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Spectral statistics for the discrete Anderson model in the localized regime

By

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Abstract

We report on recent results on the spectral statistics of the discrete Anderson model in the localized phase obtained in [6]. In particular, we describe the

- locally uniform Poisson behavior of the rescaled eigenvalues,
- independence of the Poisson processes obtained as such limits at distinct energies,
- locally uniform Poisson behavior of the joint distributions of the rescaled energies and rescaled localization centers in a large range of scales.
- the distribution of the rescaled level spacings, locally and globally in energy,
- the distribution of the rescaled localization centers spacings.

Our results show, in particular, that, for the discrete Anderson Hamiltonian with smoothly distributed random potential at sufficiently large coupling, the limit of the level spacing distribution is that of i.i.d. random variables distributed according to the density of states of the random Hamiltonian.

§ 1. Introduction

On $\ell^2(\mathbb{Z}^d)$, consider the random Anderson model

$$H_\omega = -\Delta + V_\omega$$

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where $-\Delta$ is the discrete Laplace operator

$$(-\Delta u)_n = \sum_{|m-n|=1} u_m \quad \text{for } u = (u_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$$

and V_ω is the random potential

$$(V_\omega u)_n = \omega_n u_n \quad \text{for } u = (u_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d).$$

We assume that the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ are independent identically distributed and that their distribution admits a compactly supported bounded density, say g.
It is then well known (see e.g. [9]) that

- there exists $\Sigma := [S_-, S_+] = [-2d, 2d] + \operatorname{supp} g \subset \mathbb{R}$ such that, for almost every $\omega = (\omega_n)_{n\in\mathbb{Z}^d}$, the spectrum of H_ω is equal to Σ;
- for some $S_- < s_- \leq s_+ < S_+$, the intervals $I_- = [S_-, s_-)$ and $I_+ = (s_+, S_+]$ are contained in the region of complete localization for H_ω, in particular, $I_- \cup I_+$ contains only pure point spectrum associated to exponentially decaying eigenfunctions; for the precise meaning of the region of complete localization, we refer to [1, 9, 5]; if the disorder is sufficiently large or if the dimension $d = 1$ then, one can pick $I_+ \cup I_- = \Sigma$; define $I = I_+ \cup I_-$;
- there exists a bounded density of states, say $E \mapsto \nu(E)$, such that, for any continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, one has

  $$\int_{\mathbb{R}} \varphi(E)\nu(E)dE = \mathbb{E}(\langle \delta_0, \varphi(H_\omega)\delta_0\rangle). \tag{1.1}$$

  Here, and in the sequel, $\mathbb{E}(\cdot)$ denotes the expectation with respect to the random parameters.
  Let N be the integrated density of states of H_ω i.e. N is the distribution function of the measure $\nu(E)dE$. The function ν is only defined E almost everywhere. In the sequel, unless we explicitly say otherwise, when we speak of $\nu(E)$ for some E, we mean that the non decreasing function N is differentiable at E and that $\nu(E)$ is its derivative at E.

We now describe the local level and localization center statistics, the level spacing statistics and the localization center spacings statistics in I.

§ 2. The local level statistics

For $L \in \mathbf{N}$, let $\Lambda = \Lambda_L = [-L, L]^d \cap \mathbf{Z}^d \subset \mathbf{Z}^d$ be a large box and $H_{\omega,\Lambda}$ be the operator H_ω restricted to Λ with periodic boundary conditions. Let $|\Lambda|$ be the volume of Λ i.e. $|\Lambda| = (2L+1)^d$.

$H_\omega(\Lambda)$ is an $|\Lambda| \times |\Lambda|$ real symmetric matrix. Let us denote its eigenvalues ordered increasingly and repeated according to multiplicity by $E_1(\omega,\Lambda) \le E_2(\omega,\Lambda) \le \cdots \le E_{|\Lambda|}(\omega,\Lambda)$.

Let E_0 be an energy in I such that $\nu(E_0) > 0$. The local level statistics near E_0 is the point process defined by

$$\Xi(\xi, E_0, \omega, \Lambda) = \sum_{j=1}^{|\Lambda|} \delta_{\xi_j(E_0,\omega,\Lambda)}(\xi) \tag{2.1}$$

where

$$\xi_j(E_0,\omega,\Lambda) = |\Lambda|\,\nu(E_0)\,(E_j(\omega,\Lambda) - E_0), \quad 1 \le j \le |\Lambda|. \tag{2.2}$$

The main result of [12] reads

Theorem 2.1 ([12]). *Let E_0 be an energy in I such that $\nu(E_0) > 0$. When $|\Lambda| \to +\infty$, the point process $\Xi(E_0,\omega,\Lambda)$ converges weakly to a Poisson process on $\mathbf{R}$ with intensity 1.*

