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Five-dimensional AGT Relation, $q$-$\mathcal{W}$ Algebra and Deformed $\beta$-ensemble

By

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Abstract

We define a $q$-deformation of the $\beta$-ensemble which satisfies $q$-$\mathcal{W}_N$ constraint. We also show a relation with the Nekrasov partition function of 5D $SU(N)$ gauge theory with $N_f = 2N$.

§1. Introduction

In Ref. [1], Alday Gaiotto and Tachikawa discovered remarkable relations between the 4D $\mathcal{N} = 2$ super conformal gauge theories and the 2D Liouville conformal field theories. Some explanations have been addressed from $\beta$-ensemble (generalized matrix model) [2, 3] in Ref. [4]–[7].

In the pure $SU(2)$ case, the AGT relation [8] between the instanton part of the partition functions of the gauge theory and correlation functions of the Virasoro algebra is extended naturally to 5D in Ref. [9] (see also [10]). The instanton counting [11]–[14] of the 5D gauge theory [15] can be viewed as a $q$-analog of 4D cases, [16]–[18] and there also exists a natural $q$-deformation of the Virasoro/$\mathcal{W}_N$ algebra. [19]–[22]

In this talk, we will study a 5D extension of the AGT relation with $N_f = 2N$ in terms of $\beta$-ensemble. The $A_{N-1}$ type quiver matrix model (the ITEP model) [23] was generalized as a $\beta$-ensemble [2] satisfying the $\mathcal{W}_N$ constraint by Ref. [3]. The partition function of the $A_{N-1}$ type $\beta$-ensemble is defined as the singular vector of the $\mathcal{W}_N$
algebra as follows[3]

\begin{equation}
Z_{N}^{c\ell} := \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_{a}} \frac{d_{\mathrm{Z}_{j}^{a}}}{2\pi i} \cdot \prod_{a=1}^{N-1} \Delta^{c\ell}(z^{a}) e^{W^{c\ell}(z^{a}, z^{a+1})}
\end{equation}

with \(z_{i}^{N} := 0\). Here \(\Delta^{c\ell}(z) := \prod_{i<j} (1 - z_{j}/z_{i})^{\beta} (z_{i}/z_{j} - 1)^{\beta}\) is the \(\beta\)-deformed kernel and

\begin{equation}
W^{c\ell}(z^{a}, z^{a+1}) := \sum_{i=1}^{r_{a}} \left\{ \beta \sum_{n>0} \frac{1}{n} \left( z_{i}^{a} \right)^{n} p_{n}^{(a)} - \beta \sum_{j=1}^{r_{a+1}} \log \left( 1 - z_{j}^{a+1}/z_{i}^{a} \right) - (s_{a} + 1) \log z_{i}^{a} \right\}
\end{equation}

is the Penner type potential. The partition function \(Z_{N}^{c\ell}\) is a function in coupling constants \(p_{n}^{(a)}\) and is specified by a set of integers \(r_{a}\) and \(s_{a}\) \((n \in \mathbb{N}\) and \(a = 1, 2, \cdots, N)\).

Since \(Z_{N}^{c\ell}\) is the singular vector, it satisfies the \(\mathcal{W}_{N}\) constraint \(\mathcal{W}_{c\ell n}^{a} Z_{N}^{c\ell} = 0 \ (n > 0)\) with the \(\mathcal{W}_{N}\) generators \(\mathcal{W}_{c\ell n}^{a}\), which Virasoro central charge is \(c = N - 1 - N(N^{2} - 1) \left( \sqrt{\beta} - 1/\sqrt{\beta} \right)^{2}\). Under the strategy of Ref. [3], we will introduce a \(q\)-deformed \(\beta\)-ensemble which automatically satisfies \(q\)-\(\mathcal{W}_{N}\) constraint. The partition function \(Z_{N}\) of the \(A_{N-1}\) type \(q\)-deformed \(\beta\)-ensemble will be defined as the singular vector of the \(q\)-\(\mathcal{W}_{N}\) algebra and is given by replacing \(\Delta^{c\ell}(z)\) and \(W^{c\ell}(z^{a}, z^{a+1})\) in (1.1) with

\begin{equation}
\Delta(z) := \prod_{i<j} (1 - z_{j}/z_{i}) \prod_{\ell \geq 0} \frac{1 - q^{\ell} p z_{j}/z_{i}}{1 - q^{\ell} t z_{j}/z_{i}} \cdot \prod_{i=1}^{r} z_{i}^{(r+1-2i)\beta},
\end{equation}

\begin{equation}
W(z^{a}, z^{a+1}) := \sum_{i=1}^{r_{a}} \left\{ \sum_{n>0} \frac{[\beta]_{q}^{n}}{n} \left( z_{i}^{a} \right)^{n} p_{n}^{(a)} + \sum_{j=1}^{r_{a+1}} \left( p_{j}^{\frac{1}{2}} z_{j}^{a+1}/z_{i}^{a} \right)^{n} \right\} - (s_{a} + 1) \log z_{i}^{a} \right\}.
\end{equation}

Here \([\beta]_{q} = (q^{\frac{\beta}{2}} - q^{-\frac{\beta}{2}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}}), \ t = q^{\beta}\) and \(p = q/t\). Then this satisfies the \(q\)-\(\mathcal{W}_{N}\) constraint with the \(q\)-\(\mathcal{W}_{N}\) generators defined in (3.12). If we specialize the mass parameters appropriately, the 5D Nekrasov partition function of \(SU(N)\) gauge theory with \(N_{f} = 2N\) reduces to the \(q\)-hypergeometric function. For \(N = 2\), we will show that if we specialize the coupling constants appropriately, \(Z_{2}\) also reduces to the \(q\)-hypergeometric function and coincides with the corresponding Nekrasov partition function.

This paper is organized as follows: In section 2, we start with recapitulating the result of the \(q\)-\(\mathcal{W}_{N}\) algebra and also define primary fields. In section 3, we introduce \(q\)-deformed \(\beta\)-ensemble which automatically satisfies \(q\)-\(\mathcal{W}_{N}\) constraint. Section 4 deals with the \(N = 2\) case. Finally in section 5, we explain a reduction of the 5D Nekrasov partition function to the \(q\)-hypergeometric function and show a coincidence with the partition function of our \(q\)-deformed \(\beta\)-ensemble. Appendix A contains a definition of the Macdonald polynomial and several useful formulas.
Notation. Let \([n]_p := (p^\frac{n}{2} - p^{-\frac{n}{2}})/(p^{\frac{1}{2}} - p^{-\frac{1}{2}})\). Parameters are \(q := e^{h/\sqrt{\beta}} = e^{g_s R}\), \(t := q^\beta = e^{-h(\sqrt{\beta} - 1/\sqrt{\beta})}\), \(u := t^\gamma\) and \(v := (q/t)^{\frac{1}{2}}\). We will use the same letter \(p\) also for the set of power sums \(p := (p_1, p_2, \cdots)\), but this appears only at \(P_{\lambda}(x[p])\) or \(Z_2(p)\). The integral \(\oint\frac{dz}{2\pi i z}f(z)\) denotes the constant term in \(f\).

