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Well-posedness and standing waves for the fourth-order non-linear Schrödinger-type equation

By

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Abstract

We consider the initial value problem for the fourth-order non-linear Schrödinger-type equation (4NLS) which describes the motion of an isolated vortex filament. In the first part of this note we review some recent results on the time local well-posedness of (4NLS) and give the alternative proof of those results. In the second part of this note we consider the stability of a standing wave solution to (4NLS) for the completely integrable case.

§ 1. Introduction

This is a joint work with Masaya Maeda. In this note we consider the initial value problem for the fourth-order non-linear Schrödinger-type equation (4NLS) of the form:

$$(1.1)\qquad \begin{cases} i\partial_t\psi + \partial_x^2\psi + \nu\partial_x^4\psi = \mathcal{N}(\psi,\overline{\psi},\partial_x\psi,\partial_x\overline{\psi},\partial_x^2\psi,\partial_x^2\overline{\psi}), & (t,x)\in\mathbb{R}^2, \\ \psi(0,x)=\phi(x), \quad x\in\mathbb{R}, \end{cases}$$

where $\phi(x):\mathbb{R}\to\mathbb{C}$ is a given function and $\psi(t,x):\mathbb{R}^2\to\mathbb{C}$ is the unknown function. The nonlinear term $\mathcal{N}$ is given by

$$\mathcal{N}(\psi,\overline{\psi},\partial_x\psi,\partial_x\overline{\psi},\partial_x^2\psi,\partial_x^2\overline{\psi}) = \lambda_1|\psi|^2\psi + \lambda_2|\psi|^4\psi + \lambda_3(\partial_x\psi)^2\overline{\psi} + \lambda_4|\partial_x\psi|^2\psi + \lambda_5\psi^2\partial_x^2\overline{\psi} + \lambda_6|\psi|^2\partial_x^2\psi,$$

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where $\nu \neq 0$ and λ_j $(j = 1, \cdots, 6)$ are real constants. The equation (1.1) describes the three dimensional motion of an isolated vortex filament embedded in an inviscid incompressible fluid filling an infinite region. This equation is proposed by Fukumoto-Moffatt [5] as some detailed model taking account of the effect from the higher order corrections of the Da Rios model (cubic nonlinear Schrödinger equation)

$$i\partial_t \psi + \partial_x^2 \psi = -\frac{1}{2}|\psi|^2\psi, \quad (t,x) \in \mathbb{R}^2. \tag{1.2}$$

For the physical background of (1.1), see Fukumoto [4].

In this note we consider the well-posedness of the initial value problem (1.1) and the orbital stability of standing wave solutions to (1.1). Our notion of well-poseness contains the existence and uniqueness of the solution, and the continuity of the data-to-solution map. By standing wave solutions, we mean a solution of (1.1) with the form $\psi(t,x) = e^{i\omega t}\varphi(x)$, where $\omega \in \mathbb{R}$ and φ is a real-valued function.

Concerning the local well-posedness of (1.1) in the usual Sobolev space $H^s(\mathbb{R})$, we proved in [17] that the initial value problem of (1.1) is locally well-posed in H^s with $s \geq 1/2$ under the condition that $\nu < 0$ and $\lambda_6 = 0$. Later on Huo-Jia [8] obtained the same results when $\nu > 0$ and $\lambda_6 = 0$. Furthermore, in [18] we proved the well-posedness of (1.1) in H^s with $s > 7/12$ when $\nu < 0$ without the condition $\lambda_6 = 0$. Recently Huo-Jia [9] extended the previous works to H^s with $s > 1/2$ without any restriction on the coefficients. Summing up the previous results, we have the following theorem.

Theorem 1.1 (Huo-Jia [8, 9], Segata [17, 18]). *Let $s > 1/2$. For any $\phi \in H^s(\mathbb{R})$, there exists $T = T(\|\phi\|_{H^s}) > 0$ and a unique solution ψ of (1.1) satisfying*

$$\psi \in C([0,T]; H^s(\mathbb{R})) \cap X_T,$$

where X_T is defined by (2.12). Moreover, for any $R > 0$, the map $\phi \mapsto \psi(t)$ is Lipschitz continuous from the ball $\{\phi \in H^s(\mathbb{R}); \|\phi\|_{H^s} < R\}$ to $C([0,T]; H^s(\mathbb{R}))$.

