ON THE PERSISTENCE PROPERTIES OF SOLUTIONS OF NONLINEAR DISPERSIVE EQUATIONS IN WEIGHTED SOBOLEV SPACES

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ABSTRACT. We study persistence properties of solutions to some canonical dispersive models, namely the semi-linear Schrödinger equation, the k-generalized Korteweg-de Vries equation and the Benjamin-Ono equation, in weighted Sobolev spaces $H^s(\mathbb{R}^n)\cap L^2(|x|^l dx),\ s,\,l>0$.

1. Introduction

This work is concerned with persistence properties of solutions to some nonlinear dispersive equations in weighted Sobolev spaces $H^s(\mathbb{R}^n)\cap L^2(|x|^l dx), s, l>0$. We shall consider the initial value problems (IVP) associated to the following dispersive models : the nonlinear Schrödinger (NLS) equation

$$i\partial_t u+\Delta u=\mu|u|^{a-1}u,\quad t\in\mathbb{R},\quad x\in\mathbb{R}^n,\quad \mu=\pm1,\quad a>1, \tag{1.1}$$

the k-generalized Korteweg-de Vries (k-gKdV) equations

$$\partial_t u+\partial_x^3 u+u^k\partial_x u=0,\quad t,\,x\in\mathbb{R},\ \ k\in\mathbb{Z}^+, \tag{1.2}$$

and the Benjamin-Ono (BO) equation

$$\partial_t u+\mathcal{H}\partial_x^2 u+u\partial_x u=0,\qquad t,\,x\in\mathbb{R}, \tag{1.3}$$

where $\mathcal{H}$ denotes the Hilbert transform

$$\mathcal{H}f(x)=\frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int_{|y|\ge\epsilon}\frac{f(x-y)}{y}dy=-i\,(\mathrm{sgn}(\xi)\,\widehat{f}(\xi))^{\vee}(x). \tag{1.4}$$

These models have been widely studied in several contexts. For example, the KdV $k=1$ in (1.2) was first deduced as a model for long waves propagating in a channel. Subsequently the KdV and its modified form ($k=2$ in (1.2)) were found to be relevant in a number of different physical systems. Also they have been studied because of their relation to inverse scattering theory [20]. The NLS arises as a model in several different physical phenomena (see [61] and references therein). In the particular, case $n=1$ and $a=3$ it has been shown to be completely integrable [66]. The BO equation (1.3) was first deduced in [3] and [54] as a model for long internal gravity waves in deep stratified fluids. It was also shown that it is a completely integrable system (see [2], [12] and references therein).

We recall the notion of well posedness given in [34] : the IVP is said to be locally well posed (LWP) in the function space X if for each $u_0\in X$ there exist $T>0$ and a unique solution $u\in C([-T,T]:X)\cap....=Y_T$ of the equation, with the map data $\rightarrow$ solution being locally continuous from X to Y_T. This notion of LWP includes the "persistent" property, i.e. the solution describes a continuous curve on X. In particular, it implies that the solution flow defines a dynamical system in X. When

T can be taken arbitrarily large one says that the corresponding IVP is globally well posed (GWP) in X.

First, we shall study the Schrödinger equation (1.1).

2. The Schrödinger equation (1.1)

The results in [9], [10], [21], [35], and [65] yield the following LWP theory in the classical Sobolev spaces $H^s(\mathbb{R}^n)$ for the IVP associated to the NLS equation (1.1).

Theorem A. *Let $s_c = n/2 - 2/(a-1)$.*

(I) *If $s > s_c$, $s \geq 0$, with $[s] \leq a-1$ if a is not an odd integer, then for each $u_0 \in H^s(\mathbb{R}^n)$ there exist $T = T(\|u_0\|_{s,2}) > 0$ and a unique solution $u = u(x,t)$ of the IVP associated to the NLS equation* (1.1) *with*

$$u \in C([-T,T] : H^s(\mathbb{R}^n)) \cap L^q([-T,T] : L^p_s(\mathbb{R}^n)) = Z^s_T. \tag{2.1}$$

Moreover, the map data $\to$ solution is locally continuous from $H^s(\mathbb{R}^n)$ into Z^s_T.

(II) *If $s = s_c$ and $s \geq 0$, then part (I) holds with $T = T(u_0) > 0$.*

Notations : (a) for $1 < p < \infty$ and $s \in \mathbb{R}$

$$L^p_s(\mathbb{R}^n) \equiv (1-\Delta)^{-s/2} L^p(\mathbb{R}^n) = J^{-s/2} L^p(\mathbb{R}^n), \qquad \|\cdot\|_{s,p} \equiv \|(1-\Delta)^s \cdot\|_p, \tag{2.2}$$

with $L^2_s(\mathbb{R}^n) = H^s(\mathbb{R}^n)$,

(b) the pair of indices (q,p) in (2.1) are given by the Strichartz estimates (see [60] and [21]):

$$\Big(\int_{-\infty}^{\infty} \|e^{it\Delta} u_0\|_p^q dt\Big)^{1/q} \leq c\|u_0\|_2, \tag{2.3}$$

where

$$\frac{n}{2} = \frac{2}{q} + \frac{n}{p}, \qquad 2 \leq p \leq \infty, \ \text{ if } \ n = 1, \ \ 2 \leq p < 2n/(n-2), \ \text{ if } \ n \geq 2.$$