§ 2. Quantum deformation of \(\mathcal{W}_N\) algebra

We start with recapitulating the results of the \(q-\mathcal{W}_N\) algebra [21, 22] and define primary fields.

§ 2.1. \(q-\mathcal{W}_N\) algebra

We use three kinds of basis for bosons. First we define fundamental bosons \(h_n^i\) and \(Q_h^i\) for \(i = 1, 2, \cdots, N\) and \(n \in \mathbb{Z}\) such that

\[
[h_n^i, h_m^j] = \frac{1}{n}(q^\frac{1}{2} - q^{-\frac{1}{2}})(t^\frac{1}{2} - t^{-\frac{1}{2}})\frac{[\delta_{ij} N - 1]}{[N]_p} p^{\frac{n}{2} \text{sgn}(j-i) \delta_{n+m,0}},
\]

\[
[h_n^i, Q_h^j] = (\delta_{ij} - \frac{1}{N}) \delta_{n,0}, \quad [Q_h^i, Q_h^j] = 0, \quad \sum_{i=1}^{N} p^{in} h_n^i = 0, \quad \sum_{i=1}^{N} Q_h^i = 0
\]

with \(q, t := q^\beta \in \mathbb{C}, p := q/t, [n]_p := (p^\frac{n}{2} - p^{-\frac{n}{2}})/(p^{\frac{1}{2}} - p^{-\frac{1}{2}})\) and \(\text{sgn}(i) := 1, 0 \text{ or } -1\) for \(i > 0, i = 0 \text{ or } i < 0\), respectively. Here \([A, B] := AB - BA\). This bosons correspond to the weights \(\vec{h}_i\) of the vector representation whose inner product is \((\vec{h}_i \cdot \vec{h}_j) = \delta_{ij} - 1/N\).

This algebra is invariant under the following involutions: \(\omega^2_\pm = 1\),

\[
\omega_+ : \quad \sqrt{\beta} \mapsto 1/\sqrt{\beta}, \quad (q, t) \mapsto (t, q), \quad h_n^i \mapsto h_{n}^{N-i+1}, \quad Q_h^i \mapsto Q_{h}^{N-i+1},
\]

\[
\omega_- : \quad \sqrt{\beta} \mapsto -\sqrt{\beta}, \quad (q, t) \mapsto (q^{-1}, t^{-1}), \quad h_n^i \mapsto h_{n}^{N-i+1}, \quad Q_h^i \mapsto Q_{h}^{N-i+1}
\]

We also use root type bosons \(\alpha_a^a := h_a^n - h_a^{a+1}\) and \(Q_a^a := Q_h^a - Q_h^{a+1}\) and weight type bosons \(\Lambda_a^a := \sum_{b=1}^{a} h_b^n (b-a-\frac{1}{2}) \) and \(Q_a^a := \sum_{b=1}^{a} Q_h^b\) for \(a = 1, 2, \cdots, N-1\).

Let us define fundamental vertices \(\Lambda_i(z)\) and \(q-\mathcal{W}_N\) generators \(W^i(z)\) for \(i = 1, 2, \cdots, N\) as follows:

\[
\Lambda_i(z) := \exp \left\{ \sum_{n \neq 0} h_n^i z^{-n} \right\}, \quad q \sqrt{\beta} h_0^i \frac{N+1}{2} - i
\]

\[
W^i(zp^{\frac{1-i}{2}}) := \sum_{1 \leq j_1 < \cdots < j_i \leq N} \Lambda_{j_1}(z) \Lambda_{j_2}(zp^{-1}) \cdots \Lambda_{j_i}(zp^{1-i})
\]

To obtain the \(q = 1\) limit, we need to change the normalization of bosons by

\[
h_n^i\text{old} = h_n^i\text{new} \sqrt{(q^\frac{n}{2} - q^{-\frac{n}{2}})(t^\frac{n}{2} - t^{-\frac{n}{2}})/n^2} = h_n^i\text{new} \sqrt{[\beta]_q (q^\frac{n}{2} - q^{-\frac{n}{2}})/n} \quad (n \neq 0)
\]

\[
h_0^i\text{old} = h_0^i\text{new} \text{ and } Q_h^i\text{old} = Q_h^i\text{new}\] unchanged. Letting \(q \rightarrow 1\) yields the four-dimensional case.
and $W^0(z) := 1$. Here $* : :$ stands for the usual bosonic normal ordering such that the bosons $h_n^i$ with non-negative mode $n \geq 0$ are in the right. These generators are obtained from the following quantum Miura transformation:

\begin{equation}
\sum_{i=0}^{N} (-1)^i W(zp^{\frac{1+i}{2}}) p^{(N-i)D_z} = : (p^{D_z} - \Lambda_1(z)) (p^{D_z} - \Lambda_2(zp^{-1})) \cdots (p^{D_z} - \Lambda_N(zp^{1-N})) : \\
\end{equation}

with $D_z := z \frac{\partial}{\partial z}$. Remark that $p^{D_z}$ is the $p$-shift operator such that $p^{D_z} f(z) = f(pz)$.

The mode $n$ generator $W_n^i$ is defined by $\sum_{n \in \mathbb{Z}} W_n^i z^{-n} := W^i(z)$.

By using root type bosons we define screening currents $S^\pm_a(z)$ as follows:

\begin{equation}
S^\pm_a(z) := \exp \left\{ \mp \sum_{n \in \mathbb{Z}} \frac{\alpha_n^a}{\xi^\pm_n - \xi^\mp_n} z^{-n} \right\} e^{\pm \sqrt{\beta^\pm} Q_n^a z^{\pm \sqrt{\beta^\pm} \alpha_n^a}}, \quad \xi^+ = q, \quad \xi^- = t,
\end{equation}

with $\alpha_n^a := h_n^a - h_{n+1}^a$ and $Q_n^a := Q_n^h - Q_{n+1}^h$. Note that the Langlands duality $\omega_- \omega_+ S^a_+(z) = \omega^\pm S^a_- (z)$. We denote the negative mode part of $S^a_\pm(z)$ by $(S^a_\pm(z))_- := \exp \left\{ \mp \sum_{n < 0} \frac{\alpha_n^a}{\xi^\pm_n - \xi^\mp_n} z^{-n} \right\}$. Then the screening charges defined by $\oint dz S^a_\pm(z)$ commute with any $q$-$\mathcal{W}_N$ generators

\begin{equation}
\oint dz S^a_\pm(z), W^b(w) = 0, \quad a, b = 1, 2, \ldots, N - 1.
\end{equation}

