Reducibility of steady-state bifurcations in coupled cell systems

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Abstract

A general theory for coupled cell systems was formulated recently by I. Stewart, M. Golubitsky and their collaborators. In their theory, a coupled cell system is a network of interacting dynamical systems whose coupling architecture is expressed by a directed graph called a coupled cell network. An equivalence relation on cells in a regular network (a coupled cell network with identical nodes and identical edges) determines a new network called quotient network by identifying cells in the same equivalence class and determines a quotient system as well. In this paper we develop an idea of reducibility of bifurcations in coupled cell systems associated with regular networks. A bifurcation of equilibria from subspace where states of all cells are equal is called a synchrony-breaking bifurcation. We say that a synchrony-breaking steady-state bifurcation is reducible in a coupled cell system if any bifurcation branch for the system is lifted from those for some quotient system. First, we give the complete classification of codimension-one synchrony-breaking steady-state bifurcations in 1-input regular networks (where each cell receives only one edge). Second, we show that under a mild condition on the multiplicity of critical eigenvalues, codimension-one synchrony-breaking steady-state bifurcations in generic coupled cell systems associated with an n-cell coupled cell network with D_n symmetry, a regular network, is reducible for $n > 2$.

Keywords: coupled cell network, coupled cell system, network symmetry, quotient network, synchrony-breaking bifurcation.

1 Introduction

A general theory for coupled cell systems was introduced recently in I. Stewart, et al. [1]. Since then the authors and their collaborators have been releasing many papers related to the theory. By their formulation a *coupled cell system* is a system of coupled ODEs whose coupling information is given by a *coupled cell network* that is essentially a directed graph whose nodes (or cells) represent states that evolve in time and whose edges (or couplings) represent interactions between those states. See [1], [2], [4] for more precise formulation.

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In [6] the authors considered *synchrony-breaking bifurcations* in coupled cell systems, which is an analogue of the symmetry-breaking bifurcations in systems with symmetry. Such synchrony-breaking bifurcations are the main subject of this paper. We shall recall them briefly in the following paragraphs based on [6].

In this paper we study codimension-one synchrony-breaking bifurcations of steady-state solutions in coupled cell systems. We focus on a special class of coupled cell networks called *regular networks*, in which all cells are identical and couplings are also identical, in particular each cell has the same number of incoming edges called "inputs". For a regular network, define an associated ODE called an *admissible* vector field to the regular network as follows: Since the total number of cells is finite, we can enumerate the cells and let name the cells after its numbers. Let $x_j \in \mathbb{R}^k$ be the state variable of the j-th cell (or cell j), where k is the dimension of the internal dynamics in each cell, which is assumed to be identical. Then the j-th component of the admissible vector field has the form

$$\dot{x}_j = f(x_j, \overline{x_{\sigma_j(1)}, \dots, x_{\sigma_j(v)}}) \quad j = 1, \dots, n \tag{1.1}$$

where the cell j receives inputs from the cells $\sigma_j(1), \dots, \sigma_j(v)$. The $\sigma_j(i)$'s are allowed to be equal to each other and even to j. The number v is called the *valency* of the network and it is constant for any choice of the cell j because each cell has the same number of inputs. The overbar indicates that the coupling coordinates are invariant under permutations of the coupling cells. This invariance is assumed, since we assume a unique type of coupling. Since there is only one type of node, we assume that the function $f\colon \mathbb{R}^k \times (\mathbb{R}^k)^v \to \mathbb{R}^k$ is independent of j.

Example [n-cell bidirectional ring]: Consider the following n-cell regular network with valency 2, called a *bidirectional ring*:

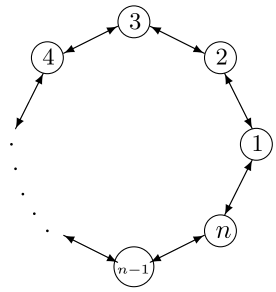

Figure 1: n-cell bidirectional ring (BR_n)

The corresponding admissible vector field takes the form

$$\begin{cases} \dot{x}_1 = f(x_1, \overline{x_2, x_n}) \\ \dot{x}_2 = f(x_2, \overline{x_3, x_1}) \\ \dots. \\ \dots. \\ \dot{x}_n = f(x_n, \overline{x_1, x_{n-1}}) \end{cases} \tag{1.2}$$

where $f\colon \mathbb{R}^k \times (\mathbb{R}^k)^2 \to \mathbb{R}^k$ satisfies $f(a, b, c) = f(a, c, b)$.

We say that a coupled cell system exhibits *synchrony*, if two or more cells behave identically. A *polydiagonal* is a subspace Δ of the phase space $(\mathbb{R}^k)^n$ of coupled cell system which is defined by equalities among some cell coordinates. A *synchrony subspace* is a polydiagonal Δ that is flow-invariant for every admissible vector field associated with the coupled cell network. It is obvious that the subspace $\Delta_0 = \{(x, \ldots, x) \in (\mathbb{R}^k)^n\}$ given by setting all coordinates equal in a regular network yields a synchrony subspace. This Δ_0 is called the completely synchronous subspace.

We assume that an admissible vector field F has a completely synchronous equilibrium $X_0 \in \Delta_0$. Let $E^c = E^c_F(X_0)$ be the center subspace of $(dF)_{X_0}$. We say that the equilibrium X_0 has a *synchrony-breaking bifurcation*, if $E^c \setminus \Delta_0 \neq \emptyset$.

For a given coupled cell network and a polydiagonal Δ of an associated admissible vector field, we color cells so that any two cells i, j have the same color when $x_i = x_j$ in Δ. Theorem 4.3 of [4] states that Δ is a synchrony subspace, if and only if the coloring associated with Δ is "balanced". In the case of regular networks, a coloring of cells is called *balanced*, if cells with same color have the same number of inputs in each color. Clearly, a balanced coloring is an equivalence relation on cells, and therefore, for a coupled cell network, a balanced coloring defines a new network called the *quotient network* by identifying cells with the same color. Observe that a network can have different quotient networks corresponding to different choices of balanced colorings.

Given a (regular) coupled cell network Γ and its quotient network Q, a coupled cell system on the original network F_Γ uniquely determines a coupled cell system on the quotient network F_Q, which we call the quotient system. As noted above, the dynamics in the quotient system describes a synchronous dynamics in the original system, and hence it is possible to lift a solution in the quotient system to one in the original system. More explicitly, once we have a solution in the quotient system, we can designate each variable in the original system as the solution for its representative in the quotient system. The solutions obtained in such way are called lifted solutions from the quotient system.

The purpose of this paper is to study relations between bifurcations in coupled cell systems and those in their quotient systems regarding network architectures. A related work on this subject is released in [6] in which the authors showed that for two certain regular 5-cell coupled cell networks admitting 3-cell bidirectional ring as a quotient network, generically there exists an additional bifurcation branch of equilibrium in their associated coupled cell systems which is not lifted from the quotient systems associated with the 3-cell bidirectional ring. This result leads us to ask if those additional branches are lifted from any of other quotient systems of the original ones. If this is true for all additional branches, then we can conclude that generically all bifurcation branches in the original system are lifted from its quotient systems. In fact there exist coupled cell networks whose associated coupled cell systems satisfy such phenomenon. In general we ask when and how much one can understand all bifurcating solutions of original system only by studying its quotient systems and comparing these to the original system. This question is our main motivation. Let us introduce a notion of reducibility of bifurcations in coupled cell systems. Below we give a definition of reducibility for bifurcations.

