varepsilon^2) + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}+1}) + O(\varepsilon^{n+2-(n-2)p})\Bigg]
$$

$$
\times \left(\int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} + O(\varepsilon^{\frac{n(p+1)}{2}})\right)
$$

$$
- \frac{1}{2^*}\Bigg[\mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}} + 2^*\mu_s(\mathbb{R}^n_+)^{\frac{2^*-1}{2^*-2}}\Bigg(\frac{1}{2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_1 H(0)\varepsilon + \frac{1}{2^*-2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_2 H(0)\varepsilon
$$

$$
- \frac{\lambda}{2^*-2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-2-p}{2^*-2}} \int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}}\Bigg) + O(\varepsilon^2) + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}+1}) + O(\varepsilon^{n+2-(n-2)p})\Bigg]
$$

$$
\times \left(1 - \frac{2^* K_1}{2\mu_s(\mathbb{R}^n_+)} H(0)\varepsilon + O(\varepsilon^2)\right)
$$

$$
= \left(\frac{1}{2} - \frac{1}{2^*}\right)\mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}} + \frac{1}{2}\mu_s(\mathbb{R}^n_+)^{\frac{2}{2^*-2}} K_2 H(0)\varepsilon - \frac{\lambda}{p+1}\mu_s(\mathbb{R}^n_+)^{\frac{p+1}{2^*-2}} \int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}}
$$

$$
+ O(\varepsilon^2) + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}+1}) + O(\varepsilon^{n+2-(n-2)p}). \tag{2.21}
$$

By using (2.21), we can show (ii) and (iii) as follows. First, if $\lambda > 0$, $H(0) = 0$ and $1 < p < \frac{n+2}{n-2}$, i.e., $2 - \left(\frac{n+2}{2} - \frac{(n-2)p}{2}\right) > 0$, we have for small $\varepsilon > 0$,

$$
\max_{t \geq 0} \Phi_{s,p}(t\hat{v}_\varepsilon)
$$

$$
\leq \left(\frac{1}{2} - \frac{1}{2^*}\right)\mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}} - \frac{\lambda}{p+1}\mu_s(\mathbb{R}^n_+)^{\frac{p+1}{2^*-2}} \int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} + O(\varepsilon^2) + o(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}})
$$

$$
< \left(\frac{1}{2} - \frac{1}{2^*}\right)\mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}}.
$$

Thus (ii) is proved.

Next, if $\lambda > 0$ and $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, i.e., $1 - \left(\frac{n+2}{2} - \frac{(n-2)p}{2}\right) > 0$, we have for small $\varepsilon > 0$,

$$
\max_{t \geq 0} \Phi_{s,p}(t\hat{v}_\varepsilon)
$$

$$
\leq \left(\frac{1}{2} - \frac{1}{2^*}\right)\mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}} - \frac{\lambda}{p+1}\mu_s(\mathbb{R}^n_+)^{\frac{p+1}{2^*-2}} \int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} + O(\varepsilon) + o(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}})
$$

$$
< \left(\frac{1}{2} - \frac{1}{2^*}\right)\mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}}.
$$

Thus (iii) is proved. □

In the proof of the assertion (iv) of Theorem 1.1, we use the information derived from the positive solution of the equation

$$\begin{cases} \Delta v + \lambda\, v^{\frac{n+2}{n-2}} + \frac{v^{2^*-1}}{|x|^s} = 0 \quad \text{in } \mathbb{R}^n_+, \\ v = 0 \quad \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{2.22}$$

If the equation (2.22) has no positive solution, the proof of Theorem 1.1 is valid. In case (2.22) has a positive solution, we use the following lemma in our proof.

Lemma 2.3. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$, $\lambda > 0$, $\partial\Omega \in C^3$ at 0 and $H(0) < 0$. Then there exists a nonnegative function $v_0 \in H^1_0(\Omega) \setminus \{0\}$ such that $\Phi_{s,*}(v_0) < 0$ and*

$$\max_{t \geq 0} \Phi_{s,*}(tv_0) < c^*,$$

where

$$\Phi_{s,*}(u) := \Phi_{s,\frac{n+2}{n-2}}(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{(n-2)\lambda}{2n}(u^+)^{\frac{2n}{n-2}} - \frac{1}{2^*}\frac{(u^+)^{2^*}}{|x|^s} \right) dx \quad \textit{for } u \in H^1_0(\Omega),$$

$c^* := \Phi_{s,*}(v)$, and $v \in H^1_0(\mathbb{R}^n_+)$ is the positive solution of the equation (2.22).

Proof. Considering the proof of Lemma 2.1, we take $u = v \in H^1_0(\mathbb{R}^n_+)$ and define $\hat{v}_\varepsilon \in H^1_0(\Omega)$ by (2.4). From the estimates (2.15), we get for $t \geq 0$,

$$\begin{aligned}
&\Phi_{s,*}(t\hat{v}_\varepsilon) \leq \frac{t^2}{2}\int_\Omega |\nabla \hat{v}_\varepsilon|^2 dx - \frac{\lambda\, t^{p+1}}{p+1}\int_\Omega \hat{v}_\varepsilon^{\,p+1} dx - \frac{t^{2^*}}{2^*}\int_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*}}{|x|^s}dx \\
&\leq \frac{t^2}{2}\left(\int_{\mathbb{R}^n_+} |\nabla v|^2 dy - K_1' H(0)\varepsilon + K_2' H(0)\varepsilon + O(\varepsilon^2) \right) \\
&- \frac{\lambda\, t^{p+1}}{p+1}\left(\int_{\mathbb{R}^n_+} v^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}}) \right) - \frac{t^{2^*}}{2^*}\left(\int_{\mathbb{R}^n_+} \frac{v^{2^*}}{|y|^s}dy - \frac{2^* K_1'}{2}H(0)\varepsilon + O(\varepsilon^2) \right) \\
&= \frac{t^2}{2}\int_{\mathbb{R}^n_+} |\nabla v|^2 dy - \frac{\lambda\, t^{p+1}}{p+1}\int_{\mathbb{R}^n_+} v^{p+1} dy - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^n_+} \frac{v^{2^*}}{|y|^s}dy \\
&+ \frac{H(0)}{2}\left((K_2' - K_1')t^2 + K_1' t^{2^*} \right)\varepsilon + O(\varepsilon^2) \\
&:= f_1(t) + \frac{H(0)\varepsilon}{2} f_2(t) + O(\varepsilon^2),
\end{aligned} \tag{2.23}$$

where

$$K_1' = \frac{2s}{2^*}\int_{\mathbb{R}^n_+} \frac{v^{2^*}|y'|^2 y_n}{|y|^{2+s}}dy \quad \text{and} \quad K_2' = \int_{\mathbb{R}^{n-1}} |(\nabla v)(y',0)|^2 |y'|^2 dy'.$$

Since $2^* > 2$, $\frac{2n}{n-2} > 2$ and

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 dy = \lambda \int_{\mathbb{R}^n_+} v^{p+1} dy + \int_{\mathbb{R}^n_+} \frac{v^{2^*}}{|y|^s}dy,$$

we find

$$\sup_{t \geq 0} f_1(t) = f_1(1) = c^* \quad \text{and} \quad f_2(t) > 0 \text{ for } t > t_1,$$

where

$$t_1 := \begin{cases} 0 \quad \text{if } K_2' > K_1', \\ \left(\frac{K_1' - K_2'}{K_1'}\right)^{\frac{1}{2^*-2}} < 1 \quad \text{if } K_2' < K_1'. \end{cases}$$

Hence, in case $H(0) < 0$ and ε small, we conclude

$$\Phi_{s,*}(t\hat{v}_\varepsilon) < f_1(1) = c^*.$$

Finally, we take $v_0 = t_0\hat{v}_\varepsilon$ where t_0 is large enough so that $\Phi_{s,*}(v_0) < 0$. The lemma is proved. □

Lemma 2.4. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$. If (i) $n = 3$ and $0 < s < 1$ or (ii) $n \geq 4$ and $0 < s < 2$, there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $\Phi_{s,*}(v_0) < 0$ and*

$$\max_{t\geq 0} \Phi_{s,*}(tv_0) < \frac{1}{n}\lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},$$

where

$$\Phi_{s,*}(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{(n-2)\lambda}{2n}(u^+)^{\frac{2n}{n-2}} - \frac{1}{2^*}\frac{(u^+)^{2^*}}{|x|^s}\right) dx \quad \textit{for } u \in H_0^1(\Omega)$$

and

$$S_n := \inf\left\{ \int_\Omega |\nabla u|^2 dx \,\middle|\, u \in H_0^1(\Omega) \textit{ and } \int_\Omega |u|^{\frac{2n}{n-2}} dx = 1 \right\}.$$

Remark. It is well-known that when $\Omega = \mathbb{R}^n$, S_n is achieved by the function

$$g(x) = C(1 + |x|^2)^{-\frac{n-2}{2}},$$

where C is a normalization constant.