The value $s_c = n/2 - 2/(a-1)$ in Theorem A is determined by a scaling argument : if $u(x,t)$ is a solution of the IVP associated to the NLS equation (1.1), then $u_\lambda(x,t) = \lambda^{2/(a-1)} u(\lambda x, \lambda^2 t)$ satisfies the same equation with data $u_\lambda(x,0) = \lambda^{2/(a-1)} u_0(\lambda x)$. Hence, for $s \in \mathbb{R}$

$$\|D^s u_\lambda(x,0)\|_2 = c\||\xi|^s \widehat{u_\lambda}(\xi,0)\|_2 = c\lambda^{2/(a-1)+s-n/2}\|u_0\|_2, \tag{2.4}$$

is independent of λ when $s = s_c$. In Theorem A the case (I) corresponds to the sub-critical case and (II) to the critical one. In the latter, one has that if $\|D^{s_c} u_0\|_2$ is sufficiently small, then the local solution extends globally in time.

For the optimality of the results in Theorem A see [4], [11], and [40].

Formally, solutions of the NLS equation (1.1) satisfies the following conservation laws:

$$\|u(\cdot,t)\|_2 = \|u_0\|_2,$$

and

$$E(t) = \int_{\mathbb{R}^n} (|\nabla_x u(x,t)|^2 + \frac{2\mu}{a+1}|u(x,t)|^{a+1}) dx = E(0).$$

Using these conservation laws one can extend the LWP results in Theorem A to a GWP one, for details we refer to [6], [64], and references therein.

Concerning the persistence properties in weighted Sobolev spaces of solutions of the IVP associated to the NLS equation (1.1) one has the following result established in [26], [27], and [28].

Theorem B. *In addition to the hypothesis in Theorem A assume* $u_0 \in L^2(|x|^{2m}dx)$, $m \in \mathbb{Z}^+$ *with* $m \leq a-1$ *if* a *is not an odd integer.*

(I) *If* $s \geq m$, *then*

$$u \in C([-T,T] : H^s \cap L^2(|x|^{2m}dx)) \cap L^q([-T,T] : L^p_s \cap L^p(|x|^{2m}dx) = Z^{s,m}_T. \tag{2.5}$$

(II) *If* $1 \leq s < m$, *then* (2.5) *holds with* $[s]$ *instead of* m *and*

$$\Gamma^\beta u = (x_j + 2it\partial_{x_j})^\beta u \in C([-T,T] : L^2) \cap L^q([-T,T] : L^p), \tag{2.6}$$

for any $\beta \in (\mathbb{Z}^+)^n$ *with* $|\beta| \leq m$.

The proof of Theorem B (see [26], [27], [28]) combines the operators ("vector fields")

$$\Gamma_j = x_j + 2it\partial_{x_j} = e^{i|x|^2/4t} 2it\partial_{x_j}(e^{-i|x|^2/4t}\cdot) = e^{it\Delta}x_j e^{-it\Delta}\cdot, \quad j = 1,..,n, \tag{2.7}$$

their commutative relation

$$(i\partial_t + \Delta)\Gamma_j u = \Gamma_j(i\partial_t u + \Delta u), \qquad j = 1,..,n, \tag{2.8}$$

so that $e^{it\Delta}(x_j u_0) = \Gamma_j\, e^{it\Delta}u_0$, and the structure of the nonlinearity in (1.1).

It should be remarked that Theorem B shows that the amount of decay in $L^2(|x|^{2m}dx)$ preserved by the solution depends on the regularity in the Sobolev scale H^s, $s \geq 0$) of the data, and the non-preserved decay is transformed in "local regularity". In particular, (2.6) tells us that $t^\beta \partial^\beta_x u \in L^2_{loc}(\mathbb{R}^n)$, for $|\beta| \leq m$ and $t \in [-T,T] - \{0\}$.

Also one notices that the power of the weight m in Theorem B is assumed to be an integer. In [53] we were able to remove this restriction.

Theorem 1. *In addition to the hypothesis in Theorem A assume* $u_0 \in L^2(|x|^{2m}dx)$, $m > 0$ *with* $[m] \leq a-1$ *if* a *is not an odd integer.*

(I) *If* $s \geq m$,

$$u \in C([-T,T] : H^s \cap L^2(|x|^{2m}dx)) \cap L^q([-T,T] : L^p_s \cap L^p(|x|^{2m}dx) = Z^{s,m}_T. \tag{2.9}$$

(II) *If* $1 \leq s < m$, *then* (2.9) *holds with* $[s]$ *instead of* m *and*

$$\mathit{\Gamma}^b\, \Gamma^\beta u(\cdot,t) \in C([-T,T] : L^2) \cap L^q([-T,T] : L^p), \tag{2.10}$$

where $\mathit{\Gamma}^b = e^{i|x|^2/4t} 2^b t^b D^b \left(e^{-i|x|^2/4t}\cdot\right)$ *with* $|\beta| = [m]$ *and* $b = m - [m]$.