To prove the well-posedness of (1.1) they applied the Banach fixed point theorem to the corresponding integral equation in the Fourier restriction space introduced by Bourgain [1]. The Fourier restriction space is a completion of the Schwartz space with respect to the norm

$$\|\psi\|_{X_{s,b}} = \|\langle \tau + \xi^2 - \nu\xi^4\rangle^b \langle\xi\rangle^s \widehat{\psi}(\tau,\xi)\|_{L^2_\xi L^2_\tau(\mathbb{R}^2)},$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$ and $\widehat{\psi}(\tau,\xi)$ is the space-time Fourier transform of $\psi(t,x)$.

The Fourier restriction norm reflects the algebraic structure of the symbol of the associated linear operator $i\partial_t + \partial_x^2 + \nu\partial_x^4$. Thanks to this fact, the Fourier restriction space is very useful to prove the well-posedness of the nonlinear dispersive and hyperbolic equations in the lower order Sobolev spaces. However very complicate calculation is needed to obtain the nonlinear estimates in the Fourier restriction space.

The first purpose in this paper is to give an alternative proof of Theorem 1.1 simpler than the Fourier restriction method. Our proof for Theorem 1.1 also relies on the Banach fixed point theorem via the corresponding integral equation. However we use the space-time norm defined by

$$\|\psi\|_{X_T} = \|\psi\|_{L_t^\infty(0,T;H_x^s(\mathbb{R}))} + \text{auxiliary norms.}$$

instead of employing the Fourier restriction norm.

The difficulty to prove the well-posedness for (1.1) comes from the second order derivatives of ψ in the nonlinear term $\mathcal{N}$. Due to this difficulty we cannot apply the classical energy method. We overcome this "loss of a derivatives" by making use of the Kato local smoothing effect obtained by Kato [10] and Kenig-Ponce-Vega [13].

However the simple combination of the contraction principle and the Kato local smoothing effect only implies that (1.1) is locally well-posed in H^s for $s > 1$ (see Section 2 below). In order to guarantee the local well-posedness of (1.1) for H^s with $s \le 1$, we combine above two tools with the restriction lemma due to Christ-Kiselev [3] (see Lemma 2.1 below).

The Christ-Kiselev lemma allows one to reduce the "retarded" estimate from the "non-retarded" one. By using this lemma we can obtain better estimates for the inhomogeneous terms which are the crucial estimates to prove Theorem 1.1. We note that in Theorem 1.1 we do not impose any restriction condition on the coefficients. It will be clear from our proof below how to extend those results to more general nonlinear dispersive equations.

From Theorem 1.1, it is natural to consider the following question: Is (1.1) well-posed in H^s with $s \le 1/2$? To answer this question, we consider the completely integrable case of (1.1):

$$\tag{1.3} i\partial_t\psi + \partial_x^2\psi + \nu\partial_x^4\psi = -\frac{1}{2}|\psi|^2\psi - \frac{3}{8}\nu|\psi|^4\psi - \frac{3}{2}\nu(\partial_x\psi)^2\overline{\psi} - \nu|\partial_x\psi|^2\psi - \frac{1}{2}\nu\psi^2\partial_x^2\overline{\psi} - 2\nu|\psi|^2\partial_x^2\psi.$$

This corresponds to (1.1) with $\lambda_1 = -1/2$, $\lambda_2 = -3\nu/8, \lambda_3 = -3\nu/2$, $\lambda_4 = -\nu, \lambda_5 = -\nu/2$ and $\lambda_6 = -2\nu$. In this case, (1.3) has the following two parameter family of solutions

$$\psi_{N,\gamma}(t,x) = 2\gamma^{1/2}e^{it\{\gamma+\nu\gamma^2-(1+6\nu\gamma)N^2+\nu N^4\}}e^{ixN} \times\operatorname{sech}(\gamma^{1/2}(x-((2+4\nu\gamma)N-4\nu N^3)t)), \quad N\in\mathbb{R},\ \gamma>0,$$

which have been obtained by Hoseini-Marchant [7]. By using this special solution we can obtain the following theorem:

Theorem 1.2. *For $s \in (-1/2, 1/2)$, the data-to-solution map associated to (1.3) $H^s \to C([0,T]; H^s)(\phi \mapsto \psi(t))$ is not uniformly continuous. That is,*

$$\exists \epsilon > 0\ \exists R > 0 \text{ s.t. } \forall \delta > 0\ \forall T > 0\ \exists \phi_1, \phi_2 \text{ s.t.} ||\phi_1||_{H^s} \leq R,\ ||\phi_2||_{H^s} \leq R,$$
$$||\phi_1 - \phi_2||_{H^s} < \delta \text{ and } ||\psi_1(T) - \psi_2(T)||_{H^s} \geq \epsilon,$$

where $\psi_1(t)$, $\psi_2(t)$ are solutions of (1.3) with $\psi_1(0) = \phi_1$, $\psi_2(0) = \phi_2$.