For steady-state bifurcations we define the notion of reducibility as follows:

Definition 1.1. a) Let $\{Q_1, \dots, Q_m\}$ be quotient networks of $\mathit{\Gamma}$. Then a bifurcation in $\mathit{\Gamma}$ is *reduced* to bifurcations in $\{Q_1, \dots, Q_m\}$, if for any bifurcation branch in $F_{\mathit{\Gamma}}$ there exists Q_i such that it is lifted from F_{Q_i}.

b) A bifurcation in a coupled cell network $\mathit{\Gamma}$ is *reducible*, if the bifurcation in $\mathit{\Gamma}$ is reduced to bifurcations in the set of all quotient networks of $\mathit{\Gamma}$.

Otherwise, the bifurcation in a coupled cell network $\mathit{\Gamma}$ is *non-reducible*. That is, there exists a bifurcation branch which is not lifted from any of its quotient.

Remark 1.2. We always consider that any bifurcation in the trivial coupled cell network, which consists of just one cell, is non-reducible.

Remark 1.3. A similar definition to reducibility of steady-state bifurcations can be given for other bifurcations such as the Hopf bifurcation with some additional modifications, which will be a subject of future research.

Now our main motivation is to study reducibility of bifurcations in coupled cell systems and to classify coupled cell networks in terms of reducibility. Here we want to clarify that we study reducibility of bifurcations in coupled cell systems concerning only the architecture of coupled cell networks. Thus, we study how the network architecture affects the (non)reducibility of bifurcations in associated coupled cell systems. From now on, we will understand bifurcations in a coupled cell network as bifurcations in coupled cell systems associated with the coupled cell network.

In this paper we study reducibility of steady-state bifurcations in 1-input regular cell networks in general and n-cell coupled cell networks with D_n symmetry. First, for 1-input coupled cell networks, we show that any 1-input regular coupled cell network is a loop with finite number of trees attached to it. We classify all bifurcating solutions in the loop-chain networks and, as a corollary, we can understand all bifurcating solutions in any 1-input regular coupled cell network for each critical real eigenvalue for the bifurcations. Hence we conclude that steady-state bifurcations in 1-input regular coupled cell network are non-reducible only if the loop in the network contains one or two cells. Second, for n-cell coupled cell networks with D_n symmetry, we show that steady-state bifurcations are reducible for $n > 2$ if the critical eigenvalues for the bifurcations have multiplicity not more than two. Thus, we can investigate all bifurcation branches in the n-cell coupled cell networks with D_n symmetry only from its quotients.

We organize this article as follows: In Section 2 we formulate our main results. In Theorem 2.5 we give the complete classification of codimension-one synchrony-breaking steady-state bifurcations in 1-input regular networks. In Theorem 2.8 we show that the codimension-one synchrony-breaking steady-state bifurcations in generic coupled cell systems on the n-cell coupled cell network with D_n symmetry ($n > 2$) are reducible if the critical eigenvalues for the bifurcations have multiplicity one or two. The proofs of these results are given in Section 3. In Section 4 we give concluding remarks with related works.

2 Main Results

First we consider 1-input regular coupled cell networks. Let $x_i \in \mathbb{R}^k$ be the state variable of i-th cell, where k is the dimension of the internal dynamics in each cell. The i-th component of an admissible vector field of 1-input regular network takes the form

$$\dot{x_i} = f(x_i, x_j),$$

when there is only one edge from a cell x_j to the cell x_i. Since there is only one kind of cells and one kind of couplings, we assume that the function $f\colon \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is independent of i.

Let $\Gamma = (C, E)$ be a coupled cell network. For cells u and v in C, a *path* from u to v is a sequence of cells $x_0 = u, x_1, x_2, \dots, x_n = v$ in C such that there is an edge from cell x_i to cell x_{i+1} for all $i = 0, 1, \dots, n-1$.

Definition 2.1. We say Γ is *connected*, if for any two distinct cells u and v, there is either a path from u to v, or one from v to u (not necessarily both).

We say Γ is *strongly connected*, if for any two distinct cells u and v, there is a path from u to v, and also from v to u.

Proposition 2.2. *Any connected coupled cell network can be uniquely decomposed into strongly connected coupled cell networks $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ such that, for any $i = 1, \dots, k$ and from each cell in Γ_i, there does not exist an edge to a cell in Γ_j with $j < i$.*

The proof of this proposition is straightforward, hence omitted. We call this decomposition the *Morse decomposition* of Γ.

Proposition 2.3. *Let Γ be a connected 1-input regular coupled cell network. Suppose $\{\Gamma_1, \Gamma_2, \dots, \Gamma_k\}$ is its Morse decomposition. Then Γ_1 is a loop and each Γ_i with $i > 1$ is a single cell.*

Proof. (i) Since a given network is 1-input, every cell receives an edge from only one cell. Choose an arbitrary cell j_1. Let j_{s+1} be the only cell such that j_s receives an edge from it with $s \geq 1$. That the number of cells is finite implies that there exist s_1, s_2 with $j_{s_1} = j_{s_2}$, and hence the sequence $\{j_s\}_{s \geq 1}$ forms a loop with a chain attached to it. Therefore every cell is associated with a loop with a chain attached to it. Consider any two distinct cells. If the corresponding loops are different, then they do not have any common cell. As the connectivity of the graph, there must exists a connection between these two loops, say between a cell j_l in the first loop and a cell j_m in the second loop. Observe that one of j_l and j_m receives at least 2 inputs, which is a contradiction. Therefore there cannot exist two distinct loops. This shows what we want to prove. Q.E.D.

Definition 2.4. A 1-input coupled cell network consisting of a loop with a chain attached to it is called a (1-input) *loop-chain* network.

Our first main result is as follows:

Theorem 2.5. *For a 1-input regular coupled cell network Γ, assume that there occur codimension-one synchrony-breaking steady-state bifurcations in F_Γ. Then, generically, the bifurcating steady-state solutions of F_Γ restricted to every loop-chain subnetwork of Γ can be classified into the following three types:*

(i) a cascading solution of square-root type, that is, state of every cell in the loop is zero and for any other cell, there are two possibilities of state solutions. Once there is a nonzero state for some cell, for the rest of cells in the chain, states are defined recurrently and the lowest order of the parameter of solutions at zero decreases twice in each step. See (3.14) for more precise definition.

(ii) *a fully synchronous solution, that is, state of all cells are the same.*

(iii) *an alternately synchronous solution, that is, there are two types of states for cells and any two cells which are connected to each other by an edge have different states.*

Corollary 2.6. *The bifurcation corresponding to the solutions of type (i) in the theorem is non-reducible only in networks where the containing loop has one cell. Otherwise reducible.*

The bifurcation corresponding to the solutions of type (ii) is reducible in all networks, except the trivial one.

The bifurcation corresponding to the solutions of type (iii) is non-reducible only in the network where itself is a loop consisting of two cells. Otherwise reducible.

Remark 2.7. The precise formulation of non-degeneracy conditions is given in the proof. See § 3.1.

For two or more input coupled cell networks, it is almost impossible to characterize because the network structure would become more and more complex as the number of cells increases. However, we try to understand dynamics on such coupled cell networks by considering feed-forward networks since their Morse decompositions can be represented as feed-forward networks.

On the other hand, it is also important to study the first component subnetwork of the Morse decomposition, since it affects cells in all other components. Therefore we are interested in strongly connected regular coupled cell networks. In this paper we consider certain multiple input coupled cell networks, n-cell coupled cell networks with D_n symmetry, which can be considered as one of the natural and well-studied strongly connected coupled cell networks, and we have the following result, which is our second main result:

Theorem 2.8. *Generically, a codimension-one synchrony-breaking steady-state bifurcation in a coupled cell system on the n-cell coupled cell network with D_n symmetry ($n > 2$) is reducible if the critical eigenvalue for the bifurcation has multiplicity one or two.*

Remark 2.9. In case of $n = 2$ the bifurcation is non-reducible.