In particular,

$$t^m\, \partial^\beta_x\, D^b u(\cdot,t) \in L^2_{loc}(\mathbb{R}^n), \quad |\beta| = [m], \;\; b = m - [m], \;\; t \in (-T,T) - \{0\}. \tag{2.11}$$

As an application of this result we also prove that the persistence property in these weighted spaces can only hold for regular enough solutions. More precisely:

Lemma 1. *Let* u *be a solution of the IVP associated to the NLS equation* (1.1) *provided by Theorem A. If there exist two times* $t_1, t_2 \in [0,T]$, $t_1 \neq t_2$ *such that*

$$|x|^m u(t_1), \quad |x|^m u(t_2) \in L^2(\mathbb{R}^n), \qquad m > s, \tag{2.12}$$

$m\leq a-1$ if a is not an odd integer, then

$$u\in C([-T,T]:H^m\cap L^2(|x|^{2m}dx))\cap L^q([-T,T]:L^p_m\cap L^p(|x|^{2m}dx).$$

Moreover, if a is an odd integer and (2.12) holds for all $m\in\mathbb{Z}^+$, then

$$u\in C([-T,T]:\mathbb{S}(\mathbb{R}^n)). \tag{2.13}$$

A key ingredient in our proof was an appropriate version of the Leibnitz rule for homogeneous fractional derivatives of order $b\in\mathbb{R}$

$$D^bf(x)\equiv((2\pi|\xi|)^b\hat{f})^\vee(x) \tag{2.14}$$

deduced as a direct consequence of the characterization of the $L^p_s(\mathbb{R}^n)$ spaces (see (2.2)) given in [58].

Theorem D. *Let $b\in(0,1)$ and $2n/(n+2b)\leq p<\infty$. Then $f\in L^p_b(\mathbb{R}^n)$ if and only if*

$$(a)\;\; f\in L^p(\mathbb{R}^n),$$

$$(b)\quad \mathcal{D}^bf(x)=(\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^2}{|x-y|^{n+2b}}dy)^{1/2}\in L^p(\mathbb{R}^n), \tag{2.15}$$

with

$$\|f\|_{b,p}=\|(1-\Delta)^{b/2}f\|_p\simeq\|f\|_p+\|D^bf\|_p\simeq\|f\|_p+\|\mathcal{D}^bf\|_p. \tag{2.16}$$

For the proof of Theorem D we refer to [58], where the optimality of the lower bound $2n/(n+2b)$ was also established. The case $p=2n/(n+2b)$ was proven in [18]. For a detailed discussion on the different characterizations of the $L^p_s(\mathbb{R}^n)$ spaces we refer to [58] and [59].

It is easy to see that for $p=2$ and $b\in(0,1)$ one has

$$\|\mathcal{D}^bf\|_2\simeq\|D^bf\|_2, \tag{2.17}$$

$$\|\mathcal{D}^b(fg)\|_2\leq c(\|f\,\mathcal{D}^bg\|_2+\|g\,\mathcal{D}^bf\|_2), \tag{2.18}$$

and for $p>2n/(n+2b)$

$$\mathcal{D}^b(fg)(x)\leq\|f\|_\infty\,\mathcal{D}^bg(x)+|g(x)|\,\mathcal{D}^bf(x). \tag{2.19}$$

We observe that in (2.18) both terms on the right hand side are estimates on the product of functions. We do not know whether or not (2.18) still holds with D^b instead of $\mathcal{D}^b$,or for $p\neq2$.

Theorem D (i.e. the estimates (2.18)-(2.17)) allows us to get the following inequalities:

–(i) Let $b\in(0,1)$. For any $t>0$

$$\mathcal{D}^b(e^{it|x|^2})\leq c(t^{b/2}+t^b|x|^b). \tag{2.20}$$

–(ii) Let $b\in(0,1)$. Then there exists $c=c(b)>0$ such that for any $t\in\mathbb{R}$

$$\||x|^b\,e^{it\Delta}f\|_2\leq c(t^{b/2}\|f\|_2+t^b\|D^bf\|_2+\||x|^bf\|_2). \tag{2.21}$$

–(iii) Defining the operator $\varGamma^b$ for $b>0$ as in Theorem 1 (see (2.10))

$$\varGamma^b\equiv\varGamma^b(t)=e^{i|x|^2/4t}2^bt^bD^b\left(e^{-i|x|^2/4t}\cdot\right), \tag{2.22}$$

one has for $b > 0$ and $t \in \mathbb{R}$ that

$$\Gamma^b(t) e^{it\Delta} f = e^{it\Delta}(|x|^b f), \tag{2.23}$$

and consequently

$$\Gamma^b(t) f = e^{it\Delta}(|x|^b e^{-it\Delta} f). \tag{2.24}$$

In addition to the estimates (2.20)-(2.24) the following two lemmas were essential in the proof of Theorem 1 given in [53]. The first is a version of the Gagliardo-Nirenberg inequality for fractional derivatives.