A by-product of the contraction mapping principle provides the Lipschitz continuity of data-solution map. Therefore, Theorem 1.2 tells us that one cannot solve the initial value problem (1.3) by a contraction mapping principle implemented on the integral equation for the initial data in H^s with $-1/2 < s < 1/2$.

Remark. Although Theorem 1.2 does not cover the lower order case $s \leq -1/2$, we believe that same conclusion holds for this range. Furthermore, we believe that (1.1) is truly ill-posed for $s < -1/2$. Indeed, Kenig-Ponce-Vega [14] proved that the nonlinear Schrödinger equation (1.2) cannot have a unique solution starting from the delta function. This implies the ill-posedness of (1.2) in H^s with $s < -1/2$. The ill-posedness question for (1.1) in the lower order case is an issue in the future.

Remark. In [17] and [8], it was proved that if we assume the strong restriction $\lambda_6 = 0$, then (1.1) is locally well-posed in $H^{1/2}$. So far we do not know whether (1.1) is well-posed in H^s with $s = 1/2$ for the case $\lambda_6 \neq 0$.

Next we consider the orbital stability of the standing wave solution. We continue to consider the completely integrable equation (1.3). The standing waves of (1.3) is explicitly given by

$$\psi(t,x) = e^{i\omega t}\varphi_\gamma(x), \quad \varphi_\gamma(x) = 2\gamma^{1/2}\text{sech}(\gamma^{1/2}x).$$

where γ is positive solution to the quadratic equation $\nu\gamma^2 + \gamma - \omega = 0$. The equation (1.3) has further advantage that (1.3) admits infinitely many conservation quantities. Indeed, Langer and Perline [15] derived the following conservation quantities for (1.3):

$$I_0(\psi) = \frac{1}{2}\int_{\mathbb{R}} |\psi|^2 dx, \tag{1.4}$$

$$I_1(\psi) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x\psi|^2 dx - \frac{1}{8}\int_{\mathbb{R}} |\psi|^4 dx, \tag{1.5}$$

$$\begin{aligned} I_2(\psi) = &\frac{1}{2}\int_{\mathbb{R}} |\partial_x^2\psi|^2 dx + \frac{3}{4}\int_{\mathbb{R}} |\psi|^2\overline{\psi}\partial_x^2\psi dx + \frac{1}{8}\int_{\mathbb{R}} |\psi|^2\psi\partial_x^2\overline{\psi} dx \\ &+ \frac{5}{8}\int_{\mathbb{R}} (\partial_x\psi)^2\overline{\psi}^2 dx + \frac{3}{4}\int_{\mathbb{R}} |\partial_x\psi|^2|\psi|^2 dx + \frac{1}{16}\int_{\mathbb{R}} |\psi|^6 dx. \end{aligned} \tag{1.6}$$

In general, the conservation quantities for (1.3) are expressed as

$$I_m(\psi) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x^m \psi|^2 dx + \int_{\mathbb{R}} P_m(\psi, \overline{\psi}, \cdots, \partial_x^{m-1}\psi, \partial_x^{m-1}\overline{\psi})dx, \quad m \in \mathbb{N},$$

where P_m is a polynomial. Therefore combining Theorem 1.1, the conservation laws and the Gagliardo-Nirenberg inequality, we easily see that (1.3) is globally well-posed in H^m with $m \in \mathbb{N}$. We notice that when (1.3) is not completely integrable, we do not know whether (1.3) is globally well-posed or not.

There are two main approaches to prove the stability of the standing wave solutions, the method based on the study of a linearized Hamiltonian around the standing wave (see for instance, Grillakis-Shatah-Strauss [6]), and the purely variational methods which is developed by Cazenave-Lions [2].

Concerning the former approach, the linearized Hamiltonian around the φ_γ associated to (1.3) is very complicate and it is difficult to get any information on its spectrum. On the other hand, the standing wave solution $e^{i\omega t}\varphi_\gamma(x)$ has fine variational characterizations (see Proposition 4.1 below). Thanks to this characterization, we can employ the latter approach and obtain the following theorem.