§ 2.1. Uniform Poisson convergence

In [6], we obtain a uniform version of Theorem 2.1 i.e. a version that holds uniformly over an energy interval of size asymptotically infinite compared to $|\Lambda|^{-1}$.
Fix $1 > \beta > (d+1)/(d+2)$. Let $I_\Lambda(E_0,\beta)$ be the interval centered at E_0 of length $2|\Lambda|^{-\beta}$. Let the number of eigenvalues of $H_\omega(\Lambda)$ inside $I_\Lambda(E_0,\beta)$ be equal to $N_\Lambda(\omega, E_0)$. For $1 \le j \le N_\Lambda(\omega,E_0) - 1$, define the renormalized eigenvalues $\xi_j(\omega,\Lambda)$ by (2.2) for $E_j \in I_\Lambda(E_0,\beta)$. Hence, for all $1 \le j \le N_\Lambda(\omega,E_0) - 1$, one has $\xi_j(\omega,\Lambda) \in |\Lambda|^{1-\beta}\cdot[-1,1]$. We then prove

Theorem 2.2 ([6]). *Let E_0 be an energy in I such that $\nu(E_0) > 0$. Then, there exists $\delta > 0$, such that, for any sequences of intervals $I_1 = I_1^\Lambda, \dots, I_p = I_p^\Lambda$ in $|\Lambda|^{1-\beta}\cdot[-1,1]$ such that*

$$\inf_{j \ne k} \mathit{dist}(I_j, I_k) \ge e^{-|\Lambda|^\delta}, \tag{2.3}$$

one has, for any sequences of integers $k_1 = k_1^\Lambda, \cdots, k_p = k_p^\Lambda \in \mathbf{N}^p$,

$$\lim_{|\Lambda|\to+\infty} \left| \mathbb{P}\left(\left\{ \omega;\ \begin{matrix} \#\{j;\ \xi_j(\omega,\Lambda) \in I_1\} = k_1 \\ \vdots \qquad\qquad \vdots \\ \#\{j;\ \xi_j(\omega,\Lambda) \in I_p\} = k_p \end{matrix} \right\} \right) - e^{-|I_1|}\frac{|I_1|^{k_1}}{k_1!} \cdots e^{-|I_p|}\frac{|I_p|^{k_p}}{k_p!} \right| = 0.$$

Note that, in Theorem 2.2, we do not require the limits

$$\lim_{|\Lambda|\to+\infty} e^{-|I_1|}\frac{|I_1|^{k_1}}{k_1!} = \lim_{|\Lambda|\to+\infty} e^{-|I_1^\Lambda|}\frac{|I_1^\Lambda|^{k_1^\Lambda}}{k_1^\Lambda!}, \ \dots,$$
$$\lim_{|\Lambda|\to+\infty} e^{-|I_p|}\frac{|I_p|^{k_p}}{k_p!} = \lim_{|\Lambda|\to+\infty} e^{-|I_p^\Lambda|}\frac{|I_p^\Lambda|^{k_p^\Lambda}}{k_p^\Lambda!}$$

to exist.
Clearly, Theorem 2.1 is a consequence of the stronger Theorem 2.2. The main improvement over the statements found in [12] is that the interval over which the Poisson statistics holds uniformly is much larger. We also note that Theorem 2.2 gives the asymptotics of the level spacing distribution over intervals I_Λ of size $|\Lambda|^{-(d+1)/(d+2)}$ (see section 3.4 and, in particular, Theorem 3.6). It also gives the asymptotic independence of the local Poisson processes defined at energies E_Λ and E'_Λ such that

$$|E_\Lambda - E_0| + |E'_\Lambda - E_0| \le |\Lambda|^{-\beta} \quad \text{and} \quad |\Lambda|\cdot|E_\Lambda - E'_\Lambda| \underset{\Lambda\to\mathbb{Z}^d}{\to} +\infty$$

We refer to the next section for more general results on this asymptotic independence. It is natural to wonder what is the largest size of interval in which a result like Theorem 2.2. We do not know the answer to that question.

§ 2.2. Asymptotic independence of the local processes

Once Theorem 2.1 is known, it is natural to wonder how the point processes obtained at two distinct energies relate to each other. We prove the following

Theorem 2.3 ([6, 10]). *Assume that the dimension $d = 1$. Pick $E_0 \in I$ and $E'_0 \in I$ such that $E_0 \neq E'_0$, $\nu(E_0) > 0$ and $\nu(E'_0) > 0$.*
When $|\Lambda| \to +\infty$, the point processes $\Xi(E_0,\omega,\Lambda)$ and $\Xi(E'_0,\omega,\Lambda)$, defined in (2.1), converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity the Lebesgue measure. That is, for $U_+ \subset \mathbb{R}$ and $U_- \subset \mathbb{R}$ compact intervals and $\{k_+,k_-\} \in \mathbb{N}\times\mathbb{N}$, one has

$$\mathbb{P}\left(\left\{\omega;\ \left\{\begin{array}{c} \#\{j;\xi_j(E_0,\omega,\Lambda)\in U_+\} = k_+ \\ \#\{j;\xi_j(E'_0,\omega,\Lambda)\in U_-\} = k_- \end{array}\right\}\right)\right) \underset{\Lambda\to\mathbb{Z}^d}{\to} e^{-|U_+|}\frac{|U_+|^{k_+}}{k_+!}\cdot e^{-|U_-|}\frac{|U_-|^{k_-}}{k_-!}.$$