Lemma 2. *Let* $1 < q, p, r < \infty$ *and* $0 < \alpha < \beta$. *Then*

$$\|D^\alpha f\|_p \leq c \|f\|_r^{1-\theta} \, \|D^\beta f\|_q^\theta, \tag{2.25}$$

with

$$\frac{1}{p} - \frac{\alpha}{n} = (1-\theta)\frac{1}{r} + \theta\left(\frac{1}{q} - \frac{\beta}{n}\right), \qquad \theta \in [\alpha/\beta, 1]. \tag{2.26}$$

The second is an interpolation estimate, which as Lemma 2, is a consequence of the three line theorem.

Lemma 3. *Let* a, $b > 0$. *Assume that* $J^a f = (1-\Delta)^{a/2} f \in L^2(\mathbb{R})$ *and* $\langle x \rangle^b f = (1+|x|^2)^{b/2} f \in L^2(\mathbb{R})$. *Then for any* $\theta \in (0,1)$

$$\|J^{\theta a}(\langle x \rangle^{(1-\theta)b} f)\|_2 \leq c \|\langle x \rangle^b f\|_2^{1-\theta} \, \|J^a f\|_2^\theta. \tag{2.27}$$

For the study of persistence properties of the solution to the IVP associated to the NLS equation (1.1) in exponential weighted spaces we refer to [16], [17], and references therein.

Next, we shall consider the k-gKdV equation (1.2).

3. The k-generalized Korteweg-de Vries equation (1.2)

The following theorem describes the LWP theory in the classical Sobolev spaces $H^s(\mathbb{R})$ for the IVP associated to the kgKdV equation (1.2).

Theorem E. (I) *The IVP associated to the equation* (1.2) *with* $k = 1$ *is LWP in* $H^s(\mathbb{R})$ *for* $s \geq s_1^* = -3/4$.

(II) *The IVP associated to the equation* (1.2) *with* $k = 2$ *is LWP in* $H^s(\mathbb{R})$ *for* $s \geq s_2^* = 1/4$.

(III) *The IVP associated to the equation* (1.2) *with* $k = 3$ *is LWP in* $H^s(\mathbb{R})$ *for* $s \geq s_3^* = -1/6$.

(IV) *The IVP associated to the equation* (1.2) *with* $k \geq 4$ *is LWP in* $H^s(\mathbb{R})$ *for* $s \geq s_k^* = (k-4)/2k$.

The result $s > -3/4$ for the case $k = 1$ was established in [39]. The limiting value $s = -3/4$ was obtained in [11], [24], and [42]. The result for the case $k = 2$ was proven in [38]. The result $s > -1/6$ for the case $k = 3$ was given in [22]. The limiting value $s = -1/6$ was obtained in [63]. The proof of the cases $k \geq 4$ was given in [38].

The above local results apply to both real and complex valued functions.

The scaling argument described in (2.4) affirms that LWP should hold for $s \geq s_k = (k-4)/2k$. As Theorem E shows this is the case for $k \geq 3$ (where for $s_k = s_k^*$

one has $T = T(u_0)$). However, in the cases $k = 1$ and $k = 2$ the values suggested by the scaling do not seem to be reachable in the Sobolev scale, see [40], and [11]. For the sharpness of these results we refer to [4], [40], and [11].

Real valued solutions of the k-gKdV equation (1.2) formally satisfy at least three conservation laws:

$$I_1(u) = \int_{-\infty}^{\infty} u(x,t)dx, \qquad I_2(u) = \int_{-\infty}^{\infty} (u(x,t))^2 dx,$$
$$I_3(u) = \int_{-\infty}^{\infty} ((\partial_x u(x,t))^2 - \frac{2}{(k+1)(k+2)} u(x,t)^{k+2}) dx.$$

It was proven in [13] that for $k = 1$ and $k = 2$ one has global well posedness for $s > -3/4$ and $s > 1/4$, respectively. The global cases for $k = 1,\ s = -3/4$ and $k = 2,\ s = 1/4$ were proven in [24] and [42]. For the case $k = 3$ the global well posedness is known for $s > -1/42$, see [23].

For $k = 4$ blow up of "large" enough solutions was proven in [48]. Similar results for $k \geq 5$ remain an open problem.

Concerning the persistence of these solutions in weighted Sobolev spaces one has the following result found in [34].

Theorem F. *Let $m \in \mathbb{Z}^+$. Let $u \in C([-T,T] : H^s(\mathbb{R})) \cap .....$ with $s \geq 2m$ be the solution of the IVP associated to the equation* (1.2) *provided by Theorem E. If $u(x,0) = u_0(x) \in L^2(|x|^{2m}dx)$, then*

$$u \in C([-T,T] : H^s(\mathbb{R}) \cap L^2(|x|^{2m}dx)).$$

We recall that if for a solution $u \in C([0,T] : H^s(\mathbb{R}))$ of (1.2) one has that $\exists\, t_0 \in [0,T]$ such that $u(\cdot,t_0) \in H^{s'}(\mathbb{R})$, $s' > s$, then $u \in C([0,T] : H^{s'}(\mathbb{R}))$. So we shall mainly consider the most interesting case $s = 2m$ in Theorem F.