Proof of Lemma 2.4. Let x_0 be an interior point of Ω such that $B(x_0, 3r) \subset \Omega$. Take $\phi(x) \in C_c^\infty(\Omega)$ be a cut off function with $\phi|_{B(x_0,r)} = 1$ and $\phi(x)|_{\Omega\setminus B_{(x_0,2r)}} = 0$. Consider

$$g_\varepsilon(x) := \varepsilon^{-\frac{n-2}{2}} g\left(\frac{x - x_0}{\varepsilon}\right) \phi(x) \in H_0^1(\Omega).$$

For $t \geq 0$, we have

$$\Phi_{s,*}(tg_\varepsilon) = \frac{t^2}{2}\int_\Omega |\nabla g_\varepsilon|^2 dx - \frac{(n-2)\lambda\, t^{\frac{2n}{n-2}}}{2n}\int_\Omega g_\varepsilon^{\frac{2n}{n-2}} dx - \frac{t^{2^*}}{2^*}\int_\Omega \frac{g_\varepsilon^{2^*}}{|x|^s} dx. \tag{2.24}$$

Using the integration by parts and a change of the variable $y = \frac{x-x_0}{\varepsilon}$, we get

$$\begin{aligned} &\int_\Omega |\nabla g_\varepsilon(x)|^2 dx \\ &= \varepsilon^{-n}\int_{B(x_0,2r)} \left|(\nabla g)\left(\frac{x-x_0}{\varepsilon}\right)\right|^2 \phi(x)dx - \varepsilon^{2-n}\int_{B(x_0,2r)\setminus B(x_0,r)} g\left(\frac{x-x_0}{\varepsilon}\right)^2 \phi(x)\Delta\phi(x)dx \\ &= \int_{B(0,\frac{2r}{\varepsilon})} |\nabla g(y)|^2 \phi(x_0 + \varepsilon y)dy - \varepsilon^2 \int_{B(0,\frac{2r}{\varepsilon})\setminus B(0,\frac{r}{\varepsilon})} g(y)^2 \phi(x_0 + \varepsilon y)(\Delta\phi)(x_0 + \varepsilon y)dy. \end{aligned}$$

Direct calculation gives

$$\begin{cases} \int_{B(0,\frac{2r}{\varepsilon})} |\nabla g(y)|^2 dy = \int_{\mathbb{R}^n} |\nabla g(y)|^2 dy + O(\varepsilon^{n-2}), \\ \int_{\mathbb{R}^n\setminus B(0,\frac{r}{\varepsilon})} g(y)^2 dy = \begin{cases} O\left(\frac{1}{\varepsilon}\right) \quad \text{if } n = 3, \\ O(\log(\frac{1}{\varepsilon})) \quad \text{if } n = 4, \\ O(\varepsilon^{n-4}) \quad \text{if } n \geq 5. \end{cases} \end{cases}$$

Hence,

$$\int_\Omega |\nabla g_\varepsilon(x)|^2 dx = \int_{\mathbb{R}^n} |\nabla g(y)|^2 dy + \begin{cases} O(\varepsilon) & \text{if } n=3, \\ O(\varepsilon^2 \log(\frac{1}{\varepsilon})) & \text{if } n=4, \\ O(\varepsilon^{n-2}) & \text{if } n \geq 5. \end{cases}$$

Next, we have

$$\int_\Omega g_\varepsilon(x)^{\frac{2n}{n-2}} dx = \int_{\mathbb{R}^n} g(y)^{\frac{2n}{n-2}} dy + O(\varepsilon^n),$$

and

$$\int_\Omega \frac{g_\varepsilon(x)^{2^*}}{|x|^s} dx = \varepsilon^s \int_{B(0,\frac{2r}{\varepsilon})} \frac{(1+|y|^2)^{-(n-s)}}{|x_0+\varepsilon y|^s} \phi(x_0+\varepsilon y)^{2^*} dy = O(\varepsilon^s).$$

Therefore,

$$\begin{aligned} \Phi_{s,*}(tg_\varepsilon) = {} & \frac{t^2}{2} \int_{\mathbb{R}^n} |\nabla g(y)|^2 dy - \frac{(n-2)\lambda\, t^{\frac{2n}{n-2}}}{2n} \int_{\mathbb{R}^n} g(y)^{\frac{2n}{n-2}} dy \\ & - \varepsilon^s \int_{B(0,\frac{2r}{\varepsilon})} \frac{(1+|y|^2)^{-(n-s)}}{|x_0+\varepsilon y|^s} \phi(x_0+\varepsilon y)^{2^*} dy + \begin{cases} O(\varepsilon) & \text{if } n=3, \\ O(\varepsilon^2 \log(\frac{1}{\varepsilon})) & \text{if } n=4, \\ O(\varepsilon^{n-2}) & \text{if } n \geq 5. \end{cases} \end{aligned}$$

Elementary calculus gives

$$\max_{t \geq 0} \left(\frac{t^2}{2} \int_{\mathbb{R}^n} |\nabla g(y)|^2 dy - \frac{(n-2)\lambda\, t^{\frac{2n}{n-2}}}{2n} \int_{\mathbb{R}^n} g(y)^{\frac{2n}{n-2}} dy \right) = \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}}.$$

To sum up,

$$\max_{t \geq 0} \Phi_{s,*}(tg_\varepsilon) < \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},$$

if either (i) $n=3$ and $0<s<1$ or (ii) $n \geq 4$ and $0<s<2$. This completes the proof. □

We need the following lemma in the proof of Theorem 1.2.

Lemma 2.5. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$, $\lambda > 0$, $\partial\Omega \in C^3$ at 0, $H(0) < 0$ and $1 \leq p < \frac{n}{n-2}$. Then there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $\Psi_{s,p}(v_0) < 0$ and*

$$\max_{0 \leq t \leq 1} \Psi_{s,p}(tv_0) < \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}},$$

where

$$\Psi_{s,p}(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{\lambda}{p+1}(u^+)^{p+1} - \frac{1}{2^*} \frac{(u^+)^{2^*}}{|x|^s} \right) dx \quad \text{for } u \in H_0^1(\Omega).$$

Proof. As in the proof of Lemma 2.1, we use the entire solution $u \in H_0^1(\mathbb{R}^n_+)$ of the equation (2.2), and define $\hat{v}_\varepsilon \in H_0^1(\Omega)$ by (2.4). Then from the estimates (2.15), we obtain for $t \geq 0$,

$$\begin{aligned} \Psi_{s,p}(t\hat{v}_\varepsilon) & \leq \frac{t^2}{2} \int_\Omega |\nabla \hat{v}_\varepsilon|^2 dx + \frac{\lambda\, t^{p+1}}{p+1} \int_\Omega \hat{v}_\varepsilon^{\,p+1} dx - \frac{t^{2^*}}{2^*} \int_\Omega \frac{v_\varepsilon^{2^*}}{|x|^s} dx \\ & \leq \frac{t^2}{2} \left(\mu_s(\mathbb{R}^n_+) - K_1 H(0)\varepsilon + K_2 H(0)\varepsilon + O(\varepsilon^2) \right) \\ & + \frac{\lambda\, t^{p+1}}{p+1} \left(\varepsilon^{\frac{n+2}{2} - \frac{(n-2)p}{2}} \int_{\mathbb{R}^n_+} u^{p+1} dy + O(\varepsilon^{\frac{n(p+1)}{2}}) \right) - \frac{t^{2^*}}{2^*} \left(1 - \frac{2^* K_1}{2\mu_s(\mathbb{R}^n_+)} H(0)\varepsilon + O(\varepsilon^2) \right). \end{aligned} \tag{2.25}$$