Theorem 1.3. *The standing wave solution $e^{i\omega t}\varphi_\gamma(x)$ to (1.3) is orbitally stable in H^m with $m \in \mathbb{N}$ in the following sense: for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\phi \in H^m$ with*

$$\inf_{\theta\in\mathbb{R}, y\in\mathbb{R}} ||\phi - e^{i\theta}\varphi_\gamma(\cdot + y)||_{H^m} < \delta,$$

the solution ψ to (1.3) with the initial data $\psi(0, x) = \phi(x)$ exists globally in time and satisfies

$$\sup_{t\in(0,\infty)} \inf_{\theta\in\mathbb{R}, y\in\mathbb{R}} \|\psi(t) - e^{i\theta}\varphi_\gamma(\cdot + y)\|_{H^m} < \varepsilon.$$

The plan of this note is as follows. In sections 2 and 3, we give outline of proof for Theorem 1.1 and Theorem 1.2, respectively. In Section 4, we describe the sketch of the proof for Theorem 1.3.

We close this section by introducing several notations and function spaces which will be used throughout this note. $D_x^s = (-\partial_x^2)^{s/2}$ and $\langle D_x\rangle^s = (I - \partial_x^2)^{s/2}$ denote the Riesz and Bessel potentials of order $-s$, respectively. Let $\{W(t)\}_{t\in\mathbb{R}}$ be the unitary group generated by the linear operator $i\partial_x^2 + i\nu\partial_x^4$:

$$W(t)\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi - it\xi^2 + i\nu t\xi^4}\hat{\phi}(\xi)d\xi,$$

where $\hat{\phi}(\xi)$ is the Fourier transform of ϕ with respect to x variable. $\|\cdot\|_{L_T^p L_x^q}$ and $\|\cdot\|_{L_x^q L_T^p}$ denote the space-time norms defined by

$$\|\psi\|_{L_T^p L_x^q} = \|\|\psi(t,\cdot)\|_{L_x^q(\mathbb{R})}\|_{L_t^p(0,T)},$$
$$\|\psi\|_{L_x^q L_T^p} = \|\|\psi(\cdot,x)\|_{L_t^p(0,T)}\|_{L_x^q(\mathbb{R})}.$$

§ 2. Well-posedness for 4NLS

In this section we guarantee the time local well-posedness for (1.1). For simplicity of the exposition we consider the initial value problem

$$\begin{cases} i\partial_t \psi + \partial_x^2 \psi + \nu \partial_x^4 \psi = |\psi|^2 \partial_x^2 \psi, & (t,x) \in \mathbb{R}^2, \\ \psi(0,x) = \phi(x), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

The problem (2.1) is rewritten as the integral equation

$$\psi(t) = W(t)\phi - i \int_0^t W(t-\tau)(|\psi|^2 \partial_x^2 \psi)(\tau) d\tau, \tag{2.2}$$

where $\{W(t)\}_{t\in\mathbb{R}}$ be the unitary group generated by the linear operator $i\partial_x^2 + i\nu\partial_x^4$. We prove the existence of the solution to (2.2) in the short interval $[0,T]$ by using the Banach fixed point theorem for suitable function space. For the simplicity, we asume $0 < T < 1$.

As already mentioned in Section 1, the difficulty to prove the local well-posedness comes from the second derivatives of the unknown function in the nonlinear term. To overcome this difficulty we employ the Kato local smoothing effect due to Kato [10] and Kenig-Ponce-Vega [13]:

$$\|D_x^{3/2} W(t)\phi\|_{L_x^\infty L_T^2} \le C\|\phi\|_{L_x^2} \qquad \text{(homogeneous type)}, \tag{2.3}$$

$$\|D_x^3 \int_0^t W(t-\tau)F(\tau)d\tau\|_{L_x^\infty L_T^2} \le C\|F\|_{L_x^1 L_T^2} \qquad \text{(inhomogeneous type)}. \tag{2.4}$$

Interpolating (2.3) with (2.4), we obtain

$$\|D_x^2 \int_0^t W(t-\tau)F(\tau)d\tau\|_{L_x^\infty L_T^2} \le CT^{1/3}\|F\|_{L_x^{3/2} L_T^2}. \tag{2.5}$$

The estimate (2.5) tells us that the inhomogeneous term gains extra smoothness in x of order 2 in $L_x^\infty L_T^2$. Thanks to this gain of the regularity, we can overcome the loss of second order derivatives in the nonlinearity. Therefore combining the contraction mapping principle and the Kato local smoothing effect we guarantee the time local well-posedness for (2.1). To this end, we prove that the map

$$\Phi(\psi)(t) = W(t)\phi - i \int_0^t W(t-\tau)(|\psi|^2 \partial_x^2 \psi)(\tau) d\tau \tag{2.6}$$

is a contraction on suitable function space. Taking into account of the smoothing effect we introduce the function space

$$X_T = \{\psi \in C([0,T]; H^s(\mathbb{R})); \|\psi\|_{X_T} < \infty\},$$
$$\|\psi\|_{X_T} = \|\psi\|_{L_T^\infty H_x^s} + \|D_x^{s+3/2}\psi\|_{L_x^\infty L_T^2} + \text{auxiliary norms}.$$