So we see that, in the localized regime, in dimension 1, at distinct energies, the local eigenvalues behave independently from each other. Theorem 2.3 is a consequence of a decorrelation estimate for distinct energies that is proved in [10]. It is natural to expect that this decorrelation estimate stays true and, hence, that Theorem 2.3 stays true, for arbitrary dimensions. Nevertheless, we are only able to prove

Theorem 2.4 ([6, 10]). *Pick $E_0 \in I$ and $E_0' \in I$ such that $|E_0 - E_0'| > 2d$, $\nu(E_0) > 0$ and $\nu(E_0') > 0$.*
When $|\Lambda| \to +\infty$, the point processes $\Xi(E_0, \omega, \Lambda)$ and $\Xi(E_0', \omega, \Lambda)$, defined in (2.1), *converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity 1.*

Theorems 2.3 and 2.4 naturally lead to wonder how far the energies E_0 and E_0' need to be from each other with respect to the scaling used to renormalize the eigenvalues for the asymptotic independence to still hold.
We prove

Theorem 2.5 ([6]). *Pick $E_0 \in I$ such that $\nu(E_0) > 0$. Assume moreover that the density of states ν is continuous at E_0.*
Consider two sequences of energies, say $(E_\Lambda)_\Lambda$ and $(E_\Lambda')_\Lambda$ such that

1. *one has $E_\Lambda \underset{\Lambda \to \mathbb{Z}^d}{\to} E_0$ and $E_\Lambda' \underset{\Lambda \to \mathbb{Z}^d}{\to} E_0$,*
2. *one has $|\Lambda| \cdot |E_\Lambda - E_\Lambda'| \underset{\Lambda \to \mathbb{Z}^d}{\to} +\infty$.*

Then, the point processes $\Xi(E_\Lambda, \omega, \Lambda)$ and $\Xi(E_\Lambda', \omega, \Lambda)$, defined in (2.1), *converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity 1.*

A crucial tool in proving Theorem 2.5 are the generalized Minami estimates proved in [4] that can also be interpreted as local decorrelation estimates. Theorem 2.5 shows that, in the localized regime, eigenvalues that are sufficiently far away from each other but still close, i.e. that are separated by a distance that is asymptotically infinite when compared to the mean spacing between the eigenlevels, behave as independent random variables. There are no interactions except at very short distances.
Assumption (2) can clearly not be omitted in Theorem 2.5; it suffices to consider e.g. $E_\Lambda = E_\Lambda' + a|\Lambda|^{-1}$ to see that the two limit random processes are obtained as a shift from one another.
To complete this section, we note again that, when $|E_\Lambda' - E_\Lambda| = o(|\Lambda|^{-d/(d+2)})$, Theorem 2.5 is a consequence of Theorem 2.2.

§ 3. Localization center statistics

Recall that $E_1(\omega, \Lambda) \leq E_2(\omega, \Lambda) \leq \cdots \leq E_{|\Lambda|}(\omega, \Lambda)$ denote the eigenvalues of $H_{\omega,\Lambda}$ ordered increasingly and repeated according to multiplicity.

To $E_j(\omega, \Lambda)$, we associate a normalized eigenvector of $H_{\omega,\Lambda}$, say $\varphi_j(\omega, \Lambda)$. The components of the vector $\varphi_j(\omega, \Lambda)$ are denoted by $(\varphi_j(\omega, \Lambda; \gamma))_{\gamma \in \Lambda}$.

For $\varphi \in \ell^2(\Lambda)$, define the set of localization centers for φ as

$$C(\varphi) = \{\gamma \in \Lambda;\ \varphi(\gamma) = \max_{\gamma' \in \Lambda} |\varphi(\gamma')|\}.$$

One has

Lemma 3.1. *For any $p > 0$, there exists $C_p > 0$ such that, with probability at least $1 - |\Lambda|^{-p}$, if $E_j(\omega, \Lambda)$ is in the localized regime i.e. if $E_j(\omega, \Lambda) \in I$ then the diameter of $C(\varphi_j(\omega, \Lambda))$ is less than $C_p \log |\Lambda|$.*

Hence, in the localized regime, localization centers for an eigenfunction can be at most as far as $C \log |\Lambda|$ from each other. From now on, a localization center for a function φ will denote any point in the set of localization centers $C(\varphi)$ and let $x_j(\omega, \Lambda)$ be a localization center for $\varphi_j(\omega, \Lambda)$.