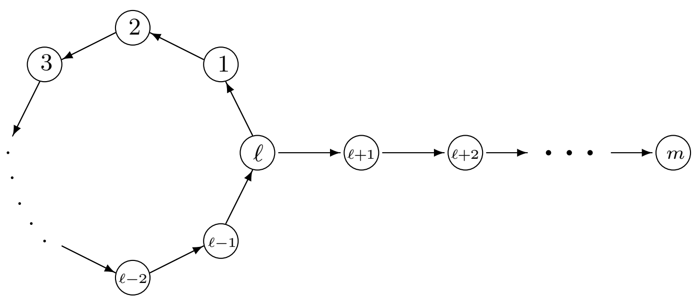

Figure 2: 1-input loop-chain network

3 Proof of Main Results

3.1 Proof of Theorem 2.5

Here we give a proof of Theorem 2.5.

First of all, let us recall the Lyapunov-Schmidt reduction briefly which plays a main role in the proofs for theorems in this paper.

Consider a system of n equations

$$F_i(x, \alpha) = 0, \qquad i = 1, \dots, n, \tag{3.1}$$

where $F = (F_1, \dots, F_n) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth mapping. We assume that $F_i(0, 0) = 0$ for all i and we attempt to find solutions $x = (x_1, \dots, x_n)$ of the system as a function of α locally, near the origin. Let $L = D_x F(0, 0)$ be the $n \times n$ Jacobian matrix. There are natural decompositions of $\mathbb{R}^n$ where $\mathbb{R}^n = \operatorname{Ker} L \oplus M$, and $\mathbb{R}^n = N \oplus \operatorname{Range} L$. Let E be the projection of $\mathbb{R}^n$ onto $\operatorname{Range} L$ with $\operatorname{Ker} E = N$. Suppose that $dim \operatorname{Ker} L = m$. If $m = 0$ the system can be solved uniquely by the Implicit Function Theorem. If $m > 0$ one can obtain a system of m equations

$$g_i(y, \alpha) = 0, \qquad i = 1, \dots, m \tag{3.2}$$

where $\{y_1, \dots, y_m\} \subset \{x_1, \dots, x_n\}$ and $G = (g_1, \dots, g_m) : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is a smooth mapping, and satisfies that whose solutions $y = (y_1(\alpha), \dots, y_m(\alpha))$ near the origin can be put in one-to-one correspondence with solutions of the system (3.1). This is the Lyapunov-Schmidt reduction for (3.1). For more detail, see [5].

Lower order terms of the reduced system can be found as follows:

$$a) \quad \frac{\partial g_i}{\partial y_j} = 0, \tag{3.3}$$

$$b) \quad \frac{\partial^2 g_i}{\partial y_j \partial y_k} = \langle v_i^*, d^2F(v_j, v_k) \rangle, \tag{3.4}$$

$$c) \quad \frac{\partial^3 g_i}{\partial y_j \partial y_k \partial y_\ell} = \langle v_i^*, V \rangle, \tag{3.5}$$

$$d) \quad \frac{\partial g_i}{\partial \alpha} = \langle v_i^*, F_\alpha \rangle, \tag{3.6}$$

$$e) \quad \frac{\partial^2 g_i}{\partial y_j \partial \alpha} = \langle v_i^*, (dF_\alpha) \cdot v_j - d^2F(v_j, L^{-1}EF_\alpha) \rangle, \tag{3.7}$$

where $\{v_i\}_{i=1,\dots,m}$ is a basis for $\operatorname{Ker} L$, $\{v_i^*\}_{i=1,\dots,m}$ is a basis for $(\operatorname{Range} L)^\perp$, and

$$V = d^3F(v_j, v_k, v_\ell) - d^2F(v_j, w_{\ell k}) - d^2F(v_k, w_{\ell j}) - d^2F(v_\ell, w_{kj}), \tag{3.8}$$

where $w_{st} = L^{-1}Ed^2F(v_s, v_t)$.

Also note that the procedure for obtaining a reduced system G by the Lyapunov-Schmidt reduction in [5] clearly shows that if $F(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$, then $G(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$.

Proof of Theorem 2.5. Suppose we have a loop-chain network as illustrated in Figure 2, embedded in Γ. A one-parameter family of admissible vector fields F associated with the loop-chain network has the form

$$\begin{cases} \dot{x}_1 = f(x_1, x_\ell, \lambda) \\ \dot{x}_2 = f(x_2, x_1, \lambda) \\ \dots. \\ \dot{x}_\ell = f(x_\ell, x_{\ell-1}, \lambda) \\ \dot{x}_{\ell+1} = f(x_{\ell+1}, x_\ell, \lambda) \\ \dots. \\ \dot{x}_{\ell+m} = f(x_{\ell+m}, x_{\ell+m-1}, \lambda), \end{cases} \tag{3.9}$$

where $x_i \in \mathbb{R}^k,\ i = 1, \dots, \ell + m$, and $\lambda \in \mathbb{R}$ is the bifurcation parameter.

We can write the equation (3.9) simply as $\dot{X} = F(X, \lambda)$, where $X = (x_1, \dots, x_{\ell+m})^T$. In order to describe steady-state bifurcations we must solve the equation

$$F(X, \lambda) = 0. \tag{3.10}$$

As discussed in Section 2.3 in [6] there is no loss of generality in assuming that the phase space of each cell is one-dimensional, that is $k = 1$.

The adjacency matrix associated with the loop-chain network is of the form

$$\mathcal{A}_{\mathrm{LCN}} = \begin{bmatrix} A & 0_{\ell\times m} \\ C & B \end{bmatrix}$$

where A is an $\ell \times \ell$ matrix corresponding to the loop, B is an $m \times m$ matrix corresponding to the chain, and C is an $m \times \ell$ matrix whose all entries are 0 except the upper right one. More precisely,

$$A = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & \mathbf{0} & \\ & 1 & 0 & & \\ & \mathbf{0} & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & \mathbf{0} & \\ & 1 & 0 & & \\ & \mathbf{0} & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \tag{3.11}$$

It follows that the characteristic polynomial of $\mathcal{A}_{\mathrm{LCN}}$ is

$$\chi(\mathcal{A}_{\mathrm{LCN}}) = ((-\mu)^\ell - 1) \times (-\mu)^m. \tag{3.12}$$

Since we study steady-state bifurcations in F, we look for only real eigenvalues. The possible real eigenvalues of $\mathcal{A}_{\mathrm{LCN}}$ are $\mu = \{-1, 0, 1\}$. By Proposition 2.14 in [6] the critical real eigenvalues of the linearization of F at the origin are $f_u(0) - f_v(0), f_u(0), f_u(0) + f_v(0)$, where $f_u(0), f_v(0)$ denote the derivatives of f with respect to the first and second variable, respectively, evaluated at the origin. We also denote by $f_{uu}(0), f_{uv}(0), f_{uuu}(0)$, etc., the second and third derivatives of f with respect to the first and the second variable at the origin, and so on.

Observe that, in view of Proposition 2.14 in [6], $f_u(0)-f_v(0), f_u(0)+f_v(0)$ are the eigenvalues of $DF(0)$ restricted to the loop part, since ±1 are the eigenvalues of the adjacency matrix A. Similarly, the eigenvalue $f_u(0)$ corresponds to the remaining tree part.

Assume that $f(0, 0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$. If we have a critical eigenvalue, generically we have a codimension-one synchrony-breaking steady-state bifurcation from the trivial equilibrium.

(i) Assume that $f_u(0) = 0$. Since the critical eigenvalue $f_u(0)$ does not correspond to the loop, as explained above, there can not occur any bifurcation in the loop, and hence $x_i = 0$ for $i = 1, 2 \ldots, \ell$. We then find $x_{\ell+1}, x_{\ell+2}, \ldots$ successively as explained below.