For parameters $u$ and $\gamma$ with $u := t^\gamma$, let us define the following vertex operators

\begin{equation}
V^a_u(z) := \exp \left\{ \sum_{n \neq 0} \frac{(u^{\frac{n}{2}} - u^{-\frac{n}{2}}) \Lambda_n^a}{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})} z^{-n} \right\} e^{-\gamma \sqrt{\beta} \Lambda_0^a},
\end{equation}

with $\Lambda_n^a := \sum_{b=1}^{a} h_n^b p^{(b-a-\frac{1}{2})n}$ and $Q_{\Lambda}^a := \sum_{b=1}^{a} Q_n^h$. They satisfy

\begin{equation}
g_{u,p}^{a,L}(x) \Lambda_i(z)V^a_u(w) - V^a_u(w) \Lambda_i(z) g_{u,p}^{a,R}(x) = (u^{-1} - 1) \sum_{b=1}^{a} \delta_{i,b} \delta(\frac{w}{zu^\frac{1}{2}}) : \Lambda_i(z)V^a_u(w) :,
\end{equation}

where $g_{u,p}^{a,L}(x)$ and $g_{u,p}^{a,R}(x)$ are inverse of the OPE factors,

\begin{align}
g_{u,p}^{a,L}(x) := & \frac{\Lambda_j(z)V^a_u(w)}{\Lambda_j(z)V^a_u(w)} = \exp \left\{ \sum_{n \in \mathbb{Z}} \frac{u^{\frac{n}{2}} - u^{-\frac{n}{2}}}{n} \frac{[a]_p^n}{[N]_p^n} \right\} u^{-\frac{a}{N}}, \quad a = 1, 2, \ldots, N - 1. \\
g_{u,p}^{a,R}(x) := & \frac{V^a_u(w) \Lambda_j(z)}{V^a_u(w) \Lambda_j(z)} = \exp \left\{ \sum_{n \in \mathbb{Z}} \frac{u^{\frac{n}{2}} - u^{-\frac{n}{2}}}{n} \frac{[a]_p^n}{[N]_p^n} \right\} u^{-\frac{a-N}{N}}.
\end{align}
for any \( j > a \).

### §2.2. Highest weight module of \( q\)-\( \mathcal{W}_N \) algebra

Next we refer to the representation of the \( q\)-\( \mathcal{W}_N \) algebra. Let \( \mathcal{F}_\alpha \) be the boson Fock space generated by the highest weight state \( |\alpha\rangle \) such that \( \alpha_n|\alpha\rangle = 0 \) for \( n \geq 0 \) and \( |\alpha\rangle := \exp\{\sum_{n=1}^{N-1} \alpha^n Q_n^\alpha\}|0\rangle \). Note that \( \alpha_0^\alpha|\alpha\rangle = \alpha^\alpha|\alpha\rangle \). The dual module \( \mathcal{F}_\alpha^* \) is generated by \( \langle \alpha| \) such that \( \langle \alpha | \alpha_n \rangle = 0 \) for \( n \geq 0 \) and \( \langle \alpha | := \langle 0 | \exp\{-\sum_{n=1}^{N-1} \alpha^n Q_n^\alpha\} \). The bilinear form \( \mathcal{F}_\alpha^* \otimes \mathcal{F}_\alpha \to \mathbb{C} \) is uniquely defined by \( \langle 0 | 0 \rangle = 1 \).

Let \( |\lambda\rangle \) be the highest weight vector of the \( q\)-\( \mathcal{W}_N \) algebra which satisfies \( W_n^a |\lambda\rangle = 0 \) for \( n > 0 \) and \( a = 1, 2, \cdots, N-1 \) and \( \mathcal{W}_0^a |\lambda\rangle = \lambda^a |\lambda\rangle \) with \( \lambda^a \in \mathbb{C} \). Let \( M_\lambda \) be the Verma module over the \( q\)-\( \mathcal{W}_N \) algebra generated by \( |\lambda\rangle \). The dual module \( M_\lambda^* \) is generated by \( |\lambda\rangle \) such that \( \langle \lambda |W_n^a = 0 \) for \( n < 0 \) and \( \langle \lambda |W_0^a = \lambda^a \langle \lambda | \). The bilinear form \( M_\lambda^* \otimes M_\lambda \to \mathbb{C} \) is uniquely defined by \( \langle \lambda |\lambda\rangle = 1 \). A singular vector \( |\chi\rangle \in M_\lambda \) is defined by \( W_0^a |\chi\rangle = 0 \) for \( n > 0 \) and \( \mathcal{W}_0^a |\chi\rangle = (\lambda^a + N^a) |\chi\rangle \) with \( N^a \in \mathbb{C} \).

The highest weight vector \( |\alpha\rangle \in F_\alpha \) of the boson algebra is also that of the \( q\)-\( \mathcal{W}_N \) algebra, i.e., \( \mathcal{W}_n^a |\alpha\rangle = 0 \) for \( n > 0 \) and \( a = 1, 2, \cdots, N-1 \). Note that \( \mathcal{W}_0^a |0\rangle = [N]_p^a |0\rangle \) with \([N]_p := (p^\frac{N}{2} - p^{-\frac{N}{2}})/(p^\frac{1}{2} - p^{-\frac{1}{2}}) \).

For a set of non-negative integers \( s_a \) and \( r_a \geq r_{a+1} \geq 0 \) with \( a = 1, \cdots, N-1 \), let

\[
\pm \alpha_{r,s}^\pm := (1 + r_a - r_{a-1}) \sqrt{\beta}^\pm - (1 + s_a) \sqrt{\beta}^\mp, \quad r_0 := 0, \tag{2.14}
\]

\[
\pm \tilde{\alpha}_{r,s}^\pm := (1 - r_a + r_{a+1}) \sqrt{\beta}^\pm - (1 + s_a) \sqrt{\beta}^\mp, \quad r_N := 0. \tag{2.15}
\]