§ 3.1. Uniform Poisson convergence for the joint (energy,center)-distribution

We now place ourselves in the same setting as in section 2.1. We prove

Theorem 3.2. *Let E_0 be an energy in I such that $\nu(E_0) > 0$.*
Then, there exists $\delta > 0$, such that,

- *for any sequences of intervals $I_1 = I_1^\Lambda, \dots, I_p = I_p^\Lambda$ in $|\Lambda|^{1-\beta} \cdot [-1, 1]$ satisfying (2.3),*
- *for any sequences of cubes $C_1 = C_1^\Lambda, \dots, C_p = C_p^\Lambda$ in $[-1/2, 1/2]^d$*

one has, for any sequences of integers $k_1 = k_1^\Lambda, \cdots, k_p = k_p^\Lambda \in \mathbb{N}^p$,

$$\lim_{|\Lambda| \to +\infty} \left| \mathbb{P}\left(\left\{ \omega;\ \begin{matrix} \# \left\{ n;\ \begin{matrix} \xi_n(\omega, \Lambda) \in I_1 \\ x_n/L \in C_1 \end{matrix} \right\} = k_1 \\ \vdots \qquad\qquad \vdots \\ \# \left\{ n;\ \begin{matrix} \xi_n(\omega, \Lambda) \in I_p \\ x_n/L \in C_p \end{matrix} \right\} = k_p \end{matrix} \right\} \right) - \prod_{j=1}^{p} e^{-|I_j||C_j|} \frac{(|I_j||C_j|)^{k_j}}{k_j!} \right| = 0$$

where $x_n(\omega) = x_n(\omega, \Lambda_L)$ is the localization center associated to the eigenvalue $E_n(\omega, \Lambda_L) = E_0 + L^d \xi_n(\omega, \Lambda)$.

This result generalizes the results of [8, 14].

§ 3.2. Covariant scaling joint (energy,center)-distribution

Fix a sequence of scales $\ell = (\ell_\Lambda)_\Lambda$ such that

$$\frac{\ell_\Lambda}{\log|\Lambda|} \underset{|\Lambda|\to+\infty}{\to} +\infty \quad \text{and} \quad \ell_\Lambda \leq |\Lambda|^{1/d}. \tag{3.1}$$

Pick $E_0 \in I$ so that $\nu(E_0) > 0$. Consider the point process

$$\Xi^2_\Lambda(\xi, x; E_0, \ell) = \sum_{j=1}^{|\Lambda|} \delta_{\nu(E_0)(E_j(\omega,\Lambda)-E_0)\ell^d_\Lambda}(\xi) \otimes \delta_{x_j(\omega)/\ell_\Lambda}(x).$$

The process is valued in $\mathbb{R} \times \mathbb{R}^d$; actually, if $c\,\ell_\Lambda \geq |\Lambda|^{1/d}$, it is valued in $\mathbb{R} \times (-c, c)^d$. Assuming that the scales $(\ell_\Lambda)_\Lambda$ are chosen so that the limits exists, define

$$c_\ell := \lim_{|\Lambda|\to+\infty} |\Lambda|^{1/d} \ell^{-1}_\Lambda \in [1, +\infty]. \tag{3.2}$$

We prove

Theorem 3.3 ([6]). *The point process $\Xi^2_\Lambda(\xi, x; E_0, \ell)$ converges weakly to a Poisson process on $\mathbb{R} \times (-c_\ell, c_\ell)^d$ with intensity 1.*

In the case $\ell_\Lambda = |\Lambda|^{1/d}$, the result of Theorem 3.3 was obtained in [8] and it is a consequence of Theorem 3.2 (see also [15, 14]). In general, we see that, once the energies and the localization centers are scaled covariantly, the convergence to a Poisson process is true at any scale that is essentially larger than the localization width. The scaling we introduce is very natural; it is the one prescribed by the Heisenberg uncertainty principle: the more precision we require in the energy variable, the less we can afford in the space variable. In this respect, the energies behave like a homogeneous symbol of degree d. This is quite different from what one has in the case of the Laplace operator.

Let us note that for the process $\Xi^2_\Lambda(\xi, x; E_0, \ell)$ one can prove analogues of Theorem 2.2, 2.3, 2.4, 2.5 and 3.2.