The proof of Theorem F combines the operator

$$\Gamma = x + 3t\partial_x^2,$$

and its commutative relation with the linear part $L = \partial_t + \partial_x^3$ of the equation (1.2) i.e.

$$\Gamma(\partial_t + \partial_x^3)v = (\partial_t + \partial_x^3)\Gamma v.$$

As in the case of the NLS equation (1.1) we would like to extend Theorem F where $m \in \mathbb{Z}^+$ to the case $m \in \mathbb{R}$, $m > 0$. Our first result in this direction is the following:

Theorem 2. *Let $m \geq 0$. Let $u \in C([-T,T] : H^m(\mathbb{R})) \cap .....$ with $m \geq \max\{s_k^*; 0\}$ be the solution of the IVP associated to the equation* (1.2) *provided by Theorem E. If $u(x,0) = u_0(x) \in L^2(|x|^m dx)$, then*

(I) *If $m < 1$, then for any $\epsilon > 0$*

$$u \in C([-T,T] : H^m(\mathbb{R}) \cap L^2(|x|^{m-\epsilon}dx)).$$

(II) *If $m \geq 1$, then*

$$u \in C([-T,T] : H^m(\mathbb{R}) \cap L^2(|x|^m dx)).$$

In [51] and [52] the loss of power $\epsilon > 0$ in the weight when $m < 1$ was removed for the equation (1.2) with non-linearity $k = 2, 4, 5, ...$ More precisely, the following optimal result was established in [52]:

Theorem 3. *Let $m \geq max\{s_k^*; 0\}$ with $k = 2,4,5,\dots$. Let $u \in C([-T,T] : H^m(\mathbb{R})) \cap \dots$ be the solution of the IVP associated to the equation* (1.2) *provided by Theorem E. If $u(x,0) = u_0(x) \in L^2(|x|^m dx)$, then*

$$u \in C([-T,T] : H^m(\mathbb{R}) \cap L^2(|x|^m dx)).$$

It should be remarked that in the cases $k = 1$ and $k = 3$ the proof of the local theory in Theorem E is based on the spaces $X_{s,b}$ introduced in the context of dispersive equations in [5]. For all the other powers k one has a local existence theory based on a contraction principle in a spaces defined by mixed norms of the type $L^p(\mathbb{R} : L^q([0,T]))$ or $L^q([0,T] : L^p(\mathbb{R}))$ (see [38]). This is the main difficulty in extending the optimal result in Theorem 3 to the powers $k = 1$ and $k = 3$ in (1.2).

Proof of Theorem 2

We shall sketch the ideas in the proof of Theorem 2 and refer to [51] and [52] for the justification of the argument and further details.

Following Kato's idea in [34] to establish the local smoothing effect (i.e. multiplying the equation (1.2) by $u(x,t)\phi(x)$, integrating the result, and using integration by parts) one formally gets the identity

$$\frac{d}{dt}\int u^2\phi dx + 3\int (\partial_x u)^2\phi' dx - \int u^2\phi^{(3)}dx - \frac{2}{k+2}\int u^{k+2}\phi' dx = 0. \tag{3.1}$$

Let us consider first the case $max\{s_k^*; 0\} \leq m < 1$.

From the local theory one has the following estimates for the solution $u = u(x,t)$

$$\sup_{x\in\mathbb{R}} \Big(\int_0^T |\partial_x D_x^m u(x,t)|^2 dt\Big)^{1/2} < c_T \|J^m u_0\|_2 = c_T\|u_0\|_{m,2}, \tag{3.2}$$

(the sharp form of the local smoothing effect found in [37]-[38]), and

$$\begin{aligned} \|D_x^m u\|_{L^2_x L^2_T} &= \Big(\int_{-\infty}^{\infty}\int_0^T |D_x^m u(x,t)|^2 dt dx\Big)^{1/2} \\ &\leq T^{1/2} \sup_{t\in[0,T]} \|D_x^m u(t)\|_2 < c_T \|D^m u_0\|_2 \leq c_T\|u_0\|_{m,2}. \end{aligned} \tag{3.3}$$

Now, we consider the extensions of the estimates in (3.2)-(3.3) to the operators D_x^{1+m+iy} and D_x^{m+iy}, $y \in \mathbb{R}$ respectively. First, in the linear case one has the estimates

$$\begin{aligned} &\|D_x^{m+1+iy} v\|_{L^\infty_x L^2_T} \leq c_T \|D^m v_0\|_2, \\ &\|D_x^{m+iy} v\|_{L^2_x L^2_T} \leq c_T \|D^m v_0\|_2, \end{aligned} \tag{3.4}$$

for

$$v(x,t) = U(t)v_0(x) = c\int_{-\infty}^{\infty} e^{ix\xi} e^{it\xi^3}\widehat{v}_0(\xi)d\xi. \tag{3.5}$$