It is easy to see that for each small $\varepsilon > 0$, there exists $t_0 > 0$ such that $\Psi_{s,p}(t_0\hat{v}_\varepsilon) < 0$. Moreover, let t_m be the maximum point of the right-hand side of (2.25) in $(0, t_0)$. Then as in the same way to the proof of Lemma 2.1, we can find the expression of t_m by

$$\begin{aligned} t_m &= \mu_s(\mathbb{R}^n_+)^{\frac{1}{2^*-2}} + \frac{1}{2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_1 H(0)\varepsilon + \frac{1}{2^*-2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_2 H(0)\varepsilon + O(\varepsilon^2) \\ &+ \frac{\lambda}{2^*-2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-2-p}{2^*-2}} \int_{\mathbb{R}^n_+} u^{p+1} dy\, \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}+1}) + O(\varepsilon^{n+2-(n-2)p}) \\ &= \mu_s(\mathbb{R}^n_+)^{\frac{1}{2^*-2}} + \frac{1}{2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_1 H(0)\varepsilon + \frac{1}{2^*-2}\mu_s(\mathbb{R}^n_+)^{-\frac{2^*-3}{2^*-2}} K_2 H(0) + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}}), \quad (2.26) \end{aligned}$$

where note that $1 \le p < \frac{n}{n-2}$ implies $\frac{n+2}{2} - \frac{(n-2)p}{2} \le 2 < \min\{\frac{n+2}{2} - \frac{(n-2)p}{2} + 1, n+2-(n-2)p\}$. By substituting (2.26) into (2.25), we eventually get

$$\begin{aligned} \max_{0 \le t \le t_0} \Psi_{s,p}(t\hat{v}_\varepsilon) &\le \left(\frac{1}{2} - \frac{1}{2^*}\right)\mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}} + \frac{1}{2}\mu_s(\mathbb{R}^n_+)^{\frac{2}{2^*-2}} K_2 H(0)\varepsilon + O(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}}) \\ &< \left(\frac{1}{2} - \frac{1}{2^*}\right)\mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}}, \end{aligned}$$

which is possible since $H(0) < 0$ and $\frac{n+2}{2} - \frac{(n-2)p}{2} - 1 > 0$, i.e., $p < \frac{n}{n-2}$. Thus Lemma 2.5 is proved. □

3 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1 by applying Lemma 2.1 and the mountain pass lemma of the following type:

Theorem C. *Let Φ be a C^1-function on a Banach space E. Assume that there exist an open set $0 \in U \subset E$ and $\rho \in \mathbb{R}$ such that*

$$\begin{cases} \Phi(u) \ge \rho \quad \text{for all } u \in \partial U, \\ \Phi(0) < \rho, \quad \Phi(v) < \rho \quad \text{for some } v \ne U. \end{cases} \tag{3.1}$$

Set

$$c := \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi(w) \ge \rho,$$

where $\mathcal{P}$ denotes the class of continuous paths joining 0 to v. Then there exists a sequence $\{u_j\} \subset E$ such that

$$\begin{cases} \Phi(u_j) \to c, \\ \Phi'(u_j) \to 0 \quad \text{in } E^*. \end{cases}$$

Remark. Suppose $v_i \in H^1_0(\Omega)$ with $\Phi_{s,*}(v_i) < 0$ for $i = 1, 2$, where $\Phi_{s,*}$ is as defined in Lemma 2.3. Then $\Phi_{s,*}(tv_i) < 0$ for $t > 1$, and $\Phi_{s,*}(tv_i) \to -\infty$ as $t \to \infty$. Hence, there exists a continuous path $v(\eta) \in H^1_0(\Omega)$ with $v(0) = v_1$, $v(1) = v_2$ and $\Phi_{s,*}(v(\eta)) < 0$. Therefore, if we take

$$c := \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi_{s,*}(w) > 0, \tag{3.2}$$

then c is independent of the choice of v as long as $\Phi_{s,*}(v) < 0$.

Proof of the Subcritical Case for Theorem 1.1. In what follows, take $E = H_0^1(\Omega)$ in Theorem C and check the condition (3.1). By the Sobolev inequality and the Sobolev-Hardy inequality, we have

$$\begin{cases} \int_\Omega (u^+)^{p+1}dx \le \begin{cases} \frac{1}{\lambda_1(\Omega)}\|\nabla u\|_{L^2(\Omega)}^2 & \text{if } p=1, \\ C\|\nabla u\|_{L^2(\Omega)}^{p+1} & \text{if } p>1, \end{cases} \\ \int_\Omega \frac{(u^+)^{2^*}}{|x|^s}dx \le C\|\nabla u\|_{L^2(\Omega)}^{2^*}. \end{cases}$$

Thus we see that

$$\begin{aligned} \Phi_{s,p}(u) &= \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{\lambda}{p+1}(u^+)^{p+1} - \frac{1}{2^*}\frac{(u^+)^{2^*}}{|x|^s}\right)dx \\ &\ge \|\nabla u\|_{L^2(\Omega)}^2 \times \begin{cases} \left(\frac{1}{2}\left(1-\frac{\lambda}{\lambda_1(\Omega)}\right) - C\|\nabla u\|_{L^2(\Omega)}^{2^*-2}\right) & \text{if } p=1, \\ \left(\frac{1}{2} - C\|\nabla u\|_{L^2(\Omega)}^{p-1} - C\|\nabla u\|_{L^2(\Omega)}^{2^*-2}\right) & \text{if } p>1. \end{cases} \end{aligned}$$

Noting that $1-\frac{\lambda}{\lambda_1(\Omega)} > 0$, i.e., $\lambda < \lambda_1(\Omega)$ in the case $p=1$, and taking $U := B_{r_0}(0)$ with small $r_0 > 0$, we have

$$\Phi_{s,p}(u) \ge \rho > 0 \quad \text{for all } u \in \partial\Omega,$$

where

$$\rho := \begin{cases} r_0^2\left(\frac{1}{2}\left(1-\frac{\lambda}{\lambda_1(\Omega)}\right) - Cr_0^{2^*-2}\right) & \text{if } p=1, \\ r_0^2\left(\frac{1}{2} - Cr_0^{p-1} - Cr_0^{2^*-2}\right) & \text{if } p>1. \end{cases}$$

By Lemma 2.1, there exists a nonnegative function $v_0 \in H_0^1(\Omega)\setminus\{0\}$ such that

$$\sup_{t\ge 0}\Phi_{s,p}(tv_0) < \left(\frac{1}{2}-\frac{1}{2^*}\right)\mu_s(\Omega)^{\frac{2^*}{2^*-2}},$$

and set $v =: t_0v_0$, where $t_0 > 0$ is chosen large enough so that $v \notin U$ and $\Phi_{s,p}(v) \le 0$. Here, note that

$$\rho \le c := \inf_{P\in\mathcal{P}}\max_{w\in P}\Phi_{s,p}(w) \le \sup_{t\ge 0}\Phi_{s,p}(tv) < \left(\frac{1}{2}-\frac{1}{2^*}\right)\mu_s(\Omega)^{\frac{2^*}{2^*-2}}. \tag{3.3}$$

Since $\Phi_{s,p}(0) = 0$, by applying Theorem C, there exists a sequence $\{u_j\} \subset H_0^1(\Omega)$ such that

$$\Phi_{s,p}(u_j) \to c \quad \text{and} \quad \Phi'_{s,p}(u_j) \to 0 \quad \text{in } H^{-1}(\Omega), \quad \text{i.e.,}$$

$$\int_\Omega \left(\frac{1}{2}|\nabla u_j|^2 - \frac{\lambda}{p+1}(u_j^+)^{p+1} - \frac{1}{2^*}\frac{(u_j^+)^{2^*}}{|x|^s}\right)dx = c + o(1), \tag{3.4}$$

and

$$-\Delta u_j - \lambda(u_j^+)^p - \frac{(u_j^+)^{2^*-1}}{|x|^s} =: \zeta_j \quad \text{with } \zeta_j \to 0 \quad \text{in } H^{-1}(\Omega). \tag{3.5}$$