§ 3.3. Non-covariant scaling joint (energy,center)-distribution

One can also study what happens when the energies and localization centers are not scaled covariantly. Consider two sequences of scales, say $\ell = (\ell_\Lambda)_\Lambda$ and $\ell' = (\ell'_\Lambda)_\Lambda$. Pick $E_0 \in I$ so that $\nu(E_0) > 0$. Consider the point process

$$\Xi^2_\Lambda(\xi, x; E_0, \ell, \ell') = \sum_{j=1}^{|\Lambda|} \delta_{\nu(E_0)(E_j(\omega,\Lambda)-E_0)\ell^d_\Lambda}(\xi) \otimes \delta_{x_j(\omega)/\ell'_\Lambda}(x).$$

Then, one proves

Theorem 3.4 ([6]). *Assume the sequences of increasing scales $\ell = (\ell_\Lambda)_\Lambda$ and $\ell' = (\ell'_\Lambda)_\Lambda$ satisfy* (3.1) *and* (3.2) *for respectively the constants c_ℓ and $c_{\ell'}$. Assume that*

$$\text{if } \ell_L = o(L) \text{ then } \frac{\ell_{\Lambda_{L+\ell_L}}}{\ell_{\Lambda_L}} \underset{|\Lambda|\to+\infty}{\to} 1 \text{ and } \frac{\ell'_{\Lambda_{L+\ell_L}}}{\ell'_{\Lambda_L}} \underset{|\Lambda|\to+\infty}{\to} 1. \tag{3.3}$$

Let J and C be bounded measurable sets respectively in $\mathbb{R}$ and $(-c_{\ell'}, c_{\ell'})^d \subset \mathbb{R}^d$. One has

1. *if, for some $\rho > 0$, one has $\frac{\ell_\Lambda}{\ell'_\Lambda} \le |\Lambda|^{-\rho}$, then ω-almost surely, for Λ sufficiently large,*
$$\int_{J\times C} \Xi^2_\Lambda(\xi, x; E_0, \ell, \ell')d\xi dx = 0.$$
2. *if, for some $\rho > 0$, one has $\frac{\ell_\Lambda}{\ell'_\Lambda} \ge |\Lambda|^{\rho}$, then ω-almost surely,*
$$\left(\frac{\ell_\Lambda}{\ell'_\Lambda}\right)^{-d} \int_{J\times C} \Xi^2_\Lambda(\xi, x; E_0, \ell, \ell')d\xi dx \underset{|\Lambda|\to+\infty}{\to} |J|\cdot|C|.$$

Theorem 3.4 proves that the local energy levels and the localization centers become uniformly distributed in large energy windows if one conditions the localization centers to a cube of much smaller side-length. On the other hand, for a typical sample, if one looks for eigenvalues in an energy interval much smaller than the correctly scaled one with localization centers in a cube, then, asymptotically, there are none.
Under assumption (3.1), if one replaces the polynomial growth or decay conditions on the ratio of scales by the condition that they tend to 0 or ∞, or if one omits condition (3.3), the results stays valid except for the fact that the convergence is not almost sure anymore but simply holds in some L^p norm.

§ 3.4. The level spacing statistics

Our goal is now to understand the level spacing statistics for eigenvalues near $E_0 \in I$. Pick I_Λ a compact interval containing E_0 such that its Lebesgue measure $|I_\Lambda|$ stays bounded.
First, let us note that, by the existence of the density of states and also Theorem 2.1, if $\nu(E_0) > 0$, the mean spacing between eigenvalues of $H_\omega(\Lambda)$ near E_0 is of size $\{\nu(E_0)|\Lambda|\}^{-1}$. Hence, to study the statistics of level spacings in I_Λ, I_Λ should contain asymptotically infinitely many energy levels of $H_{\omega,\Lambda}$. Let us study the number of these levels.

3.4.1. A large deviation principle for the eigenvalue counting function Define the random numbers

$$N(I_\Lambda,\omega,\Lambda) := \#\{j;\ E_j(\omega,\Lambda) \in I_\Lambda\}. \tag{3.4}$$

Write $I_\Lambda = [a_\Lambda, b_\Lambda]$. We show that $N(I_\Lambda,\omega,\Lambda)$ satisfies a large deviation principle

Theorem 3.5. *Fix $\rho' \in (0, 1/(1+2d))$. Then, there exists $\delta > 0$ small such that, if $(I_\Lambda)_\Lambda$ is a sequence of compact intervals in the localization region I satisfying*

$$\begin{aligned} &N(I_\Lambda)\,(\log|\Lambda|)^{1/\delta} \underset{|\Lambda|\to+\infty}{\to} 0, \quad N(I_\Lambda)\,|\Lambda|^{1-\nu} \underset{|\Lambda|\to+\infty}{\to} +\infty, \\ &N(I_\Lambda)\,|I_\Lambda|^{-1-\rho'} \underset{|\Lambda|\to+\infty}{\to} +\infty. \end{aligned} \tag{3.5}$$

then, for any $p > 0$, for $|\Lambda|$ sufficiently large (depending on ρ' and ν but not on the specific sequence $(I_\Lambda)_\Lambda$), one has

$$\mathbb{P}\left(|N(I_\Lambda,\Lambda,\omega) - N(I_\Lambda)|\Lambda|| \geq N(I_\Lambda)|\Lambda|(\log|\Lambda|)^{-\delta}\right) \leq |\Lambda|^{-p}. \tag{3.6}$$

