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Introduction to the focus issue: Fifty years of chaos: Applied and theoretical

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The discovery of deterministic chaos in the late nineteenth century, its subsequent study, and the development of mathematical and computational methods for its analysis have substantially influenced the sciences. Chaos is, however, only one phenomenon in the larger area of dynamical systems theory. This Focus Issue collects 13 papers, from authors and research groups representing the mathematical, physical, and biological sciences, that were presented at a symposium held at Kyoto University from November 28 to December 2, 2011. The symposium, sponsored by the International Union of Theoretical and Applied Mechanics, was called *50 Years of Chaos: Applied and Theoretical*. Following some historical remarks to provide a background for the last 50 years, and for chaos, this Introduction surveys the papers and identifies some common themes that appear in them and in the theory of dynamical systems. *© 2012 American Institute of Physics*.
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Deterministic chaos is, at first glance, surprising. How can the solution of a nonlinear differential equation, or a sequence generated by iterating a smooth map, be so unpredictable? Both are uniquely determined by the initial conditions and, respectively, the equation or the map. We now know that the apparent unpredictability is due to sensitive dependence on initial conditions caused by rapid divergence of neighboring solutions, a property that is quite common in nonlinear differential equations with three or more variables, invertible maps in two or more dimensions, and all non-invertible maps. (The logistic equation is a prime example.) Chaotic dynamics is locally expansive in one or more directions in phase space and contractive in the remaining dimensions. Chaos was first discovered and studied by the mathematician Henri Poincaré in 1889–1912 and it remained a primarily mathematical phenomenon until the 1960s. During this "first period" a substantial theory, built on analysis, geometry and 200 years of work in differential equations and classical mechanics, was developed to characterize invariant sets that live in the phase spaces of nonlinear systems, and to describe bifurcations in which they appear, disappear, or change their stability types. Chaos occupies an important place in this theory of dynamical systems, but it is only a part of it. As described below, some mathematicians drew on examples from the physical sciences and engineering during the first period, but vigorous interactions throughout the physical and mathematical sciences began to develop only in the 1960s and 1970s. The last 50 years has seen a second period flourish, and applications expand to include the biological and life sciences. Chaotic invariant sets can form boundaries for basins of attraction of simple equilibria or periodic orbits, and they can be attracting sets themselves. They can describe irregular time series and spatial structures observed in nature, they can generate quasirandom bit sequences for technological applications, and they appear in analog to digital converters. Analytical, geometrical, and computational methods have been developed to detect and characterize chaotic sets, and experiments have confirmed that they appear in a variety of real systems. The papers in this Focus Issue provide examples of these and other manifestations of chaos.

A BRIEF HISTORY OF CHAOS: 1889–1961 AND A LITTLE BIT BEYOND

This focus issue grew out of a symposium sponsored by the International Union of Theoretical and Applied Mechanics (IUTAM) that brought together researchers from the physical and biological sciences, engineering, and mathematics to discuss recent developments in nonlinear dynamics and chaos theory. The meeting was timed to celebrate a remarkable discovery, and held 50 years to the week after it was made. In the Department of Electrical Engineering at Kyoto University, on November 27, 1961, a graduate student named Yoshisuke Ueda noticed that orbits of a periodically forced nonlinear oscillator displayed a "randomly transitional" behavior in certain parameter ranges, instead of the periodic, sub- or superharmonic and quasi-periodic motions that he (and his supervisors) expected.[1] Before surveying the papers that follow, we provide a background by outlining some key work on dynamical systems prior to Ueda's discovery.

Dynamical systems theory began with the work of Poincaré (1854–1912) on the three-body problem of celestial mechanics,[2] and specifically in a massive paper,[3] which won a prize celebrating the 60th birthday of King Oscar II of

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Sweden and Norway. In this and his earlier papers, Poincaré proposed new methods for studying nonlinear ordinary differential equations (ODEs). He described the use of first return (Poincaré) maps for the study of periodic motions, defined stable and unstable manifolds, discussed stability issues, developed perturbation methods, and proved the (Poincaré) recurrence theorem. While revising his prize paper,[3] he realised that certain differential equations describing mechanical systems with two or more degrees of freedom were not integrable in the classical sense, due to the presence of "doubly asymptotic" points, now called homo- and heteroclinic orbits. Moreover, he saw that these orbits had profound implications for the stability of motion in general, and realized that his previous claim that a version of the restricted three-body problem of celestial mechanics had only stable behavior was false. In December 1889 and January 1890, he created the first explicit example of deterministic chaos.[4,5]