The Taylor expansion of f at the origin is

$$\begin{aligned} f(u, v, \lambda) =& f_u(0)u + f_v(0)v + \frac{1}{2}f_{uu}(0)u^2 + \frac{1}{2}f_{vv}(0)v^2 + f_{uv}(0)uv \\ &+ f_{u\lambda}(0)u\lambda + f_{v\lambda}(0)v\lambda + \mathcal{O}(3). \end{aligned} \tag{3.13}$$

In the equation $f(x_{\ell+1}, x_\ell, \lambda) = 0$, we set $x_\ell = 0$ and obtain

$$x_{\ell+1} \times \left\{ \frac{1}{2}f_{uu}(0)x_{\ell+1} + f_{u\lambda}(0)\lambda + \mathcal{O}(2) \right\} = 0.$$

It implies that $x_{\ell+1} = 0$ or $x_{\ell+1} = h(\lambda)$. In case $x_{\ell+1} = 0$, the same holds for the equation $f(x_{\ell+2}, x_{\ell+1}, \lambda) = 0$, and we conclude that $x_{\ell+2} = 0$ or $x_{\ell+2} = h(\lambda)$. As long as we choose $x_{\ell+j} = 0$ for all $j = 1, 2, \ldots,$ we have the same conclusion for $f(x_{\ell+j+1}, x_{\ell+j}, \lambda) = 0$. If we choose $x_{\ell+j} = h(\lambda)$ for some j, then the solution at the next step depends on $h(\lambda)$. This is the basic idea for finding all the equilibrium solutions, and below we give a precise argument based on this idea.

Lemma 3.1. *For a natural number r, let $y(\lambda)$ be a smooth function defined on an open interval $(0, \lambda_0)$ for some $\lambda_0 > 0$ which has the lowest order $\lambda^{2^{-r}}$ at 0. Let $\phi(u, v, \eta) : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Assume that $\phi(0, 0, \eta) \equiv 0$ for all $\eta \in \mathbb{R}$ and $\phi_u(0) = 0$, $\phi_v(0)\phi_{uu}(0) < 0$ where $\phi_u(0), \phi_v(0), \phi_{uu}(0)$ are derivatives of ϕ at 0. Then, for sufficiently small λ, the equation $\phi(x, y(\lambda), \lambda) = 0$ has exactly two solutions $x_i(\lambda)$ $(i = 1, 2)$, both of which have the lowest order $\lambda^{2^{-(r+1)}}$ at 0.*

Proof. From the Taylor expansion of ϕ at the origin we obtain

$$\begin{aligned} \phi(x, y(\lambda), \lambda) =& \phi_v(0)y(\lambda) + \frac{1}{2}\phi_{uu}(0)x^2 + \frac{1}{2}\phi_{vv}(0)y(\lambda)^2 + \phi_{uv}(0)xy(\lambda) \\ &+ \phi_{u\lambda}(0)x\lambda + \phi_{v\lambda}(0)y(\lambda)\lambda + \mathcal{O}(3). \end{aligned}$$

Let us change the coordinate as $x = x, \lambda = \mu^{2^r}$ $(\mu > 0)$ in $\phi(x, y(\lambda), \lambda) = 0$, and we get $\tilde{\phi}(x, \mu) = 0$. Since $y(\lambda)$ has the lowest order $\lambda^{2^{-r}}$ at 0, it follows that

$$\frac{\partial \tilde{\phi}(x, \mu)}{\partial \mu}(0, 0) = \phi_v(0).$$

Since $\phi_v(0) \neq 0$, the Implicit Function Theorem guarantees existence of a unique solution $\mu = \Lambda(x)$ satisfying

$$\tilde{\phi}(x, \Lambda(x)) \equiv 0$$

with $\Lambda(0) = 0$, $\Lambda'(0) = 0$ and $\Lambda''(0) = -\phi_{uu}(0)/2\phi_v(0)$. Therefore there exist $\tilde{x}_1(\mu), \tilde{x}_2(\mu)$ with the lowest order $\mu^{1/2}$. Hence there exist $x_1(\lambda), x_2(\lambda)$ with the lowest order $\lambda^{2^{-(r+1)}}$ at 0 satisfying $\phi(x, y(\lambda), \lambda) = 0$. Q.E.D.
Note that if $\phi_v(0)\phi_{uu}(0) > 0$ in the lemma, then there is no solution for x.

Let j be the smallest natural number such that $x_{\ell+j} \neq 0$. Obviously

$$x_{\ell+j} = h(\lambda) = -\frac{2f_{u\lambda}(0)}{f_{uu}(0)}\lambda + \mathcal{O}(2).$$

Here we assume that, as non-degeneracy conditions, $f_{uu}(0) \neq 0$, $f_{u\lambda}(0) \neq 0$ and $f_v(0) \neq 0$ as well. We can easily show that the equation

$$f(x_{\ell+j+1}, x_{\ell+j}, \lambda) = f(x_{\ell+j+1}, h(\lambda), \lambda) = 0$$

has only two solutions for $x_{\ell+j+1}$ with the lowest order $|\lambda|^{1/2}$ at 0 and these solutions are defined on either a positive or a negative sided neighborhood of $\lambda = 0$ depending on the signs of the derivatives $f_v(0)$, $f_{u\lambda}(0)$. If the solutions for $x_{\ell+j+1}$ are defined on a negative sided neighborhood of $\lambda = 0$, we can replace λ as $-(-\lambda)$ in the system and consider $-\lambda$ as λ. Therefore we can assume that the solutions for $x_{\ell+j+1}$ are defined on a positive sided neighborhood of $\lambda = 0$.

Once $x_{\ell+j+1}$ is chosen as $h_1(\lambda)$, we use the Lemma 3.1 repeatedly to find $x_{\ell+j+s}$ with $s > 1$. By setting $h_0(\lambda) = h(\lambda)$, we obtain a final solution of (3.10) of the form

$$X(\lambda) = (\underbrace{0, \ldots, 0}_{\ell}, 0, \ldots, 0, h_0(\lambda), h_1(\lambda), \ldots, h_p(\lambda)) \tag{3.14}$$

where $h_i(\lambda)$ has the lowest order $\lambda^{2^{-i}}$ at 0.

Note that, from the above argument, we can conclude that the equation (3.10) has exactly $2m$ solutions, each of which is of the form (3.14) for a suitable choice of $x_{\ell+j}$.

(ii) Assume that $f_u(0) + f_v(0) = 0$. This assumption leads to the occurrence of a steady-state bifurcation in the loop. Now we assume the system restricted to the loop, which is

$$\begin{cases} \dot{x}_1 = f(x_1, x_\ell, \lambda) \\ \dot{x}_2 = f(x_2, x_1, \lambda) \\ \ldots. \\ \dot{x}_\ell = f(x_\ell, x_{\ell-1}, \lambda), \end{cases} \tag{3.15}$$

or simply we can write in the form $\dot{X} = F_L(X, \lambda)$, where $X = (x_1, \ldots, x_\ell)^T$.

We need to solve the equation

$$F_L(X, \lambda) = 0. \tag{3.16}$$

The linearization of F_L at 0 is

$$DF_L(0) = f_u(0)I + f_v(0)A \tag{3.17}$$

where A is defined in (3.11) and I is the identity matrix of size ℓ.