The singular vectors \( |\chi_{r,s}^\pm\rangle \in \mathcal{F}_{\alpha_{r,s}^\pm} \) are realized by the screening currents as follows:

\[
|\chi_{r,s}^\pm\rangle = \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{d_{\mathrm{z}_j^a}}{2 \pi i} S_{\pm}^{1}(z_1^a) \cdots S_{\pm}^{r_a}(\alpha_{1}^{a}) \cdots S_{\pm}^{N-1}(z_{N-1}^{a}) |\alpha_{r,s}^\pm\rangle \tag{2.16}
\]

with \( z^N := 0, \zbar := 1/z, \xi_+ := q \) and \( \xi_- := t \). Note that \( \omega_- \omega_+ |\chi_{r,s}^+\rangle = |\chi_{r,s}^-\rangle \). Here

\[
\Pi(z, w) := \Pi(z, w; q, t) := \prod_{i,j} \exp \left\{ \sum_{n>0} \frac{[\beta]_q^n}{n} p^{-\frac{n}{2}} z_i^n w_j^n \right\} = \prod_{i,j} \prod_{\ell \geq 0} \frac{1 - q^\ell z_i w_j}{1 - q^\ell z_i}, \tag{2.17}
\]

\[
\Delta(z) := \Delta(z; q, t) := \prod_{i<j} \exp \left\{ - \sum_{n>0} \frac{[2]_q^n}{n} \frac{z_i^n z_j^n}{z_i^n z_i^n} \right\} \prod_{i=1}^{r} z_i^{(r+1-2i)\beta} \tag{2.18}
\]

with \( \beta := \log t / \log q \). Note that \( \Delta(cz) = \Delta(z) \).
§ 3. Quantum deformation of $\beta$-ensemble

Note that the singular vector in (2.16) is naturally mapped to the Macdonald polynomial [24] defined in the appendix A. [22] As a generalization of this map one can define, under the strategy of Ref. [3], a quantum deformation of the generalized matrix model, i.e., $q$-deformed $\beta$-ensemble.

§ 3.1. $q$-deformed $\beta$-ensemble

With a new parameters $p^{(a)} := (p_1^{(a)}, p_2^{(a)}, \ldots)$ let us define the following vertex operator

\begin{equation}
V_N := \prod_{a=1}^{N-1} \exp \left\{ \sum_{n>0} \frac{\Lambda_n^a}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} p_n^{(a)} \right\},
\end{equation}

with $\Lambda_n^a := \sum_{b=1}^{a} h_n^b (b-a-\frac{1}{2})n$ and $Q_n^a := \sum_{b=1}^{a} Q_h^b$. Note that $[\Lambda_n^a, \Lambda_m^b] = 0$ for $n, m > 0$. Then $\langle \alpha | V_N \rangle$ defines the isomorphism between the boson algebras $\langle h_{n/n \in \mathbb{Z}}^{a/1 \leq a < N} \rangle$ and $\langle p_n^{(a)}, \alpha^a, \frac{\partial}{\partial p_n^{(a)}} \rangle_{n \in \mathbb{N}}^{1 \leq a < N}$ by

\begin{equation}
\langle \alpha | V_N h_{-n}^i \rangle = \frac{t^{\frac{n}{2}} - t - \frac{n}{2}}{n} \sum_{b=1}^{N-1} A^{i,b}(p^{-n}) p_n^{(b)} \langle \alpha | V_N \rangle,
\end{equation}

\begin{equation}
\langle \alpha | V_N h_n^i \rangle = (q^{\frac{n}{2}} - q^{-\frac{n}{2}}) \sum_{b=1}^{N-1} B^{i,b}(p^n) \frac{\partial}{\partial p_n^{(b)}} \langle \alpha | V_N \rangle
\end{equation}

for $n > 0$ and $\langle \alpha | V_N h_0^i \rangle = h^i \langle \alpha | V_N \rangle$ with $h^i = \left[ \sum_{b=i}^{N-1} - \sum_{b=1}^{N-1} b/N \right] \alpha^b$. Here

\begin{equation}
A^{i,b}(p) := \frac{[N \theta(i \leq b) - i]}{[N]_p} p^{\frac{1}{2} (b-N \theta(i>b))},
\end{equation}

\begin{equation}
B^{i,b}(p) := p^{\frac{1}{2}} \delta_{i,b} - p^{-\frac{1}{2}} \delta_{i-1,b}
\end{equation}

with $\theta(P) := 1$ or 0 if the proposition $P$ is true or false, respectively.

The vector $|S_{r,s}^+\rangle := \prod_{a=1}^{N-1} \prod_{k=1}^{r_a} (S_+^{a}(z_k^a))_{-} \cdot |\alpha_{r,s}^+\rangle$ in (2.16) also defines another linear map from $\langle h_{n/n \in \mathbb{N}}^{a/1 \leq a < N} \rangle$ to $\langle \sum_{k=1}^{r_a} (z_k^a)^n \rangle_{n \in \mathbb{N}}^{1 \leq a < N}$ by

\begin{equation}
h_n^i |S_{r,s}^+\rangle = |S_{r,s}^+\rangle \frac{t^{\frac{n}{2}} - t - \frac{n}{2}}{n} \sum_{b=1}^{N-1} B^{i,b}(p^n) \sum_{k=1}^{r_b} (z_k^b)^n, \quad n > 0.
\end{equation}

Let us define the following partition function

**Definition 3.1.** Let $Z_N := Z_N \left( \{p^{(a)}\}_{a=1}^{N-1}\right) := \langle \alpha_{r,s}^+ | V_N | \chi_{r,s}^+ \rangle$. 

Then by (2.16), (2.8) and (3.3), we have

\[ Z_N = \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \langle \alpha_{r,s}^+ | V_N S_+^1(z_1^1) \cdots S_+^{N-1}(z_1^{N-1}) | \alpha_{r,s}^+ \rangle \]

\[ = \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \exp \left\{ \sum_{n>0} \frac{[\beta]_q^n}{n} (z_j^a)^n p_n^{(a)} \right\} \Delta(z^a) \prod_{a=1}^{N-1} \Delta(z^a) e^{W(z^a, z^{a+1})} \]

with

\[ W(z^a, z^{a+1}) := \sum_{i=1}^{r_a} \left\{ \sum_{n>0} \frac{[\beta]_q^n}{n} \left( z_i^a \right)^n p_n^{(a)} + \sum_{j=1}^{r_a+1} \left( p_{\frac{j-1}{2} z_i^{a+1}} - \frac{z_i^a}{z_i^a} \right)^n \right\} (s_a + 1) \log z_i^a \]

Here \( z^N := 0 \). This \( Z_N \) is regarded as a \( q \)-deformation of the partition function of the generalized matrix model\([3]\) i.e., \( \beta \)-ensemble. One can define other type of partition functions by acting involutions (2.3), (2.4) and (A.8).

We can calculate this integral by using the Macdonald polynomials \( P_\lambda(x) \) with the Young diagram \( \lambda \), their fusion coefficient \( f_{\lambda, \mu}^{\nu} \) and the inner products \( \langle *, * \rangle \) and \( \langle *, * \rangle_r \) defined in the appendix A.