G. D. Birkhoff (1884–1944) was one of relatively few mathematicians to continue on Poincaré's path in the early 20th century.[6,7] Birkhoff's work on iterated mappings of the annulus[8,9] was especially relevant to the study of periodically forced oscillators to which Ueda's advisor, C. Hayashi, had directed him. Indeed, the van der Pol equation, a model of the vacuum tube diode, played a central role in the development of dynamical systems theory. This began in the 1920s with a brief paper by van der Pol and van den Mark,[10] engineers at the Phillips Laboratories in Eindhoven, who were interested in subharmonic solutions and who noted in passing that their experimental apparatus produced "an irregular noise" in certain frequency ranges: perhaps an early observation of chaos? Cartwright and Littlewood alluded to this paper in their proof of "discontinuous recurrent" orbits in the van der Pol equation,[11] and drew on Birkhoff's proof that annulus maps with coexisting stable orbits of distinct periods also possessed complicated invariant sets. Their analysis was later simplified by Levinson.[12] In the same period, Soviet researchers defined structurally stable systems[13] (roughly speaking, those that preserve their qualitative properties under small perturbations of the defining ODEs) and began to study bifurcations in planar systems.[14,15]

When Smale became interested in dynamical systems in 1959–1960,[16] he conjectured that a structurally stable ODE could possess only finite sets of periodic orbits in any bounded region of its state space. Levinson suggested that Cartwright–Littlewood paper might provide a counterexample. Smale's geometric interpretation of a Poincaré map for the forced van der Pol equation led to his construction of the "horseshoe map,"[17] and more generally contributed to the formulation of a broad research program in dynamics.[18] Subsequently, Melnikov[19] and Arnold[20] provided rather general perturbative methods for proving the existence of homoclinic tangles such as those recognized by Poincaré and Smale.

This work, which was almost all done by mathematicians, brings us to the 1960s. In that decade, a few engineers and physical scientists became interested in chaos. Ueda's discovery in November 1961 was an early example, predating by 2 years Lorenz's better known paper on a strange (=chaotic) attractor in a truncated model for convection in a fluid layer.[21] (Both Ueda and Lorenz acknowledged the importance of Birkhoff's work in enabling them to interpret their observations.) Throughout the 1960s Ueda continued to think about mathematical aspects of his findings, drawing on Levinson's work on second order ODEs as well as Poincaré's book.[2] However, apart from a brief section in one paper [Ref. 22, §3.2, Figs. 6–8], some conference proceedings and a research report,[23] he did not publish them for over 10 years.[24] Lorenz's work also remained unnoticed by mathematicians until the 1970s, when J. A. Yorke was given a copy by a colleague in the Department of Meteorology at the University of Maryland, which he passed on to Smale.[25] Soon thereafter, dynamical systems theory was percolating throughout the sciences and motivations and examples were flowing back to mathematics. By 1985, a bibliography of dynamical systems listed over 4400 papers and books.[26]

This brief history highlights only one thread within the rich tapestry of dynamical systems. More extensive treatments, along with comments on recent developments, can be found in Refs. 5 and 27, but advances have been so rapid and widespread that an adequate historical perspective on the past 50 years is still lacking. Chaos theory—as a part of nonlinear dynamics—has fostered a globally interconnected vision of the sciences in a time of strongly developing technologies. It has affected not only emerging interdisciplinary fields but also classical ones such as mechanics, within which the Kyoto Symposium was conceived and partially nurtured. In spite of some inflated claims and misuses of concepts and tools, which can augment but not replace careful mathematical modeling, the past fifty years of chaos have brought us much good sense, and a measure of order.

A detailed account of Poincaré's work on dynamics appears in Ref. 4. For more on the discoveries of Ueda and Lorenz, and their sometimes difficult paths to publication and acceptance, see Refs. 1 and 25. For a discussion of the sociological and cultural contexts of nonlinear dynamics and chaos, see Ref. 28.

THE PAPERS IN THIS ISSUE

The 13 papers that follow form a varied baker's dozen, representing several of the classical and more recent areas of nonlinear dynamics and ranging from basic theory to applied technology.