It is already known that the only critical eigenvalue of $DF_L(0)$ is $f_u(0) + f_v(0) = 0$, so $dim\,\mathrm{Ker}\, DF_L(0) = 1$. Now we use the Lyapunov-Schmidt reduction and obtain a single reduced equation, say $g(x_j, \lambda) = 0$ for some j. Let $v = v^* = (1, \ldots, 1)$. Then v and v^* are bases for $\operatorname{Ker} DF_L(0)$ and $(\operatorname{Range} DF_L(0))^{\perp}$ respectively. Since we have the Taylor expansion (3.13), we can easily obtain:

$$\frac{\partial g}{\partial x_j} = 0,$$

$$\frac{\partial^2 g}{\partial x_j^2} = \langle v^*, d^2 F_L(v, v) \rangle = \ell(f_{uu}(0) + f_{vv}(0) + 2f_{uv}(0)),$$

$$\frac{\partial^2 g}{\partial x_j \partial \lambda} = \langle v^*, (dF_{L_\lambda}) \cdot v - d^2 F(v, L^{-1} E F_{L_\lambda}) \rangle = \ell(f_{u\lambda}(0) + f_{v\lambda}(0)).$$

We have the assumption $f(0, 0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$. Hence, as noted before, $g(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$. This shows that $g(x_j, \lambda) = x_j \times \widetilde{g}(x_j, \lambda)$, where

$$\frac{\partial \widetilde{g}}{\partial x_j} = \frac{1}{2}\ell(f_{uu}(0) + f_{vv}(0) + 2f_{uv}(0)), \quad \frac{\partial \widetilde{g}}{\partial \lambda} = \ell(f_{u\lambda}(0) + f_{v\lambda}(0)).$$

As non-degeneracy conditions we assume that $f_{uu}(0) + f_{vv}(0) + 2f_{uv}(0) \neq 0$ and $f_{u\lambda}(0) + f_{v\lambda}(0) \neq 0$. Then by the Implicit Function Theorem, we directly conclude that there exists a unique non-trivial solution $x_j = X_j(\lambda)$, and hence it shows that there exists a unique non-trivial solution $X(\lambda)$ of $F_L(X, \lambda) = 0$.

On the other hand, if we restrict the system to $x_1 = x_2 = \ldots = x_\ell = x$, then we have the equation

$$f(x, x, \lambda) = 0.$$

By (3.13), we obtain

$$x \times \Big(\frac{1}{2}\big(f_{uu}(0) + f_{vv}(0) + 2f_{uv}(0)\big)x + \big(f_{u\lambda}(0) + f_{v\lambda}(0)\big)\lambda + \mathcal{O}(2) \Big) = 0.$$

From the above non-degeneracy conditions, the Implicit Function Theorem guarantees the existence of a unique non-trivial transcritical solution $x(\lambda)$ for the equation $f(x, x, \lambda) = 0$. Since this certainly gives a solution of the original equation $F_L(X, \lambda) = 0$, by uniqueness, the only non-trivial solution of $F_L(X, \lambda) = 0$ is given by

$$X(\lambda) = \big(x(\lambda), \ldots, x(\lambda)\big).$$

(iii) Assume that $f_u(0) - f_v(0) = 0$ (and ℓ is even). The idea of the proof of this part is almost the same as before.

Since $f_u(0) - f_v(0) = 0$ is the only critical eigenvalue of $DF_L(0)$, where F_L is the restricted system to the loop, again we obtain a single reduced equation $g(x_j, \lambda) = 0$ for some j by the Lyapunov-Schmidt reduction. Let $v = v^* = (1, -1, \ldots, 1, -1)$ be bases for $\operatorname{Ker} DF_L(0)$ and $(\operatorname{Range} DF_L(0))^{\perp}$ respectively. Let us define the followings:

$$A = f_{uu}(0) + f_{vv}(0) - 2f_{uv}(0), \qquad C = f_{uu}(0) - f_{vv}(0),$$
$$B = f_{u\lambda}(0) - f_{v\lambda}(0), \qquad D = f_{uuu}(0) - 3f_{uuv}(0) + 3f_{uvv}(0) - f_{vvv}(0).$$

Easy observation shows that

$$\begin{aligned}
&d^2F_L(v,v) = (A,\dots,A),\\
&d^3F_L(v,v,v) = (D,-D,\dots,D,-D),\\
&L^{-1}Ed^2F_L(v,v) = (w_1,w_2,\dots,w_1,w_2)
\end{aligned}$$

where $w_1 + w_2 = B/f_u(0)$. Then we can obtain the following:

$$\frac{\partial g}{\partial x_j} = 0,$$

$$\frac{\partial^2 g}{\partial x_j^2} = \langle v^*, d^2F_L(v,v)\rangle = 0,$$

$$\frac{\partial^2 g}{\partial x_j \partial\lambda} = \langle v^*, (dF_{L_\lambda})\cdot v - d^2F_L(v, L^{-1}EF_{L_\lambda})\rangle = \ell B,$$

$$\frac{\partial^3 g}{\partial x_j^3} = \langle v_i^*, d^3F_L(v,v,v) - 3d^2F_L(v, L^{-1}Ed^2F_L(v,v))\rangle = \ell\left(D - \frac{3AC}{2f_u(0)}\right).$$

Similarly to the previous case, the assumption that $f(0,0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$ implies $g(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$, which shows $g(x_j,\lambda) = x_j \times \widetilde{g}(x_j,\lambda)$. Hence

$$\frac{\partial \widetilde{g}}{\partial x_j} = 0, \quad \frac{\partial \widetilde{g}}{\partial \lambda} = \ell B, \quad \frac{\partial^2 \widetilde{g}}{\partial x_j^2} = \ell\left(D - \frac{3AC}{2f_u(0)}\right).$$

From the above, under the non-degeneracy conditions

$$B \neq 0, \quad D - \frac{3AC}{2f_u(0)} \neq 0,$$

we can conclude that the steady-state bifurcation is of pitchfork type. Hence there are two non-trivial solutions for $F_L(X,\lambda) = 0$.

On the other hand, if we restrict the system to $x_1 = x_3 = \ldots = x_{\ell-1} = x$ and $x_2 = x_4 = \ldots = x_\ell = y$, then we have equations

$$\begin{cases} f(x,y,\lambda) = 0\\ f(y,x,\lambda) = 0.\end{cases}$$

This is a special case of the case (iii) when $\ell = 2$, and hence, just as above, we obtain two non-trivial solutions, say $(x(\lambda), y(\lambda)) = (x_j(\lambda), y_j(\lambda))$ with $j = 1,2$. Therefore we have two non-trivial solutions of the original problem in the case (iii) which take the form $X_j(\lambda) = (x_j(\lambda), y_j(\lambda), \dots, x_j(\lambda), y_j(\lambda))$ with $j = 1,2$. This means $X_1(\lambda)$ and $X_2(\lambda)$ are the only solutions for $F_L(X,\lambda) = 0$. Q.E.D.

One can easily show the Corollary 2.6 of the Theorem 2.5 by using the following Lemma 3.2.

Lemma 3.2. *If, generically, coupled cell systems on a fixed coupled cell network have a bifurcation branch of equilibrium for which some cells are synchronous, then this steady-state bifurcation is reducible.*

Proof. Let $X_0 = (x_1^0(\lambda), \ldots, x_n^0(\lambda))$ be a bifurcation branch of equilibrium where λ is the bifurcation parameter defined on some interval $[\lambda_1, \lambda_2]$. Then

$$\{(x_1^0(\lambda), \ldots, x_n^0(\lambda)) \mid \lambda \in [\lambda_1, \lambda_2]\}$$

defines a sub-polydiagonal in which i-th and j-th coordinates are equal if $x_i^0(\lambda) = x_j^0(\lambda)$ on $[\lambda_1, \lambda_2]$ for $i \neq j$. Then the sub-polydiagonal is invariant under every vector field on the coupled cell network because of the genericity and continuation. Hence, by Theorem 6.5 in [1], the coloring corresponding to the sub-polydiagonal is balanced. By the assumption there exist two cells with same color and hence the quotient network associated with the coloring is smaller than the original network. This shows that the steady-state bifurcation is reducible. Q.E.D.