To apply the three line theorem we consider the function $F(z)$ defined on $\mathcal{S} = \{z \in \mathbb{C} : \Re(z) \in [0,1]\}$

$$F(z) = \int_{-\infty}^{\infty}\int_0^T D_x^{s(z)} v(x,t)\,\phi(x,z)\,f(t)\,dtdx,$$

where

$$s(z)=(1-z)(1+m)+zm,\quad 1/q(z)=(1-z)+z/2,\quad q=2/(2-m),$$

$$\phi(x,z)=|g(x)|^{q/q(z)}\,\frac{g(x)}{|g(x)|},\quad \text{with}\quad \|g\|_{L^{2/(2-m)}_x}=\|f\|_{L^2([0,T])}=1,$$

which is analytic on the interior of $\mathcal{S}$. So using that

$$\|\phi(\cdot,0+iy)\|_1=\|\phi(\cdot,1+iy)\|_2=1,$$

one gets that

$$\begin{aligned}\|\partial_x v\|_{L^{2/m}_xL^2_T}&\le c\|D_xv\|_{L^{2/m}_xL^2_T}\\ &\le c\sup_{y\in\mathbb{R}}\|D_x^{1+m+iy}v\|^{1-m}_{L^\infty_xL^2_T}\sup_{y\in\mathbb{R}}\|D_x^{m+iy}v\|^m_{L^2_xL^2_T}\le c_T\|D^mv_0\|_2.\end{aligned}\tag{3.6}$$

Inserting the estimate (3.6) in the proof of the local well posedness one obtains that

$$\|\partial_x u\|_{L^{2/m}_xL^2_T}\le c_T\|u_0\|_{m,2},\tag{3.7}$$

for $u=u(x,t)$ solution of the k-gKdV equation (1.2).

Now taking $\phi(x)=\langle x\rangle^{m-\epsilon}$, $\epsilon>0$ sufficiently small in (3.1), (we recall that $m<1$) and integrating in the time interval $[0,T]$ one finds that

$$\begin{aligned}&\int_0^T\int_{-\infty}^{\infty}(\partial_xu(x,t))^2\phi'(x)dxdt=c\|\partial_xu\,\langle x\rangle^{\frac{m}{2}-\frac{1}{2}-\frac{\epsilon}{2}}\|^2_{L^2_xL^2_T}\\ &\le c\|\langle x\rangle^{m/2-1/2-\epsilon/2}\|_{L^{2/(1-m)}_x}\|\partial_xu\|_{L^{2/m}_xL^2_T}\le c_{m,\epsilon}\|\partial_xu\|_{L^{2/m}_xL^2_T},\end{aligned}\tag{3.8}$$

which combined with (3.6) and (3.1) shows that $\langle x\rangle^{m/2-\epsilon/2}u(\cdot,t)\in L^2(\mathbb{R})$ for $t\in[0,T]$. This basically completes the proof of the case $m<1$.

Next, we shall consider the case $m\ge 1$.

We take in (3.1) $\phi(x)=\langle x\rangle^m$ in (3.1), so we need to estimate the term

$$\int_{-\infty}^{\infty}|\partial_xu(x,t)|^2\langle x\rangle^{m-1}dx=\|\partial_xu(\cdot,t)\langle\cdot\rangle^{(m-1)/2}\|^2_{L^2_x}.$$

Thus, combining Lemma 3 in the previous section, the preservation of the L^2-norm of the solution, and Lemma 3 it follows that

$$\begin{aligned}&\|\partial_xu(\cdot,t)\langle\cdot\rangle^{(m-1)/2}\|_2\\ &\le\|\partial_x(u(\cdot,t)\langle\cdot\rangle^{(m-1)/2})\|_2+c\|u(\cdot,t)\langle\cdot\rangle^{(m-3)/2}\|_2\\ &\le\|\partial_xJ^{-1}J(u(\cdot,t)\langle\cdot\rangle^{(m-1)/2})\|_2+c\|u(\cdot,t)\langle\cdot\rangle^{m/2}\|_2\\ &\le c\|J(u(\cdot,t)\langle\cdot\rangle^{(m-1)/2})\|_2+c\|u(\cdot,t)\langle\cdot\rangle^{m/2}\|_2\\ &\le c\|J^mu(\cdot,t)\|_2^{1/m}\|u(\cdot,t)\langle\cdot\rangle^{m/2}\|_2^{1-1/m}+c\|u(\cdot,t)\langle\cdot\rangle^{m/2}\|_2.\end{aligned}\tag{3.9}$$

Hence, inserting (3.9) in (3.1), using Young and Gronwall inequalities, the hypothesis $m\ge1$, and the fact that the H^m-norm of the solution is bounded in the time interval $[0,T]$ one obtains the desired result

$$\sup_{t\in[0,T]}\|\langle x\rangle^{m/2}u(\cdot,t)\|_{L^2}<\infty.$$

This completes the sketch of the proof of Theorem 2.