**Proposition 3.2.**

\[ Z_N = \prod_{a=1}^{N-1} \sum_{\lambda_a, \mu_a, \nu_a} f_{\mu_a+(\nu_a)}^{\mu_a+(s_a^0)} P_{\mu_a+(s_a^0)}(z^a) P_{\nu_a}(x[p]) \frac{r_a! \langle \mu_a + (s_a^0) \rangle_{r_a}}{\langle \mu_a \rangle} \]

with \( \langle 0 \rangle := 1 \). Here \( \lambda_a, \mu_a \) and \( \nu_a \) are Young diagrams such that \( \lambda_a \geq \lambda_a, i+1 \), and so on. \( P_\lambda(x[p]) \) denotes the Macdonald function in power sums \( p := (p_1, p_2, \cdots) \).

One can show that (3.8) is summed over \((N-2) + (N-3)\) Young diagrams for \( N \geq 3 \).

For any function \( \mathcal{O} \) in \( z_j^a \)'s, the correlation function with respect to \( \mathcal{O} \) is defined by

\[ \langle \langle \mathcal{O} \rangle \rangle := \frac{1}{Z_N} \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot \mathcal{O} \prod_{a=1}^{N-1} \Delta(z^a) e^{W(z^a, z^{a+1})} \]

The effective action \( S_{\text{eff}} \) defined by \( Z_N = \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot e^{S_{\text{eff}}} \) is now

\[ S_{\text{eff}} = \sum_{a=1}^{N-1} W(z^a, z^{a+1}) - \sum_{n>0} [2]_p^n \frac{[\beta]_q^n}{n} \sum_{a=1}^{N-1} \sum_{i<j} \left( \frac{z_j^a}{z_i^a} \right)^n + \beta \sum_{a=1}^{N-1} \sum_{i=1}^{r_a} (r_a + 1 - 2i) \log z_i^a. \]
§ 3.2. $q$-$\mathcal W_N$ constraint, Loop equation and quantum spectral curve

Next let us define $\hat \Lambda_i(z)$ and $\mathcal W^i(z)$ by the isomorphism (3.3) as follows:

\begin{align}
\hat \Lambda_i(z)\langle \alpha_{r,s}|V_N &:= \langle \alpha_{r,s}|V_N \Lambda_i(z), \\
\mathcal W^i(z)\langle \alpha_{r,s}|V_N &:= \langle \alpha_{r,s}|V_N W^i(z)
\end{align}

and $\sum_{n \in \mathbb Z} \mathcal W^i_n z^{-n} := \mathcal W^i(z)$, which are the power sum realization of fundamental vertices $\Lambda_i(z)$ and $q$-$\mathcal W_N$ generators $W^i(z)$, respectively. Then the highest weight condition for the singular vector $W^{a}_n |\chi\rangle = 0$ for $n > 0$ is equivalent to the following $q$-$\mathcal W_N$ constraint:

**Theorem 3.3.**

\begin{equation}
W^a_n Z_N = 0, \quad n > 0.
\end{equation}

Let us define $\tilde \Lambda_i(z)$ and $\tilde \mathcal W^i(z)$ by linear maps (3.3) and (3.6) as follows:

\begin{align}
\langle \alpha_{r,s}^+|V_N S_{r,s}^+ \tilde \Lambda_i(z) &:= \langle \alpha_{r,s}^+|V_N \Lambda_i(z)|S_{r,s}^+), \\
\langle \alpha_{r,s}^+|V_N S_{r,s}^+ \tilde \mathcal W^i(z) &:= \langle \alpha_{r,s}^+|V_N W^i(z)|S_{r,s}^+
\end{align}

and $\sum_{n \in \mathbb Z} \tilde \mathcal W^i_n z^{-n} := \tilde \mathcal W^i(z)$. Hence

\begin{equation}
\left\langle \tilde \mathcal W^i(z) \right\rangle = \frac{1}{Z_N} \langle \alpha_{r,s}^+|V_N W^i(z)|\chi_{r,s}^+\rangle.
\end{equation}

Therefore the highest weight condition for the singular vector $W^{a}_n |\chi\rangle = 0$ for $n > 0$ is equivalent to the following loop equation:

**Theorem 3.4.**

\begin{equation}
\left\langle \tilde \mathcal W^a_n \right\rangle = 0, \quad n > 0.
\end{equation}

The quantum spectral curve should be

\begin{equation}
\left\langle \left(\frac{D_z - \hat \Lambda_1(z)}{D_z - \hat \Lambda_2(zp^{-1})} \cdots \left(\frac{D_z - \hat \Lambda_N(zp^{1-N}}{D_z - \hat \Lambda_1(z))\right) \right) \right\rangle = 0
\end{equation}

which regularity in $z$ is guaranteed by the loop equation (3.17).

Let $(g, t) = (e^{R\epsilon_2}, e^{-R\epsilon_1}) = (e^{g_s R}, e^{g_s \beta R})$ with the radius $R \in \mathbb R$ of the 5th dimensional circle $S^1$. Let us rescale the variables as $\bar p^{(a)}_n := g_s p^{(a)}_n$, $\bar r_a := g_s r_a$ and $\bar s_a := g_s s_a$. Under the limit $g_s \to 0$ and $r_a s_a \to \infty$ with fixed $\bar r_a$ and $\bar s_a$, the sift operator $p^D_z$ tends to a commutative variable and the quantum spectral curve reduces to the usual one.
§ 4. N = 2 case

Here we give an example when N = 2, i.e., the q-deformed Virasoro case. The partition function $Z_2$ is now

$$Z_2(p) = \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} z_j^{-s} \exp \left\{ \sum_{n>0} \frac{[\beta]_q^n}{n} z_j^n p_n \right\} \Delta(z) = p^{\frac{r^2}{2}} \left( \frac{s^r}{s^r} \right) P_{(s^r)}(x[p]).$$

Then we have

**Proposition 4.1.** The partition function $Z_2(p)$ substituting $p_n = \sum_i x_i^n + \frac{1-u^n}{1-t^n} y^n$ and $\frac{1-t^n}{1-q^n} p_n = (-1)^{n-1} \left( \sum_i x_i^n + \frac{1-u^n}{1-t^n} y^n \right)$ are

$$\frac{Z_2\left( \sum_i x_i + \frac{1-u}{1-t} y \right)}{Z_2\left( \frac{1-u}{1-t} y \right)} = 2 \varphi_1^{(q,t)} \left[ \frac{q^{-s}, t^r}{q^{1-s} t^{r-1}/u}, \frac{qx}{uy} \right],$$

$$\omega_{q,t} \frac{Z_2\left( \sum_i x_i + \frac{1-u}{1-t} y \right)}{Z_2\left( \frac{1-u}{1-t} y \right)} = 2 \varphi_1^{(t,q)} \left[ \frac{t^{-r}, q^s}{t^{1-r} q^{s-1}/u}, \frac{tx}{uy} \right].$$

with $\omega_{q,t}$ in (A.8). Here $2 \varphi_1^{(q,t)} \left[ \frac{a, b}{c}; x \right]$ is the multivariate q-hypergeometric function [28]

$$2 \varphi_1^{(q,t)} \left[ \frac{a, b}{c}; x \right] := \sum_{\lambda: t^{\lambda} \leq M} P_{\lambda}(x) \prod_{i, j \in \lambda} \frac{(t^{i-1} - aq^{j-1})(t^{i-1} - bq^{j-1})}{(t^{i-1} - cq^{j-1})(1 - q^{\lambda_i-j+1} t^{\lambda_j'-i})}.$$  