3.2 Proof of Theorem 2.8

Let G be a coupled cell network with n identical nodes and one kind of couplings (edges) between them. Assume that any edge is bidirectional, that is, if a cell i interacts with a cell j then the cell j interacts with the cell i exactly in the same way. Also we can always assume that the nodes form vertices of a convex regular n-sided polygon. Then one can obtain a new regular coupled cell network Γ with D_n symmetry whose nodes are those of G and edges are the union of all edges of G rotated by the angles $2\pi k/n$ with $k = 1, \ldots, n$. We call G a *generator* of Γ.

Proposition 3.3. *Any regular coupled cell network with D_n symmetry has a generator.*

Proof. For any two cells there is a reflection that switches the two cells. Therefore any edge is bidirectional.

Let θ be the action of the D_n symmetry corresponding to the rotation. For any edge e there are edges $\theta e, \ldots, \theta^{n-1} e$ obtained by the rotation. Therefore the set of edges of the coupled cell network is divided into subsets

$$E_i = \{e_i, \theta e_i, \ldots, \theta^{n-1} e_i\}.$$

It is obvious that a coupled cell subnetwork with the n nodes and edges chosen only one from each E_i is a generator of the coupled cell network. Q.E.D.

From the Proposition 3.3 we can obtain that the adjacency matrix of the coupled cell network Γ is

$$A_\Gamma = A_G + B^{-1} A_G B + \cdots + B^{-(n-1)} A_G B^{n-1}$$

where A_G is the adjacency matrix of G and

$$B = \begin{pmatrix} 0 & 1 & & \Large 0 \\ & 0 & \ddots & \\ & \Large 0 & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}.$$

In order to find eigenvalues of A_Γ let us complexify the original space, that is A_Γ acts on $\mathbb{C}^n$. Let $v_\ell = (1, \zeta^\ell, \ldots, \zeta^{(n-1)\ell})^T$ where $\zeta = \exp(2\pi i/n)$ and $\ell \in \{0, 1, \ldots, n-1\}$. Then easy computation shows that

$$A_\Gamma\, v_\ell = (\overline{v_\ell}^T A_G\, v_\ell) \cdot v_\ell.$$

Hence all eigenvalues of A_Γ are $\overline{v_\ell}^T A_G\, v_\ell$ with $\ell = 0, 1, \ldots, n-1$.

Note that

i) A_Γ is a symmetric matrix, i.e., $A_\Gamma^T = A_\Gamma$. Hence its eigenvalues $\overline{v_\ell}^T A_G\, v_\ell$ are all real for $\ell = 0, 1, \ldots, n-1$.

ii) Since $v_\ell = \overline{v}_{n-\ell}$,

$$\overline{v_\ell}^T A_G\, v_\ell = \overline{v}_{n-\ell}^T A_G\, v_{n-\ell}.$$

Hence when $\ell \neq 0, n/2$, every eigenvalue of A_Γ is with multiplicity at least two and the only possibilities for an eigenvalue being with multiplicity one are when $\ell = 0, n/2$.

Proof of Theorem 2.8. Let an admissible vector field associated with an n-cell coupled cell network with D_n symmetry be given by

$$\dot{X} = F(X, \lambda),\ X = (x_1, \ldots, x_n)^T. \tag{3.18}$$

Here $\lambda \in \mathbb{R}$ is a bifurcation parameter. As discussed in Section 2.3 in [6] there is no loss of generality in assuming that the phase space of each cell is one-dimensional. So we assume that $x_i \in \mathbb{R},\ i = 1, \ldots, n$.

Note that each component of F is of the form

$$\dot{x}_i = f(x_i, \overline{x_{i_1}, \ldots, x_{i_k}}, \lambda) \tag{3.19}$$

with an identical $f : \mathbb{R}^{k+1} \times \mathbb{R} \to \mathbb{R}$ for $i = 1, \ldots, n$.

Suppose the linearization of F at 0, $D_xF(0)$, has a simple critical eigenvalue. Then, by Proposition 2.14 in [6], the critical eigenvalue is $f_u(0) + f_v(0)$ or $f_u(0) - f_v(0)$, when n is even. Observe that these are also eigenvalues of the linearizations of the quotient systems of (3.18) with one cell and two cells, for even n. If we consider synchrony-breaking steady-state bifurcation problem, then there is no additional bifurcation branch in the original system other than those in the quotient systems because the multiplicity of the critical eigenvalue is one. Therefore the synchrony-breaking steady-state bifurcation is reducible.

Suppose now $D_xF(0)$ has a critical eigenvalue with multiplicity two. By Proposition 2.14 in [6], we can consider that the eigenvalue is of the form $f_u(0) + \mu f_v(0)$ where μ is an eigenvalue of A_Γ with multiplicity two. That is,

$$f_u(0) + \mu f_v(0) = 0 \text{ (bifurcation condition).}$$

Here we assume that $f_v(0) \neq 0$ as a non-degeneracy condition. Since we are interested in synchrony-breaking steady-state bifurcations, we have to solve the following equation:

$$F(X, \lambda) = 0 \tag{3.20}$$

under the above bifurcation condition and $F(0, \lambda) \equiv 0$ for all $\lambda \in \mathbb{R}$.

It is easy to check that $dim\,\mathrm{Ker} D_xF(0) = 2$. By the Lyapunov-Schmidt reduction and Theorem 1.28 in [3], we can obtain a reduced equation

$$g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$$

which is D_n - equivariant and whose bifurcating solutions are in one-to-one correspondence with those of F.

For the moment we ignore the parameter because of its irrelevance along the proof.

We identify $\mathbb{R}^2 \cong \mathbb{C}$. By Theorem 2.24 in [3], if $h : \mathbb{C} \to \mathbb{C}$ is D_n - equivariant under the standard action, then there exists $p, q : \mathbb{R}^2 \to \mathbb{R}$ such that

$$h(z) = p(u, v)z + q(u, v)\overline{z}^{n-1} \tag{3.21}$$

where $u = z\overline{z}$ and $v = z^n + \overline{z}^n$. Here the standard action means

$$\begin{cases} \theta z = e^{i\theta} z \quad (\theta = \frac{2\pi}{n}) \\ \kappa z = \overline{z}. \end{cases}$$

The Theorem 2.24 in [3] also tells that if $q(0) \neq 0$, then there is no solution other than those predicted by the Equivariant Branching Lemma (see [3] in detail). Here, of course, we consider the eigenvalue crossing condition and it is obviously a non-degeneracy condition for f.

We shall now prove that generically $q(0) \neq 0$. In order to prove this it is enough to show that the condition holds only for one specific g, or equivalently f. Let Υ be the set of all admissible functions f in the right hand side of (3.19), namely, let

$$\Upsilon = \{f(u, \overline{v_1, \ldots, v_k}) : \mathbb{R}^{k+1} \to \mathbb{R} \mid f \text{ is smooth}\}.$$

For a non-negative integer vector $\boldsymbol{i} = (i_0, i_1, \ldots, i_k)$, define the $\boldsymbol{i}$-th Taylor coefficient of f at 0 as

$$\varphi_{\boldsymbol{i}}(f) = \Big(\frac{\partial}{\partial u}\Big)^{i_0} \Big(\frac{\partial}{\partial v_1}\Big)^{i_1} \cdots \Big(\frac{\partial}{\partial v_k}\Big)^{i_k} f(0).$$

Note that $\varphi_{\boldsymbol{i}}$ can be considered as a map $\varphi_{\boldsymbol{i}} : \Upsilon \to \mathbb{R}$ which is linear (see [5], Chapter I, §3). Since $q(0)$ is a coefficient of the reduced equation obtained by the Lyapunov-Schmidt reduction it is expressed by a polynomial of some lower degree Taylor coefficients of f. Thus,

$$[q(0)](f) = \sum_{j:\text{finite}} a_j \cdot \varphi_{j_1}(f) \cdots \varphi_{j_{\ell(j)}}(f) \tag{3.22}$$

for some $a_j \in \mathbb{R}$ and $\ell(j) \in \mathbb{N}$. Let us denote $\psi(f) = [q(0)](f) : \Upsilon \to \mathbb{R}$. If $\psi(f) \neq 0$ for some f, then there exists nonzero $a_j \in \mathbb{R}$.