We show the boundedness of $\{u_j\}$. Multiplying (3.5) by u_j, we obtain

$$\int_\Omega \left(|\nabla u_j|^2 - \lambda(u_j^+)^{p+1} - \frac{(u_j^+)^{2^*}}{|x|^s}\right)dx = <\zeta_j, u_j>, \tag{3.6}$$

and (3.4)$-\frac{1}{2}$(3.6) yields that

$$\int_\Omega \left(\lambda\left(\frac{1}{2}-\frac{1}{p+1}\right)(u_j^+)^{p+1} + \left(\frac{1}{2}-\frac{1}{2^*}\right)\frac{(u_j^+)^{2^*}}{|x|^s}\right)dx = c + o(1) - \frac{1}{2} <\zeta_j, u_j>$$

$$\leq c + o(1) + \frac{1}{2}\|\zeta_j\|_{H^{-1}(\Omega)}\|u_j\|_{H_0^1(\Omega)} \leq C\left(1 + \|u_j\|_{H_0^1(\Omega)}\right). \tag{3.7}$$

By (3.4) and (3.7), we have

$$\|u_j\|^2_{H_0^1(\Omega)} \leq C\left(1 + \|u_j\|_{H_0^1(\Omega)}\right),$$

which implies $\|u_j\|_{H_0^1(\Omega)} \leq C$. Then extracting a subsequence, still denoted by u_j, we see

$$\begin{cases} u_j \rightharpoonup u \quad \text{weakly in } \ H_0^1(\Omega), \\ u_j^+ \to u^+ \quad \text{strongly in } \ L^{p+1}(\Omega), \\ \frac{u_j^+}{|x|^{\frac{s}{2^*}}} \rightharpoonup \frac{u^+}{|x|^{\frac{s}{2^*}}} \quad \text{weakly in } \ L^{2^*}(\Omega). \end{cases}$$

Thus passing to the limit in (3.5) yields that

$$\Delta u + \lambda\,(u^+)^p + \frac{(u^+)^{2^*-1}}{|x|^s} = 0,$$

and then from the maximum principle, we obtain $u \geq 0$ in Ω.

Finally, we shall prove $u \neq 0$ in $H_0^1(\Omega)$, and suppose $u \equiv 0$ in Ω. Since $\int_\Omega |\nabla u_j|^2 dx$ and $\int_\Omega \frac{(u_j^+)^{2^*}}{|x|^s} dx$ are both bounded, we may assume

$$\lim_{j\to\infty} \int_\Omega |\nabla u_j|^2 dx =: C_1 \quad \text{and} \quad \lim_{j\to\infty} \int_\Omega \frac{(u_j^+)^{2^*}}{|x|^s} dx =: C_2.$$

Thus passing to the limit in (3.4) and (3.6), we get

$$\frac{C_1}{2} - \frac{C_2}{2^*} = c \quad \text{and} \quad C_1 - C_2 = 0,$$

and then we have

$$c = \left(\frac{1}{2} - \frac{1}{2^*}\right) C_1. \tag{3.8}$$

Here, we see that

$$\int_\Omega |\nabla u_j|^2 dx \geq \mu_s(\Omega) \left(\int_\Omega \frac{|u_j|^{2^*}}{|x|^s} dx\right)^{\frac{2}{2^*}} \geq \mu_s(\Omega) \left(\int_\Omega \frac{(u_j^+)^{2^*}}{|x|^s} dx\right)^{\frac{2}{2^*}},$$

and then

$$C_1 \geq \mu_s(\Omega)\, C_1^{\frac{2}{2^*}}, \quad \text{i.e.,} \quad C_1 \geq \mu_s(\Omega)^{\frac{2^*}{2^*-2}}. \tag{3.9}$$

By (3.8) and (3.9), we have

$$c \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}}. \tag{3.10}$$

Thus combining (3.3) with (3.10) yields

$$\left(\frac{1}{2} - \frac{1}{2^*}\right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}} \leq c < \left(\frac{1}{2} - \frac{1}{2^*}\right) \mu_s(\Omega)^{\frac{2^*}{2^*-2}},$$

which is a contradiction. Thus $v \neq 0$ in $H_0^1(\Omega)$. □

Proof of the Critical Case for Theorem 1.1. We now consider the case that $p = \frac{n+2}{n-2}$. We divide the proof into two steps.

Step 1. By Lemma 2.4, there exists a nonnegative function $v_0 \in H_0^1(\Omega)\backslash\{0\}$ with $\Phi_{s,*}(v_0) < 0$ such that

$$\max_{t\geq 0} \Phi_{s,*}(tv_0) < \frac{1}{n}\lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}}.$$

Hence, it is easy to see there exists $\varepsilon_0 > 0$ such that $\Phi^\varepsilon_{s,*}(tv_0) < 0$ and

$$\max_{0\leq t\leq 1} \Phi^\varepsilon_{s,*}(tv_0) < \frac{1}{n}\lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}} \quad \text{for } \ 0 \leq \varepsilon \leq \varepsilon_0,$$

where

$$\Phi^\varepsilon_{s,*}(u) := \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 - \frac{\lambda}{p_\varepsilon}(u^+)^{p_\varepsilon} - \frac{1}{2^*(s) - \varepsilon}\frac{(u^+)^{2^*-\varepsilon}}{|x|^s}\Big)dx \quad \text{for } \ u \in H_0^1(\Omega),$$

and

$$p_\varepsilon := \frac{2n}{n-2} - \frac{2\varepsilon}{2-s}.$$

Note that $\Phi^0_{s,*} = \Phi_{s,*}$.

Taking ε_0 small such that $p_{\varepsilon_0} - 2 > 0$ and $2^* - 2 - \varepsilon_0 > 0$, by the Sobolev-Hardy inequality,

$$\begin{aligned}
\Phi^\varepsilon_{s,*}(u) &= \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 - \frac{\lambda}{p_\varepsilon}(u^+)^{p_\varepsilon} - \frac{1}{2^*(s) - \varepsilon}\frac{u^{2^*-\varepsilon}}{|x|^s}\Big)dx \\
&\geq \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - C\|\nabla u\|^{p_\varepsilon}_{L^2(\Omega)} - C\|\nabla u\|^{2^*-\varepsilon}_{L^2(\Omega)} \\
&= \|\nabla u\|^2_{L^2(\Omega)}\Big(\frac{1}{2} - C\|\nabla u\|^{p_\varepsilon-2}_{L^2(\Omega)} - C\|\nabla u\|^{2^*-2-\varepsilon}_{L^2(\Omega)}\Big),
\end{aligned}$$

where the constant $C > 0$ is independent of ε. Therefore, there exist $r_0 > 0$ and $\rho_0 > 0$ such that

$$\Phi^\varepsilon_{s,*}(u) \geq r_0^2\Big(\frac{1}{2} - Cr_0^{p_\varepsilon-2} - Cr_0^{2^*-2-\varepsilon}\Big) =: \rho_\varepsilon \geq \rho_0 > 0$$

for any $u \in \partial B_{r_0}(0)$, where $B_{r_0}(0) = \{u \in H_0^1(\Omega) | \|\nabla u\|_{L^2(\Omega)} < r_0\}$. It is obvious that $v_0 \notin \overline{B_{r_0}(0)}$. Hence, we obtain

$$\rho_0 \leq c_\varepsilon := \inf_{P\in\mathcal{P}} \max_{u\in P} \Phi^\varepsilon_{s,*}(u) \leq \max_{0\leq t\leq 1} \Phi^\varepsilon_{s,*}(tv_0) < \frac{1}{n}\lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},$$

where $\mathcal{P}$ denotes the class of continuous paths in $H_0^1(\Omega)$ joining 0 to v_0.

