# Composition Operators on Function Spaces with Fractional Order of Smoothness

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**Abstract**

In this paper we will give an overview concerning properties of composition operators $T_f(g) := f \circ g$ in the framework of Besov-Lizorkin-Triebel spaces. Boundedness and continuity will be discussed in a certain detail. In addition we also give a list of open problems.

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# 1 Introduction

Let $E$ denote a normed space of functions. Composition operators $T_f:\ g \mapsto f \circ g$, $g \in E$, are simple examples of nonlinear mappings. It is a little bit surprising that the knowledge about these operators is rather limited. One reason is, of course, that the properties of $T_f$ strongly depend on $f$ and $E$. Here in this paper we are concerned with $E$ being either a Besov or a Lizorkin-Triebel space (for definitions of these classes we refer to the appendix at the end of this article). These scales of spaces generalize Sobolev spaces $W^m_p(\mathbb{R}^n)$, Bessel potential spaces $H^s_p(\mathbb{R}^n)$, Slobodeckij spaces $W^s_p(\mathbb{R}^n)$ as well as Hölder spaces $C^s(\mathbb{R}^n)$ in view of the identities

- $W^m_p(\mathbb{R}^n) = F^m_{p,2}(\mathbb{R}^n)$, $1 < p < \infty$, $m \in \mathbb{N}$;
- $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$, $1 < p < \infty$, $s \in \mathbb{R}$;

- $W^s_p(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n)$, $1 \le p < \infty$, $s > 0$, $s \notin \mathbb{N}$;
- $C^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n)$, $s > 0$, $s \notin \mathbb{N}$.

In our opinion there is a very interesting interplay between integrability and regularity properties of $f, g$ and of the composition $f \circ g$. It is our aim to describe this in detail and to give a survey on the state of the art. Many times we will not give proofs but discuss illustrating examples. The theory is far from being complete. However we believe that a discussion of these operators in spaces of fractional order of smoothness $s > 0$ is appropriate and leads to a better understanding of the various phenomenons which occur.
Convention: If there is no need for a distinction between Lizorkin-Triebel spaces and Besov spaces we will simply write $E^s_{p,q}(\mathbb{R}^n)$ instead of $F^s_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$, respectively. In the same spirit, $E^s_{p,q}(\mathbb{R}^n)$ can denote $W^s_p(\mathbb{R}^n)$ in case $s \in \mathbb{N}$ and $p = 1$ or $+\infty$, although those spaces are not Lizorkin-Triebel nor Besov spaces.

## An illustrating example

Let us have a look at the following boundary value problem:

$$\begin{aligned} \Delta u(x) + f(u(x)) &= h(x)\,, \quad x \in \Omega\,, \\ u &= 0\,, \quad x \in \partial\Omega\,. \end{aligned}$$

Here $\Omega$ is an open and bounded subset of $\mathbb{R}^n$ with smooth boundary. Let $L$ denote the solution operator for the Dirichlet problem for the Laplacian with respect to $\Omega$. Then our boundary value problem can be reformulated as a fixed point problem

$$u = L(h - f(u))\,. \tag{1}$$

By $\mathring{E}^s_{p,q}(\Omega)$ we denote the collection of all functions in $E^s_{p,q}(\Omega)$ having vanishing boundary values (this makes sense if $s > 1/p$). Since $L : E^{s-2}_{p,q}(\Omega) \to \mathring{E}^s_{p,q}(\Omega)$ is an isomorphism a discussion of (1) requires

$$T_f(E^s_{p,q}(\Omega)) \subset E^{s-2}_{p,q}(\Omega)$$

including some estimates which relate the norms $\| T_f(u) \|_{E^{s-2}_{p,q}(\Omega)}$ and $\| u \|_{E^s_{p,q}(\Omega)}$.

**Remark 1** There are many papers dealing with problems as in our illustrating example. We refer e.g. to [28], [34], [49], where boundedness and continuity of composition operators are treated in connection with the Schrödinger equation.

## The main problem

From a little bit more abstract point of view the above example indicates that we have to study the following problem.

**Problem 1:**
*Suppose* $E^{s_0,\ell oc}_{p_0,q_0}(\mathbb{R}^n) \subset E^{s_1,\ell oc}_{p_1,q_1}(\mathbb{R}^n)$. *Find necessary and sufficient conditions on* $f : \mathbb{R} \to \mathbb{R}$ *s.t.*

$$T_f(E^{s_0}_{p_0,q_0}(\mathbb{R}^n)) \subset E^{s_1}_{p_1,q_1}(\mathbb{R}^n)\,.$$

**Remark 2** (i) For a given space of functions $E$ on $\mathbb{R}^n$, the associated space $E^{\ell oc}$ is the collection of all functions $g$ s.t. $u \cdot g \in E$, for all $u \in \mathcal{D}(\mathbb{R}^n)$.
(ii) Our assumption $E^{s_0,\ell oc}_{p_0,q_0}(\mathbb{R}^n) \subset E^{s_1,\ell oc}_{p_1,q_1}(\mathbb{R}^n)$ indicates that there is no hope for an increase of the local regularity of the whole set $T_f(E^{s_0}_{p_0,q_0}(\mathbb{R}^n))$.
(iii) The theory of the operators $T_f$, as we know it at this moment, does not depend very much on the underlying domain $\Omega$. So we discuss $T_f$ on function spaces defined on $\mathbb{R}^n$.
(iv) Of course, it would make sense to replace $T_f$ by more general mappings like

$$N(g_1, \ldots, g_d)(x) := f(x, g_1(x), \ldots, g_d(x))\,, \qquad x \in \mathbb{R}^n\,,$$

where $f : \mathbb{R}^{n+d} \to \mathbb{R}$. In this generality these mappings $N$ are called Nemytskij operators. Much less is known for these general mappings. In our survey only very few remarks will be made concerning this general situation.

There are hundred's of references dealing with *Problem 1* and its generalizations. However, only in very few cases, e.g. if $f(t) := t^m$, $m \in \mathbb{N}$, one knows the final answer. We will add a few remarks later on. For the moment we will turn to a simplified problem.

**Problem 2:**
*Find necessary and sufficient conditions on* $f : \mathbb{R} \to \mathbb{R}$ *s.t.*

$$T_f(E^s_{p,q}(\mathbb{R}^n)) \subset E^s_{p,q}(\mathbb{R}^n)\,. \tag{2}$$

We will call the property (2) the acting property. In applications one needs more than the acting condition. In general one also needs boundedness and continuity. This justifies to consider the following modified problems.

**Problem 2':**

*Find necessary and sufficient conditions on $f : \mathbb{R} \to \mathbb{R}$ s.t.*

$$T_f : \quad E^s_{p,q}(\mathbb{R}^n) \to E^s_{p,q}(\mathbb{R}^n)$$

*is bounded.*

**Remark 3** (i) Generally speaking, a mapping $T$ of a metric space $E$ to itself is said to be bounded if $T(A)$ is bounded for all bounded set $A \subset E$.
(ii) Coming back to our *illustrating example* it makes sense to ask for optimal inequalities describing the boundedness of $T_f$. We will take care of this problem as well in our survey.
(iii) We do not know whether *Problems* 2 and 2' have the same solution. Indeed, we do not have a counterexample in the framework of composition operators showing that the acting condition can occur without boundedness. Furthermore, let us mention that it is classically known that the acting condition implies at least a weak form of boundedness, see e.g. [39, pp. 275-276], [55, Lem. 5.2.4, 5.3.1/1] and [14, Section 4.2].
(iv) If we slightly generalize the class of operators by considering $N(g)(x) := f(x, g(x))$, $x \in \mathbb{R}^n$, i.e. Nemytskij operators, then it is well-known that acting conditions and boundedness conditions may be different. For simplicity we only consider $n = 1$ and $\mathbb{R}$ replaced by $[0, 1]$. Then the classical example is given by

$$f_s(x,u) := \begin{cases} 0 & \text{if} \quad u \le x^{s/2}\,, \\ \frac{1}{u^{2/s}} - \frac{x}{u^{4/s}} & \text{if} \quad u > x^{s/2}\,, \end{cases}$$

where $0 < s < 1$. The associated Nemytskij operator maps $B^s_{\infty,\infty}[0, 1]$ into itself, but is not bounded. We refer to [5].

**Problem 2":**

*Find necessary and sufficient conditions on $f : \mathbb{R} \to \mathbb{R}$ s.t.*

$$T_f : \quad E^s_{p,q}(\mathbb{R}^n) \to E^s_{p,q}(\mathbb{R}^n)$$

*is continuous.*

**Remark 4** Problem 2 ( Problem 2') and Problem 2" have different answers in general. Below we will show that for some Lizorkin-Triebel spaces there exist noncontinuous bounded composition operators, see Corollary 1.

In our opinion also the following problem is of interest.

**Problem 3:**

*Characterize all function spaces $E$, where the following assertions are equivalent*

- $T_f(E) \subset E$,
- $T_f \,:\, E \to E$ *is bounded,*
- $T_f \,:\, E \to E$ *is continuous,*

*whatever be* $f : \mathbb{R} \to \mathbb{R}$.

At the end of our paper, see Section 7, we will give a list of spaces for which the acting property is equivalent to the boundedness and to the continuity. We will also produce a negative list, it means, we will collect also those spaces, for which such equivalences do not hold.

Both authors have written surveys with respect to this topic in earlier times, see [11], [55, Chapt. 5] and [15]. To increase the readibility of this survey we allow some overlap with those articles. However, we mainly concentrate on the progress since 1995. In addition, a number of open problems is formulated within this text.

This survey is organized as follows. After the Introduction we collect a number of necessary conditions for the acting property to hold. This will make the splitting in the discussion of sufficient conditions more transparent. We start with this discussion in Section 3 by concentrating on Sobolev spaces. In Section 4 we will discuss the case of fractional order of smoothness. The results of this section motivate to study composition operators on intersections which we will do in Section 5. Some generalizations to the vector-valued situation are considered in Section 6 (but here we concentrate on Sobolev spaces). As mentioned above, there will be a section devoted to *Problem 3*, namely Section 7. In the very short final section we make some concluding remarks.

## Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. If $E$ and $F$ are two Banach spaces, then the symbol $E \hookrightarrow F$ indicates that the embedding is continuous. The symbol $c$ denotes a positive constant which depends only on the fixed parameters $n, s, p, q$ and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols " $\lesssim$ " and " $\gtrsim$ " instead of "$\leq$" and "$\geq$", respectively. The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \leq c\,B$. Similarly $\gtrsim$ is defined. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$.

We denote by $C_b^m(\mathbb{R}^n)$ the Banach space of functions on $\mathbb{R}^n$ which are continuous and bounded, together with their derivatives up to order $m$, and by $C_{ub}(\mathbb{R}^n)$ the Banach space of bounded and uniformly continuous functions on $\mathbb{R}^n$. The classical Sobolev spaces are denoted by $W_p^m(\mathbb{R}^n)$, and their homogeneous counterparts by $\dot{W}_p^m(\mathbb{R}^n)$, for $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. Inhomogeneous Besov and Lizorkin-Triebel spaces are denoted by $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, respectively. We use the notation $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and $\dot{B}^s_{p,q}(\mathbb{R}^n)$ for the homogeneous Lizorkin-Triebel and Besov spaces. For their definition, we refer to Section 9. General information about these function spaces, as well as the Wiener classes $BV_p$, can be found e.g. in [55, 63, 64] ($F^s_{p,q}(\mathbb{R}^n)$), [33, 37, 63] ($\dot{F}^s_{p,q}(\mathbb{R}^n)$), and [20, 21, 67] ($BV_p$). Let us mention that we exclude the spaces $F^s_{\infty,q}(\mathbb{R}^n)$, even in case when we write $E^s_{p,q}(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$.
If an equivalence class $[f]$, for the a.e. equality, contains a continuous representative, then we call the class continuous and speak of values of $f$ at any point (by taking the values of the continuous representative).
If $f$ is a function defined on $\mathbb{R}^n$, and if $h \in \mathbb{R}^n$, we put $\Delta_h f(x) := f(x+h) - f(x)$. Throughout the paper $\psi \in \mathcal{D}(\mathbb{R}^n)$ denotes a specific cut-off function, i.e., $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 3/2$.

# 2 Necessary conditions

We begin with a collection of necessary conditions for the acting property. They are all found essentially before 1990. However, they give a good idea concerning the expectable solution of *Problem* 2.

**Proposition 1** *Let $s > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. The acting property $T_f(\mathcal{D}(\mathbb{R}^n)) \subset E^s_{p,q}(\mathbb{R}^n)$ implies $f \in E^{s,\ell oc}_{p,q}(\mathbb{R})$.*

The proof follows easily by testing $T_f$ on functions $u \in \mathcal{D}(\mathbb{R}^n)$ such that $u(x) = x_1$ on some ball of $\mathbb{R}^n$, see e.g. [4, Thm. 3.5] or [55, Thm. 5.3.1/2].

**Proposition 2** *Let $s > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function.*
(i) *The acting property $T_f(E^s_{p,q} \cap L_\infty(\mathbb{R}^n)) \subset B^s_{p,\infty}(\mathbb{R}^n)$ implies $f \in W^{1,\ell oc}_\infty(\mathbb{R})$.*
(ii) *In case $E^s_{p,q}(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$, the acting property $T_f(E^s_{p,q}(\mathbb{R}^n)) \subset B^s_{p,\infty}(\mathbb{R}^n)$ implies $f \in \dot{W}^1_\infty(\mathbb{R})$.*

**Remark 5** (i) The necessity of (local) Lipschitz continuity for the acting condition in the framework of function spaces with fractional order of smoothness has been observed for the first time by Igari [36] ($p = q = 2$, $0 < s < 1$, $s \neq 1/2$). Then the result has been extended to Besov and Lizorkin-Triebel spaces by the first named author and some co-authors. We refer to [12] for more detailed references and for the proof of Prop. 2. Let us also mention the recent publication [4, Thm. 3.1 and 3.2], where a proof in a more general context is given.
(ii) Observe that $E^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^n)$. Hence, the Nikol'skij-Besov space $B^s_{p,\infty}(\mathbb{R}^n)$ is the largest space within the family $E^s_{p,q}(\mathbb{R}^n)$.