Suppose $\psi(\widetilde{f}) = 0$ for some $\widetilde{f}$. Then we can perturb $\widetilde{f}$ by $cf + (1-c)\widetilde{f}$ and $\psi(cf + (1-c)\widetilde{f}) \neq 0$ for almost all $c \in \mathbb{R}$, except finite number of points in $\mathbb{R}$, since the right hand side of the equation (3.22) will be a non-constant polynomial of c. This shows denseness of the set $\psi^{-1}(\mathbb{R}\backslash\{0\})$ in Υ. Since $\varphi_{\boldsymbol{i}}(f)$ are continuous, so is $\psi(f)$ and hence the set $\psi^{-1}(\mathbb{R}\backslash\{0\})$ is open and nonempty. Therefore the set $\Upsilon\backslash\psi^{-1}(\{0\}) = \psi^{-1}(\mathbb{R}\backslash\{0\})$ is open and dense, which is generic.

Now we give examples of f for which the non-degeneracy condition $q(0) \neq 0$ holds. Consider

$$f(u, \overline{v_1, \ldots, v_k}) = -\mu u + v_1 + \cdots + v_k + u^{n-1}.$$

Recall that $(dF)_0$ has 0 as an eigenvalue with multiplicity two. Let

$$w_1 := \frac{v+\overline{v}}{2} = (1, \cos\alpha, \dots, \cos(n-1)\alpha)^T,$$
$$w_2 := \frac{v-\overline{v}}{2i} = (0, \sin\alpha, \dots, \sin(n-1)\alpha)^T$$

be eigenvectors corresponding to the eigenvalue 0 where $\mu = 2\cos\alpha$. Since $(dF)_0$ is symmetric, the eigenvectors of $(dF)_0^T$ corresponding to the eigenvalue 0 are the same.

Suppose that the action of the D_n symmetry of the original system F is defined as the following:

$$\begin{cases} \gamma(r_1, \dots, r_n)^T = (r_n, r_1, \dots, r_{n-1})^T \\ \tau(r_1, \dots, r_n)^T = (r_1, r_n, \dots, r_3, r_2)^T \end{cases}$$

for $(r_1, \dots, r_n)^T \in \mathbb{R}^n$. Then it is easy to check that for any $a, b \in \mathbb{R}$

$$\gamma^s(aw_1 + bw_2) = (a\cos\alpha - b\sin\alpha)\gamma^{s-1}w_1 + (a\sin\alpha + b\cos\alpha)\gamma^{s-1}w_2$$

for $s = 1, \dots, n$ where γ^s denotes s times iterated actions of γ and

$$\tau(aw_1 + bw_2) = aw_1 - bw_2.$$

Hence we can conclude that if we choose w_1, w_2 as a basis for $\mathrm{Ker} D_x F(0)$, then

$$g(z) = g_1(x,y) + ig_2(x,y) \qquad (z = x + iy \text{ and } x, y \in \mathbb{R})$$

is D_n-equivariant under the standard action where $g_1(x,y), g_2(x,y)$ are the reduced equations of the equation $F(X) = 0$ after using the Lyapunov-Schmidt reduction. So $g(z)$ is of the form as in the equation (3.21).

Note that since all coefficients of f with degree between two and $n-2$ are 0, we obtain that

$$\begin{aligned} a_{n-1-\ell} &:= \frac{\partial^{n-1} g_1(x,y)}{\partial x^{n-1-\ell}\,\partial y^{\ell}}(0) = \langle w_1, d^{n-1}F(\underbrace{w_1, \dots, w_1}_{n-1-\ell}, \underbrace{w_2, \dots, w_2}_{\ell})\rangle, \\ b_{n-1-\ell} &:= \frac{\partial^{n-1} g_2(x,y)}{\partial x^{n-1-\ell}\,\partial y^{\ell}}(0) = \langle w_2, d^{n-1}F(\underbrace{w_1, \dots, w_1}_{n-1-\ell}, \underbrace{w_2, \dots, w_2}_{\ell})\rangle \end{aligned} \tag{3.23}$$

for $\ell = 0, 1, \dots, n-1$. Then

$$\begin{aligned} g_1(x,y) &= \sum_{j=0}^{n-1} \binom{n}{n-1-\ell} a_{n-1-j} x^{n-1-j} y^j + \mathcal{O}(n) \\ g_2(x,y) &= \sum_{j=0}^{n-1} \binom{n}{n-1-\ell} b_{n-1-j} x^{n-1-j} y^j + \mathcal{O}(n). \end{aligned} \tag{3.24}$$

Denote that

$$\begin{aligned} R_1(x,y) &:= \sum_{j=0}^{n-1} \binom{n}{n-1-\ell} a_{n-1-j} x^{n-1-j} y^j, \\ R_2(x,y) &:= \sum_{j=0}^{n-1} \binom{n}{n-1-\ell} b_{n-1-j} x^{n-1-j} y^j. \end{aligned} \tag{3.25}$$

Let us recall the Definition 1.15 and the Theorem 1.17 from [3] for fixed point subspace. The *fixed point subspace* of Σ is

$$\mathrm{Fix}(\Sigma) = \{v \in \mathbb{R}^n : \sigma v = v \quad \text{for all} \quad \sigma \in \Sigma\}$$

where $\Sigma \subsetneqq \Gamma$ is a subgroup and Γ is a Lie group that acts on $\mathbb{R}^n$. Then $\mathrm{Fix}(\Sigma)$ is invariant under any Γ-equivariant $\phi : \mathbb{R}^n \to \mathbb{R}^n$. That is, $\phi(\mathrm{Fix}(\Sigma)) \subsetneqq \mathrm{Fix}(\Sigma)$.

Since fixed point subspace is invariant and is a projection of the original space, $q(0) \neq 0$ on a fixed point subspace will directly imply $q(0) \neq 0$ on the original space, which is our claim. Let us observe $q(0)$ along fixed point subspaces. Take $z = x \in \mathbb{R}\backslash\{0\}$ in the equation (3.21) and we have

$$g(x) = p(x^2, 0)x + q(0)x^{n-1} \pmod{|x|^n}.$$

If n is odd, $q(0)$ is precisely the coefficient of the $(n-1)$-th degree term of g. If n is even, the coefficient of the $(n-1)$-th degree term of g is $A + q(0)$ for some $A \in \mathbb{R}$. If we take $z = \exp(\pi i/2^m)x$ where $x \in \mathbb{R}\backslash\{0\}$ and $n = 2^m s$ for some odd integer s, we get that the coefficient of the $(n-1)$-th degree term of g is $\exp(\pi i/2^m)(A - q(0))$.

(i) Suppose n is odd. When $y = 0$ the coefficient of the $(n-1)$-th term of g is only $(g_1)_{x^{n-1}}$ at zero. From (3.23) we obtain that

$$(g_1)_{x^{n-1}} = \langle w_1, d^{n-1}F(w_1, \ldots, w_1)\rangle.$$

Hence

$$(g_1)_{x^{n-1}} = \frac{1}{(n-1)!}\sum_{j=0}^{n-1}(\cos j\alpha)^n = \frac{1}{(n-1)!}\frac{n}{2^{n-1}}$$

which is nonzero.