Since $P_{\lambda}(x; q, t) = P_{\lambda}(x; q^{-1}, t^{-1}), 2 \varphi_1^{(q,t)} \left[ \frac{a, b}{c}; x \right]$ satisfies

$$2 \varphi_1^{(q,t)} \left[ \frac{a, b}{c}; x \right] = 2 \varphi_1^{(q^{-1}, t^{-1})} \left[ \frac{a^{-1}, b^{-1}}{c^{-1}}; \frac{ab}{qc} x \right].$$

When $M = \infty$,

$$\omega_{q,t} 2 \varphi_1^{(q,t)} \left[ \frac{a, b}{c}; x \right] = 2 \varphi_1^{(t,q)} \left[ \frac{a, b}{c}; x \right], \quad M = \infty.$$  

When $M = 1, 2 \varphi_1^{(q,t)} \left[ \frac{a, b}{c}; x \right]$ reduces to the usual q-hypergeometric function

$$2 \varphi_1^{(q,t)} \left[ \frac{a, b}{c}; x \right] := 2 \varphi_1 \left[ \frac{a, b}{c}; q, x \right] := \sum_{n \geq 0} x^n \prod_{t=0}^{n-1} \frac{(1 - aq^t)(1 - bq^t)}{(1 - cq^t)(1 - q^{t+1})}, \quad M = 1.$$  

In the next section we will show a relation between our $Z_2\left( \sum_i x_i^n + \frac{1-u^n}{1-t^n} y^n \right)$ and the 5-dimensional $SU(2)$ Nekrasov partition function.
§5. Five-dimensional Nekrasov partition function

Let $Q = (Q_1, \cdots, Q_N)$ with $\prod_{i=1}^{N} Q_i = 1$ and $Q^\pm = (Q_1^\pm, \cdots, Q_N^\pm)$ be sets of complex parameters. The instanton part of the five-dimensional $SU(N)$ Nekrasov partition function with $N_f = 2N$ fundamental matters\footnote{The parameters $(q, t)$ are related with those $(\epsilon_1, \epsilon_2)$ of the $\Omega$ background through $(q, t) = (e^{R\epsilon_2}, e^{-R\epsilon_1})$ where $R$ is the radius of the 5th dimensional circle. The parameter $Q$ is related with the vacuum expectation value $a$ of the scalar fields in the vector multiplets and the mass $m$ of the fundamental matter as $Q_i = q^{a_i}, Q_i^+ = q^{-m_i}$ and $Q_i^- = q^{-m_{N+i}}$.} is written by a sum over $N$ Young diagrams $\lambda_i$ $(i = 1, 2, \cdots, N)$ as follows (double-sign corresponds):\cite{25, 18}

\begin{equation}
Z^{\text{inst}}(Q) = \sum_{\{\lambda_i\}} \prod_{i,j} \frac{N_{\lambda_i\ast}(vQ_i/Q_j^\pm) N_{\lambda_i\ast}(vQ_j^\mp/Q_i)}{N_{\lambda_i\lambda_j}(Q_i/Q_j)} \cdot \prod_{i} \left( \frac{\Lambda_{\alpha}^\pm}{v^N} \right)^{|\lambda_i|}
\end{equation}

with $v := (q/t)^{1/2}$, $N_{\lambda\mu}(Q) := N_{\lambda\mu}(Q; q, t)$, $\Lambda_{\alpha}^\pm := \Lambda^{2N} \prod_{j=1}^{N} \left( \frac{Q_j^\pm}{Q_j^\mp} \right)^{1/2}$ and

\begin{equation}
N_{\lambda\mu}(Q; q, t) := \prod_{(i,j)\in \lambda} \left( 1 - Q q^\lambda_{i-j} t^{\mu_j-i+1} \right) \prod_{(i,j)\in \mu} \left( 1 - Q q^{-\mu_{i+j-1}} t^{-\lambda_j+i} \right) = \prod_{(i,j)\in \mu} \left( 1 - Q q^\lambda_{i-j} t^{\mu_j-i+1} \right) \prod_{(i,j)\in \lambda} \left( 1 - Q q^{-\mu_{i+j-1}} t^{-\lambda_j+i} \right).
\end{equation}

Here $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a Young diagram such that $\lambda_i \geq \lambda_{i+1}$. $\lambda'$ is its conjugate Young diagram and $|\lambda| = \sum_{i} \lambda_i$. $Z^{\text{inst}}(Q; Q^+; Q^-)$ is symmetric in masses $Q^\pm_{-j}$’s. Note that $N_{\lambda\mu}(Q; q, t)$ satisfies

\begin{equation}
N_{\lambda\mu}(vQ; q, t) = N_{\mu\lambda}(Q/v; q^{-1}, t^{-1}) = N_{\mu'\lambda'}(Q/v; v, q),
\end{equation}

\begin{equation}
N_{\lambda\ast}(vQ)N_{\lambda}(vQ') = N_{\lambda}(v/Q)N_{\lambda\ast}(v/Q')(Q/Q')^{|\lambda|}.
\end{equation}

There exists $Q$ such that $N_{\lambda\ast}(Q)$ vanishes except for $\lambda = (0), (n)$ or $(1^n)$. Hence one can adjust $N$ out of $N_f = 2N$ parameters $Q^\pm_i$’s so that (5.1) reduces to all $\lambda_i = (0)$ but a $\lambda_j = (n)$ or $(1^n)$ with $n \in \mathbb{Z}_{\geq 0}$ same as Ref. [26]. For example, if $(Q_1, \cdots, Q_{N-1}, Q_N) = (Q_1^\pm, \cdots, Q_N^\pm, tQ_N^\pm)/v$ with $\prod_{i=1}^{N} Q^\pm = v^N/t$ then the right hand side of (5.1) is summed over only $(\lambda_1, \cdots, \lambda_{N-1}, \lambda_N) = ((0), \cdots, (0), (n))$ with $n \in \mathbb{Z}_{\geq 0}$. On the other hand, if $(Q_1, \cdots, Q_{N-1}, Q_N) = (Q_1^\pm, \cdots, Q_N^\pm, Q_N^+/q)/v$ with $\prod_{i=1}^{N} Q^\pm = qv^N$ then only $(\lambda_1, \cdots, \lambda_{N-1}, \lambda_N) = ((0), \cdots, (0), (1^n))$ contributes. Therefore we obtain
Proposition 5.1.