Since $\Phi^\varepsilon_{s,*}(0) = 0$, by applying Theorem C, there exists a double sequence $\{u_{\varepsilon,j}\} \subset H_0^1(\Omega)$ such that

$$\Phi^\varepsilon_{s,*}(u_{\varepsilon,j}) \to c_\varepsilon \quad \text{and} \quad (\Phi^\varepsilon_{s,*})'(u_{\varepsilon,j}) \to 0 \quad \text{in } \ H^{-1}(\Omega), \quad \text{i.e.,}$$

$$\int_\Omega \left(\frac{1}{2}|\nabla u_{\varepsilon,j}|^2 - \frac{\lambda}{p_\varepsilon}(u^+_{\varepsilon,j})^{p_\varepsilon} - \frac{1}{2^* - \varepsilon}\frac{(u^+_{\varepsilon,j})^{2^*-\varepsilon}}{|x|^s}\right) dx = c_\varepsilon + o(1), \tag{3.11}$$

and

$$-\Delta u_{\varepsilon,j} - \lambda(u^+_{\varepsilon,j})^{p_\varepsilon-1} - \frac{(u^+_{\varepsilon,j})^{2^*-1-\varepsilon}}{|x|^s} =: \zeta_{\varepsilon,j} \quad \text{with } \ \zeta_{\varepsilon,j} \to 0 \quad \text{in } \ H^{-1}(\Omega) \quad \text{as } \ j \to \infty. \tag{3.12}$$

Applying (3.12) on $u_{\varepsilon,j}$, we get

$$\int_\Omega \Big(|\nabla u_{\varepsilon,j}|^2 - \lambda(u_{\varepsilon,j}^+)^{p_\varepsilon} - \frac{(u_{\varepsilon,j}^+)^{2^*-\varepsilon}}{|x|^2}\Big)dx =< \zeta_{\varepsilon,j}, u_{\varepsilon,j} > . \tag{3.13}$$

Similar to the subcritical case, for $0 \le \varepsilon \le \varepsilon_0$ with ε_0 small, we get the uniform boundedness of $\{u_{\varepsilon,j}\}$,

$$\|u_{\varepsilon,j}\|_{H_0^1(\Omega)} \le C. \tag{3.14}$$

Hereafter, we assume $\varepsilon > 0$. Then, up to a subsequence, there exists a function $u_\varepsilon \in H_0^1(\Omega)$ such that as $j \to \infty$,

$$\begin{cases} u_{\varepsilon,j} \rightharpoonup u_\varepsilon \quad \text{weakly in } H_0^1(\Omega), \\ u_{\varepsilon,j}^+ \to u_\varepsilon^+ \quad \text{strongly in } L^{p_\varepsilon}(\Omega), \\ \frac{u_{\varepsilon,j}^+}{|x|^{\frac{s}{2^*-\varepsilon}}} \to \frac{u_\varepsilon^+}{|x|^{\frac{s}{2^*-\varepsilon}}} \quad \text{strongly in } L^{2^*-\varepsilon}(\Omega). \end{cases}$$

We claim that $u_\varepsilon^+ \neq 0$ in $H_0^1(\Omega)$. Suppose, on the contrary, $u_\varepsilon^+ = 0$ in $H_0^1(\Omega)$. Then passing to the limit $j \to \infty$ in (3.11) and (3.13) yields that

$$c_\varepsilon = \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 dx = 0,$$

which contradicts to $c_\varepsilon \ge \rho_0 > 0$. Thus, $u_\varepsilon^+ \neq 0$ in $H_0^1(\Omega)$. Hence, taking the limit $j \to \infty$ in (3.12), we see that

$$\Delta u_\varepsilon + \lambda\,(u_\varepsilon^+)^{p_\varepsilon - 1} + \frac{(u_\varepsilon^+)^{2^*-1-\varepsilon}}{|x|^s} = 0 \quad \text{in } \Omega. \tag{3.15}$$

By the maximum principle, we find that $u_\varepsilon \ge 0$ in Ω. Inferring from (3.11) and (3.15), this positive solution u_ε satisfies

$$\begin{cases} \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 dx - \frac{\lambda}{p_\varepsilon}\int_\Omega u_\varepsilon^{p_\varepsilon} dx - \frac{1}{2^*-\varepsilon}\int_\Omega \frac{{u_\varepsilon}^{2^*-\varepsilon}}{|x|^s} dx = c_\varepsilon, \\ \int_\Omega |\nabla u_\varepsilon|^2 dx - \lambda\int_\Omega u_\varepsilon^{p_\varepsilon} dx - \int_\Omega \frac{u_\varepsilon^{2^*-\varepsilon}}{|x|^s} dx = 0. \end{cases} \tag{3.16}$$

Note that by (3.14), we have

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \le C.$$

Thus, by extracting a subsequence $\{u_j := u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ with $\varepsilon_j \to 0$ as $j \to \infty$, there exists a function $u \in H_0^1(\Omega)$ such that

$$\begin{cases} u_j \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \\ u_j^+ \rightharpoonup u^+ \quad \text{weakly in } L^{\frac{2n}{n-2}}(\Omega), \\ \frac{u_j^+}{|x|^{\frac{s}{2^*}}} \rightharpoonup \frac{u^+}{|x|^{\frac{s}{2^*}}} \quad \text{weakly in } L^{2^*}(\Omega). \end{cases}$$

Now, passing to the limit in (3.15) yields that

$$\Delta u + \lambda\,(u^+)^{\frac{n+2}{n-2}} + \frac{(u^+)^{2^*-1}}{|x|^s} = 0 \quad \text{in } \Omega.$$

By the maximum principle, we see that $u \ge 0$ in Ω. Therefore, u satisfies

$$\Delta u + \lambda\, u^{\frac{n+2}{n-2}} + \frac{u^{2^*-1}}{|x|^s} = 0 \quad \text{in } \Omega.$$

The rest part of the proof is to show $u \neq 0$ in $H_0^1(\Omega)$. Assume $u = 0$ in $H_0^1(\Omega)$. Then the blow-up occurs, i.e.,

$$m_j := u_j(x_j) = \max_{\Omega} u_j(x) \to \infty \quad \text{as } j \to \infty.$$

Otherwise, by (3.16) and the Lebesgue theorem, we have

$$0 = \lim_{j\to\infty} \frac{1}{2} \int_{\Omega} |\nabla u_j|^2 dx = c_0,$$

which contradicts to $c_0 > 0$.

We consider the scaling

$$v_j(y) := m_j^{-1} u_j(x_j + k_j y) \quad \text{for } y \in \Omega_j := \Big\{ z \in \mathbb{R}^n \Big| x_j + k_j z \in \Omega \Big\},$$

where $k_j = m_j^{-\frac{p_j-2}{2}}$ and $p_j = \frac{2n}{n-2} - \frac{2\varepsilon_j}{2-s}$. By (3.15), v_j satisfies

$$\begin{cases} \Delta v_j + \lambda\, v_j^{p_j-1} + \dfrac{v_j^{2^*-1-\varepsilon_j}}{\left|\frac{x_j}{k_j}+y\right|^s} = 0 \quad \text{in } \Omega_j, \\ v_j = 0 \quad \text{on } \partial\Omega_j. \end{cases} \tag{3.17}$$

Let $\Omega_\infty = \lim_{j\to\infty} \Omega_j$. We distinguish into the following cases.