The mechanics of elastic structures and rigid bodies is treated in the papers of Lenci *et al.*,[29] Strzalko *et al.*,[30] and Kapitaniak *et al.*[31] Issues of mechanical modeling (e.g., of friction and impacts in Ref. 31), synchronization, control of chaos, imperfections, and symmetries of the mechanical systems play important roles in these studies. Lenci *et al.* exploit chaos properties to control the global nonlinear dynamics of simplified models of a large class of structures exhibiting interacting buckling phenomena, thus increasing their practical load carrying capacities. Strzalko *et al.* study synchronous behavior in an experimental set of two pairs of double pendula, with a view to converting base oscillations into rotational motions exploitable for energy production. Kapitaniak *et al.* present simulations and experiments on die throwing, addressing the theoretical predictability of outcomes in contrast with the difficulty of practical implementation.

Kreilos and Eckhardt[32] investigate stability and bifurcations in transitional and weakly turbulent Couette flow. The model—the incompressible Navier-Stokes equation—is not in question here; the issue is to extract useful information from direct numerical simulations of a very high-dimensional system, and by following branches of equilibria representing steady flow patterns, most of which are unstable. Chaotic saddles (homoclinic tangles) are found to produce transient turbulent bursts. Cvitanovič *et al.*,[33] building on earlier work on similar channel flows, show that a proper understanding of symmetries imposed by the governing equations is essential to visualizing and decomposing the global structure of high-dimensional state spaces.

Mathematical methods are developed in the paper of Sabuco *et al.*[34] and at substantially greater lengths in those of Bush *et al.*,[35] Lingala *et al.*,[36] and Budišić *et al.*[37] Sabuco *et al.* extend their earlier work on safe sets in the context of chaos control to asymptotically safe sets, providing an algorithm that approximates the set of initial conditions that eventually enters a safe set. Bush *et al.* review an extensive program that uses algebraic-topological and combinatorial methods to deduce rigorous global information on iterated nonlinear maps, proving connections from saddle-type invariant sets to attracting sets. Lingala *et al.* describe particle filtering methods that approximate distributions of state variables observed in chaotic systems. Budišić *et al.* review the Koopman operator (an infinite-dimensional linear map that advances observable functions along orbits of a dynamical system) and explain how its eigenfunctions preserve global information; they also introduce continuous quantifications or ergodicity and mixing behaviors. These "data-driven" papers all illustrate theories and methods by means of multiple examples.

Hirata *et al.*[38] revisit a data set collected from the giant axon of the squid (the preparation used in Hodgkin and Huxley's Nobel Prize winning work in which the dynamics of action potentials were first modeled). The authors show that a relaxed, numerically adapted version of Devaney's criteria for chaos[39] identifies the neuron's voltage time series as chaotic, and also briefly describe a simple mapping that produces deterministic chaos in a neural network model of memory recall.

Finally, three papers describe interesting electronic systems with technological implications. Kohda *et al.*[40] show that expanding attractors (originally, purely abstract mathematical objects) can be used to build analog-to-digital converters with high bit-rate accuracies. Sunada *et al.*[41] take a complementary view, using a chaotically oscillating laser to produce bit sequences that pass stringent statistical tests for random number generators. In *et al.*[42] construct an integrated circuit containing an array of bistable oscillators that can be made to entrain to distinct frequency ranges, enabling a "circuit on a chip" to rapidly lock on components of an arbitrary radio-frequency spectrum.

While most of these papers were prepared independently (one pair[30,31] does share three authors and another[38,40] shares one), they nicely illustrate common interests in nonlinear dynamics. In particular, computational methods (rigorously based, as in Refs. 35 and 37, or more formal) play an important part. This is likely to last for some time: 123 years after Poincaré's prize paper it is still embarassingly difficult to extract global information on ODEs defined in 3 or more dimensions, or invertible maps of 2 or more dimensions. Symmetries and bifurcations, cross sections and Poincaré maps are prevalent, and increasingly high dimensional phase spaces are being considered. Several papers discuss multiple applications, or present an approach or methods that apply beyond their specific examples, those of Bush *et al.*,[35] Cvitanovič *et al.*,[33] Hirata *et al.*,[38] Lingala *et al.*,[36] and Budišić *et al.*[37] provide examples.

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