The large deviation principle (3.6) is meaningful only if $N(I_\Lambda)|\Lambda| \to +\infty$; as N is Lipschitz continuous as a consequence of (W), this implies that

$$|\Lambda| \cdot |I_\Lambda| \to +\infty \quad \text{when} \quad |\Lambda| \to +\infty.$$

In this case, if $N(I_\Lambda)|\Lambda|$ satisfies (3.5), one has

$$\mathbb{E}(N(I_\Lambda,\omega,\Lambda)) = N(I_\Lambda)|\Lambda| + o\left(N(I_\Lambda)|\Lambda|\right).$$

So (3.6) also says

$$\mathbb{P}\left(|N(I_\Lambda,\omega,\Lambda) - \mathbb{E}(N(I_\Lambda,\omega,\Lambda))| \geq \varepsilon_\Lambda\,\mathbb{E}(N(I_\Lambda,\omega,\Lambda))\right) \leq e^{-\mathbb{E}(N(I_\Lambda,\omega,\Lambda))^\delta/\delta}.$$

Remark 3.1. *Notice that the condition* (3.5) *allows for I_Λ to be centered at a point E_0 where $\nu(E_0) = 0$ as long as the rate of vanishing of ν near E_0 is not too fast. Actually, all the results presented in this paper can be extended to this setting i.e. in Theorems 2.1, 2.2, 2.3, 2.4, 2.5, 3.2, 3.3, 3.4, 3.6, 4.1 and 5.2, one can replace the assumption $\nu(E_0) > 0$ by* (3.5) *(see [6]). Of course, for the results to remain valid, in the definition of the points processes or the empirical distributions, one has to replace the normalization constant $|\Lambda|\nu(E_0)$ by $|\Lambda|\,N(I_\Lambda)/|I_\Lambda|$.*

3.4.2. The level spacing statistics near a given energy Define $\mathcal{E}$ to be the set of energies E such that $\nu(E) = N'(E)$ exists and

$$\lim_{|x|+|y|\to 0} \frac{N(E+x) - N(E+y)}{x-y} = \nu(E).$$

The requirement on the points in $\mathcal{E}$ is somewhat stronger than asking for the simple existence of $\nu(E)$. Nevertheles, one proves that the set $\mathcal{E}$ is of full Lebesgue measure. It clearly contains the continuity points of $\nu(E)$.

Fix $E_0 \in \mathcal{E}$. If $I_\Lambda = [a_\Lambda, b_\Lambda]$ is such that $\sup_{I_\Lambda} |x| \underset{|\Lambda|\to+\infty}{\to} 0$, then

$$N_\Lambda(E_0 + I_\Lambda) = \nu(E_0)|I_\Lambda||\Lambda|(1 + o(1)) \quad \text{as} \quad |\Lambda| \to +\infty.$$

Consider the renormalized eigenvalue spacings: for $1 \leq j \leq N$,

$$\delta E_j(\omega, \Lambda) = |\Lambda|\, \nu(E_0)(E_{j+1}(\omega, \Lambda) - E_j(\omega, \Lambda)) \geq 0.$$

Define the empirical distribution of these spacings to be the random numbers, for $x \geq 0$

$$DLS(x; I_\Lambda, \omega, \Lambda) = \frac{\#\{j;\ E_j(\omega, \Lambda) \in I_\Lambda,\ \delta E_j(\omega, \Lambda) \geq x\}}{N(I_\Lambda, \omega, \Lambda)}.$$

We first study the level spacings distributions of the energies inside an interval that shrink to a point.
We prove

Theorem 3.6 ([6]). *Fix $E_0 \in \mathcal{E}$ such that $\nu(E_0) > 0$ and pick $(I_\Lambda)_\Lambda$ a sequence of intervals centered at E_0 such that $\sup_{I_\Lambda} |x| \underset{|\Lambda|\to+\infty}{\to} 0$.*
Assume that, for some $\delta > 0$, one has

$$|\Lambda|^{1-\delta} \cdot |I_\Lambda| \underset{|\Lambda|\to+\infty}{\to} +\infty \quad \text{and} \quad \text{if } \ell_L = o(L) \text{ then } \frac{|I_{\Lambda_{L+\ell_L}}|}{|I_{\Lambda_L}|} \underset{L\to+\infty}{\to} 1. \tag{3.7}$$

Then, with probability 1 , as $|\Lambda| \to +\infty$, $DLS(x; I_\Lambda, \omega, \Lambda)$ converges uniformly to the distribution $x \mapsto e^{-x}$, that is, with probability 1,

$$\sup_{x\geq 0} \left| DLS(x; I_\Lambda, \omega, \Lambda) - e^{-x} \right| \underset{|\Lambda|\to+\infty}{to} 0.$$