**Proposition 3** *Suppose* $1 + 1/p < s < n/p$. *Then the acting property* $T_f(E^s_{p,q}(\mathbb{R}^n)) \subset B^s_{p,\infty}(\mathbb{R}^n)$ *implies* $f(t) = c\,t$ *for some constant* $c$.

The phenomenon described in Proposition 3 is well known since Dahlberg [30] had published his short note in 1979. He had proved that the implication $T_f(W^m_p(\mathbb{R}^n)) \subset W^m_p(\mathbb{R}^n)$ requires $f(t) = c\,t$ for some constant $c$. Extensions to Besov and Triebel-Lizorkin spaces have been given by Bourdaud [8, 11]. Extensions to values $p, q < 1$ can be found in Runst [54] and in [59]. Runst has also been the first who had investigated implications of $T_f(E^s_{p,q}(\mathbb{R}^n)) \subset B^s_{p,\infty}(\mathbb{R}^n)$. A proof of Prop. 3 can be also found in [55, 5.3.1] and [4, Thm. 3.3].
As many times in the theory of Besov-Lizorkin-Triebel spaces in limiting situations the third index $q$ has some influence.

**Proposition 4** *Suppose* $1 + 1/p = s < n/p$.
(i) *Let* $q > 1$. *Then the acting property* $T_f(B^s_{p,q}(\mathbb{R}^n)) \subset B^s_{p,q}(\mathbb{R}^n)$ *implies* $f(t) = c\,t$ *for some constant* $c$.
(ii) *Let* $p > 1$. *Then the acting property* $T_f(F^s_{p,q}(\mathbb{R}^n)) \subset F^s_{p,q}(\mathbb{R}^n)$ *implies* $f(t) = c\,t$ *for some constant* $c$.

This has been proved by the first named author in [10] and [11], see also [55, Lem. 5.3.1/2] and [4, Thm. 3.3]. The existence of nontrivial composition operators on $B^{1+(1/p)}_{p,1}(\mathbb{R}^n)$, $n > p+1$, and on $F^2_{1,q}(\mathbb{R}^n)$, $n > 2$, is an open problem.

**Remark 6** The degeneracy is connected to the existence of unbounded functions in $E^s_{p,q}(\mathbb{R}^n)$. Instead, if we consider $E^s_{p,q} \cap L_\infty(\mathbb{R}^n)$, the Dahlberg phenomenon disappears. Indeed $E^s_{p,q} \cap L_\infty(\mathbb{R}^n)$ is known to be a Banach algebra for the pointwise product. Hence, all entire functions, vanishing at 0, act on it. For this problem, we refer also to Section 5.

**Remark 7** Notice that the embedding $W^{1,\ell oc}_\infty(\mathbb{R}) \hookrightarrow E^{s,\ell oc}_{p,q}(\mathbb{R})$ holds if $0 < s < 1$, while the reverse embedding holds for $s > 1 + (1/p)$. In view of the four preceding propositions, the values $s = 1$, $s = 1 + (1/p)$ and $s = n/p$ appear as the critical ones for *Problem* 2.

# 3 Sobolev spaces

For an easier reading of the paper we first discuss *Problems 2, 2', 2"* for Sobolev spaces. According to Remark 3, we present boundedness in the same section than the acting condition.

## 3.1 The acting condition on Sobolev spaces

In view of Section 2, in particular Remarks 6 and 7, it makes sense to separate the discussion into the cases

- $m = 1$;
- $m \ge 2$ and $W_p^m(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$;
- $m \ge 2$ and $W_p^m(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$.

**Theorem 1** *Suppose* $1 \le p \le \infty$*. Let* $f : \mathbb{R} \to \mathbb{R}$ *be a Borel measurable function s.t.* $f(0) = 0$*. Then it holds*

$$T_f(W_p^1(\mathbb{R}^n)) \subset W_p^1(\mathbb{R}^n) \quad \Longleftrightarrow \quad \begin{cases} f' \in L_\infty^{\ell oc}(\mathbb{R}) & p > n \\ & \text{or} \quad p = 1 = n \\ & \\ f' \in L_\infty(\mathbb{R}) & \text{otherwise.} \end{cases}$$

*In either case we have*

$$\| f \circ g \|_{W_p^1(\mathbb{R}^n)} \le \| f' \|_{L_\infty(\mathbb{R})} \, \| g \|_{W_p^1(\mathbb{R}^n)} \, . \tag{3}$$

**Remark 8** (i) For a proof we refer to Marcus and Mizel [42], see also [6, Chapt. 9].
(ii) If $f$ and $g$ are $C^1$ functions, and if $f(0) = 0$, then

$$\int_{\mathbb{R}^n} |f(g(x))|^p \, dx \;=\; \int_{\mathbb{R}^n} |f(g(x)) - f(0)|^p \, dx \le \| f' \|^p_{L_\infty(\mathbb{R})} \int_{\mathbb{R}^n} |g(x)|^p \, dx \, , \tag{4}$$

$$\int_{\mathbb{R}^n} \left| f'(g(x)) \, \frac{\partial g}{\partial x_i}(x) \right|^p dx \;\le\; \| f' \|^p_{L_\infty(\mathbb{R})} \int_{\mathbb{R}^n} \left| \frac{\partial g}{\partial x_i}(x) \right|^p dx \, , \quad i = 1, \ldots, n \, .$$

Hence, (3) follows under these extra conditions. By (4) also the role of the condition $f(0) = 0$ becomes clear. The extension of these estimates to the general case is more complicated.

**Theorem 2** *Let* $m = 2, 3, \ldots,$ *and let* $1 \le p < \infty$*. We suppose that* $W_p^m(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$*. Let* $f : \mathbb{R} \to \mathbb{R}$ *be a Borel measurable function s.t.* $f(0) = 0$*.*

*(i) Then the composition operator $T_f$ maps $W^m_p(\mathbb{R}^n)$ into itself if, and only if, $f \in W^{m,\ell oc}_p(\mathbb{R})$.*
*(ii) For $f \in W^{m,\ell oc}_p(\mathbb{R})$ we have*

$$\| f \circ g \|_{W^m_p(\mathbb{R}^n)} \le C(f,g) \Big( \| g \|_{W^m_p(\mathbb{R}^n)} + \| g \|^m_{W^m_p(\mathbb{R}^n)} \Big) , \tag{5}$$

*where*

$$\begin{aligned} C(f,g) &:= c \Big( \| f \|_{W^{m-1}_\infty(I_g)} + \| f^{(m)} \|_{L_p(I_g)} \Big) \\ I_g &:= \{ t \in \mathbb{R} : \, |t| \le \|g\|_{L_\infty(\mathbb{R}^n)} \} , \end{aligned}$$

*and $c$ is independent of $f$ and $g$.*

**Remark 9** (i) For a proof we refer to [9], see also [55, 5.2.4].
(ii) The estimate (5) will be typical for the supercritical case, i.e., the case $E^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$. Recall the chain rule

$$D^\gamma (f \circ g) = \sum_{\ell=1}^{|\gamma|} \sum_{\substack{\alpha^1+\cdots+\alpha^\ell=\gamma \\ |\alpha^i|\neq 0}} c_{\gamma,\ell,\alpha^1,\ldots,\alpha^\ell} \, (f^{(\ell)} \circ g) \, D^{\alpha^1} g \, \cdots \, D^{\alpha^\ell} g \, , \tag{6}$$

where $\gamma := (\gamma_1, \ldots, \gamma_n)$, $\alpha^i := (\alpha^i_1, \ldots, \alpha^i_n)$, $i = 1, \ldots, \ell$, are multi-indices and $c_{\gamma,\ell,\alpha^1,\ldots,\alpha^\ell}$ are certain combinatorical constants. Let $\gamma = (m, 0, \ldots, 0)$. Then the sum on the right-hand side contains the terms

$$(f^{(m)} \circ g) \Big( \frac{\partial g}{\partial x_1} \Big)^m \qquad \text{and} \qquad (f' \circ g) \, \frac{\partial^m g}{\partial x_1^m} \, .$$

From this point of view an estimate as given in (5) looks natural.

We need a further class of functions. If $E$ is a normed space of functions on $\mathbb{R}^n$, then the space $E_{unif}$ is the collection of all $g \in E^{\ell oc}$ s.t.

$$\| g \|_{E_{unif}} := \sup_{a \in \mathbb{R}^n} \| g \, \psi(\, \cdot - a) \|_E < \infty \, ,$$

where $\psi$ is the cut-off function from our list of conventions.

**Theorem 3** *Let $m = 2, 3, \ldots,$ and let $1 \le p < \infty$. We suppose that $W^m_p(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function s.t. $f(0) = 0$.*
(i) *If $m = n/p \ge 2$, then $T_f(W^m_p(\mathbb{R}^n)) \subset W^m_p(\mathbb{R}^n)$ holds if, and only if, $f' \in W^{m-1}_{p,unif}(\mathbb{R})$.*
(ii) *Let $m = n/p \ge 2$ and $f' \in W^{m-1}_{p,unif}(\mathbb{R})$. Then*

$$\| f \circ g \|_{W^m_p(\mathbb{R}^n)} \le C_f \Big( \| g \|_{W^m_p(\mathbb{R}^n)} + \| g \|^m_{W^m_p(\mathbb{R}^n)} \Big) \tag{7}$$

*where*

$$C_f := c\left( \| f \|_{\dot{W}^1_\infty(\mathbb{R})} + \sup_{a\in\mathbb{R}} \left( \int_a^{a+1} |f^{(m)}(t)|^p \, dt \right)^{1/p} \right) \tag{8}$$

*and $c$ is independent of $f$ and $g$.*
(iii) *Let either $1 < p < \infty$ and $2 \le m < n/p$, or $p = 1$ and $3 \le m < n$. Then $T_f(W^m_p(\mathbb{R}^n)) \subset W^m_p(\mathbb{R}^n)$ holds if, and only if, $G(t) = c\,t$, $t \in \mathbb{R}$, for some $c \in \mathbb{R}$.*
(iv) *If $n \ge 3$, then $T_f(W^2_1(\mathbb{R}^n)) \subset W^2_1(\mathbb{R}^n)$ holds if, and only if, $f'' \in L_1(\mathbb{R})$.*
(v) *Let $f'' \in L_1(\mathbb{R})$. Then*

$$\| f \circ g \|_{W^2_1(\mathbb{R}^n)} \le c\,(\, \| f' \|_{L_\infty(\mathbb{R})} + \| f'' \|_{L_1(\mathbb{R})}\,)\, \| g \|_{W^2_1(\mathbb{R}^n)} \tag{9}$$

*and $c$ is independent of $f$ and $g$.*

**Remark 10** (i) For a proof we refer to [9], see also [55, 5.2.4] and [14]. The crucial idea in the proofs of Thm. 2 and Thm. 3 (except part (iii)) consists in an integration by parts involving the norm of the Sobolev space. Part (iii) in Thm. 3 is a particular case of Proposition 3, see some further comments in Subsection 4.3.
(ii) The class $W^m_{p,unif}(\mathbb{R})$ plays a crucial role in the composition problem, not only in the critical case $m = n/p$. This becomes clear if we consider instead of the usual Sobolev space $W^m_p(\mathbb{R}^n)$ the so-called Adams-Frazier space $W^m_p \cap \dot{W}^1_{mp}(\mathbb{R}^n)$, see Theorem 25.

We turn back to the problem touched in Remark 9(ii). Here is an improvement of the estimate (5) in the supercritical case.

**Theorem 4** *Let $m = 2, 3, \ldots,$ and let $1 \le p < \infty$. We suppose that $W^m_p(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$. Let $f : \mathbb{R} \to \mathbb{R}$ be in $W^m_p(\mathbb{R})$ s.t. $f(0) = 0$. Then there exists a constant $c$ s.t.*

$$\| f \circ g \|_{W^m_p(\mathbb{R}^n)} \le \; c \, \| f \|_{W^m_p(\mathbb{R})} \left( \| g \|_{W^m_p(\mathbb{R}^n)} + \| g \|^{m-1/p}_{W^m_p(\mathbb{R}^n)} \right). \tag{10}$$

*Here $c$ is independent of $f$ and $g \in W^m_p(\mathbb{R}^n)$.*

**Remark 11** A proof of (10) has been given in [14]. The exponent $m - 1/p$ is the optimal one, see Proposition 6 in Section 4. However, the result does not extend to all $f$ in $W^{m,\ell oc}_p(\mathbb{R})$. A counterexample is given by $f(t) = \sin t$, $t \in \mathbb{R}$, see [14].

## 3.2 Continuity of composition operators on Sobolev spaces

**Theorem 5** *(i) Let $1 \le p < \infty$. Every composition operator $T_f$, which maps $W^1_p(\mathbb{R}^n)$ into itself, is continuous.*
*(ii) Let $1 < p < \infty$, $m \in \mathbb{N}$ and $m > n/p$. Then every composition operator $T_f$, which maps $W^m_p(\mathbb{R}^n)$ into itself, is continuous.*

**Proof** We concentrate on (ii). Under the given restrictions we have $W^m_p(\mathbb{R}^n) = F^m_{p,2}(\mathbb{R}^n)$ in the sense of equivalent norms. From Thm. 2 we derive the equivalence of the acting condition $T_f(W^m_p(\mathbb{R}^n)) \subset W^m_p(\mathbb{R}^n)$ and $f \in W^{m,\ell oc}_p(\mathbb{R})$, $f(0) = 0$. Furthermore, the boundedness of $T_f$ follows from the estimate (5). In this situation we can apply Proposition 7, see paragraph 4.2.3, and obtain that $T_f$ must be continuous as well.

**Remark 12** Part (i) is a famous result of Marcus and Mizel [43]. Part (ii) seems to be a novelty.

# 4 Spaces of fractional order of smoothness

According to Section 2, it makes sense to separate the discussion into the cases

- $0 < s < 1$;
- $1 < s < 1 + 1/p$;
- $1 + 1/p < s < n/p$;
- $\max(1 + 1/p, n/p) < s$.

## 4.1 The case of low smoothness

In case $0 < s < 1$, the only known necessary condition, see Remark 7, turns out to be also sufficient.