By using the inequality (2.5), we have

$$
\begin{aligned}
&\|\Phi\psi\|_{L_T^\infty H_x^s} + \|D_x^{s+3/2}\Phi\psi\|_{L_x^\infty L_T^2} \\
&\leq C\|\psi\|_{H_x^s} + CT^{1/3}\|D_x^{s-1/2}(|\psi|^2\partial_x^2\psi)\|_{L_x^{3/2}L_T^2} + \ell.o.t. \\
&\leq C\|\psi\|_{H_x^s} + CT^{1/3}\||\psi|^2 D_x^{s+3/2}\psi\|_{L_x^{3/2}L_T^2} + \ell.o.t. \\
&\leq C\|\psi\|_{H_x^s} + CT^{1/3}\|\psi\|^2_{L_x^3 L_T^\infty}\|D_x^{s+3/2}\psi\|_{L_x^\infty L_T^2} + \ell.o.t.
\end{aligned}
$$

To close this estimate, we have to evaluate $L_x^3 L_T^\infty$ norm for a solution. This type of the estimates are called the estimates for "maximal functions". As in the proof of Kenig-Ponce-Vega [12] and [13] we can obtain $L_x^2 L_T^\infty$ and $L_x^4 L_T^\infty$ estimates

$$\|W(t)\phi\|_{L_x^2 L_T^\infty} \leq C\|\phi\|_{H_x^{1+}}, \tag{2.7}$$

$$\|W(t)\phi\|_{L_x^4 L_T^\infty} \leq C\|\phi\|_{H_x^{1/4}}. \tag{2.8}$$

Interpolating (2.7) with (2.8), we obtain $L_x^3 L_T^\infty$ estimate

$$\|W(t)\phi\|_{L_x^3 L_T^\infty} \leq C\|\phi\|_{H_x^{1/2+}}. \tag{2.9}$$

Especially, we have

$$\|\int_0^t W(t-\tau)F(\tau)d\tau\|_{L_x^3 L_T^\infty} \leq C\|F\|_{L_T^1 H_x^{1/2+}}. \tag{2.10}$$

With that in mind, we modify X_T as

$$\|\psi\|_{X_T} = \|\psi\|_{L_T^\infty H_x^s} + \|D_x^{s+3/2}\psi\|_{L_x^\infty L_T^2} + \|\psi\|_{L_x^3 L_T^\infty} + \text{auxiliary norms.}$$

By the inequality (2.10), we have

$$
\begin{aligned}
\|\Phi\psi\|_{L_x^3 L_T^\infty} &\leq C\|\psi\|_{H_x^{1/2+}} + C\|D_x^{1/2+}(|\psi|^2\partial_x^2\psi)\|_{L_T^1 L_x^2} + \ell.o.t. \\
&\leq C\|\psi\|_{H_x^{1/2+}} + C\||\psi|^2 D_x^{5/2+}\psi\|_{L_T^1 L_x^2} + \ell.o.t. \\
&\leq C\|\psi\|_{H_x^{1/2+}} + CT^{1/2}\||\psi|^2 D_x^{5/2+}\psi\|_{L_T^2 L_x^2} + \ell.o.t. \\
&\leq C\|\psi\|_{H_x^{1/2+}} + CT^{1/2}\||\psi|^2 D_x^{5/2+}\psi\|_{L_x^2 L_T^2} + \ell.o.t. \\
&\leq C\|\psi\|_{H_x^{1/2+}} + CT^{1/2}\|\psi\|^2_{L_x^4 L_T^\infty}\|D_x^{5/2+}\psi\|_{L_x^\infty L_T^2} + \ell.o.t.
\end{aligned}
$$

Therefore if $5/2+ \leq s+3/2$, that is $s > 1$, then we can control the last term by X_T norm. Hence above method yields the time local well-posedness for H^s with $s > 1$. To guarantee the well-posedness for H^s with $1/2 < s \leq 1$, we need to refine the method mentioned above. To this end we employ the restriction lemma essentially due to Christ-Kiselev [3]. The version of this lemma that we use is the one presented in Kenig-Koenig [11, p.883].