Case 1. If, up to a subsequence, $\frac{|x_j|}{k_j} \to \infty$, then $v_j(y)$ converges to some $v(y)$ uniformly in every compact subset of $\overline{\Omega_\infty}$, where $v(y) \in H_0^1(\Omega_\infty)$ with $v(0) = 1$ is the solution of the equation

$$\begin{cases} \Delta v + \lambda v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \Omega_\infty, \\ v = 0 \quad \text{on } \partial\Omega_\infty. \end{cases}$$

It is well-known that the above equation is only solvable for $\Omega_\infty = \mathbb{R}^n$. We easily see that

$$\begin{cases} C_1 := \lim\limits_{j\to\infty} \displaystyle\int_{\Omega} |\nabla u_j|^2 dx = m_j^{(\frac{n-2}{2-s})\varepsilon_j} \lim\limits_{j\to\infty} \displaystyle\int_{\Omega_j} |\nabla v_j|^2 dy \geq \int_{\mathbb{R}^n} |\nabla v|^2 dy =: A_1, \\ C_2 := \lim\limits_{j\to\infty} \displaystyle\int_{\Omega} u_j^{\frac{2n}{n-2}} dx = m_j^{(\frac{n}{2-s})\varepsilon_j} \lim\limits_{j\to\infty} \displaystyle\int_{\Omega_j} v_j^{\frac{2n}{n-2}} dy \geq \int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dy =: A_2, \\ C_3 := \lim\limits_{j\to\infty} \displaystyle\int_{\Omega} \frac{u_j^{2^*}}{|x|^s} dx = m_j^{(\frac{n-s}{2-s})\varepsilon_j} \lim\limits_{j\to\infty} \displaystyle\int_{\Omega_j} \frac{v_j^{2^*}}{|\frac{x_j}{k_j} + y|^s} dy. \end{cases} \tag{3.18}$$

Furthermore, note that

$$\frac{C_1}{2} - \frac{(n-2)\lambda}{2n} C_2 - \frac{C_3}{2^*} = c_0, \quad C_1 - \lambda\, C_2 - C_3 = 0, \quad \text{and} \quad A_1 = \lambda\, A_2. \tag{3.19}$$

By (3.18) and (3.19), we have

$$c_0 = \left(\frac{1}{2} - \frac{1}{2^*}\right) C_1 + \lambda \left(\frac{1}{2^*} - \frac{n-2}{2n}\right) C_2 \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) A_1 + \lambda \left(\frac{1}{2^*} - \frac{n-2}{2n}\right) A_2 = \frac{\lambda}{n} A_2.$$

On the other hand, by the Sobolev inequality, we see that

$$S_n A_2^{\frac{n-2}{n}} \leq A_1.$$

This leads to

$$A_2 \geq \lambda^{-\frac{n}{2}} S_n^{\frac{n}{2}}.$$

Hence,

$$c_0 \geq \frac{\lambda}{n} A_2 \geq \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},$$

which contradicts to

$$c_0 \leq \max_{0 \leq t \leq 1} \Phi_{s,*}(tv_0) < \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}}.$$

Case 2. If, up to a subsequence, $\frac{x_j}{k_j} \to y_0 \in \mathbb{R}^n$, then Ω_∞ is a half space. Therefore, up to a linear transformation, v_j converges to some v uniformly in any compact set of $\overline{\mathbb{R}^n_+}$, where v is a positive solution of the equation

$$\begin{cases} \Delta v + \lambda\, v^{\frac{n+2}{n-2}} + \frac{v^{2^*-1}}{|y|^s} = 0 \quad \text{in } \mathbb{R}^n_+, \\ v = 0 \quad \text{on } \partial \mathbb{R}^n_+ \end{cases} \tag{3.20}$$

with $v(y_0) = 1$ for some $y_0 \in \mathbb{R}^n_+$. If the equation (3.20) has no positive solution, this leads to a contradiction. Hence, the sequence $u_j(x)$ does not blow-up.

Step 2. However, if the equation (3.20) admits a positive solution, by Lemma 2.3 and Lemma 2.4, there exists a nonnegative function $v_0 \in H^1_0(\Omega)\backslash\{0\}$ with $\Phi_{s,*}(v_0) < 0$ such that

$$\max_{t \geq 0} \Phi_{s,*}(tv_0) < \min\left\{c^*, \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}}\right\}.$$

Redoing Step 1, since the min-max value c_0 is independent of the choice of v_0, we only need to deal with Case 2, see Remark after Theorem C. Let

$$B_1 = \int_{\mathbb{R}^n_+} |\nabla v(y)|^2 dy, \quad B_2 = \int_{\mathbb{R}^n_+} v^{\frac{2n}{n-2}} dy \quad \text{and} \quad B_3 = \int_{\mathbb{R}^n_+} \frac{v^{2^*}}{|y|^s} dy.$$

Noting that

$$\begin{cases} C_1 \geq B_1, \quad C_2 \geq B_2, \quad C_3 \geq B_3, \\ C_1 - \lambda C_2 - C_3 = 0, \quad B_1 - \lambda B_2 - B_3 = 0, \\ c_0 = \frac{C_1}{2} - \frac{(n-2)\lambda}{2n} C_2 - \frac{1}{2^*} C_3, \end{cases}$$

we see that

$$c_0 = \frac{\lambda}{n} C_2 + \left(\frac{1}{2} - \frac{1}{2^*}\right) C_3 \geq \frac{\lambda}{n} B_2 + \left(\frac{1}{2} - \frac{1}{2^*}\right) B_3 = \frac{B_1}{2} - \frac{(n-2)\lambda}{2n} B_2 - \frac{1}{2^*} B_3 = c^*.$$

This contradicts to $c_0 \leq \max_{0 \leq t \leq 1} \Phi_{s,*}(tv_0) < c^*$. Hence, we have proved $u \neq 0$ in $H^1_0(\Omega)$. The positivity of u is achieved by the strong maximum principle. The proof is complete. □

4 Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. However, since $H(0) < 0$ implies $\mu_s(\Omega) < \mu_s(\mathbb{R}^n_+)$, we cannot apply Lemma 2.5 according to the argument of Theorem 1.1 to show the existence of a solution for the equation (1.3).

Proof of Theorem 1.2. For any small $\varepsilon \ge 0$, we let

$$\Psi^{\varepsilon}_{s,p}(u) := \int_{\Omega} \left(\frac{1}{2}|\nabla u|^2 + \frac{\lambda}{p+1}(u^+)^{\,p+1} - \frac{1}{2^*-\varepsilon}\frac{(u^+)^{2^*-\varepsilon}}{|x|^s} \right) dx \quad \text{for } u \in H^1_0(\Omega),$$

where $\Psi^0_{s,p} = \Psi_{s,p}$. By $\lambda > 0$ and the Sobolev-Hardy inequality, we see that

$$\Psi^{\varepsilon}_{s,p}(u) \ge \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - C\|\nabla u\|^{2^*-\varepsilon}_{L^2(\Omega)} = \|\nabla u\|^2_{L^2(\Omega)} \left(\frac{1}{2} - C\|\nabla u\|^{2^*-2-\varepsilon}_{L^2(\Omega)} \right),$$

where $C > 0$ is independent of $\varepsilon \ge 0$. Then there exist positive constants r and ρ such that

$$\Psi^{\varepsilon}_{s,p}(u) \ge r^2 \left(\frac{1}{2} - C\, r^{2^*-2-\varepsilon} \right) =: \rho_\varepsilon \ge \rho > 0 \tag{4.1}$$

for all $u \in \partial B_r(0)$. On the other hand, by Lemma 2.5, there exists a nonnegative function $v_0 \in H^1_0(\Omega) \setminus \{0\}$ such that $\Psi^0_{s,p}(v_0) < 0$ and

$$\max_{0 \le t \le 1} \Psi^0_{s,p}(tv_0) < \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}}. \tag{4.2}$$

By (4.1), (4.2) and the continuity of $\Psi^{\varepsilon}_{s,p}$ at $\varepsilon = 0$, we see that $\Psi^{\varepsilon}_{s,p}(v_0) < 0$ and

$$0 < \rho \le c_\varepsilon := \inf_{P \in \mathcal{P}} \max_{w \in P} \Psi^{\varepsilon}_{s,p}(w) \le \max_{0 \le t \le 1} \Psi^{\varepsilon}_{s,p}(tv) < \left(\frac{1}{2} - \frac{1}{2^*} \right) \mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}} \tag{4.3}$$

for any small $\varepsilon \ge 0$. Since $\Psi^{\varepsilon}_{s,p}(0) = 0$, by applying Theorem C, there exists a sequence $\{u_{\varepsilon,j}\} \subset H^1_0(\Omega)$ such that

$$\Psi^{\varepsilon}_{s,p}(u_{\varepsilon,j}) \to c_\varepsilon \quad \text{and} \quad (\Psi^{\varepsilon}_{s,p})'(u_{\varepsilon,j}) \to 0 \quad \text{in } H^{-1}(\Omega) \quad \text{as } j \to \infty, \quad \text{i.e.,}$$

$$\int_{\Omega} \left(\frac{1}{2}|\nabla u_{\varepsilon,j}|^2 + \frac{\lambda}{p+1}(u^+_{\varepsilon,j})^{\,p+1} - \frac{1}{2^*-\varepsilon}\frac{(u^+_{\varepsilon,j})^{2^*-\varepsilon}}{|x|^s} \right) dx = c_\varepsilon + o(1) \quad \text{as } j \to \infty, \tag{4.4}$$

and

$$-\Delta u_{\varepsilon,j} + \lambda\,(u^+_{\varepsilon,j})^{\,p} - \frac{(u^+_{\varepsilon,j})^{2^*-1-\varepsilon}}{|x|^s} =: \zeta_{\varepsilon,j} \quad \text{with } \zeta_{\varepsilon,j} \to 0 \quad \text{in } H^{-1}(\Omega) \quad \text{as } j \to \infty. \tag{4.5}$$