To finish this section concerning the k-gKdV equation (1.2) we will make some comments concerning the proof of Theorem 3 given in [51] and [52]. One of the key element in that proof is the following commutator estimate:

Lemma 4. *Let $0<\alpha<1$ and $1<p<\infty$. Then for functions $f,g:\mathbb{R}\to\mathbb{C}$ one has that*

$$\|D^\alpha(fg)-fD^\alpha g\|_p\le c\|Q_N(D^\alpha f)\|_{L^\infty l^1_N}\|g\|_2, \tag{3.10}$$

where

$$\|Q_N(f)\|_{L^\infty l^1_N}\equiv\|\sum_{N\in\mathbb{Z}}|Q_N(f)|\,\|_{L^\infty},$$

and

$$Q_N(f)(x)=((\eta\left(\frac{\xi}{2^N}\right)+\eta\left(-\frac{\xi}{2^N}\right))\widehat{f}(\xi))^\vee(x),$$

where $\eta\in C_0^\infty(\mathbb{R})$ with $supp(\eta)\subseteq[1,2,2]$ so that

$$\sum_{N\in\mathbb{Z}}(\eta\left(\frac{x}{2^N}\right)+\eta\left(-\frac{x}{2^N}\right))=1,\quad \text{for } x\neq 0.$$

In the proof of Theorem 3 for the case $k=2$ and $m=1/4$ (extremal case) given in [51] Lemma 4 was combined with the inequality

$$\|D_\xi^{1/8}Q_N\left(\frac{e^{it\xi^3}}{(1+\xi^2)^{1/8}}\right)\|_{L^\infty_\xi l^1_N}<\infty,$$

to establish the main estimate in the proof.

For the study of persistence properties of the solution to the IVP associated to the k-gKdV equation (1.2) in exponential weighted spaces we refer to [41] and [15] and references therein.

Finally, we shall consider the BO equation (1.3).

4. The Benjamin-Ono equation (1.3)

The LWP in the Sobolev spaces $H^s(\mathbb{R})$ of the IVP associated to the BO equation (1.3) has been largely considered : in [1] and [32] LWP was established for $s>3/2$, in [56] for $s\ge 3/2$, in [44] for $s>5/4$, in [36] for $s>9/8$, in [62] for $s\ge 1$, in [7] for $s>1/4$, and in [31] LWP was proven in $H^s(\mathbb{R})$ for $s\ge 0$.

Real valued solutions of the IVP (1.3) satisfy infinitely many conservation laws (time invariant quantities), the first three are the following:

$$\begin{aligned} &I_1(u)=\int_{-\infty}^{\infty}u(x,t)dx,\quad I_2(u)=\int_{-\infty}^{\infty}u^2(x,t)dx,\\ &I_3(u)=\int_{-\infty}^{\infty}(|D_x^{1/2}u|^2-\frac{u^3}{3})(x,t)dx,\end{aligned} \tag{4.1}$$

where $D_x=\mathcal{H}\,\partial_x$.

The k-conservation law I_k provides an *a priori* estimate of the L^2-norm of the derivatives of order $(k-2)/2,\ k>2$ of the solution, i.e. $\|D_x^{(k-2)/2}u(t)\|_2$. This allows one to deduce GWP from LWP results.

In the BO equation the dispersive effect is described by a non-local operator and is significantly weaker than that exhibited by the Korteweg-de Vries (KdV) equation, i.e. $k = 1$ in (1.2). Indeed, it was proven in [49] that for any $s \in \mathbb{R}$ the map data-solution from $H^s(\mathbb{R})$ to $C([0,T] : H^s(\mathbb{R}))$ is not locally C^2, and in [45] that it is not locally uniformly continuous. In particular, this implies that no LWP results can be obtained by an argument based only on a contraction method.

Consider the weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \text{ and } \dot{Z}_{s,r} = \{f \in Z_{s,r} : \widehat{f}(0) = 0\} \quad s,\, r \in \mathbb{R}. \tag{4.2}$$

In [32] the following results were obtained:

Theorem G. (I) *The IVP associated to the BO equation* (1.3) *is GWP in* $Z_{2,2}$.

(II) *If* $\widehat{u}_0(0) = 0$*, then the IVP associated to the BO equation* (1.3) *is GWP in* $\dot{Z}_{3,3}$.

(III) *If* $u(x,t)$ *is a solution of the IVP associated to the BO equation*(1.3) *such that* $u \in C([0,T] : Z_{4,4})$ *for arbitrary* $T > 0$*, then* $u(x,t) \equiv 0$.

We observe that the linear part of the equation in (1.3) $L = \partial_t + \mathcal{H}\partial_x^2$ commutes with the operator $\Gamma = x - 2t\mathcal{H}\partial_x$, i.e.

$$[L; \Gamma] = L\Gamma - \Gamma L = 0.$$

Also, the solution $v(x,t)$ of the associated IVP

$$v(x,t) = U(t)v_0(x) = e^{-it\mathcal{H}\partial_x^2}v_0(x) = \left(e^{-it\xi|\xi|}\,\widehat{v}_0\right)^{\vee}(x), \tag{4.3}$$

satisfies that $v(\cdot,t) \in L^2(|x|^{2k}dx)$, $t \in [0,T]$, when $v_0 \in Z_{k,k},\ k \in \mathbb{Z}^+$ for $k = 1, 2, ...$ and

$$\int_{-\infty}^{\infty} x^j\, v_0(x)dx = 0, \quad j = 0, 1, ..., k-3, \quad \text{if} \quad k \geq 3.$$

In [33] the unique continuation result in $Z_{4,4}$ in Theorem G was improved:

Theorem I. *Let* $u \in C([0,T] : H^2(\mathbb{R}))$ *be a solution of the IVP* (1.3)*. If there exist three different times* $t_1, t_2, t_3 \in [0,T]$ *such that*

$$u(\cdot, t_j) \in Z_{4,4}, \quad j = 1, 2, 3, \quad \textit{then} \quad u(x,t) \equiv 0. \tag{4.4}$$

As in the previous cases, the goal was to extend the results in Theorem G and Theorem I from integer values to the continuum optimal range of indices (s,r). In this direction one finds the following results established in [19]:

Theorem 4.