**Theorem 6** *Let $1 \le p, q \le \infty$ and $0 < s < 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function s.t. $f(0) = 0$.*
*(i) The following assertions are equivalent:*

*(a) $T_f(E^s_{p,q}(\mathbb{R}^n)) \subset E^s_{p,q}(\mathbb{R}^n)$;*

*(b) $T_f : E^s_{p,q}(\mathbb{R}^n) \to E^s_{p,q}(\mathbb{R}^n)$ is bounded;*

*(c) Either $f' \in L_\infty(\mathbb{R})$ if $E^s_{p,q}(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$ or $f' \in L^{\ell oc}_\infty(\mathbb{R})$ if $E^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$.*

*(ii) Let $f'$ be as in (c). Then*

$$\| f \circ g \|_{E^s_{p,q}(\mathbb{R}^n)} \le \| f' \|_{L_\infty(I_g)} \, \| g \|_{E^s_{p,q}(\mathbb{R}^n)}$$

*holds for all $g \in E^s_{p,q}(\mathbb{R}^n)$ and $I_g$ is defined as in Thm. 2.*

**Proof** The space $E^s_{p,q}(\mathbb{R}^n)$ is defined by first order differences, see Definition 2 in Section 9. For those differences we have the obvious inequality $|\Delta_h(f\circ g)| \le \|f'\|_\infty\, |\Delta_h g|$. For the estimate of the $L_p$-term we refer to (4). Both inequalities together prove (ii) and at the same time the implications $(c) \Rightarrow (b), (a)$. The nontrivial implication (a) $\Rightarrow$ (c) follows by Proposition 2. ■

By using real interpolation for Lipschitz-continuous operators, see [52], one can add continuity to the list in part (i) of the theorem.

**Theorem 7** *Let $1 \le p, q < \infty$ and $0 < s < 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function s.t. $f(0) = 0$ and either $f' \in L_\infty(\mathbb{R})$ if $B^s_{p,q}(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$ or $f' \in L^{\ell oc}_\infty(\mathbb{R})$ if $B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$. Then $T_f : B^s_{p,q}(\mathbb{R}^n) \to B^s_{p,q}(\mathbb{R}^n)$ is continuous.*

The proof in [55, Thm. 5.5.2/3] uses the real interpolation formula

$$\Big(L_p(\mathbb{R}^n), W^1_p(\mathbb{R}^n)\Big)_{s,q} = B^s_{p,q}(\mathbb{R}^n)\,, \qquad 0 < s < 1\,, \quad 1 \le q \le \infty\,,$$

in connection with a result of Maligranda [41] concerning the continuity of a nonlinear operator $T$ with respect to real interpolation, see [55, Sect. 2.5].

## 4.2 The case of high smoothness

In view of the results presented in Section 2, there is a natural conjecture.

**Conjecture 1** *Let $s > 1 + (1/p)$. The composition operator $T_f$, associated to a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, maps $E^s_{p,q} \cap L_\infty(\mathbb{R}^n)$ to $E^s_{p,q}(\mathbb{R}^n)$ if, and only if, $f \in E^{s,\ell oc}_{p,q}(\mathbb{R})$ and $f(0) = 0$.*

The validity of the above Conjecture is known in case of Sobolev spaces (recall $W^m_p(\mathbb{R}^n) = F^m_{p,2}(\mathbb{R}^n)$, $1 < p < \infty$, $m \in \mathbb{N}$), see Thm. 1 and [14, Thm. 2]. For $n = 1$, it is also valid for any Lizorkin-Triebel space and for some Besov spaces, see Subsection 4.2.1. The extension to dimensions $n > 1$ is an open question.

### 4.2.1 The acting condition in the one-dimensional situation

To begin with we deal with a simplified situation where we give a sketch of the proof. However, the used arguments are also typical for the more general results which will be mentioned below.

We need some more classes of functions. For a function $g : \mathbb{R} \to \mathbb{R}$, we denote by $\|g\|_{BV_p}$ the supremum of numbers

$$\left( \sum_{k=1}^{N} |g(b_k) - g(a_k)|^p \right)^{1/p} ,$$

taken over all finite sets $\{]a_k, b_k[\, ;\, k = 1, \ldots, N\}$ of pairwise disjoint open intervals. A function $g$ is said to be of *bounded p-variation* if $\|g\|_{BV_p} < +\infty$. The collection of all such functions is called the Wiener class and denoted by $BV_p$. We refer to [20, 21, 67] for a discussion of these classes. Their importance in the composition problem is related to the Peetre's embedding

$$B^{1/p}_{p,1}(\mathbb{R}) \hookrightarrow BV_p(\mathbb{R})\,, \tag{11}$$

see e.g. [53] and [20, Thm. 5].

**Proposition 5** *Let $1 \le p < \infty$, $p \le q \le \infty$ and $1 + 1/p < s < 2$. Let $f' \in B^{s-1}_{p,q}(\mathbb{R})$ s.t. $f(0) = 0$. Then there exists a constant $c > 0$ such that the inequality*

$$\| f \circ g \|_{B^s_{p,q}(\mathbb{R})} \le c \, \| f' \|_{B^{s-1}_{p,q}(\mathbb{R})} \left( \| g \|_{B^s_{p,q}(\mathbb{R})} + \| g' \|^{s-(1/p)}_{BV_{sp-1}} \right) \tag{12}$$

*holds for all real analytic functions $g$ in $B^s_{p,q}(\mathbb{R})$.*

**Remark 13** By (11) and by assumption $s > 1 + (1/p)$, a weaker version of (12) is given by

$$\| f \circ g \|_{B^s_{p,q}(\mathbb{R})} \le c \, \| f' \|_{B^{s-1}_{p,q}(\mathbb{R})} \left( \| g \|_{B^s_{p,q}(\mathbb{R})} + \| g' \|^{s-(1/p)}_{B^{s-1}_{p,q}(\mathbb{R})} \right) . \tag{13}$$

**Proof** Because of

$$B^{s-1}_{p,q}(\mathbb{R}) \hookrightarrow B^{s-1-1/p}_{\infty,\infty}(\mathbb{R}) \hookrightarrow C^{s'}_{ub}(\mathbb{R}) \hookrightarrow L_\infty(\mathbb{R})\,, \qquad 0 < s' < s - 1 - \frac{1}{p}\,,$$

the function $f'$ is a continuous. Hence we may apply the chain rule and obtain

$$(f \circ g)' = (f' \circ g)\, g'$$

in the pointwise sense. Furthermore, we will use that

$$\| u \|_{B^s_{p,q}(\mathbb{R})} \asymp \| u \|_{L_p(\mathbb{R})} + \| u' \|_{B^{s-1}_{p,q}(\mathbb{R})} \tag{14}$$

for all distribution $u$ if $s > 0$, see [63, 2.3.8].

*Step 1.* Since the estimate of the $L_p$-term is as in (4), we only have to deal with the homogeneous part of the Besov norm. For brevity we put

$$\omega_p(f,h) := \left( \int_{\mathbb{R}} |\Delta_h f(x)|^p \, dx \right)^{1/p} , \qquad h \in \mathbb{R}\ .$$

Using (14) we have to estimate

$$\left(\int_{-1}^{1}\left(\frac{\omega_p((f\circ g)',h)}{|h|^{s-1}}\right)^q \frac{dh}{|h|}\right)^{1/q}.$$

Since

$$\omega_p((f'\circ g)\,g',h) \le \|\, f'\,\|_\infty\, \omega_p(g',h) + U(h)\,,$$

where

$$U(h) := \left(\int_{\mathbb{R}} |\Delta_h(f'\circ g)(x)|^p |g'(x)|^p\, dx\right)^{1/p},$$

we are reduced to prove that

$$\left(\int_{-1}^{1}\left(\frac{U(h)}{|h|^{s-1}}\right)^q \frac{dh}{|h|}\right)^{1/q}$$

can be estimated by the right-hand side of (12).

*Step 2.* Without loss of generality we may assume $h > 0$. The set of zeros of $g'$ is discrete, and its complement in $\mathbb{R}$ is the union of a family $(I_l)_l$ of nonempty open disjoint intervals. For any $h > 0$ we denote by $I_l'$ the (possibly empty) set of $x \in I_l$ whose distance to the right endpoint of $I_l$ is greater than $h$, and we set

$$I_l'' := I_l \setminus I_l'\,, \qquad a_l := \sup_{I_l} |g'|\,.$$

By $g_l$ we mean the restriction of $g$ to $I_l$, hence a strictly monotone smooth function. If $I_l' \neq \emptyset$, then we have

$$|g(g_l^{-1}(y)+h) - y| \le a_l\, h \quad \text{for} \quad y \in g_l(I_l')\,, \tag{15}$$

where $g_l^{-1}$ denotes the inverse function of $g_l$.

*Substep 2.1.* Let

$$\Omega_p(f,t) := \left(\int_{\mathbb{R}} \sup_{|h|\le t} |\Delta_h f(x)|^p\, dx\right)^{1/p}, \qquad t > 0\,. \tag{16}$$

By (15) and by a change of variable we find

$$\int_{I_l'} |\Delta_h(f'\circ g)(x)|^p |g'(x)|^p\, dx \le a_l^{p-1}\, \Omega_p^p(f', a_l h)\,. \tag{17}$$

By the Minkowski inequality w.r.t. $L_{q/p}$, and by Proposition 8 in Section 9, we obtain

$$\Big(\int_0^\infty \Big(\frac{1}{h^{(s-1)p}} \quad \sum_l a_l^{p-1}\, \Omega_p^p(f', a_l\, h)\Big)^{q/p} \frac{dh}{h}\Big)^{1/q}$$

$$\begin{aligned}
&\leq \quad \left(\sum_l \left(\int_0^\infty \left(\frac{1}{h^{p(s-1)}}\ a_l^{p-1}\, \Omega_p^p(f', a_l\, h)\right)^{q/p} \frac{dh}{h}\right)^{p/q}\right)^{1/p} \\
&= \quad \Big(\sum_l a_l^{p-1+p(s-1)}\Big)^{1/p} \Big(\int_0^\infty \Big(\frac{\Omega_p(f',t)}{t^{s-1}}\Big)^q \frac{dt}{t}\Big)^{1/q} \\
&\leq \quad c\, \|\, f'\,\|_{B^{s-1}_{p,q}(\mathbb{R})} \Big(\sum_l \Big(\sup_{I_l}\ |g'|\Big)^{sp-1}\Big)^{1/p} .
\end{aligned}$$

Now we follow [20, proof of Thm. 7]. By definition, the function $g'$ vanishes at the endpoints of $I_l$. Let $\beta_l$ be one of these endpoints. Furthermore, there is at least one point $\xi_l \in I_l$ such that

$$|g'(\xi_l)| = \sup_{I_l}\ |g'|\, .$$

Hence

$$\sum_l \sup_{I_l}\ |g'|^{sp-1} = \sum_l |g'(\xi_l) - g'(\beta_l)|^{sp-1} \leq \|g'\|^{sp-1}_{BV_{sp-1}}\, .$$

By (17), we conclude that

$$\begin{aligned}
&\Big(\int_0^\infty \Big(\frac{1}{h^{(s-1)p}} \sum_l \quad \int_{I_l'} |\Delta_h(f'\circ g)(x)|^p |g'(x)|^p\, dx\Big)^{q/p} \frac{dh}{h}\Big)^{1/q} \\
&\qquad\qquad \leq \quad c\, \|\, f'\,\|_{B^{s-1}_{p,q}(\mathbb{R})}\ \|\, g'\,\|^{s-(1/p)}_{BV_{sp-1}}\, . \qquad (18)
\end{aligned}$$

*Substep 2.2.* Since $g'$ vanishes at the right endpoint of $I_l''$, it holds

$$|g'(x)| \leq \sup_{|v|\leq h} |g'(x) - g'(x+v)|$$

for all $x \in I_l''$. Thus we obtain

$$\sum_l \int_{I_l''} |\Delta_h(f'\circ g)(x)|^p |g'(x)|^p\, dx \leq (2\|f'\|_\infty)^p\, \Omega_p(g',h)^p\, . \qquad (19)$$

By Proposition 8, we conclude that

$$\begin{aligned}
&\Big(\int_0^\infty \Big(\frac{1}{h^{(s-1)p}} \sum_l \quad \int_{I_l''} |\Delta_h(f'\circ g)(x)|^p |g'(x)|^p\, dx\Big)^{q/p} \frac{dh}{h}\Big)^{1/q} \\
&\qquad\qquad \leq \quad c\, \|\, f'\,\|_\infty\, \|g\|_{B^s_{p,q}(\mathbb{R})}\, . \qquad (20)
\end{aligned}$$

Putting (4), (17), (18) and (20) together we obtain (12). The proof is complete. ■

To derive the acting property with respect to $B^s_{p,q}(\mathbb{R})$ one uses a specific density argument (to get rid of the restriction to real analytic $g$), the Fatou property, see, e.g., [22] and [32]. To obtain the acting property for $B^s_{p,q}(\mathbb{R})$ for higher values of $s$, $m + 1/p < s < m + 1$, $m \geq 2$, one can use an induction argument in combination with

$$\| u \|_{B^s_{p,q}(\mathbb{R})} \asymp \| u \|_{L_p(\mathbb{R})} + \| u^{(m)} \|_{B^{s-m}_{p,q}(\mathbb{R})} \tag{21}$$

if $s > 0$, see [63, 2.3.8]. However, to get an optimal inequality for those values of $s$ and to deal with $m \leq s \leq m + 1/p$, $m \geq 2$, more effort is needed. To describe this we have to introduce a further space of functions. Since from now on we do not give proofs and the arguments in the Lizorkin-Triebel case are similar (however, a bit more sophisticated) we turn to the general notation. The outcome is the following, see [23], [24], [25] and [48] for all details.

**Theorem 8** *Let $1 < p < \infty$, $1 \leq q \leq \infty$ ($p \leq q$ if $E = B$), and $s > 1 + (1/p)$. The composition operator $T_f$, associated to a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, acts on $E^s_{p,q}(\mathbb{R})$ if, and only if, $f(0) = 0$ and $f \in E^{s,\ell oc}_{p,q}(\mathbb{R})$.*

**Remark 14** We believe that the restriction $p \leq q$ in the $B$-case is connected with the method of proof but not with the problem itself.