Multiplying (4.5) by $u_{\varepsilon,j}$, we obtain

$$\int_{\Omega} \left(|\nabla u_{\varepsilon,j}|^2 + \lambda\,(u^+_{\varepsilon,j})^{\,p+1} - \frac{(u^+_{\varepsilon,j})^{2^*-\varepsilon}}{|x|^s} \right) dx = < \zeta_{\varepsilon,j}, u_{\varepsilon,j} > . \tag{4.6}$$

We show the boundedness of $\{u_{\varepsilon,j}\}$ in $H^1_0(\Omega)$. (4.4) $- \frac{1}{2}$(4.6) yields that

$$\int_{\Omega} \left(\lambda \left(\frac{1}{p+1} - \frac{1}{2} \right) (u^+_{\varepsilon,j})^{\,p+1} + \left(\frac{1}{2} - \frac{1}{2^*-\varepsilon} \right) \frac{(u^+_{\varepsilon,j})^{2^*-\varepsilon}}{|x|^s} \right) dx = c_\varepsilon + o(1) - \frac{1}{2} < \zeta_{\varepsilon,\,k}, u_{\varepsilon,j} >$$
$$\le C(1 + \|u_{\varepsilon,j}\|_{H^1_0(\Omega)}),$$

and then we have

$$\int_{\Omega} \frac{(u^+_{\varepsilon,j})^{2^*-\varepsilon}}{|x|^s} dx \le \frac{\lambda(p-1)(2^*-\varepsilon)}{(p+1)(2^*-2-\varepsilon)} \int_{\Omega} (u^+_{\varepsilon,j})^{\,p+1} dx + C(1 + \|u_{\varepsilon,j}\|_{H^1_0(\Omega)}), \tag{4.7}$$

where $C > 0$ can be taken independently of $j \in \mathbb{N}$ and $\varepsilon \geq 0$. Then by (4.4) and (4.7), we have

$$\begin{aligned}\frac{1}{2}\|\nabla u_{\varepsilon,j}\|^2_{L^2(\Omega)} &\leq -\frac{\lambda}{p+1}\left(1-\frac{p-1}{2^*-2-\varepsilon}\right)\int_\Omega (u^+_{\varepsilon,j})^{p+1}dx + C(1+\|u_{\varepsilon,j}\|_{H^1_0(\Omega)}) \\ &\leq C(1+\|u_{\varepsilon,j}\|_{H^1_0(\Omega)}), \end{aligned} \tag{4.8}$$

where we used that $1-\frac{p-1}{2^*-2-\varepsilon} > 0$ for any small $\varepsilon \geq 0$ by virtue of $1-\frac{p-1}{2^*-2} > 0$, i.e., $p < 2^*-1$. (4.8) implies

$$\|u_{\varepsilon,j}\|_{H^1_0(\Omega)} \leq C, \tag{4.9}$$

where $C > 0$ is independent of $j \in \mathbb{N}$ and also $\varepsilon \geq 0$.

Hereafter, we take $\varepsilon > 0$. Extracting a subsequence, still denoted by $u_{\varepsilon,j}$, we see that

$$\begin{cases} u_{\varepsilon,j} \rightharpoonup u_\varepsilon & \text{weakly in } H^1_0(\Omega), \\ u^+_{\varepsilon,j} \to u^+_\varepsilon & \text{strongly in } L^{p+1}(\Omega), \\ \frac{u^+_{\varepsilon,j}}{|x|^{\frac{s}{2^*-\varepsilon}}} \to \frac{u^+_\varepsilon}{|x|^{\frac{s}{2^*-\varepsilon}}} & \text{strongly in } L^{2^*-\varepsilon}(\Omega) \end{cases} \tag{4.10}$$

as $j \to \infty$. Then by passing to the limit $j \to \infty$ in (4.5), we get

$$-\Delta u_\varepsilon + \lambda\,(u^+_\varepsilon)^p - \frac{(u^+_\varepsilon)^{2^*-1-\varepsilon}}{|x|^s} = 0 \quad \text{in } \Omega,$$

and then by the maximum principle, we obtain $u_\varepsilon \geq 0$ in Ω.

It is easy to see that $u_\varepsilon \not\equiv 0$ in $H^1_0(\Omega)$. As a consequence, for any small $\varepsilon > 0$, we get a positive solution $u_\varepsilon \in H^1_0(\Omega)$ satisfying

$$\Delta u_\varepsilon - \lambda\, u^p_\varepsilon + \frac{u_\varepsilon^{2^*-1-\varepsilon}}{|x|^s} = 0. \tag{4.11}$$

Next, for $\varepsilon > 0$, passing to the limit $j \to \infty$ in (4.4) and multiplying (4.11) by u_ε yield that

$$\begin{cases} \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 dx + \frac{\lambda}{p+1}\int_\Omega u_\varepsilon^{p+1}dx - \frac{1}{2^*-\varepsilon}\int_\Omega \frac{u_\varepsilon^{2^*-\varepsilon}}{|x|^s}dx = c_\varepsilon, \\ \int_\Omega |\nabla u_\varepsilon|^2 dx + \lambda\int_\Omega u_\varepsilon^{p+1}dx - \int_\Omega \frac{u_\varepsilon^{2^*-\varepsilon}}{|x|^s}dx = 0. \end{cases} \tag{4.12}$$

Moreover, by taking a limit $j \to \infty$ in (4.9), we have

$$\|u_\varepsilon\|_{H^1_0(\Omega)} \leq C,$$

where $C > 0$ is independent of $\varepsilon > 0$. Thus by extracting a subsequence $\{u_j := u_{\varepsilon_j}\}$ with $\varepsilon_j \to 0$ as $j \to \infty$, we get

$$\begin{cases} u_j \rightharpoonup u & \text{weakly in } H^1_0(\Omega), \\ u_j \to u & \text{strongly in } L^{p+1}(\Omega), \\ \frac{u_j}{|x|^{\frac{s}{2^*}}} \rightharpoonup \frac{u}{|x|^{\frac{s}{2^*}}} & \text{weakly in } L^{2^*}(\Omega). \end{cases}$$

Thus passing to the limit $j \to \infty$ in (4.11) yields that

$$\Delta u - \lambda\, u^p + \frac{u^{2^*-1}}{|x|^s} = 0.$$

We shall prove $u \neq 0$ in $H_0^1(\Omega)$. Suppose $u = 0$ in $H_0^1(\Omega)$. Up to a subsequence, we let

$$C_1 := \lim_{j\to\infty} \int_\Omega |\nabla u_j|^2 dx \quad \text{and} \quad C_2 := \lim_{j\to\infty} \int_\Omega \frac{u_j^{2^*-\varepsilon_j}}{|x|^s} dx,$$

and then letting $j \to \infty$ in (4.12), we get

$$\frac{C_1}{2} - \frac{C_2}{2^*} = c_0 \quad \text{and} \quad C_1 - C_2 = 0, \quad \text{i.e.,} \quad \left(\frac{1}{2} - \frac{1}{2^*}\right) C_1 = c_0. \tag{4.13}$$

We easily see that $u_j(x_j) = \max_\Omega u_j \to \infty$ as $j \to \infty$. In what follows, we divide the proof into three steps. We let

$$\kappa_j := u_j(x_j)^{-\frac{2(2^*-2-\varepsilon_j)}{(n-2)(2^*-2)}}. \tag{4.14}$$

Step 1. We claim $|x_j| = O(\kappa_j)$ as $j \to \infty$.