Hence, the rescaled level spacings behave as if the eigenvalues were i.i.d. uniformly distributed random variables (see [18] or section 7 of [16]). This distribution for the level spacings is the one predicted by physical heuristics in the localized regime ([7, 11, 13, 17]). It is also in accordance with Theorem 2.1. In [12, 3], the domains in energy where the statistics could be studied were much smaller than the ones considered in Theorem 3.6. Indeed, the energy interval was of order $|\Lambda|^{-1}$ whereas, here, it is assumed to tend to 0 but be large when compared to $|\Lambda|^{-1}$. In particular, in [12, 3], the intervals were not large enough to enable the computation of statistics of levels as not enough levels were involved: the intervals typically contained only finitely many intervals.

The first condition in (3.7) ensures that I_Λ contains sufficiently many eigenvalues of $H_\omega(\Lambda)$. The second condition in (3.7) is a regularity condition of the decay of $|I_\Lambda|$. If one omits either or both of these two conditions and only assumes that $|\Lambda| \cdot |I_\Lambda| \to +\infty$, one still gets convergence in probability of $DLS(x; I_\Lambda, \omega, \Lambda)$ to e^{-x} i.e.

$$\mathbb{P}\left(\sup_{x\geq 0}\left|DLS(x; I_\Lambda, \omega, \Lambda) - e^{-x}\right| \geq \varepsilon\right) \underset{|\Lambda|\to+\infty}{\to} 0.$$

3.4.3. The level spacing statistics on macroscopic energy intervals Theorem 3.6 seems optimal as the density of states at E_0 enters into the correct rescaling to obtain a universal result. Hence, the distribution of level spacings on larger intervals needs to take into account the variations of the density of states on these intervals. Indeed, on intervals of non vanishing size, we compute the asymptotic distribution of the level spacings when one omits the local density of states in the spacing and obtain

Theorem 3.7 ([6]). *Pick $J \subset I$ a compact interval such $\lambda \mapsto \nu(\lambda)$ be continuous on J and $N(J) := \int_J \nu(\lambda)d\lambda > 0$. Define the renormalized eigenvalue spacings, for $1 \leq j \leq N$,*

$$\delta_J E_j(\omega, \Lambda) = |\Lambda| N(J)(E_{j+1}(\omega, \Lambda) - E_j(\omega, \Lambda)) \geq 0$$

and the empirical distribution of these spacing to be the random numbers, for $x \geq 0$

$$DLS'(x; J, \omega, \Lambda) = \frac{\#\{j;\ E_j(\omega, \Lambda) \in J,\ \delta_J E_j(\omega, \Lambda) \geq x\}}{N(J, \omega, \Lambda)}.$$

Then, as $|\Lambda| \to +\infty$, with probability 1, $DLS'(x; J, \omega, \Lambda)$ converges uniformly to the distribution $x \mapsto g_{\nu,J}(x)$ where

$$g_{\nu,J}(x) = \int_J \mathrm{e}^{-\nu_J(\lambda)x}\nu_J(\lambda)d\lambda \text{ where } \nu_J = \frac{1}{N(J)}\nu. \tag{3.8}$$

We see that, in the large volume limit, the rescaled level spacings behave as if the eigenvalues were i.i.d. random variables distributed according to the density $\frac{1}{N(J)}\nu(\lambda)$ i.e. to the density of states normalized to be a probability measure on J (see section 7 of [16]).

In Theorem 3.7, we assumed the density of states to be continuous. This is known to hold in the large coupling limit if the density of the distribution of the random variables is sufficiently smooth (see [2]).

§ 4. The localization center spacing statistics

Pick E_0 as above. Inside the cube Λ, the number of centers that corresponds to energies in I_Λ is roughly equal to $\nu(E_0)|I_\Lambda|N$. Thus, if we assume that the localization centers are uniformly distributed as is suggested by Theorems 3.3 and 3.4, the reference mean spacing between localization centers is of size $(|\Lambda|/(\nu(E_0)|I_\Lambda||\Lambda|)^{1/d} =$

$(\nu(E_0)|I_\Lambda|)^{-1/d}$. This motivates the following definition.
Define the empirical distribution of center spacing to be the random number
(4.1)

$$DCS(s; I_\Lambda, \omega, \Lambda) = \frac{\#\left\{j;\ E_(\omega,\Lambda) \in I_\Lambda,\ \sqrt[d]{\nu(E_0)|I_\Lambda|}\cdot \min_{i\neq j} |x_j(\omega) - x_i(\omega)| \geq s\right\}}{N(I_\Lambda, \omega, \Lambda)}$$

where $N(I_\Lambda, \omega, \Lambda)$ is defined in (3.4).
We prove an analogue of Theorems 3.6, namely

Theorem 4.1 ([6]). *Pick $E_0 \in I$ such that $\nu(E_0) > 0$. Assume*

$$|I_\Lambda| = o\left(\frac{1}{\log^d |\Lambda|}\right).$$

Then, as $|\Lambda| \to +\infty$, in probability, $DCS(s; I_\Lambda, \omega, \Lambda)$ converges uniformly to the distribution $x \mapsto e^{-s^d}$, that is, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\left\{\omega;\ \sup_{s\geq 0}\left|DCS(s; I_\Lambda, \omega, \Lambda) - e^{-s^d}\right| \geq \varepsilon\right\}\right) \underset{\Lambda\nearrow\mathbb{R}^d}{\to} 0.$$

Of course, Theorem 3.7 also has an analogue for localization centers.