Acting property and boundedness are equivalent in this situation. To describe this, we use the smooth cut-off function $\psi$ and define $\psi_t(x) := \psi(x/t)$, $t > 0$. We denote by $\dot{\mathcal{E}}^s_{p,q}(\mathbb{R})$ the space of all functions in $L_\infty(\mathbb{R})$ which belong to the *homogeneous* Besov-Lizorkin-Triebel space $\dot{E}^s_{p,q}(\mathbb{R})$, and endow it with the natural norm

$$\|f\|_{\dot{\mathcal{E}}^s_{p,q}(\mathbb{R})} := \|f\|_{\dot{E}^s_{p,q}(\mathbb{R})} + \| f \|_{L_\infty(\mathbb{R})} \, .$$

**Theorem 9** *Let $1 < p < \infty$, $1 \leq q \leq \infty$ ($p \leq q$ if $E = B$), and $s > 1 + (1/p)$.*
(i) *Let $f \in E^{s,\ell oc}_{p,q}(\mathbb{R})$ and $f(0) = 0$. Then there exists a constant $c > 0$ such that the inequality*

$$\| f \circ g \|_{E^s_{p,q}(\mathbb{R})} \leq c \, \| (f \, \psi_{\| g \|_{L_\infty(\mathbb{R})}})' \|_{\dot{\mathcal{E}}^{s-1}_{p,q}(\mathbb{R})} \Big( \|g\|_{E^s_{p,q}(\mathbb{R})} + \|g\|^{s-(1/p)}_{E^s_{p,q}(\mathbb{R})} \Big) \tag{22}$$

*holds for all such functions $f$ and all $g \in E^s_{p,q}(\mathbb{R})$.*
(ii) *There exists a constant $c > 0$ such that the inequality*

$$\| f \circ g \|_{E^s_{p,q}(\mathbb{R})} \leq c \, \| f \|_{E^s_{p,q}(\mathbb{R})} \Big( \|g\|_{E^s_{p,q}(\mathbb{R})} + \|g\|^{s-(1/p)}_{E^s_{p,q}(\mathbb{R})} \Big) \tag{23}$$

*holds for all functions $f \in E^s_{p,q}(\mathbb{R})$, $f(0) = 0$, and all functions $g \in E^s_{p,q}(\mathbb{R})$.*

### 4.2.2 Optimal inequalities (I)

We would like to discuss the quality of the estimate (23). However, for doing that there is no need to concentrate on $n = 1$. The following simple arguments work for any $n$.

**Proposition 6** *Let $1 \le p, q \le \infty$ and $s > 1+1/p$. Let $h : [0,\infty[ \to [0,\infty[$ be a nondecreasing function satisfying $h(x) = o\left(x^{s-(1/p)}\right)$ for $x \to +\infty$. Let $N$ be any semi-norm on $\mathcal{D}(\mathbb{R}^n)$. If $f$ is a continuous function such that, for some constant $c > 0$, the inequality*

$$\| f \circ g \|_{B^s_{p,\infty}(\mathbb{R}^n)} \le c \Big( N(g) + h\left(N(g)\right) \Big) \tag{24}$$

*holds for all $g \in \mathcal{D}(\mathbb{R}^n)$, then $f$ is a polynomial of degree $\le s$.*

**Proof** We take the natural number $m \ge 1$ such that $m - 1 \le s < m$. Let $\varphi$ be a function in $\mathcal{D}(\mathbb{R}^n)$ s.t. $\varphi(x) = x_1$ for all $x \in [-1,1]^n$. Then it holds $(f \circ a\varphi)(x) = f(ax_1)$ for all $x \in [-1,1]^n$ and all $a > 0$. Hence

$$\Delta^m_{te_1}(f \circ a\varphi)(x) = \Delta^m_{at} f(ax_1)\,, \quad \forall x \in [-1/2, 1/2]^n\,, \forall t \in [0, 1/2m]\,, \forall a > 0\,.$$

Using this identity and (24), we deduce

$$\Big( \int_{-a/2}^{a/2} |\Delta^m_t f(x)|^p\, dx \Big)^{1/p} \le c\, t^s \left( a^{(1/p)+1-s} N(\varphi) + h(a\, N(\varphi))\, a^{(1/p)-s} \right)$$

for all $t \in ]0,1]$ and $a > 2m$. By taking $a$ to $+\infty$, and by applying the assumption on $h$, we deduce

$$\Delta^m_t f(x) = 0 \quad \text{a.e.}\,, \qquad \forall t \in ]0,1]\,.$$

Then by a standard argument, we deduce that $f$ is a polynomial of degree at most $m - 1$.

**Remark 15** Proposition 6 yields that the exponent $s-1/p$ in (13), (22), and (23) is optimal. It also implies the optimality of the estimate (10).

### 4.2.3 Continuity of composition operators in the one-dimensional situation

There is a general continuity theorem in [18, Cor. 2] which can be applied for all $n$, if Conjecture 1 is valid.

**Proposition 7** *Let $s > 1 + 1/p$. If Conjecture 1 is valid for $E^s_{p,q}(\mathbb{R}^n)$, then a composition operator $T_f$ is continuous if, and only if, $f(0) = 0$ and $f$ belongs to the closure of the smooth functions in $E^{s,\ell oc}_{p,q}(\mathbb{R})$.*

In consequence of Thm. 8 and this proposition we immediately get the following.

**Corollary 1** *Let $1 < p < \infty$ and $s > 1 + (1/p)$. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable.*
(i) *Let $1 \le q < \infty$ ($p \le q$ if $E = B$). Then the following assertions are equivalent :*

- *$T_f$ satisfies the acting condition $T_f(E^s_{p,q}(\mathbb{R})) \subset E^s_{p,q}(\mathbb{R})$.*
- *$T_f$ is a bounded mapping of $E^s_{p,q}(\mathbb{R})$ to itself.*
- *$T_f : E^s_{p,q}(\mathbb{R}) \to E^s_{p,q}(\mathbb{R})$ is continuous.*
- *$f \in E^{s,\ell oc}_{p,q}(\mathbb{R})$ and $f(0) = 0$.*

(ii) *The following assertions are equivalent :*

- *$T_f$ satisfies the acting condition $T_f(E^s_{p,\infty}(\mathbb{R})) \subset E^s_{p,\infty}(\mathbb{R})$.*
- *$T_f$ is a bounded mapping of $E^s_{p,\infty}(\mathbb{R})$ to itself.*
- *$f \in E^{s,\ell oc}_{p,\infty}(\mathbb{R})$ and $f(0) = 0$.*

(iii) *There exist functions $f$ in $E^s_{p,\infty}(\mathbb{R})$, $f(0) = 0$, such that $T_f : E^s_{p,\infty}(\mathbb{R}) \to E^s_{p,\infty}(\mathbb{R})$ is not continuous.*

### 4.2.4 Acting conditions in the general $n$-dimensional situation

In dimension $n > 1$, Conjecture 1 has not been proved up to now. We give here two sufficient acting conditions. In the first one, we try to approach the minimal assumptions on $f$. In the second one, we obtain an optimal estimate, but with stronger regularity assumptions on $f$.

**Theorem 10** *Let $1 \le p \le \infty$, $1 \le q \le \infty$, $\max(n/p, 1) < s < \mu$. We suppose $f \in C^\mu(\mathbb{R})$ and $f(0) = 0$. Then $T_f$ maps $E^s_{p,q}(\mathbb{R}^n)$ into $E^s_{p,q}(\mathbb{R}^n)$.*

**Remark 16** (i) Of course, we have $C^\mu(\mathbb{R}) \subset E^{s,\ell oc}_{p,q}(\mathbb{R})$. For a proof of Thm. 10 we refer to [55, Sect. 5.3.6, Thm. 2].
(ii) Let $E = B$. Under some extra conditions on $s$, it is proved in [13] that one can replace $C^\mu(\mathbb{R})$ by $B^{s+\varepsilon,\ell oc}_{p,q}(\mathbb{R})$ with $\varepsilon > 0$ arbitrary.

**Theorem 11** *Let $1 \le p \le \infty$, $1 \le q \le \infty$, $m \in \mathbb{N}$ and $\max(m, n/p) < s < m + 1$. We suppose $f \in C^{m+1}(\mathbb{R})$ and $f(0) = 0$. Then there exists a constant $c$ s.t.*

$$\| f \circ g \|_{E^s_{p,q}(\mathbb{R}^n)} \le c \, \| f \|_{C^{m+1}_b(\mathbb{R})} \left(\|g\|_{E^s_{p,q}(\mathbb{R}^n)} + \|g\|^s_{E^s_{p,q}(\mathbb{R}^n)}\right) \tag{25}$$

*holds for all such $f$ and all $g \in E^s_{p,q}(\mathbb{R}^n)$.*

**Proof** The theorem has been proved essentially in [55, 5.3.7, Thm. 1 and 2]. Only one further remark is needed. Because of $s > \max(1, n/p)$ we have

$$s - \frac{n}{p} = s\left(1 - \frac{n}{ps}\right) > 1 - \frac{n}{ps}$$

and therefore, the continuous embeddings

$$B^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^1_{ps,v}(\mathbb{R}^n) \qquad \text{and} \qquad F^s_{p,q}(\mathbb{R}^n) \hookrightarrow F^1_{ps,v}(\mathbb{R}^n) \tag{26}$$

for arbitrary $v \in ]0, \infty]$. ■

**Remark 17** (i) Probably the exponent $s$ in (25) can be improved to $s - 1/p$ under additional assumptions on $f$, see Proposition 6 and Thm. 4.
(ii) The estimation of the norm of $\|f \circ g\|$ in Theorem 11 is better than in Theorem 10. In the detailed statement of Theorem 10, the norm $\|f \circ g\|$ is controlled by $\|g\|^\mu$ instead of $\|g\|^s$ for $\|g\| \geq 1$.
(iii) Both theorems have a long list of forerunners. Let us mention at least a few: Moser 1960 (Sobolev spaces) [47], Mizohata 1965 (Bessel potential spaces with $p = 2$) [46], Peetre 1970 (Besov spaces, nonlinear interpolation) [52], Adams 1976 (Bessel potential spaces) [1], Meyer 1981 (Bessel potential spaces) [45], Runst 1986 (Besov-Lizorkin-Triebel spaces) [54], Adams & Frazier 1992 (Bessel potential spaces) [2, 3].

### 4.3 The intermediate case (I)

We consider the case $1 + 1/p < s < n/p$. By Proposition 3, the composition operator necessarily lowers the regularity. The study of $T_f$ for $f(t) := \sin t$, $t \in \mathbb{R}$, is particularly enlightening in this respect. Let us define

$$g_\alpha(x) = \psi(x)\, |x|^{-\alpha}\,, \qquad x \in \mathbb{R}^n\,, \quad \alpha > 0. \tag{27}$$

Then it is well known (and not very complicated to prove) that in case $s > 0$

$$g_\alpha \in F^s_{p,q}(\mathbb{R}^n) \qquad \Longleftrightarrow \qquad 0 < \alpha < \frac{n}{p} - s \tag{28}$$

and

$$g_\alpha \in B^s_{p,q}(\mathbb{R}^n) \quad \Longleftrightarrow \quad \text{either} \quad 0 < \alpha < \frac{n}{p} - s \quad \text{or} \quad \alpha = \frac{n}{p} - s \quad \text{and} \quad q = \infty \tag{29}$$

hold, cf. [55, Lem. 2.3.1/1]. Then

$$\frac{\partial^m}{\partial x_1^m}\sin(g_\alpha(x)) \asymp \sin^{(m)}(|x|^{-\alpha})\,(|x|^{-\alpha-2}x_1)^m + \text{ lower order terms}\,,$$

in a certain neighborhood of the origin. Compare this with the m-th order derivative of the original function. It turns out that

$$\frac{\partial^m}{\partial x_1^m}\,g_\alpha(x) \asymp |x|^{-\alpha-2m}\,x_1^m + \text{ lower order terms} \qquad (|x|\to 0)\,.$$

Since we have $(\alpha+1)m > \alpha+m$ if $m \geq 2$ this shows that the local singularity of the composition becomes stronger than the singularity of the original function $g_\alpha$.
Going back to the general situation, a natural question consists in asking for the best possible image space, hence we turn to *Problem 1* for the rest of this subsection. The following strange number will play a certain role

$$\varrho^* = \varrho^*(s,p,n) := \frac{\frac{n}{p}+\frac{1}{p}\left(\frac{n}{p}-s\right)}{\frac{n}{p}-s+1}\,. \tag{30}$$

**Theorem 12** *Suppose $1+1/p < s < n/p$. Let $f$ be a non-polynomial Borel measurable function. Then for every $r > \varrho^*(s,p,n)$ there exists a compactly supported function $g_r \in E^s_{p,q}(\mathbb{R}^n)$ such that the composition $f\circ g_r$ does not belong to $B^r_{p,\infty}(\mathbb{R}^n)$.*

**Remark 18** (i) A proof in case of smooth $f$ can be found in [58, 59, 55]. However, by the same arguments one can deal with non-smooth functions.
(ii) If $1+1/p < s < n/p$, then $1+1/p < \varrho^* < s$. This indicates a certain loss of smoothness. Whether there exists a nonlinear function $f$ such that $T_f(E^s_{p,q}(\mathbb{R}^n)) \subset B^{\varrho^*}_{p,\infty}(\mathbb{R}^n)$ is still an open question.
(iii) Also the following observation is of some interest. We study the difference $d(s) := s-\varrho^*(s,p,n)$ for fixed $n$ and $p$ ($n > p+1$). Obviously, $d(1+1/p) = d(n/p) = 0$ and $d(s) > 0$ if $1+1/p < s < n/p$. Moreover, the function $d(s)$ is concave on this interval, hence, it attains a maximal value $d(s_0)$ there. We have

$$s_0 := \frac{n}{p}+1-\sqrt{\frac{n-1}{p}} \qquad \text{and} \qquad d(s_0) = \left(\sqrt{\frac{n-1}{p}}-1\right)^2 + \frac{p-1}{\sqrt{p(n-1)}}$$

This shows that $d(s)$ has a bound depending on $p$ and $n$, but it does not have an a priori bound for fixed $p$ and independent of $n$.

We will use the following abbreviation:

$$\| f \|_{\dot{a}_p^r([-w,w])} := \left( \int_0^w |h|^{-rp-1} \int_{-w}^w |\, \Delta_h^{m+1} f(y)\, |^p \, dy \, dh \right)^{1/p} \tag{31}$$

with $m \in \mathbb{N}$ and $m+1 > r > 0$. The notation $\| f \|_{\dot{a}_p^r([-w,w])}$ reminds on a norm in a homogeneous Besov space. In some sense it is an incomplete one.