\begin{align*}
\text{(5.5)} & \quad Z^\text{inst}(Q_1^\pm/v, \ldots, Q_{N-1}^\pm/v, tQ_N^\pm/v) = N \varphi_{N-1} \\
& \quad \quad = N \varphi_{N-1} \left[ \begin{array}{c}
\frac{Q_1^\pm}{vQ_N}, \ldots, \frac{Q_{N-1}^\pm}{vQ_N} \\
\frac{tQ_1}{qQ_N}, \ldots, \frac{tQ_{N-1}}{qQ_N}
\end{array} \right] ; q^{-1}, \frac{\Lambda_N^\pm}{vN},
\end{align*}

\begin{align*}
\text{(5.6)} & \quad Z^\text{inst}(Q_1^\pm/v, \ldots, Q_{N-1}^\pm/v, tQ_N^\pm/qv) = N \varphi_{N-1} \\
& \quad \quad = N \varphi_{N-1} \left[ \begin{array}{c}
\frac{Q_1^\pm}{vQ_N}, \ldots, \frac{Q_{N-1}^\pm}{vQ_N} \\
\frac{tQ_1}{qQ_N}, \ldots, \frac{tQ_{N-1}}{qQ_N}
\end{array} \right] ; q, vN \Lambda_N^\pm,
\end{align*}

with $\prod_{i=1}^{N} Q_i^\pm = v^N/t$ for (5.5) and $\prod_{i=1}^{N} Q_i^\pm = qv^N$ for (5.6) and

\begin{align*}
\text{(5.7)} & \quad r\varphi_s \left[ a_1, \ldots, a_r \mid b_1, \ldots, b_s ; q, x \right] := \sum_{n \geq 0} x^n \prod_{\ell=0}^{n-1} \frac{(-q^{\ell})^{s+1-r} \prod_{i=1}^{r}(1-q^{\ell}a_i)}{(1-q^{\ell+1}) \prod_{i=1}^{s}(1-q^{\ell}b_i)}.
\end{align*}

Note that

\begin{align*}
\text{(5.8)} & \quad r\varphi_{r-1} \left[ a_1, \ldots, a_r \mid b_1, \ldots, b_{r-1} ; q, x \right] = r\varphi_{r-1} \left[ a_1^{-1}, \ldots, a_r^{-1} \mid b_1^{-1}, \ldots, b_{r-1}^{-1} ; q^{-1}, \tilde{x} \right],
\end{align*}

with $\tilde{x} := \frac{x \prod_{i=1}^{r} a_i}{q \prod_{i=1}^{r-1} b_i}$.

When $N = 2$, $Z^\text{inst}$ coincides with the $M = 1$ case of the partition function $Z_2$ of the $q$-deformed $\beta$-ensemble (4.2) similar to Ref. [6]

\begin{align*}
\text{(5.9)} & \quad Z^\text{inst}(Q_1^\pm/v, tQ_2^\pm/v) = 2\varphi_1 \left[ \frac{Q_2^\pm}{Q_1^\pm} ; q, v^2 \Lambda_2^\mp \right] = \frac{Z_2 \left( x + \frac{1-u}{1-t} y \right)}{Z_2 \left( \frac{1-u}{1-t} y \right)},
\end{align*}

\begin{align*}
\text{(5.10)} & \quad Z^\text{inst}(Q_1^\pm/v, Q_2^\pm/qv) = 2\varphi_1 \left[ \frac{Q_2^\pm}{Q_1^\pm} ; t, v \right] = \omega_{q,t} \frac{Z_2 \left( \sum_{i} x_i + \frac{1-u}{1-t} y \right)}{Z_2 \left( \frac{1-u}{1-t} y \right)}
\end{align*}

with

\begin{align*}
\text{(5.11)} & \quad Q_1^\pm Q_2^\pm = \frac{q}{t^2}, \quad q^s = \frac{Q_1 Q_1^\mp}{v}, \quad t^{-r} = \frac{Q_1 Q_2^\mp}{v}, \quad u^{-1} = \frac{tQ_1^\mp Q_2^\mp}{q}, \quad \frac{q x}{y} = Q_1^\mp Q_2^\mp \Lambda_2^\mp
\end{align*}

for (5.9) and

\begin{align*}
\text{(5.12)} & \quad Q_1^\pm Q_2^\pm = \frac{q^2}{t}, \quad q^s = \frac{Q_1 Q_1^\mp}{v}, \quad t^{-r} = \frac{Q_1 Q_2^\mp}{v}, \quad u = \frac{tQ_1^\mp Q_2^\mp}{q}, \quad \frac{y}{t x} = \frac{Q_1^\mp Q_2^\mp}{\Lambda_2^\mp}
\end{align*}

for (5.10). In the $SU(N)$ case, the Nekrasov partition function (5.5) may coincide with our partition function $Z_N$ by using the formulas (A.11) and the Cor. 1.6 in Ref. [27].
§ A. Macdonald polynomial

Here we recapitulate basic properties of the Macdonald polynomial.\[24\] Let $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_r)$ with $\lambda_i \geq \lambda_{i+1} \geq 0$ be a Young diagram. $\lambda'$ is its conjugate. For any $\lambda$ with $\lambda_1 \leq s$, $|\lambda| := \sum_{i=1}^{r} \lambda_i$ with $\lambda_i \geq \lambda_{i+1} \geq 0$ be a Young diagram. $\lambda'$ is its conjugate. For any $\lambda$ with $\lambda_1 \leq s$, $|\lambda| := \sum_{i=1}^{r} \lambda_i$. Let $x := (x_1, \cdots, x_r)$ and $p := (p_1, p_2, \cdots)$ with the power sum $p_n := p_n(x) := \sum_{i=1}^{r} x_i^n$. For any symmetric function $f$ in $x$ with $r = \infty$, $f(x[p])$ stands for the function $f$ expressed in the power sums $p$.

The Macdonald polynomials $P_\lambda(x) := P_\lambda(x; q, t)$ are degree $|\lambda|$ homogeneous symmetric polynomials in $x$ defined as eigenfunctions of the Macdonald operator $H$ as follows:

(A.1) \[ H P_\lambda(x) = \varepsilon_\lambda P_\lambda(x), \]

(A.2) \[ H := \sum_{i=1}^{r} \prod_{j(\neq i)} \frac{tx_i - x_j}{x_i - x_j} \cdot q^{D_{x_i}}, \quad \varepsilon_\lambda := \sum_{i=1}^{r} q^{\lambda_i} t^{r-i} \]

with a normalization condition $P_\lambda(x) = x_{1}^{\lambda_{1}} x_{2}^{\lambda_{2}} \cdots x_{r}^{\lambda_{r}} + \cdots$. Where $q^{D_x}$ with $D_x := x \frac{\partial}{\partial x}$ is the $q$-shift operator such that $q^{D_x} f(x) = f(qx)$. Note that $P_\lambda(x) := P_{(0)}(x) = 1$.