§ 5. Another point of view

In the present section, we want to adopt a different point of view on the spectral statistics. Instead of discussing the statistics of the eigenvalues of the random system restricted to some finite box in the large box limit, we will describe the spectral statistics of the infinite system in the localized phase. Let I be an interval in the region of complete localization. Then, it is well known ([9, 1, 5]) that, in this region, the following property holds

(Loc') there exists $\gamma > 0$ such that, with probability 1, if $E \in I \cap \sigma(H_\omega)$ and φ is a normalized eigenfunction associated to E then, for $x(E) \in \mathbb{Z}^d$, a maximum of $x \mapsto \|\varphi\|_x$, for some $C_\omega > 0$, one has, for $x \in \mathbb{R}^d$,

$$|\varphi(x)| \leq C_\omega(1 + |x(E)|^2)^{q/2} e^{-\gamma|x-x(E)|};$$

moreover, one has $\mathbb{E}(C_\omega) < +\infty$.

As above $x(E)$ is called *a center of localization* for the energy E or for the associated eigenfunction φ.
Without restriction on generality, we assume that $\sigma(H_\omega) \cap I = I$ ω-almost surely. Hence, any sub-interval of I contains infinitely many eigenvalues and to define statistics, we need a way to enumerate these eigenvalues. To do this, we use the localization centers; namely, we prove

Proposition 5.1 ([6]). *Fix $q > 2d$. Then, there exists $\gamma > 0$ such that, ω-almost surely, there exists $C_\omega > 1$ such that*

1. *if $x(E)$ and $x'(E)$ are two centers of localization for $E \in I$ then*
$$|x(E) - x'(E)| \le \gamma^{-2}(\log\langle x(E)\rangle + \log C_\omega)^{1/\xi}.$$
2. *for $L \ge 1$, pick $I_L \subset I$ such that $L^d N(I_L) \to +\infty$ (see Theorem **??**); if $N(I_L, L)$ denotes the number of eigenvalues of H_ω having a center of localization in Λ_L, then*
$$N(I_L, L) = N(I_L)\,|\Lambda_L|\,(1 + o(1)).$$

Point (1) is proved in [5] (see Corollary 3 and its proof). Point (2) is proved in [6].
For $L \ge 1$, pick $I_L \subset I$ such that $L^d\, N(I_L) \to +\infty$. In view of Proposition 5.1, we can consider the level spacings for the eigenvalues of H_ω having a localization center in Λ_L; indeed, for L large, there are only finitely many such eigenvalues, let us enumerate them as $E_1(\omega, L) \le E_2(\omega, L) \le \cdots \le E_{|\Lambda|}(\omega, L)$ where we repeat them according to multiplicity. Consider the renormalized eigenvalue spacings, for $1 \le j \le |\Lambda|$,

$$\delta E_j(\omega, L) = |\Lambda_L|\,(E_{j+1}(\omega, L) - E_j(\omega, L)) \ge 0.$$

Define the empirical distribution of these spacing to be the random numbers, for $x \ge 0$

$$DLS(x; I_L, \omega, L) = \frac{\#\{j;\ E_j(\omega, L) \in I_L,\ \delta E_j(\omega, L) \ge x\}}{N(I_L, L)}.$$

Then, we prove

Theorem 5.2 ([6]). *One has*

- *if $E_0 \in \mathcal{E} \cap I_L$ s.t. $\nu(E_0) > 0$ and $|I_L| \to 0$ and satisfies* (3.7)*, then, ω-almost surely, for $x \ge 0$*
$$\lim_{L\to+\infty} \sup_{x\ge 0} \left| DLS(x; I_L, \omega, L) - e^{-x/\nu(E_0)} \right| = 0;$$
- *if, for all L large, $I_L = J$ such that $\nu(J) > 0$ and ν is continuous on J then, ω-almost surely, one has*
$$\lim_{L\to+\infty} \sup_{x\ge 0} |DLS(x; I_L, \omega, L) - g_{\nu,J}(N(J)\,x)| = 0$$
where $g_{\nu,J}$ is defined in (3.8).

In the first part of Theorem 5.2, if (3.7) is not satisfied, then the convergence still holds in probability.

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