**Theorem 13** *Suppose $T_f(E^s_{p,q}(\mathbb{R}^n)) \subset E^r_{p,p}(\mathbb{R}^n)$ and $0 < r \le s < n/p$. Let $f \in L_\infty(\mathbb{R})$. Then it follows that*

$$\sup_{w \ge 1} w^{-\gamma} \| f \|_{\dot{a}_p^r([-w,w])} < \infty \tag{32}$$

*for all*

$$\gamma > \gamma_0(s,n,p,r) := \frac{\frac{n}{p} + \frac{1}{p}\left(\frac{n}{p} - s\right) - r\left(\frac{n}{p} - s + 1\right)}{\frac{n}{p} - s} \, . \tag{33}$$

**Remark 19** (i) For the proof we refer to [61].
(ii) Let us comment on the condition (33). Fix $n$ and $p$ and consider $s \uparrow n/p$. Then the lower bound for $\gamma$ tends to infinity, which means that the necessary condition (32) becomes less restrictive. This is connected with the fact that local singularities in spaces with $s \uparrow n/p$ become weaker and weaker, cf. (27) and (28). If we fix also $s$ and consider $r \downarrow 0$, then the necessary condition (32) becomes again weaker (since the lower bound of $\gamma$ increases). Clearly, that corresponds to the fact that the spaces $E^r_{p,p}(\mathbb{R}^n)$ become larger. A similar observation gives a converse result if $r \uparrow s$.
(iii) A first essential conclusion of Theorem 13 is obtained by observing that $\| f \|_{\dot{a}_p^r([-w,w])}$ is a non-decreasing function in $w$. So, whenever $f$ is not a polynomial of low degree, $\| f \|_{\dot{a}_p^r([-w,w])}$ is bounded from below by a positive constant. Then Theorem 13 says (in the case that $f$ is bounded) that $\gamma_0 \ge 0$ if, and only if, $r \le \rho^*$, which also follows from Theorem 3. Hence, Theorem 13 represents an extension of Theorem 3.

Replacing the space $E^r_{p,p}(\mathbb{R}^n) = B^r_{p,p}(\mathbb{R}^n) = F^r_{p,p}(\mathbb{R}^n)$ by the Sobolev space $W^r_p(\mathbb{R}^n)$, $r \in \mathbb{N}$, we have necessary and sufficient conditions. Let

$$\mathcal{P}_m = \Big\{ g : \quad g(t) = \sum_{\ell=0}^m a_\ell \, t^\ell \, , \quad a_\ell \in \mathbb{R}, \quad \ell = 0, \ldots, m \Big\} , \quad m \in \mathbb{N} \, ,$$

and

$$A_{\gamma,p}(f) := \sup_{w \ge 1} w^{-\gamma} \left( \int_{-w}^w |\, f^{(m)}(y)\, |^p \, dy \right)^{1/p} , \tag{34}$$

which replaces (31).

**Theorem 14** *Suppose* $0 \leq r \leq s < n/p$*. Recall,* $\rho^*$ *and* $\gamma_0$ *are defined in (30) and (33), respectively. Let* $f : \mathbb{R} \to \mathbb{R}$ *be a Borel measurable function if* $r \geq 1$ *and a continuous function if* $r = 0$*, but not an element of* $\mathcal{P}_{r-1}$*. Then* $T_f(E^s_{p,p}(\mathbb{R}^n)) \subset W^r_p(\mathbb{R}^n)$ *implies* $r \leq \rho^*$*,* $f(0) = 0$*,* $f \in W^{r,\ell oc}_p(\mathbb{R})$ *and* $A_{\gamma_0,p}(f) < \infty$*.*

Now we turn to the sufficient conditions. We concentrate on $r \geq 2$ (for $r = 0, 1$ we refer to [60]).

**Theorem 15** *Suppose* $1 < p < \infty$*,* $2 \leq r \leq s < n/p$ *and define again* $\gamma_0$ *by (33). Let* $f(0) = 0$*. If* $A_{\gamma_0,p}(f) < \infty$*, then* $T_f(E^s_{p,p}(\mathbb{R}^n)) \subset W^r_p(\mathbb{R}^n)$ *holds. Moreover, there exists some constant* $c$ *such that*

$$\| f \circ g \|_{W^r_p(\mathbb{R}^n)} \leq c \, A_{\gamma_0,p}(f) \left( \| g \|_{E^s_{p,p}(\mathbb{R}^n)} + \| g \|^{\gamma_0 + r - \frac{1}{p}}_{E^s_{p,p}(\mathbb{R}^n)} \right) \tag{35}$$

*holds with* $c$ *independent of* $f$ *and* $g$*.*

**Remark 20** Both theorems are proved in [60]. However, the used arguments are as in [30, 8, 11, 54] (Thm. 14) and [9] (Thm. 15).

Various examples are treated in [59, 60, 61]. Here we concentrate on smooth periodic functions. If $f \not\equiv 0$ is periodic and smooth, then

$$\int_{-w}^{w} | f^{(m)}(y) |^p \, dy \asymp w \, , \qquad w \geq 1 \, .$$

Hence $A_{\gamma,p}(f) < \infty$ if, and only if, $\gamma \geq 1/p$. In such a situation Theorems 14 and 15 yield final results. Here another strange number occurs. Let

$$\varrho = \varrho(s,n,p) := \frac{\frac{n}{p}}{\frac{n}{p} - s + 1} \, . \tag{36}$$

Obviously, $\varrho < \varrho^*$ if $p < \infty$.

**Corollary 2** *Let* $1 < p < \infty$ *and let* $2 \leq r \leq s < n/p$*. Suppose that* $f$ *is periodic,* $f \in C^\infty(\mathbb{R})$*, and* $f \not\equiv 0$*. Then* $T_f(E^s_{p,p}(\mathbb{R}^n)) \subset W^r_p(\mathbb{R}^n)$ *holds if, and only if,* $f(0) = 0$ *and* $r \leq \varrho$*.*

This has a fractional counterpart, see [55, 5.3.6] and [61].

**Theorem 16** *Let* $1 \leq p < \infty$ *and* $1 < s < n/p$*. Suppose that* $f$ *is periodic,* $f(0) = 0$*,* $f \in C^\infty(\mathbb{R})$*, and* $f \not\equiv 0$*. Then the following assertions are equivalent:*
*(i)* $T_f(F^s_{p,q}(\mathbb{R}^n)) \subset B^r_{p,\infty}(\mathbb{R}^n)$*. (ii)* $T_f(F^s_{p,\infty}(\mathbb{R}^n)) \subset F^r_{p,q}(\mathbb{R}^n)$*. (iii)* $r \leq \varrho$ *.*

In [55, Thm. 5.3.6/3] we have been able also to prove the following associated inequality.

**Theorem 17** *Let $f' \in C^\infty(\mathbb{R})$ and let $f(0) = 0$. Suppose $1 \le q, r \le \infty$ and $1 < s < n/p$. Let $\varrho$ be as in (36). Then there exists a constant $c$ such that*

$$\| f \circ g \|_{F^{\varrho}_{p,r}(\mathbb{R}^n)} \le c \, (\| g \|_{F^{s}_{p,q}(\mathbb{R}^n)} + \| g \|^{\varrho}_{F^{s}_{p,q}(\mathbb{R}^n)}) \tag{37}$$

*holds for all $g \in F^s_{p,q}(\mathbb{R}^n)$.*

**Remark 21** The proof of Thm. 17 relies on a specific estimate of products in Lizorkin-Triebel spaces, see [55, Thm. 4.6.2/5]. There is also counterpart for Besov spaces but less satisfactory.

## 4.4 The intermediate case (II)

Now we assume that $1 < s < 1+1/p$. Since $W^{1,\ell oc}_\infty(\mathbb{R}) \not\subset E^{s,\ell oc}_{p,q}(\mathbb{R})$ and $E^{s,\ell oc}_{p,q}(\mathbb{R}) \not\subset W^{1,\ell oc}_\infty(\mathbb{R})$, we have two independent necessary acting conditions, but we do not know if these two conditions are sufficient. Indeed, at this moment we do not have a conjecture how does the solution of *Problem 2* looks like. The best sufficient condition obtained so far is connected with a new class of functions which has been introduced for the first time by Bourdaud and Kateb.

We define $U^1_p(\mathbb{R})$ as the set of Lipschitz continuous functions $f$ on $\mathbb{R}$ such that

$$\|f'\|_{U_p} := \sup_{t>0} t^{-1/p} \, \Omega_p(f', t) < +\infty \, , \tag{38}$$

see (16) for the definition of $\Omega_p$.

**Theorem 18** *Let $1 \le p < +\infty$ and $0 < s < 1 + (1/p)$. If $f \in U^1_p(\mathbb{R})$ and $f(0) = 0$, then $T_f(B^s_{p,q}(\mathbb{R}^n)) \subset B^s_{p,q}(\mathbb{R}^n)$. Moreover, the inequality*

$$\|f \circ g\|_{B^s_{p,q}(\mathbb{R}^n)} \;\le\; c \, \left(\|f'\|_\infty + \|f'\|_{U_p}\right) \, \|g\|_{B^s_{p,q}(\mathbb{R}^n)}$$

*holds for all $g \in B^s_{p,q}(\mathbb{R}^n)$.*

**Remark 22** (i) A first proof of this theorem was found by Bourdaud and Kateb [16]. For $n = 1$, Kateb [38] improved Theorem 18 by obtaining the acting property under the condition $f' \in L_\infty \cap \dot{B}^{1/p}_{p,\infty}(\mathbb{R})$. Observe,

$$U^1_p(\mathbb{R}) \hookrightarrow \dot{W}^1_\infty \cap \dot{B}^{1+1/p}_{p,\infty}(\mathbb{R})$$

and the embedding is proper, see [20]. An extension of this result of Kateb to the general $n$-dimensional case is still open.
(ii) Whether Theorem 18 has a counterpart for Lizorkin-Triebel spaces is an open question.
(iii) Of course, smooth periodic functions $f$ do not act by composition on $E^s_{p,q}(\mathbb{R}^n)$, for $1 < s \le 1 + 1/p$, see Theorem 16.

A second proof of Thm. 18 has been given in [20]. There we first investigated the limiting situation $s = 1 + 1/p$, which we will recall below, and afterwards we used nonlinear interpolation (more exactly, real interpolation of Lipschitz continuous operators) to derive the result for $0 < s < 1 + 1/p$.

**Theorem 19** *Let $1 < p < \infty$. If $f \in U^1_p(\mathbb{R})$ and $f(0) = 0$, then $T_f(B^{1+1/p}_{p,1}(\mathbb{R}^n)) \subset B^{1+1/p}_{p,\infty}(\mathbb{R}^n)$. Moreover, the inequality*

$$\|f \circ g\|_{B^{1+1/p}_{p,\infty}(\mathbb{R}^n)} \le c \left(\|f'\|_\infty + \|f'\|_{U_p}\right) \|g\|_{B^{1+1/p}_{p,1}(\mathbb{R}^n)}$$

*holds for all $g \in B^{1+1/p}_{p,1}(\mathbb{R}^n)$.*

There is a further result, rather close to the one-dimensional case of Thm. 19, we wish to mention. Recall, the Wiener classes $BV_p(\mathbb{R})$ have been introduced in Subsection 4.2.1. We will need the class of all primitives.

**Definition 1** *Let $p \in [1, +\infty]$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ belongs to $BV^1_p(\mathbb{R})$ if $f$ is Lipschitz continuous and if its distributional derivative belongs to $BV_p(\mathbb{R})$.*

We endow $BV^1_p(\mathbb{R})$ with the norm

$$\|f\|_{BV^1_p(\mathbb{R})} := |f(0)| + \|f'\|_{BV_p(\mathbb{R})} \qquad \forall f \in BV^1_p(\mathbb{R}),$$

which renders $BV^1_p(\mathbb{R})$ a Banach space. Concerning composition of functions belonging to $BV^1_p(\mathbb{R})$ we have proved in [20] the following satisfactory result.

**Theorem 20** *Let $1 \le p < \infty$. Then the following statements hold.*

(i) *If $f, g \in BV^1_p(\mathbb{R})$, then $f \circ g \in BV^1_p(\mathbb{R})$, and*

$$\|f \circ g\|_{BV^1_p(\mathbb{R})} \le \|f\|_{BV^1_p(\mathbb{R})} \left(1 + 2^{1/p}\|g\|_{BV^1_p(\mathbb{R})}\right).$$

(ii) *Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then the operator $T_f$ maps $BV^1_p(\mathbb{R})$ to itself if, and only if, $f \in BV^1_p(\mathbb{R})$.*

**Remark 23** Also the inclusion

$$BV^1_p(\mathbb{R}) \hookrightarrow U^1_p(\mathbb{R})$$

is proper, see [20] or [40].

## The main example

There is one example which is of particular importance, mainly for historic reasons. We consider $f(t) := |t|$, $t \in \mathbb{R}$. This function belongs to the Besov space $B^{1+1/p,\ell oc}_{p,\infty}(\mathbb{R})$ and is Lipschitz continuous, of course. It is also immediate that it belongs to $BV^1_p(\mathbb{R})$ and therefore to $U^1_p(\mathbb{R})$. By employing Thm. 19 and Thm. 18 we obtain the following corollary.

**Corollary 3** *Let $1 < p < \infty$. The operator $g \mapsto |g|$ maps the Besov space $B^{1+1/p}_{p,1}(\mathbb{R}^n))$ into the Besov space $B^{1+1/p}_{p,\infty}(\mathbb{R}^n)$. Moreover, the inequality*

$$\| \, |g| \, \|_{B^{1+1/p}_{p,\infty}(\mathbb{R}^n)} \le c \, \|g\|_{B^{1+1/p}_{p,1}(\mathbb{R}^n)}$$

*holds for all $g \in B^{1+1/p}_{p,1}(\mathbb{R}^n)$.*

**Remark 24** The corollary can be derived also from a result of Savaré [57] who had investigated the mapping $g \mapsto |g|$ with respect to a Banach space $Z^{1+1/p,p}(\mathbb{R})$, where

$$B^{1+\frac{1}{p}}_{p,1}(\mathbb{R}) \hookrightarrow Z^{1+\frac{1}{p},p}(\mathbb{R}) \hookrightarrow B^{1+\frac{1}{p}}_{p,\infty}(\mathbb{R}) \, .$$

For Besov spaces with $0 < s < 1 + 1/p$ we can argue by using nonlinear interpolation to obtain the boundedness of $g \mapsto |g|$ considered as a mapping of $B^s_{p,q}(\mathbb{R}^n)$ into itself. This method can not be applied to Lizorkin-Triebel spaces. By employing a different method the outcome is the following.