Lemma 2.1. *Let $K : \mathcal{S}(\mathbb{R}^2) \to C(\mathbb{R}^3)$. Assume that*

$$\left\| \int_0^T K(t,\tau)F(\tau)d\tau \right\|_{L_x^{q_1} L_T^\infty} \leq C\|F\|_{L_x^{q_2} L_T^{p_2}}$$

for some $1 \leq p_2, q_1, q_2 \leq \infty$ with $p_2 \neq \infty, q_2 \neq \infty$. Then

$$\left\| \int_0^t K(t,\tau)F(\tau)d\tau \right\|_{L_x^{q_1} L_T^\infty} \leq C\|F\|_{L_x^{q_2} L_T^{p_2}}.$$

By using the Christ-Kiselev lemma and the duality argument we obtain the refined version of (2.10) for the inhomogeneous term.

Lemma 2.2.

$$\left\| \int_0^t W(t-\tau)F(\tau)d\tau \right\|_{L_x^3 L_T^\infty} \leq CT^{1/3}(\|\langle D_x\rangle^{0+}F\|_{L_x^{3/2} L_T^2} \tag{2.11}$$

The constants in (2.11) are independent of T.

Proof of Lemma 2.2. See [19, Lemma 2.4] $\square$

Combining the local smoothing effect and Lemma 2.2 we obtain the desired result.

Although we omit the estimate for the lower order terms, to justify all estimates, we need to modify X_T as follows:

$$\begin{aligned} \|\psi\|_{X_T} = &\|\psi\|_{L_T^\infty H_x^s} + \|D_x^{s+3/2}\psi\|_{L_x^\infty L_T^2} + \|\partial_x^2\psi\|_{L_x^\infty L_T^2} \\ &+\|D_x^2\psi\|_{L_x^{\frac{3(2s+3)}{2s-1}} L_T^{\frac{2s+3}{2}}} + \|D_x^{s+1/2}\psi\|_{L_x^{\frac{3(2s+3)}{2}} L_T^{\frac{2(2s+3)}{2s+1}}} + \|\partial_x\psi\|_{L_x^{\frac{3(2s+3)}{2s+1}} L_T^{2s+3}} \\ &+\|D_x^{s-1/2}\psi\|_{L_x^{\frac{3(2s+3)}{4}} L_T^{\frac{2(2s+3)}{2s-1}}} + \|\psi\|_{L_x^{\frac{3(2s+3)}{4}} L_T^{\frac{2(2s+3)}{2s-1}}} + \|\psi\|_{L_x^3 L_T^\infty}. \end{aligned} \tag{2.12}$$

See [19, Section 3] for the detail. The continuous dependence on the initial data follows from the standard argument. This completes the proof of Theorem 1.1. $\square$

§ 3. Lack of uniform continuity of solution map

In this section we prove the lack of uniform continuity of solution map associated to (1.3) following Kenig-Ponce-Vega [14].

Proof of Theorem 1.2. We construct two solutions ψ_1 and ψ_2 to (1.3) satisfying the following properties: There exists $\epsilon > 0$ such that for arbitrary $T > 0$ and $\delta > 0$, $||\psi_1(0) - \psi_2(0)||_{H^s} < \delta$ and $||\psi_1(T) - \psi_2(T)||_{H^s} \geq \epsilon > 0$.

To construct such functions we introduce the two parameter family of solutions to (1.3) which have been obtained by Hoseini-Marchant [7]:

$$\psi_{N,\gamma}(t,x) = 2\gamma^{\frac{1}{2}} e^{it\{\gamma+\nu\gamma^2-(1+6\nu\gamma)N^2+\nu N^4\}} e^{ixN} \times \operatorname{sech}(\gamma^{\frac{1}{2}}(x-((2+4\nu\gamma)N-4\nu N^3)t)).$$

We take

$$\psi_j(t,x) = \psi_{N_j,\gamma}(t,x), \qquad j=1,2,$$

where

$$\gamma = N^{-4s},\ N_1 = N,\ N_2 = N-\delta N^{-2s},\ N \gg 1.$$

Firstly we give an upper bound for $||\psi_1(0)-\psi_2(0)||_{H^s}$. From the definition of ψ_j,

$$\psi_j(0,x) = 2N_j^{-2s} e^{ixN_j}\operatorname{sech}(N_j^{-2s}x), \qquad j=1,2.$$

By this expression we obtain

$$\|\psi_1(0)-\psi_2(0)\|_{H^s(\mathbb{R})} \sim N^s|N_1-N_2|\gamma^{-1/4} = \delta.$$