Suppose that up to a subsequence, $\lim_{j\to\infty} \frac{|x_j|}{\kappa_j} = \infty$. By scaling, set

$$v_j(y) := \frac{u_j(x_j + \kappa_j\, y)}{u_j(x_j)} \quad \text{for } y \in \Omega_j, \tag{4.15}$$

where

$$\Omega_j := \{y \in \mathbb{R}^n \,|\, x_j + \kappa_j y \in \Omega\}. \tag{4.16}$$

By (4.11) and (4.14), v_j satisfies

$$\begin{cases} \Delta v_j - \lambda\, \kappa_j^2\, u_j(x_j)^{p-1} v_j^p + \left(\frac{\kappa_j}{|x_j|}\right)^s \frac{v_j^{2^*-1-\varepsilon_j}}{\left|\frac{x_j}{|x_j|} + \frac{\kappa_j}{|x_j|} y\right|^s} = 0 \quad \text{in } \Omega_j, \\ v_j = 0 \quad \text{on } \partial\Omega_j. \end{cases}$$

Furthermore, we have

$$\kappa_j^2\, u_j(x_j)^{p-1} = \kappa_j^{2-\frac{(n-2)(2^*-2)(p-1)}{2(2^*-2-\varepsilon_j)}} \to 0 \quad \text{as } j \to \infty,$$

where note that $\kappa_j \to 0$ and $2 - \frac{(n-2)(p-1)}{2} > 0$, i.e., $p < \frac{n+2}{n-2}$. Thus v_j converges to some v smoothly in any compact set, and v satisfies $v(0) = 1$ and

$$\Delta v = 0 \quad \text{in } \mathbb{R}^n \tag{4.17}$$

provided that $\Omega_j \to \mathbb{R}^n$, or

$$\begin{cases} \Delta v = 0 \quad \text{in some half space } H, \\ v = 0 \quad \text{on } \partial H \end{cases} \tag{4.18}$$

provided that up to a linear transformation $\Omega_j \to H := \{y \in \mathbb{R}^n \,|\, y_n > -a\}$ for some $a > 0$. On the other hand, we have

$$\int_{\Omega_j} v_j^{\frac{2n}{n-2}} dy = \kappa_j^{\frac{n\varepsilon_j}{2^*-2-\varepsilon_j}} \int_\Omega u_j^{\frac{2n}{n-2}} dx \le C,$$

and then $\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dy$ is finite. This contradicts to $v(0) = 1$. The proof of Step 1 is complete.

Note that Step 1 implies that the origin is the only blow-up point.

Step 2. We claim that up to a subsequence, $\frac{x_j}{\kappa_j} \to y_0 \neq 0$ as $j \to \infty$.

Suppose that $\frac{x_j}{\kappa_j} \to 0$ as $j \to \infty$. As in the proof of Step 1, we define v_j and Ω_j in (4.15) and (4.16), respectively. Then by (4.11), v_j satisfies

$$\begin{cases} \Delta v_j - \lambda\, \kappa_j^2\, u_j(x_j)^{\,p-1} v_j^p + \frac{v_j^{2^*-1-\varepsilon_j}}{\left|\frac{x_j}{\kappa_j}+y\right|^s} = 0 \quad \text{in } \Omega_j, \\ v_j = 0 \quad \text{on } \partial\Omega_j. \end{cases}$$

Since we already proved that $\kappa_j^2\, u_j(x_j)^{\,p-1} \to 0$, v_j converges to some v smoothly in any compact set in $\overline{\mathbb{R}^n_+}$, and v satisfies

$$\begin{cases} \Delta v + \frac{v^{2^*-1}}{|y|^s} = 0 \quad \text{in } \mathbb{R}^n_+, \\ v = 0 \quad \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

which is a contradiction to $v(0) = 1$. Thus Step 2 is proved.

Step 3. We complete the proof of Theorem 1.2 in this step. We note after a linear transformation, v_j converges to some v smoothly in any compact set in $\overline{\mathbb{R}^n_+}$, and v satisfies

$$\begin{cases} \Delta v + \frac{v^{2^*-1}}{|y|^s} = 0 \quad \text{in } \mathbb{R}^n_+, \\ v = 0 \quad \text{on } \partial\mathbb{R}^n_+ \quad \text{and} \quad v(y_0) = \max\limits_{\mathbb{R}^n_+} v = 1 \quad \text{for some } y_0 \in \mathbb{R}^n_+. \end{cases} \tag{4.19}$$

By (4.19), we have

$$\frac{\int_{\mathbb{R}^n_+} |\nabla v|^2 dy}{\left(\int_{\mathbb{R}^n_+} \frac{v^{2^*}}{|y|^s} dy\right)^{\frac{2}{2^*}}} = \left(\int_{\mathbb{R}^n_+} \frac{v^{2^*}}{|y|^s} dy\right)^{\frac{2^*-2}{2^*}} \geq \mu_s(\mathbb{R}^n_+),$$

and then

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 dy = \int_{\mathbb{R}^n_+} \frac{v^{2^*}}{|y|^s} dy \geq \mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}}. \tag{4.20}$$

Furthermore, note that

$$C_1 = \lim_{j\to\infty} \int_\Omega |\nabla u_j|^2 dx = \lim_{j\to\infty} \kappa_j^{-\frac{(n-2)\varepsilon_j}{2^*-2-\varepsilon_j}} \int_{\Omega_j} |\nabla v_j|^2 dy \geq \lim_{j\to\infty} \int_{\Omega_j} |\nabla v_j|^2 dy \geq \int_{\mathbb{R}^n_+} |\nabla v|^2 dy. \tag{4.21}$$

Then by (4.13), (4.20) and (4.21), we have

$$c_0 \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) C_1 \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \mu_s(\mathbb{R}^n_+)^{\frac{2^*}{2^*-2}},$$

which yields a contradiction to (4.3). Thus $u \neq 0$ in $H_0^1(\Omega)$, and Theorem 1.2 is proved. □

5 Proof of Theorem 1.3

In order to prove Theorem 1.3, we first prove the following lemma.

Lemma 5.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 3$, $0 \in \partial\Omega$ and $0 < s < 2$. Then for $p = \frac{n+2}{n-2}$, the equation (1.2) has no positive solution provided that Ω is star-shaped with respect to the origin.*

Proof. Multiplying (1.2) by $x \cdot \nabla u$ and ∇u, respectively, and taking integrations, we obtain

$$\begin{cases} \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 dS_x + \frac{n-2}{2}\int_\Omega |\nabla u|^2 dx = \frac{n-2}{2}\left(\lambda\int_\Omega u^{\frac{2n}{n-2}}dx + \int_\Omega \frac{u^{2^*}}{|x|^s}dx\right), \\ \int_\Omega |\nabla u|^2 dx = \lambda\int_\Omega u^{\frac{2n}{n-2}}dx + \int_\Omega \frac{u^{2^*}}{|x|^s}dx, \end{cases}$$

where ν denotes the outward normal to $\partial\Omega$. Thus we derive the following Pohozaev identity

$$\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 dS_x = 0.$$

Since Ω is star-shaped with respect to the origin, we deduce that $\frac{\partial u}{\partial\nu} \equiv 0$ on $\partial\Omega$. Hence,

$$\lambda\int_\Omega u^{\frac{n+2}{n-2}}dx + \int_\Omega \frac{u^{2^*-1}}{|x|^s}dx = -\int_\Omega \Delta u dx = 0,$$

which implies $u \equiv 0$ in Ω. □

Next, we take $\Omega = B_1(e_n)$, the unit ball centered at $e_n = (0, \cdots, 1)$. It is obvious that Ω is star-shaped about the origin. Proceeding the same variational method as that in the proof for the assertion (iv) of Theorem 1.1, due to the nonexistence of a positive solution for the equation (1.2), Case 2 holds. Therefore, we get a positive solution of the equation (1.4). This proves Theorem 1.3.

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