**Theorem 21** *Let $1 \le p, q \le \infty$ ($1 < p < \infty$, $1 \le q \le \infty$ if $E = F$). In addition we assume $0 < s < 1 + 1/p$ ($s \ne 1$ if $p = 1$ in case $E = F$). Then the operator $g \mapsto |g|$ maps the space $E^s_{p,q}(\mathbb{R}^n)$ into itself. Moreover, the inequality*

$$\| \, |g| \, \|_{E^s_{p,q}(\mathbb{R}^n)} \le c \, \|g\|_{E^s_{p,q}(\mathbb{R}^n)}$$

*holds for all $g \in E^s_{p,q}(\mathbb{R}^n)$.*

**Remark 25** (i) Completely different methods have been used by Bourdaud, Meyer [22] and Oswald [51] to prove Thm. 21 with $E = B$. Whereas in the first reference the proof is based on Hardy's inequality the second reference is using spline techniques and Marchaud's inequality.
(ii) A first proof of Thm. 21 with $E = F$ has been given in [11], but with the extra condition $s \ne 1$. Here a similar method as in [22] is applied. A second proof has been published by Triebel [65, Thm. 25.8]. It relies on atomic decompositions and allows to deal with $s = 1$ if $1 < p < \infty$.

Let us turn to the continuity of the mapping $g \mapsto |g|$. The Lipschitz continuity of $f(t) = |t|$ yields the continuity of $g \mapsto |g|$ considered as a mapping of $B^s_{p,q}(\mathbb{R}^n)$ into itself if $0 < s < 1$ and $1 \le p, q \le \infty$, see Thm. 7. Also the continuity with respect to $W^1_p(\mathbb{R}^n)$ is well-known, see Thm. 5. There is nothing known on the continuity in all other cases where we know $T_f(E^s_{p,q}(\mathbb{R}^n)) \subset E^s_{p,q}(\mathbb{R}^n)$.

## 4.5 Optimal inequalities (II)

All estimates in Subsection 4.4 do not reflect the nonlinearity of $T_f$, they are all of the form $\le c(f)\, \|g\|$. Hence, the norm of $g$ enters with the power 1. This is in sharp contrast to the estimates given in case $s > 1 + 1/p$. However, by using essentially the same type of arguments as in Prop. 6, one can also prove the following, see [20] for all details.

**Lemma 1** *Let $1 < p \le +\infty$, and $s > 1 + (1/p)$. Let $N$ be a norm on $\mathcal{D}(\mathbb{R}^n)$. Let $E$ be a normed function space such that $\mathcal{D}(\mathbb{R}^n) \subseteq E \subseteq W_1^{1,\ell oc}(\mathbb{R}^n)$ and such that there exists a positive constant $A$ such that*

$$\sup_{h \neq 0} |h|^{1-s} \left( \int_{\mathbb{R}^n} \Big| \frac{\partial g}{\partial x_i}(x+h) - \frac{\partial g}{\partial x_i}(x) \Big|^p dx \right)^{1/p} \le A \, \| g \|_E \qquad \forall g \in E \tag{39}$$

*for all $i = 1, \ldots, n$.*
*If there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ and a constant $B > 0$ such that $T_f$ maps $\mathcal{D}(\mathbb{R}^n)$ into $E$, and such that the inequality*

$$\| f \circ g \|_E \le B\, (N(g) + 1) \qquad \forall g \in \mathcal{D}(\mathbb{R}^n) \tag{40}$$

*holds, then $f$ must be an affine function.*

**Remark 26** (i) The spaces $E^t_{p,q}(\mathbb{R}^n)$, $t > 1 + 1/p$, satisfy the assumptions of Lemma 1 with $1 + 1/p < s < \min(t, 2)$.
(ii) A mapping $T : E \to E$ is called *sublinear*, if there exists a constant $c$ such that

$$\| Tg \|_E \le c\, (1 + \| g \|_E)$$

holds for all $g \in E$. Hence, by Lemma 1, a composition operator $T_f$, satisfying the acting property $T_f(E^s_{p,q}(\mathbb{R}^n)) \subset E^s_{p,q}(\mathbb{R}^n)$, can be sublinear only in case $s \le 1 + 1/p$.
(iii) It is of a certain surprise that the boundary between sublinear and superlinear estimates is given by $s = 1 + 1/p$ and not simply $s = 1$ (think on a fractional version of the chain rule).

# 5 Composition operators defined on intersections

Adams and Frazier [2, 3] have been the first who have seen that it makes sense to consider composition operators on intersections of the type $H^s_p \cap W^1_{sp}(\mathbb{R}^n)$, where $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$ is a Bessel potential space. The main observation is the following: the Dahlberg phenomenon, see Prop. 3, disappears.

**Theorem 22** *Let $1 < p, q < \infty$ and $s > 1$, $s \notin \mathbb{N}$. Let $m > s$ be a natural number and let $f \in C^m_b(\mathbb{R})$ s.t. $f(0) = 0$. Then $T_f$ maps $F^s_{p,q} \cap W^1_{sp}(\mathbb{R}^n)$ into $F^s_{p,q}(\mathbb{R}^n)$. In either case the mapping is bounded and continuous.*

**Remark 27** (i) For the first moment it is surprising that the relation of $s$ to $n/p$ does not have an influence. However, if $s > \max(1, n/p)$, then we have $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow W^1_{sp}(\mathbb{R}^n)$, see (26), and we are back in the situation discussed in Subsection 4.2.4. If $1 < s < n/p$, then $F^s_{p,q}(\mathbb{R}^n) \not\subset W^1_{sp}(\mathbb{R}^n)$ and consequently $F^s_{p,q} \cap W^1_{sp}(\mathbb{R}^n)$ is strictly smaller than $F^s_{p,q}(\mathbb{R}^n)$.
(ii) The theorem, as stated here, can be found in Brezis and Mironescu [26]. A different proof, but restricted to the case $p = q$, has been given by Maz'ya and Shaposnikova [44]. With some restrictions in $q$ the boundedness of $T_f$ is also proved in [55, 5.3.7]. For a similar result involving Besov spaces we also refer to [55, 5.3.7].

Of course, one may ask for larger or simply different subspaces of $F^s_{p,q}(\mathbb{R}^n)$ such that all functions $f \in C^m_b(\mathbb{R})$ generate a composition operator $T_f$ s.t. this subspace is mapped into $F^s_{p,q}(\mathbb{R}^n)$ by $T_f$. In [26] one can find a simple argument which explains that $F^s_{p,q} \cap W^1_{sp}(\mathbb{R}^n)$ is nearly optimal. Let $g \in F^s_{p,q}(\mathbb{R}^n)$. Since $f_1(t) = \cos t - 1$ and $f_2(t) = \sin t$ are admissible functions it follows

$$(\cos g - 1 + i\,\sin g) \in\ F^s_{p,q} \cap L_\infty(\mathbb{R}^n)\,.$$

Since

$$F^s_{p,q} \cap L_\infty(\mathbb{R}^n) \hookrightarrow F^1_{sp,2}(\mathbb{R}^n) = W^1_{ps}(\mathbb{R}^n)\,,$$

see [26] (but traced there to Oru), we obtain $e^{i\,g} - 1 \in W^1_{ps}(\mathbb{R}^n)$. This implies

$$i\, e^{i\,g}\, \frac{\partial g}{\partial x_\ell} \in L_{ps}(\mathbb{R}^n) \qquad \text{for all} \quad \ell = 1, \ldots, n\,.$$

Thus $g \in \dot{W}^1_{ps}(\mathbb{R}^n)$, the homogeneous Sobolev space. This space consists of all regular real-valued distributions s.t. all derivatives of the first order belong to $L_{ps}(\mathbb{R}^n)$ and is endowed with the seminorm

$$\|g\|_{\dot{W}^1_{ps}(\mathbb{R}^n)} := \sum_{|\alpha|=1} \|g^{(\alpha)}\|_{L_{ps}(\mathbb{R}^n)}\,.$$

Hence, the optimal subspace is contained in $F^s_{p,q} \cap \dot{W}^1_{sp}(\mathbb{R}^n)$. In case of Bessel potential and Slobodeckij spaces, Thm. 22 can be improved to become optimal with this respect. Recall, $F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n)$ in the sense of equivalent norms.

**Theorem 23** *Let $1 < p < \infty$ and $s > 1$, $s \not\in \mathbb{N}$. Let $m > s$ be a natural number and let $f \in C^m_b(\mathbb{R})$ s.t. $f(0) = 0$. Then $T_f$ maps $F^s_{p,2} \cap \dot{W}^1_{sp}(\mathbb{R}^n)$ into $F^s_{p,2}(\mathbb{R}^n)$. Furthermore, there exists a constant $c_f$ such that*

$$\|f \circ g\|_{F^s_{p,2}(\mathbb{R}^n)} \le c_f \left(\|g\|_{F^s_{p,2}(\mathbb{R}^n)} + \|g\|^s_{\dot{W}^1_{ps}(\mathbb{R}^n)}\right)$$

*holds for all $g \in F^s_{p,2} \cap \dot{W}^1_{sp}(\mathbb{R}^n)$.*

**Remark 28** Thm. 23 has been proved by Adams and Frazier in [3].

Now we turn to Slobodeckij spaces. Recall, if $s > 0$ is not an natural number, then $F^s_{p,p}(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n)$ holds in the sense of equivalent norms.

**Theorem 24** *Let $1 \le p < \infty$ and $s > 1$, $s \not\in \mathbb{N}$. Let $m > s$ be a natural number and let $f \in C^m_b(\mathbb{R})$ s.t. $f(0) = 0$. Then $T_f$ maps $F^s_{p,p} \cap \dot{W}^1_{sp}(\mathbb{R}^n)$ into $F^s_{p,p}(\mathbb{R}^n)$. Furthermore, there exists a constant $c$ such that*

$$\|f \circ g\|_{F^s_{p,p}(\mathbb{R}^n)} \le c\, \|f\|_{C^m_b(\mathbb{R})} \left(\|g\|_{F^s_{p,p}(\mathbb{R}^n)} + \|g\|^s_{\dot{W}^1_{ps}(\mathbb{R}^n)}\right)$$

*holds for all $g \in F^s_{p,p} \cap \dot{W}^1_{sp}(\mathbb{R}^n)$ and all $f \in C^m_b(\mathbb{R})$ s.t. $f(0) = 0$. In either case the mapping $T_f$ is continuous.*

**Remark 29** (i) Thm. 24 has been proved by Maz'ya and Shaposnikova in [44].
(ii) Open problem: prove Theorems 22, 23, 24 for the maximal range of $s, p, q$ and under minimal regularity conditions on $f$. Up to now, only the case of Sobolev spaces has a complete answer, given by the following statements, see [14, Thm. 1 and 2].

**Theorem 25** *Let $m$ be an integer $\ge 2$, $1 \le p < +\infty$ — with the* exception *of $m = n$ and $p = 1$ — and let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function.*
*(i) $T_f$ takes $W^m_p \cap \dot{W}^1_{mp}(\mathbb{R}^n)$ to itself if, and only if, $f(0) = 0$ and $f' \in W^{m-1}_{p,unif}(\mathbb{R})$.*
*(ii) The inequality*

$$\|f \circ g\|_{W^m_p(\mathbb{R}^n)} \le c\, C_f \left(\|g\|^m_{\dot{W}^1_{mp}(\mathbb{R}^n)} + \|g\|_{W^m_p(\mathbb{R}^n)}\right) \tag{41}$$

*holds for all $f$ s.t. $f(0) = 0$ and $f' \in W^{m-1}_{p,unif}(\mathbb{R})$, and all $g \in W^m_p \cap \dot{W}^1_{mp}(\mathbb{R}^n)$, where $C_f$ is defined by (8).*

**Theorem 26** *Let $m$ be an integer $\geq 2$, $1 \leq p < +\infty$, and let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then $T_f$ takes $W^m_p \cap L_\infty(\mathbb{R}^n)$ to itself if, and only if, $f \in W^{m,\ell oc}_p(\mathbb{R})$ and $f(0) = 0$.*

**Remark 30** The exceptional case of Thm. 25 is covered by Thm. 26 since $W^n_1 \cap \dot{W}^1_n(\mathbb{R}^n)$ is embedded into $L_\infty(\mathbb{R}^n)$.