Next we give a lower bound for $||\psi_1(T)-\psi_2(T)||_{H^s}$. We notice that $\psi_{N_j,\gamma}$ concentrates on $\{x; |x-((2+4\nu\gamma)N_j-4\nu N_j^3)t| < \gamma^{-1/2}\}$. Therefore if

$$|((2+4\nu\gamma)N_1-4\nu N_1^3)T-((2+4\nu\gamma)N_2-4\nu N_2^3)T| \gg \gamma^{-1/2}, \tag{3.1}$$

then the interaction between $\psi_1(T)$ and $\psi_2(T)$ is negligible and we can evaluate as

$$\|\psi_1(T)-\psi_2(T)\|_{H^s} \sim \|\psi_1(T)\|_{H^s} + \|\psi_2(T)\|_{H^s} \sim N^s\gamma^{1/4} = C \equiv \epsilon. \tag{3.2}$$

On the other hand, a simple calculation yields

$$\begin{aligned}
&|((2+4\nu\gamma)N_1-4\nu N_1^3)T-((2+4\nu\gamma)N_2-4\nu N_2^3)T| \\
&= T|N_1-N_2||2-4\nu(N_1^2+N_1N_2+N_2^2)+4\nu\gamma| \\
&\sim CT|N_1-N_2|N^2 = \delta T N^{2-2s}.
\end{aligned}$$

Since $s<1/2$, for any $T>0$ and $\delta>0$, we can choose N so that

$$T \gg \frac{N^{4s-2}}{\delta}$$

holds. This implies

$$\delta T N^{2-2s} \gg N^{2s} = \gamma^{-1/2}$$

and (3.1) holds. Therefore we obtain (3.2). This violates the uniform continuity and completes the proof of Theorem 1.2. □

§ 4. Stability of standing waves

In this section, we prove the orbital stability of the standing waves for $e^{i\omega t}\varphi_\gamma(x)$. To prove the stability, we use the variational characterization of $e^{i\omega t}\varphi_\gamma(x)$ which is due to Cazenave-Lions [2].

Proposition 4.1. *Let*

$$\widetilde{\mathcal{G}}_\alpha = \{\psi \in H^1; I_0(\psi) = \alpha,\ I_1(\psi) = \inf_{I_0(\tilde{\psi})=\alpha} I_1(\tilde{\psi})\},$$

where I_1 and I_2 are defined by (1.4) and (1.6), respectively. Then, we have

$$\widetilde{\mathcal{G}}_{4\gamma^{1/2}} = \{e^{i\theta}\varphi_\gamma(\cdot + y); \theta, y \in \mathbb{R}\}.$$

Furthermore, let ψ_n satisfy $I_0(\psi_n) = 4\gamma^{1/2}$ and $I_1(\psi_n) \to I_1(\varphi_\gamma)$. Then, there exist a subsequence ψ_{n_k} and $\theta, y \in \mathbb{R}$ such that $\psi_{n_k} \to e^{i\theta}\varphi_\gamma(\cdot + y)$ in H^1.

We prove the H^1 stability of the standing wave $e^{i\omega t}\varphi_\gamma(x)$ by contradiction argument due to [2].

Proof of Theorem 1.3. Firstly, we prove the H^1 stability. Suppose, $e^{i\omega t}\varphi_\gamma$ is unstable in H^1. Then, there exist $\delta > 0$, $\phi_n \in H^1$ and $t_n \in \mathbb{R}_+$ such that

$$\inf_{\theta,y\in\mathbb{R}} ||\phi_n - e^{i\theta}\varphi_\gamma(\cdot + y)||_{H^1} \to 0,\ \inf_{\theta,y\in\mathbb{R}} ||\psi_n(t_n) - e^{i\theta}\varphi_\gamma(\cdot + y)||_{H^1} > \delta,$$

where ψ_n is the solution of (1.3) with the initial data $\psi_n(0) = \phi_n$. Since I_0, I_1 are conserved under the flow of (1.3), we have $||\psi_n(t_n)||_{L^2} = ||\phi_n||_{L^2}$ and $I_1(\psi_n(t_n)) = I_1(\phi_n)$. Therefore, by Gagliardo-Nirenberg's inequality, we have

$$\begin{aligned} ||\partial_x\psi_n(t_n)||_{L^2}^2 &= 2I_1(\psi_n(t_n)) + \frac{1}{4}||\psi_n(t_n)||_{L^4}^4 \\ &\leq 2I_1(\phi_n) + C||\partial_x\psi_n(t_n)||_{L^2}||\phi_n||_{L^2}^3. \end{aligned}$$

This implies that $||\psi_n(t_n)||_{H^1}$ is bounded.