(I) *Let* $s \geq 1,\ r \in [0,s]$*, and* $r < 5/2$*. If* $u_0 \in Z_{s,r}$*, then the solution* $u(x,t)$ *of the IVP associated to the BO equation* (1.3) *satisfies that* $u \in C([0,\infty) : Z_{s,r})$.

(II) *For* $s > 9/8$ *(*$s \geq 3/2$*),* $r \in [0,s]$*, and* $r < 5/2$ *the IVP associated to the BO equation*(1.3) *is LWP (GWP resp.) in* $Z_{s,r}$.

(III) *If* $r \in [5/2, 7/2)$ *and* $r \leq s$*, then the IVP associated to the BO equation* (1.3) *is GWP in* $\dot{Z}_{s,r}$.

Theorem 5. *Let* $u \in C([0,T] : Z_{2,2})$ *be a solution of the IVP associated to the BO equation* (1.3)*. If there exist two different times* $t_1, t_2 \in [0,T]$ *such that*

$$u(\cdot, t_j) \in Z_{5/2,5/2}, \quad j = 1, 2, \quad \textit{then} \quad \widehat{u}_0(0) = 0\,, \quad (\textit{so} \ \ u(\cdot,t) \in \dot{Z}_{5/2,5/2}). \tag{4.5}$$

Theorem 6. *Let $u \in C([0,T] : \dot{Z}_{3,3})$ be a solution of the IVP* (1.3). *If there exist three different times $t_1, t_2, t_3 \in [0,T]$ such that*

$$u(\cdot, t_j) \in Z_{7/2,7/2}, \quad j = 1,2,3, \quad \textit{then} \quad u(x,t) \equiv 0. \tag{4.6}$$

We also refer readers to the related works [47], [25], and [43].

Remarks : Theorem 5 and Theorem 6 show that the upper values of r for the persisitence properties in $Z_{s,r}$ and $\dot{Z}_{s,k}$ in Theorem 4 are optimal. We recall that if $u \in C([0,T] : H^s(\mathbb{R}))$ is a solution of the BO equation (1.3) such that $\exists\, t_0 \in [0,T]$ for which $u(x,t_0) \in H^{s'}(\mathbb{R})$, $s' > s$, then $u \in C([0,T] : H^{s'}(\mathbb{R}))$. So it suffices to consider the most interesting case $s = r$ in (4.2).

The proof of Theorems 6 is based on weighted energy estimates and involves several inequalities concerning the Hilbert transform $\mathcal{H}$.

Among them one finds the A_p condition introduced in [50].

Definition 1. A non-negative function $w \in L^1_{loc}(\mathbb{R})$ satisfies the A_p inequality with $1 < p < \infty$ if

$$\sup_{Q \text{ interval}} \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} = c(w) < \infty, \tag{4.7}$$

where $1/p + 1/p' = 1$.

It was proven in [30] that this is a necessary and sufficient condition for the Hilbert transform $\mathcal{H}$ to be bounded in $L^p(w(x)dx)$ (see [30],), i.e. $w \in A_p$, $1 < p < \infty$ if and only if

$$\left(\int_{-\infty}^{\infty} |\mathcal{H}f|^p w(x)dx\right)^{1/p} \leq c^* \left(\int_{-\infty}^{\infty} |f|^p w(x)dx\right)^{1/p}, \tag{4.8}$$

In the case $p = 2$, a previous characterization of w in (4.7) was found in [29]. However, even though the main case is for $p = 2$, the characterization (4.7) will be the one used in the proof. In particular, one has that in $\mathbb{R}$

$$|x|^\alpha \in A_p \quad \Leftrightarrow \quad \alpha \in (-1, p-1). \tag{4.9}$$

In order to justify some of the arguments in the proofs one need some further continuity properties of the Hilbert transform. More precisely, the proof requires the constant c^* in (4.8) to depend only on $c(w)$ the constant describing the A_p condition (see (4.7)) and on p. In [55] precise bounds for the constant c^* in (4.7) were given which are sharp in the case $p = 2$ and sufficient for the purpose in [19].

It will be essential in the arguments in [19] that some commutator operators involving the Hilbert transform $\mathcal{H}$ are of "order zero". More precisely, one shall use the following estimate: $\forall\, p \in (1,\infty)$, l, $m \in \mathbb{Z}^+ \cup \{0\}$, $l + m \geq 1$ $\exists\, c = c(p;l;m) > 0$ such that

$$\|\partial_x^l [\mathcal{H}; a] \partial_x^m f\|_p \leq c \|\partial_x^{l+m} a\|_\infty \|f\|_p. \tag{4.10}$$

In the case $l + m = 1$, (4.10) is Calderón's first commutator estimate [8]. The case $l + m \geq 2$ of the estimate (4.10) was proved in [14].

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