# 6 Composition operators on vector-valued spaces

We turn to the vector-valued situation but restrict ourselves to Sobolev spaces.
Let $k \in \mathbb{N}$, $k \geq 2$. Let $f : \mathbb{R}^k \to \mathbb{R}$. We study the associated composition operator

$$T_f(g) := f \circ g\,, \qquad g = (g_1, \ldots, g_k)$$

under the assumption $g \in W^m_p(\mathbb{R}^n, \mathbb{R}^k)$. To describe a sufficient condition for the acting property we need a further class of functions. We need the following simple mappings: for $1 \leq j \leq k$ we define

$$\sigma_j f(x_1, \ldots, x_k) := f(x_1, \ldots, x_{j-1}, x_k,\, x_{j+1}, \ldots, x_{k-1}, x_j)\,, \qquad x \in \mathbb{R}^n\,.$$

Let $1 \leq p < \infty$ and $k > 1$. Then the space $E_p(\mathbb{R}^k)$ is the collection of all measurable functions $f$ s.t.

$$\| f \|_{E_p(\mathbb{R}^k)} := \sum_{j=1}^k \Big( \sup_{a \in \mathbb{R}} \int_a^{a+1} \| \sigma_j f(\,\cdot\,, t)\|^p_{L_\infty(\mathbb{R}^{k-1})}\, dt\Big)^{1/p} < \infty\,.$$

By $\dot{W}^m_{E_p}(\mathbb{R}^k)$ we denote the homogeneous Sobolev space built on $E_p$, i.e., the semi-norm is generated by

$$\| f \|_{\dot{W}^m_{E_p}(\mathbb{R}^k)} := \sum_{|\alpha| = m} \| f^{(\alpha)} \|_{E_p(\mathbb{R}^k)}\,.$$

Then we have the following partial generalization of Thm. 25 :

**Theorem 27** *Let $1 < p < \infty$ and $m, k \geq 2$. Let $f \in \dot{W}^1_\infty \cap \dot{W}^m_{E_p}(\mathbb{R}^k)$ and suppose $f(0) = 0$. Then $T_f$ maps $W^m_p \cap \dot{W}^1_{pm}(\mathbb{R}^n, \mathbb{R}^k)$ into $W^m_p(\mathbb{R}^n)$. Moreover, there exists a constant $c$ s.t.*

$$\| f \circ g \|_{W^m_p(\mathbb{R}^n)} \leq c \Big( \| f \|_{\dot{W}^m_{E_p}(\mathbb{R}^k)} + \| f \|_{\dot{W}^1_\infty(\mathbb{R}^k)} \Big) \Big( \| g \|_{W^m_p(\mathbb{R}^n)} + \| g \|^m_{\dot{W}^1_{mp}(\mathbb{R}^n)} \Big)\,, \qquad (42)$$

*holds for all $f \in \dot{W}^1_\infty \cap \dot{W}^m_{E_p}(\mathbb{R}^k)$ and all $g \in W^m_p \cap \dot{W}^1_{pm}(\mathbb{R}^n, \mathbb{R}^k)$.*

**Remark 31** (i) This theorem has been proved in [14, Thm. 3]. The estimate (42) is not found there, however, it can be derived as in the scalar case.
(ii) For $T_f(W^m_p \cap \dot{W}^1_{pm}(\mathbb{R}^n,\mathbb{R}^k)) \subset W^m_p(\mathbb{R}^n)$ it is necessary that $f$ is locally Lipschitz continuous, see [4] and [14].

Let us have a short look on Thm. 2. A naive but natural conjecture concerning an extension to the vector-valued case would consist in the following:

*Let $f \in \dot{W}^1_\infty \cap W^{m,\ell oc}_p(\mathbb{R}^k)$ such that $f(0) = 0$. Then $T_f$ maps $W^m_p \cap \dot{W}^1_{pm}(\mathbb{R}^n,\mathbb{R}^k)$ into $W^m_p(\mathbb{R}^n)$.*

We do not believe that this statement holds true. The space $\dot{W}^1_\infty \cap \dot{W}^m_{E_p}(\mathbb{R}^k)$ is much smaller than $\dot{W}^1_\infty \cap W^{m,\ell oc}_p(\mathbb{R}^k)$. This can be easily seen by studying tensor products of functions. Let

$$f(x) := f_1(x_1) \cdot \ldots \cdot f_k(x_k)\,, \qquad x = (x_1,\ldots,x_k) \in \mathbb{R}^k\,.$$

If $f \in \dot{W}^1_\infty \cap \dot{W}^m_{E_p}(\mathbb{R}^k)$, then each of the functions $f_j$ has to belong to $W^{m,\ell oc}_\infty(\mathbb{R})$, at least if all components of $f$ are nontrivial or more exactly, are not polynomials. To guarantee $f \in \dot{W}^1_\infty \cap W^{m,\ell oc}_p(\mathbb{R}^k)$ it is sufficient to have $f_j \in W^{m,\ell oc}_p(\mathbb{R})$ for all $j$.
In our understanding the extension to the vector-valued case will be not a straightforward generalization. We expect some new phenomenons.

**Remark 32** The extension of Thm. 27 to fractional order of smoothness is completely open.

# 7 On Problem 3

Recall, *Problem 3* consists in characterizing those function spaces $E$ s.t. the acting condition $T_f(E) \subset E$ is equivalent to boundedness of $T_f : E \to E$ and is also equivalent to the continuity of $T_f : E \to E$. With other words, if one has established the acting property then one gets boundedness and continuity for free. If a space $E$ has this property then we will write $E \in P_3$, otherwise $E \not\in P_3$.
Below we have made a list of more or less classical function spaces and fixed their relation to $P_3$. In addition we have given some references, sometimes inside our survey, sometimes not. We will use the convention $-\infty < a < b < \infty$.

| **Function space E** | $E \in P_3$ | References |
|---|---|---|
| $L_p(\mathbb{R}^n)$, $1 \le p < \infty$ | yes | [6, 3.4, 3.7] |
| $L_\infty(\mathbb{R}^n)$ | no | [6, 3.7] |
| $C_b^m(\mathbb{R}^n)$, $m \in \mathbb{N}$ | yes | obvious |
| $C^s(\mathbb{R}^n)$, $s > 0$, $s \notin \mathbb{N}$ | no | [31], [35], [48] |
| $W_p^1(\mathbb{R}^n)$, $1 \le p < \infty$ | yes | Thm. 5(i) |
| $W_p^m(\mathbb{R}^n)$, $1 < p < \infty$, $m \in \mathbb{N}$, $m > n/p$ | yes | Thm. 5(ii) |
| $W_p^m(\mathbb{R}^n)$, $1 \le p < \infty$, $m \in \mathbb{N}$, $1 + 1/p < m < n/p$ | yes | Prop. 3 |
| $F_{p,q}^s(\mathbb{R})$, $1 < p < \infty$, $1 \le q < \infty$, $1 + 1/p < s$ | yes | Cor. 1 |
| $F_{p,\infty}^s(\mathbb{R})$, $1 < p < \infty$, $1 + 1/p < s$ | no | Cor. 1 |
| $F_{p,q}^s(\mathbb{R}^n)$, $1 \le p < \infty$, $1 \le q \le \infty$, $1 + 1/p < s < n/p$ | yes | Prop. 3 |
| $F_{p,q}^{1+1/p}(\mathbb{R}^n)$, $1 < p < \infty$, $1 \le q \le \infty$ | yes | Prop. 4 |
| $B_{p,q}^s(\mathbb{R})$, $1 < p < \infty$, $p \le q < \infty$, $1 + 1/p < s$ | yes | Cor. 1 |
| $B_{p,\infty}^s(\mathbb{R})$, $1 < p < \infty$, $1 + 1/p < s$ | no | Cor. 1 |
| $B_{p,q}^s(\mathbb{R}^n)$, $1 \le p < \infty$, $1 \le q \le \infty$, $1 + 1/p < s < n/p$ | yes | Prop. 4 |
| $B_{p,q}^{1+1/p}(\mathbb{R}^n)$, $1 \le p < \infty$, $1 < q \le \infty$ | yes | Prop. 3 |
| $BMO\,(\mathbb{R}^n)$ | no | [19] |
| $VMO\,(\mathbb{R}^n)$ | no | [19] |
| $CMO\,(\mathbb{R}^n)$ | yes | [19] |
| $bmo\,(\mathbb{R}^n)$ | no | [19] |
| $vmo\,(\mathbb{R}^n)$ | yes | [19] |
| $cmo\,(\mathbb{R}^n)$ | yes | [19] |
| $AC\,[a,b]$ | yes | [5] |
| $BV_p\,[a,b]$ | yes | [5] |
| $Lip\,\alpha[a,b]$, $0 < \alpha \le 1$ | no | [31], [35], [5] |
| $A(\mathbb{T})$ | yes | [39, 8.6], [15], |

Definitions of all these function spaces will be given in the Appendix below.

# 8 Concluding remarks

As it becomes clear by the long list of open problems, in our opinion the theory of composition operators in function spaces of fractional order of smoothness (like Besov and Lizorkin-Triebel spaces) is just at its beginning. Even worst is the situation with respect to Nemytskij operators. There are nearly no final results in the general case up to our knowledge. A few information can be found in [55, 5.5.4]. A bit better is the situation when we restrict us to operators of the type

$$N(g)(x) := f(x, g(x)), \qquad x \in \mathbb{R}^n, \quad g \in E\,.$$

These special Nemytskij operators are studied in the monograph by Appell and Zabrejko [6], see also the recent survey [5]. However, the knowledge concentrates on either spaces with smoothness 0 (Lebesgue spaces, Orlicz spaces), or smoothness 1 (first order Sobolev-Orlicz spaces) or on spaces with $p = \infty$ (Hölder spaces).

# 9 Appendix

Here we recall the definition of the function spaces used in this survey.

## 9.1 Besov and Lizorkin-Triebel spaces

To introduce the (inhomogeneous) Besov-Triebel-Lizorkin spaces we make use of the characterizations via differences and derivatives.

**Definition 2** *Let $s > 0$, $1 \le q \le \infty$, and let $M \in \mathbb{N}$ be such that $M \le s < M + 1$.*
(i) *Suppose $1 \le p < \infty$. Then the Lizorkin-Triebel space $F^s_{p,q}(\mathbb{R}^n)$ is the collection of all real-valued functions $f \in L_p(\mathbb{R}^n)$ s.t.*

$$\| f \|_{F^s_{p,q}(\mathbb{R}^n)} := \| f \|_{L_p(\mathbb{R}^n)} + \left\| \Big( \int\limits_0^1 t^{-sq} \big( \frac{1}{t^n} \int\limits_{|h|<t} |\Delta_h^{M+1} f|\, dh \big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^n)} < \infty\,.$$

(ii) *Suppose $1 \le p \le \infty$. Then the Besov space $B^s_{p,q}(\mathbb{R}^n)$ is the collection of all real-valued functions $f \in L_p(\mathbb{R}^n)$ s.t.*

$$\| f \|_{B^s_{p,q}(\mathbb{R}^n)} := \| f \|_{L_p(\mathbb{R}^n)} + \Big( \int_{\mathbb{R}^n} |h|^{-sq} \| \Delta_h^{M+1} f \|^q_{L_p(\mathbb{R}^n)} \frac{dh}{|h|^n} \Big)^{\frac{1}{q}} < \infty\,.$$

**Remark 33** The spaces $F^s_{p,q}(\mathbb{R}^n)$, $B^s_{p,q}(\mathbb{R}^n)$ are Banach spaces. Nowadays there exists a rich literature on this subject. We refer to Frazier and Jawerth [33], Besov, Il'jin, and Nikol'skij [7], Nikol'skij [50], Peetre [53] and Triebel [63, 64, 66].

In some cases, an alternative equivalent norm in Besov spaces can be obtained as follows. We concentrate on $n = 1$. Using the functional $\Omega_p(f,t)$ defined in (16), we have the following result, see e.g. [64, Thm. 3.5.3, p. 194]:

**Proposition 8** *Let $1/p < s < 1$. Then a real-valued function $f$ belongs to $B^s_{p,q}(\mathbb{R})$ if, and only if,*

$$\| f \|_{L_p(\mathbb{R})} + \left( \int_0^\infty \left( \frac{\Omega_p(f,t)}{t^s} \right)^q \frac{dt}{t} \right)^{1/q} < +\infty \, . \tag{43}$$

*Moreover, the above expression generates an equivalent norm on $B^s_{p,q}(\mathbb{R})$.*

**Remark 34** The condition $s > 1/p$ cannot be avoided. Indeed, (43) implies that $f$ is locally bounded, a property which is not shared by all Besov functions for $s < 1/p$.

In Subsection 4.2.1 we have also used homogeneous Besov-Triebel-Lizorkin spaces. Here is a definition.

**Definition 3** *Let $s > 0$, $1 \le q \le \infty$, and let $M \in \mathbb{N}$ be such that $M \le s < M+1$.*
(i) *Suppose $1 \le p < \infty$. Then the homogeneous Lizorkin-Triebel space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ is the collection of all regular real-valued distributions $f$ s.t.*

$$\| f \|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} := \left\| \Big( \int\limits_0^\infty t^{-sq} \big( \frac{1}{t^n} \int\limits_{|h|<t} |\Delta_h^{M+1} f| \, dh \big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^n)} < \infty \, .$$

(ii) *Suppose $1 \le p \le \infty$. Then the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is the collection of all regular real-valued distributions $f$ s.t.*

$$\| f \|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \Big( \int_{\mathbb{R}^n} |h|^{-sq} \| \Delta_h^{M+1} f \|^q_{L_p(\mathbb{R}^n)} \frac{dh}{|h|^n} \Big)^{\frac{1}{q}} < \infty \, .$$

**Remark 35** Of course, we have $f = g$ in $\dot{E}^s_{p,q}(\mathbb{R}^n)$ if $f - g$ is a polynomial of degree $\le M$. Since we only deal with functions $f \in \dot{E}^s_{p,q} \cap L_\infty(\mathbb{R})$, satisfying $f(0) = 0$, this does not matter.

## 9.2 BMO-type spaces

$BMO(\mathbb{R}^n)$ is the set of real-valued locally integrable functions $g$ on $\mathbb{R}^n$ such that

$$\|g\|_{BMO} := \sup_{Q} ⨍_Q \left| g - ⨍_Q g \right| < +\infty\,,$$

where the supremum is taken on all cubes $Q$ with sides parallel to the coordinate axes and where

$$⨍_Q g$$

denotes the mean value of the function $g$ on $Q$. The quotient space of $BMO(\mathbb{R}^n)$, endowed with the above seminorm, by the subspace of constant functions, is a Banach space. Since the operator $T_f$ is clearly not defined on the quotient space, we prefer to consider $BMO(\mathbb{R}^n)$ as a Banach space of 'true' functions with the following norm:

$$\|g\|_* := \|g\|_{BMO} + ⨍_{Q_0} |g| \qquad \forall g \in BMO(\mathbb{R}^n),$$

where $Q_0$ is the unit cube $[-1/2, +1/2]^n$. We denote by $bmo(\mathbb{R}^n)$ the linear subspace of $BMO(\mathbb{R}^n)$ consisting of those functions $g$ which satisfy also the following condition

$$\sup_{|Q|\geq 1} ⨍_Q |g| < +\infty\,,$$

where $|Q|$ denotes the Lebesgue measure of $Q$ or, equivalently,

$$\sup_{|Q|=1} ⨍_Q |g| < +\infty\,,$$

see [19, Lem. 7]. It turns out that $bmo(\mathbb{R}^n)$ is a Banach space for the norm

$$\|g\|_{bmo} := \|g\|_{BMO} + \sup_{|Q|=1} ⨍_Q |g| \qquad \forall g \in bmo(\mathbb{R}^n).$$

We denote by $cmo(\mathbb{R}^n)$ the closure of $\mathcal{D}(\mathbb{R}^n)$ in $bmo(\mathbb{R}^n)$, and we endow $cmo(\mathbb{R}^n)$ with the norm of $bmo(\mathbb{R}^n)$. Similarly, we denote by $CMO(\mathbb{R}^n)$ the closure of $\mathcal{D}(\mathbb{R}^n)$ in $BMO(\mathbb{R}^n)$, and we endow $CMO(\mathbb{R}^n)$ with the norm of $BMO(\mathbb{R}^n)$.