Now, set

$$\tilde{\psi}_n := \frac{||\varphi_\gamma||_{L^2}}{||\phi_n||_{L^2}}\psi_n(t_n).$$

Then, $||\tilde{\psi}_n||_{L^2} = ||\varphi_\gamma||_{L^2}$. Furthernore

$$\begin{aligned} I_1(\tilde{\psi}_n) = I_1(\phi_n) &+ \frac{1}{2}\left(\frac{||\varphi_\gamma||_{L^2}^2}{||\phi_n||_{L^2}^2} - 1\right)\int_{\mathbb{R}} |\partial_x\psi_n(t_n)|^2 dx \\ &- \frac{1}{8}\left(\frac{||\varphi_\gamma||_{L^2}^4}{||\phi_n||_{L^2}^4} - 1\right)\int_{\mathbb{R}} |\psi_n(t_n)|^4 dx. \end{aligned}$$

Since $||\phi_n||_{L^2} \to ||\varphi_\gamma||_{L^2}$, $I_1(\phi_n) \to I_1(\varphi_\gamma)$ and $\psi_n(t_n)$ are bounded in H^1 and L^4, we have $I_1(\tilde{\psi}_n) \to I_1(\varphi_\gamma)$.

Therefore, by Proposition 4.1, there exists a subsequence $\tilde{\psi}_{n_k}$ such that $\tilde{\psi}_{n_k} \to e^{i\theta}\varphi_\gamma(\cdot + y) \in \widetilde{\mathcal{G}}_{4\gamma^{1/2}}$ in H^1 for some $\theta, y \in \mathbb{R}$. However, this is a contradiction with the assumption and completes the proof of Theorem 1.3 for $m = 1$. For $m \geq 2$, H^m stability of the standing waves for $e^{i\omega t}\varphi_\gamma(x)$ follows from the induction argument. See [16] for the detail. □

References

[1] Bourgain, J., *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I Schrödinger equations, II The KdV equation*, Geom. Funct. Anal. **3** (1993), 107-156, 209-262.

[2] Cazenave, T., Lions, P.-L., *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), no. 4, 549–561.

[3] Christ, M., Kiselev, A., *Maximal functions associated to filtrations*, J. Funct. Anal. **179** (2001) 409–425.

[4] Fukumoto, Y., *Motion of a curved vortex filament: higher-order asymptotics*, In Proc. of IUTAM Symposium on Geometry and Statistics of Turbulence (eds. T. Kambe, T. Nakano and T. Miyauchi). (2001), 211-216, Kluwer

[5] Fukumoto, Y., Moffatt, H. K., *Motion and expansion of a viscous vortex ring. Part I. A higher-order asymptotic formula for the velocity*, J. Fluid. Mech. **417** (2000), 1-45.

[6] Grillakis, M., Shatah, J., Strauss, W., *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. **74** (1987), no. 1, 160–197.

[7] Hoseini, S. M., Marchant, T. R., *Solitary wave interaction for a higher-order nonlinear Schrödinger equation*, IMA J. Appl. Math. **72** (2007) 206-222.

[8] Huo, Z., Jia, Y., *The Cauchy problem for the fourth-order nonlinear Schrödinger equation related to the vortex filament*, J. Differential Equations **214** (2005), 1–35.

[9] Huo, Z., Jia, Y., *A refined well-posedness for the fourth-order nonlinear Schrödinger equation related to the vortex filament*, Comm. Partial Differential Equations **32** (2007), 1493–1510.

[10] Kato, T., *On the Cauchy problem for the (generalized) KdV equation*, Advances in Math. Supplementary studies, Studies in Applied Mathematics **8** (1983), 93-128.

[11] Kenig, C. E., Koenig, K., *On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations*, Math. Res. Lett. **10** (2003), 879–895.

[12] Kenig, C. E., Ponce, G. Vega, L., *Well-posedness and initial value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc. **4** (1991), 323-347

[13] Kenig, C. E., Ponce, G. Vega, L., *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), 33-69.

[14] Kenig, C. E., Ponce, G. Vega, L., *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. **106** (2001), 617–633.

[15] Langer, J., Perline, R., *Poisson geometry of the filament equation*, J. Nonlinear Sci. **1** (1991), 71-93.

[16] Maeda, M., Segata, J., *Existence and Stability of standing waves of fourth order nonlinear Schrödinger type equation related to vortex filament*, to appear in Funkcial. Ekvac. 2010.

[17] Segata, J., *Well-posedness for the fourth order nonlinear Schrödinger type equation related to the vortex filament*, Differential Integral Equations **16** (2003), 841–864.

[18] Segata, J., *Remark on well-posedness for the fourth order nonlinear Schrödinger type equation*, Proc. Amer. Math. Soc. **132** (2004), 3559–3568.

[19] Segata, J., *Well-posedness and existence of standing waves for the fourth order nonlinear Schrödinger type equation*, Discrete. Contin. Dyn. Syst. **27** (2010), 1093–1105.