Two kinds of inner products are known in which the Macdonald polynomials are orthogonal each other. For any symmetric functions $f$ and $g$ in $x$, let us define inner product $\langle *, * \rangle$ and another one $\langle *, * \rangle_{r}$ as follows: \[3\]

(A.3) \[ \langle f, g \rangle := \oint \prod_{n>0} \frac{dp_n}{2\pi i p_n} \cdot f(x[p^*]) g(x[p]), \quad p_n^* := n \frac{1-q^n}{1-t^n} \frac{\partial}{\partial p_n}, \]

(A.4) \[ \langle f, g \rangle_{r} := \frac{1}{r!} \oint \prod_{j=1}^{r} \frac{dx_j}{2\pi i x_j} \cdot \Delta(x) f(\overline{x}) g(x), \quad \overline{x_j} := \frac{1}{x_j} \]

with $\Delta(x)$ in (2.18). Here we must treat the power sums $p_n$ as formally independent variables, i.e., $\frac{\partial}{\partial p_n} p_m = \delta_{n,m}$ for all $n, m > 0$. The inner products of Macdonald polynomials are given by

(A.5) \[ \langle P_\lambda, P_\mu \rangle = \delta_{\lambda, \mu} \langle \lambda \rangle, \quad \langle \lambda \rangle := \prod_{(i,j) \in \lambda} \frac{1 - q^{\lambda_i-j+1} t^{\lambda_j'-i}}{1 - q^{\lambda_i-j} t^{\lambda_j'-i+1}}, \]

(A.6) \[ \langle P_\lambda, P_\mu \rangle_{r} = \delta_{\lambda, \mu} \langle \lambda \rangle_{r}. \]

Let us denote by $f \left( x \left[ \frac{1-u}{1-t} \right] \right)$ the function $f(x[p])$ in the specialization $p_n := (1-u^n)/(1-t^n)$ with $u \in \mathbb{C}$, then [24]

(A.7) \[ P_\lambda \left( x \left[ \frac{1-u}{1-t} \right] \right) = \prod_{(i,j) \in \lambda} \frac{t^{i-1} - u q^{j-1}}{1 - q^{\lambda_i-j} t^{\lambda_j'-i+1}}. \]

\[3\]The usual another inner product $\langle *, * \rangle_r'$ is defined with a different kernel $\Delta'(x) := \prod_{i,j} \exp \left\{ - \sum_{n>0} (1-t^n)/(1-q^n) (x_j^n/x_i^n)/n \right\} = \prod_{i,j} \prod_{\ell \geq 0} (1-q^\ell x_j/x_i)/(1-tq^\ell x_j/x_i)$ ($|q| < 1$). Note that $C(x) := \Delta(x)/\Delta'(x)$ is a pseudo-constant, i.e., $q^{D_x} C(x) = C(x)$.
With the involution $\omega_{q,t}$,

\begin{equation}
\frac{1}{\langle \lambda \rangle} \omega_{q,t} P_{\lambda}(x; q, t) = P_{\lambda'}(x; t, q), \quad \omega_{q,t}(p_n) := (-1)^{n-1} \frac{1-q^n}{1-t^n} p_n.
\end{equation}

Let us denote a function $f$ in the set of variables $(x_1, x_2, \cdots, y_1, y_2, \cdots)$ by $f(x, y)$. Let $f_{\lambda,\mu}^\nu$ be the fusion coefficient $f_{\lambda,\mu}^\nu := \langle P_{\lambda} P_{\mu}, P_{\nu} \rangle / \langle P_{\nu}, P_{\nu} \rangle$, then we have

\begin{equation}
P_{\lambda}(x) P_{\mu}(x) = \sum_{\nu} f_{\lambda,\mu}^\nu P_{\nu}(x),
\end{equation}

\begin{equation}
\frac{P_{\nu}(x, y)}{\langle \nu \rangle} = \sum_{\lambda,\mu \subset \nu} \frac{P_{\lambda}(x)}{\langle \lambda \rangle} f_{\lambda,\mu}^\nu \frac{P_{\mu}(y)}{\langle \mu \rangle}.
\end{equation}

Let us denote the Young diagram decomposing into rectangles as $\lambda = \sum_{i=1}^{N-1} (s_i^{r_i})$, $r_i \geq r_{i+1}$, i.e., $\lambda' = (r_1^{s_1} r_2^{s_2} \cdots r_{N-1}^{s_{N-1}})$,

\[
\begin{array}{c}
\lambda = \\
\begin{array}{cccc}
s_1 & s_2 & \cdots & s_{N-2} & s_{N-1}
\end{array}
\end{array}
\begin{array}{c}
r_1 \\
r_2 \\
\cdots \\
r_{N-2} \\
r_{N-1}
\end{array}
\]

Then we have the following integral representation of the Macdonald polynomial [22]

\begin{equation}
P_{\lambda}(x) = C_{\lambda}^+ \int \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i z_j^a} \left( z_j^a \right)^{-s_a} \cdot \Pi (x, pz_1^a) \prod_{a=1}^{N-1} \Pi \left( \frac{z^a}{x_j^{a+1}}, \frac{p z^a}{x_j^a} \right) \Delta(z^a)
\end{equation}

\begin{equation}
= C_{\lambda}^+ \langle \alpha_{r,s}^+ | \exp \left\{ -\sum_{n>0} \frac{1}{1-q^n} \sum_{i=1}^{M} (q x^i)^n \right\} | \chi_{r,s}^+ \rangle,
\end{equation}

with a singular vector $| \chi_{r,s}^+ \rangle$ in (2.16). Here $z_i^N := 0$ and $\lambda^{(1)} := \lambda$, $\lambda^{(a)} := \sum_{i=a}^{N-1} (s_i^{r_i})$, i.e., $\lambda^{(a)'} = (r_a^{s_a} r_{a+1}^{s_{a+1}} \cdots r_{N-1}^{s_{N-1}})$. Acting $\omega_{q,t}$ on (A.11) gives

\begin{equation}
P_{\lambda'}(x) = C_{\lambda}^- \langle \alpha_{r,s}^- \rangle \exp \left\{ -\sum_{n>0} \frac{1}{1-q^n} \sum_{i=1}^{M} (-q x^i)^n \right\} | \chi_{r,s}^- \rangle,
\end{equation}

\begin{equation}
C_{\lambda}^- := \omega_{q,t} C_{\lambda}^+ \langle \lambda \rangle.
\end{equation}
References