According to Sarason [56], a function $g$ of $BMO(\mathbb{R}^n)$ which satisfies the limiting condition

$$\lim_{a\to 0} \left( \sup_{|Q|\leq a} ⨍_Q \left| g - ⨍_Q g \right| \right) = 0 \tag{44}$$

is said to be of vanishing mean oscillation. The subspace of $BMO(\mathbb{R}^n)$ consisting of the functions of vanishing mean oscillation is denoted $VMO(\mathbb{R}^n)$, and we endow $VMO(\mathbb{R}^n)$

with the norm of $BMO(\mathbb{R}^n)$. We note that the space $VMO(\mathbb{R}^n)$ considered by Coifman and Weiss [29] is different from that considered by Sarason, and it coincides with our $CMO(\mathbb{R}^n)$. As it is well known, $VMO(\mathbb{R}^n) \subsetneqq BMO(\mathbb{R}^n)$. For example, the function $\log|x|$ belongs to $BMO(\mathbb{R}^n)$, but not to $VMO(\mathbb{R}^n)$, see e.g. Stein [62, Ch. IV, §. I.1.2], and Brezis and Nirenberg [27, p. 211]. We set

$$vmo(\mathbb{R}^n) := VMO(\mathbb{R}^n) \cap bmo(\mathbb{R}^n)\,,$$

and we endow the space $vmo(\mathbb{R}^n)$ with the norm of $bmo(\mathbb{R}^n)$.
For the convenience of the reader, we display all the subspaces of $BMO(\mathbb{R}^n)$ we have introduced in the following diagram:

$$\begin{array}{ccc} bmo(\mathbb{R}^n) & \subsetneqq & BMO(\mathbb{R}^n) \\ \cup \not\parallel & & \cup \not\parallel \\ vmo(\mathbb{R}^n) & \subsetneqq & VMO(\mathbb{R}^n) \\ \cup \not\parallel & & \cup \not\parallel \\ cmo(\mathbb{R}^n) & \subsetneqq & CMO(\mathbb{R}^n) \end{array}$$

where all inclusions are proper and continuous.

## 9.3 Some further classical function spaces

A definition of the Wiener class $BV_p(\mathbb{R})$ has been given in Subsection 4.2.1. The space $BV_p[a,b]$ is obtained by restricting the intervals $[a_k, b_k]$ to subintervals of $[a,b]$.
By $AC\,[a,b]$ we denote the collection of all absolutely continuous functions on $[a,b]$ endowed with the norm

$$\|f\|_{AC\,[a,b]} := |f(a)| + \int_a^b |f'(t)|\,dt\,.$$

Finally, by $A(\mathbb{T})$ we denote the Wiener algebra on the torus, i.e., the set of all continuous, $2\pi$-periodic functions $f$ s.t.

$$\|f\|_{A(\mathbb{T})} := \sum_{k=-\infty}^{\infty} \Big| \int_{-\pi}^{\pi} f(t)\, e^{ikt}\, dt \Big| < \infty\,.$$

# References


[1] D.R. Adams, On the existence of capacitary strong type estimates in $\mathbb{R}^n$. *Ark. Mat.* **14** (1976), 125-140.

[2] D.R. Adams and M. Frazier, BMO and smooth truncation in Sobolev spaces. *Studia Math.* **89** (1988), 241-260.

[3] D.R. Adams and M. Frazier, Composition operators on potential spaces. *Proc. Amer. Math. Soc.* **114** (1992), 155-165.

[4] S.E. Allaoui, Remarques sur le calcul symbolique dans certains espaces de Besov à valeurs vectorielles. *Ann. Blaise Pascal* **16** (2009), 399-429.

[5] J. Appell, N. Guanda, N. Merentes and J.L. Sanchez, Some boundedness and continuity properties of nonlinear composition operators: a survey. *Preprint*, Würzburg, 2010, 28 pp.

[6] J. Appell and P. Zabrejko, *Nonlinear superposition operators.* Cambridge Univ. Press, Cambridge 1990.

[7] O.V. Besov, V.P. Il'yin and S.M. Nikol'skij, *Integral representations of functions and imbedding theorems. I, II.* Wiley, New York, 1979.

[8] G. Bourdaud, Fonctions qui opèrent sur les espaces de Sobolev. *Sem. Anal. Harm.*, Orsay (1980/81).

[9] G. Bourdaud, Le calcul fonctionnel dans les espaces de Sobolev. *Invent. Math.* **104** (1991), 435–446.

[10] G. Bourdaud, La trivialité du calcul fonctionnel dans l'espace $H^{3/2}(\mathbb{R}^4)$. *C. R. Acad. Sci. Paris, Ser. I* **314** (1992), 187–190.

[11] G. Bourdaud, The functional calculus in Sobolev spaces. In: *Function spaces, differential operators and nonlinear analysis.* Teubner-Texte Math. **133**, Teubner, Stuttgart, Leipzig, 1993, 127-142.

[12] G. Bourdaud, Fonctions qui opèrent sur les espaces de Besov et de Triebel. *Ann. I. H. Poincaré, Analyse non linéaire* **10** (1993), 413–422.

[13] G. Bourdaud, Une propriété de composition dans l'espace $H^s$(II). *C. R. Acad. Sci. Paris, Ser. I* **342** (2006), 243–246.

[14] G. Bourdaud, Superposition in homogeneous and vector valued Sobolev spaces. *Trans. Amer. Math. Soc.* **362** (2010), 6105–6130.

[15] G. Bourdaud, Le calcul symbolique dans certaines algèbres de type Sobolev. In: *Recent Developments in Fractals and Related Fields, J. Barral, S. Seuret (eds).* Birkhäuser, 2010, 131–144.

[16] G. Bourdaud and M.E.D. Kateb, Fonctions qui opèrent sur les espaces de Besov. *Math. Ann.* **303** (1995), 653–675.

[17] G. Bourdaud and M. Lanza de Cristoforis, Functional calculus in Hölder-Sygmund spaces. *Trans. Amer. Math. Soc.* **354** (2002), 4109–4129.

[18] G. Bourdaud and M. Lanza de Cristoforis, Regularity of the symbolic calculus in Besov algebras. *Studia Math.* **184** (2008), 271–298.

[19] G. Bourdaud, M. Lanza de Cristoforis and W. Sickel, Functional calculus on BMO and related spaces. *J. Funct. Anal.* **189** (2002), 515–538.

[20] G. Bourdaud, M. Lanza de Cristoforis and W. Sickel, Superposition operators and functions of bounded $p$-variation. *Revista Mat. Iberoamer.* **22** (2006), 455–487.

[21] G. Bourdaud, M. Lanza de Cristoforis and W. Sickel, Superposition operators and functions of bounded p-variation II. *Nonlinear Analysis*, Series A, **62** (2005), 483–518.

[22] G. Bourdaud and Y. Meyer, Fonctions qui opèrent sur les espaces de Sobolev. *J. Funct. Anal.* **97** (1991), 351-360.

[23] G. Bourdaud, M. Moussai and W. Sickel, An optimal symbolic calculus on Besov algebras. *Ann. I. H. Poincaré-AN* **23** (2006), 949–956.

[24] G. Bourdaud, M. Moussai and W. Sickel, Towards sharp superposition theorems in Besov and Lizorkin-Triebel spaces. *Nonlinear Analysis* **68** (2008), 2889–2912.

[25] G. Bourdaud, M. Moussai and W. Sickel, Composition operators in Lizorkin-Triebel spaces. *J. Funct. Anal.* **259** (2010), 1098–1128.

[26] H. Brezis and P. Mironescu, Gagliardo-Nirenberg, compositions and products in fractional Sobolev spaces. *J. Evol. Equ.* **1** (2001), 387–404.

[27] H. Brezis and L. Nirenberg, Degree theory and BMO; Part I: Compact manifolds without boundaries. *Selecta Mathematica, New Series,* **1** (1995), 197–263.

[28] T. Cazenave, D. Fang and H. Zheng, Continuous dependence for NLS in fractional order spaces. Preprint, Paris, 2010, 16 pp.

[29] R. Coifman and G. Weiss, Extension of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.* **83** (1977), 569–645.

[30] B.E.J. Dahlberg, A note on Sobolev spaces. *Proc. Symp. Pure Math.* **35**,1 (1979), 183–185.

[31] P. Drábek, Continuity of Nemytskij's operator in Hölder spaces. *Comm. Math. Univ. Carol.* **16** (1975), 37-57.

[32] J. Franke, On the spaces $F^s_{p,q}(\mathbb{R})$ of Triebel-Lizorkin type: Pointwise multipliers and spaces on domains. *Math. Nachr.* **125** (1986), 29–68.

[33] M. Frazier and B. Jawerth, A discrete transform and decomposition of distribution spaces. *J. Funct. Anal.* **93** (1990), 34–170.

[34] J. Ginibre, T. Ozawa, and G. Velo, On the existence of the wave operators for a class of nonlinear Schrödinger equations. *Ann. I. H. Poincaré*, Section A, **60** (1994), 211–239.

[35] M. Goebel and F. Sachweh, On the autonomous Nemytskij operator in Hölder spaces. *ZAA* **18** (1999), 205-229.

[36] S. Igari, Sur les fonctions qui opèrent sur l'espace $\hat{A}^2$. *Ann. Inst. Fourier (Grenoble)* **15** (1965), 525-536.

[37] B. Jawerth, Some observations on Besov and Lizorkin-Triebel spaces. *Math. Scand.* **40** (1977), 94–104.

[38] D. Kateb, Fonctions qui opèrent sur les espaces de Besov. *Proc. Amer. Math. Soc.* **128** (1999), 735-743.

[39] Y. Katznelson, *An introduction to harmonic analysis.* 3rd ed. Dover Publications, New-York, 2002.

[40] V.I. Kolyada and M. Lind, On functions of bounded p-variation. *J. Math. Anal. Appl.* **356** (2009), 582-604.

[41] L. Maligranda, Interpolation of locally Hölder operators. *Studia Math.* **78** (1984), 289-296.

[42] M. Marcus and V.J. Mizel, Complete characterization of functions which act via superposition on Sobolev spaces. *Trans. Amer. Math. Soc.* **251** (1979), 187-218.

[43] M. Marcus and V.J. Mizel, Every superposition operator mapping one Sobolev space into another is continuous. *J. Funct. Anal.* **33** (1979), 217–229.

[44] V.G. Maz'ya and T.O. Shaposnikova, An elementary proof of the Brezis and Mironescu theorem on the composition operator in fractional Sobolev spaces. *J. Evol. Equ.* **2** (2002), 113–125.

[45] Y. Meyer, Remarques sur un théorème de J.M.Bony. *Suppl. Rend. Circ. Mat. Palermo*, Serie II, **1** (1981), 1-20.

[46] S. Mizohata, *Lectures on the Cauchy problem.* Tata Institut, Bombay, 1965.

[47] J. Moser, A rapidly convergent iteration method and non-linear differential equations. I. *Ann. Scuola Norm. Sup. Pisa* **20** (1966), 265-315.

[48] M. Moussai, Composition operators on Besov algebras. *Preprint*, Paris 2010.

[49] M. Nakamura and T. Ozawa, Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces. *Rev. Math. Phys.* **9** (1997), 397-410.

[50] S.M. Nikol'skij, *Approximation of functions of several variables and imbedding theorems.* Springer, Berlin, 1975.

[51] P. Oswald, On the boundedness of the mapping $f \longrightarrow |f|$ in Besov spaces. *Comment. Univ. Carol.* **33** (1992), 57-66.

[52] J. Peetre, Interpolation of Lipschitz operators and metric spaces. *Mathematica (Cluj)* **12** (35) (1970), 1-20.

[53] J. Peetre, *New thoughts on Besov spaces.* Duke Univ. Math. Series I, Durham, N.C., 1976.

[54] T. Runst, Mapping properties of non-linear operators in spaces of Triebel-Lizorkin and Besov type. *Anal. Math.* **12** (1986), 313–346.

[55] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations.* De Gruyter, Berlin, 1996.

[56] D. Sarason, Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.,* **207** (1975), 391–405.

[57] G. Savaré, On the regularity of the positive part of functions. *Nonlinear Anal., Theory, Methods & Appl.* **27** (1996), 1055-1074.

[58] W. Sickel, On boundedness of superposition operators in spaces of Triebel-Lizorkin type. *Czechoslovak. Math. J.* **39** (114) (1989), 323-347.

[59] W. Sickel, Necessary conditions on composition operators acting on Sobolev spaces of fractional order. The critical case $1 < s < n/p$. *Forum Math.* **9** (1997), 267-302.

[60] W. Sickel, Conditions on composition operators which map a space of Triebel-Lizorkin type into a Sobolev space. The case $1 < s < n/p$. II. *Forum Math.* **10** (1998), 199-231.

[61] W. Sickel, Necessary conditions on composition operators acting between Besov spaces. The case $1 < s < n/p$. III. *Forum Math.* **10** (1998), 303-327.

[62] E.M. Stein, *Harmonic analysis.* Princeton Univ. Press, Princeton, 1993.

[63] H. Triebel, *Theory of Function Spaces.* Birkhäuser, Basel, 1983.

[64] H. Triebel, *Theory of Function Spaces II.* Birkhäuser, Basel, 1992.

[65] H. Triebel, *The structure of functions.* Birkhäuser, Basel, 2001.

[66] H. Triebel, *Theory of Function Spaces III.* Birkhäuser, Basel, 2006.

[67] N. Wiener, The quadratic variation of a function and its Fourier coefficients. *J. Math. Phys.* **3** (1924), 72–94.

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