(ii) Suppose n is even. Let $n = 2^m s$, s being odd. In order to prove that $q(0) \neq 0$ it is enough to prove that the coefficient of the $(n-1)$-th term of $g(x)$ and $g(\exp(\pi i/2^m)x)$ divided by $\exp(\pi i/2^m)$ are different for $x \in \mathbb{R}\backslash\{0\}$.

Set $y = 0$. Then the coefficient of the $(n-1)$-th term of $g(x)$ is

$$(g_1)_{x^{n-1}} = \frac{1}{(n-1)!}\sum_{j=0}^{n-1}(\cos j\alpha)^n = \frac{1}{(n-1)!}\frac{n}{2^n}\left(2 + \binom{n}{\frac{n}{2}}\right).$$

On the other hand,

$$\begin{aligned}
g(e^{i\frac{\pi}{2^m}}x) &= g(\cos\frac{\pi}{2^m}x + i\sin\frac{\pi}{2^m}x) \\
&= \left(R_1(\cos\frac{\pi}{2^m}, \sin\frac{\pi}{2^m}) + iR_2(\cos\frac{\pi}{2^m}, \sin\frac{\pi}{2^m})\right)x^{n-1} + \mathcal{O}(n).
\end{aligned}$$

To simplify the notation we write $R_i(\cos\frac{\pi}{2^m}, \sin\frac{\pi}{2^m})$ as R_i for $i = 1, 2$.

Note that

$$\frac{R_1 + iR_2}{e^{i\frac{\pi}{2^m}}} = (R_1\cos\frac{\pi}{2^m} + R_2\sin\frac{\pi}{2^m}) + i(R_2\cos\frac{\pi}{2^m} - R_1\sin\frac{\pi}{2^m})$$

and it is real. Hence the coefficient of the $(n-1)$-th term of $g(\exp(\pi i/2^m)x)$ divided by $\exp(\pi i/2^m)$ is

$$R_1\cos\frac{\pi}{2^m} + R_2\sin\frac{\pi}{2^m}.$$

We claim to prove that this is not equal to $(g_1)_{x^{n-1}}$ which is found above. From (3.23) we easily obtain that

$$
\begin{aligned}
&(n-1)!\Bigg(R_1 \cos\frac{\pi}{2^m} + R_2 \sin\frac{\pi}{2^m}\Bigg)\\
&= \frac{1}{2}\sum_{j=0}^{n-1}\Bigg(\big(\cos j\alpha \cos\frac{\pi}{2^m} + \sin j\alpha \cos\frac{\pi}{2^m}\big)^n + \big(\cos j\alpha \cos\frac{\pi}{2^m} - \sin j\alpha \cos\frac{\pi}{2^m}\big)^n\Bigg)\\
&= \sum_{j=0}^{n-1}\Bigg(\cos(j\alpha + \frac{\pi}{2^m})\Bigg)^n = \frac{n}{2^n}\Bigg(2\cos\frac{\pi}{2^m} + \binom{n}{\frac{n}{2}}\Bigg) = \frac{n}{2^n}\Bigg(-2 + \binom{n}{\frac{n}{2}}\Bigg)
\end{aligned}
$$

which is not equal to $(n-1)! \cdot (g_1)_{x^{n-1}}$. The claim is proved. Therefore $q(0) \neq 0$.

We just have shown that all steady-state solutions of the system (3.18) are predicted by the Equivariant Branching Lemma. Hence we conclude that there must exist some synchronous cells for any bifurcation branch and by the Lemma 3.2 the codimension-one synchrony-breaking steady-state bifurcation is reducible. Q.E.D.

4 Discussion

The structure of a coupled cell network would become complex as the number of cells and couplings increase in general. When we consider coupled cell systems on a coupled cell network with complex structure, one of approaches to solutions of the system is to look at its quotient systems because quotient systems are simpler than the original one. Thus arises a natural question that how much one can say about the solutions of the original systems by collecting information from its quotient systems. Sometimes it may be possible to find all solutions of the original system from those of its quotient systems, but sometimes not. In this paper we try to give an idea of reducibility of bifurcations which is to understand all bifurcation solutions of coupled cell systems on a fixed coupled cell network by considering bifurcation solutions of all its quotient systems. Our aim is to understand how the network structure affects the reducibility of bifurcations in coupled cell systems on a given coupled cell network. That is, we try to find some criterion for the network structure and classify coupled cell networks by reducibility.

We begin by considering 1-input regular coupled cell networks. Our first main result is the classification of codimension-one synchrony-breaking steady-state bifurcations in 1-input regular coupled cell networks. As a corollary we determined whether these bifurcations are reducible or non-reducible.

For multiple input regular coupled cell networks we consider regular coupled cell networks with D_n symmetry. The second main result states that codimension-one synchrony-breaking steady-state bifurcations in the n-cell coupled cell networks with D_n symmetry are generically reducible, if the multiplicity of the critical eigenvalue for the bifurcation is one or two. We show that for "most" of coupled cell networks with D_n symmetry and for a generic associated coupled cell system its linearization has eigenvalues with multiplicity one or two. Also we explain the difficulty for the bifurcation problem in which the multiplicity of the critical eigenvalue is greater than two.

It is known in [6] that eigenvalues of the linearization of coupled cell systems on a regular coupled cell network are in one-to-one relation with eigenvalues of the adjacency matrix of the coupled cell network as long as each cell has one-dimensional internal dynamics. Therefore below we show that for "most" of regular coupled cell networks with D_n symmetry its adjacency matrices have eigenvalues with multiplicity one or two.

Let Γ be a regular coupled cell network with D_n symmetry. Then we can choose G as a generator of Γ so that every edge is connected to the first node in G. Let the adjacency matrix of G be

$$A_G = (a_{ij})_{i,j=1,\dots,n} \quad (a_{ij} \in \mathbb{Z}^+).$$

Then $a_{ij} = 0$ if $i, j > 1$. Moreover we can assume that $a_{1j} = a_{j1} = 0$ if $n/2 < j \le n$ and $a_{1j} = a_{j1}$ if $1 \le j < n/2$. Hence eigenvalues of A_Γ are:

$$\mu_\ell = \frac{1}{2} \sum_{j=0}^{n-1} b_{j+1}(\zeta^{j\ell} + \zeta^{-j\ell})$$

with $\ell = 0, 1, \dots, n-1$. Here $\zeta = \exp(2\pi i/n)$ and $b_{j+1} = b_{n+1-j} = a_{1,j+1}$ for $1 \le j \le n/2$. We know that $\mu_i = \mu_{n-i}$ for all $i = 1, \dots, n-1$. Hence, if there exists an eigenvalue with multiplicity more than two then there should satisfy $\mu_i = \mu_k$ for some distinct $i, k \in \{0, \dots, n-1\}$ with $i + k \ne n$. Observe that $\mu_i = \mu_k$ if and only if

$$\sum_{j=1}^{n} r_j b_j = 0 \tag{4.1}$$

where $r_j = (\zeta^{(j-1)i} + \zeta^{-(j-1)i}) - (\zeta^{(j-1)k} + \zeta^{-(j-1)k}) \in \mathbb{R}$ for $j = 1, \dots, n$. It is easy to check that $(r_1, \dots, r_n) \ne 0$.

Unless the condition (4.1) is satisfied for some i and k with $i + k \ne n$, every eigenvalue of A_Γ is with multiplicity one or two. Hence any such regular coupled cell network with D_n symmetry has an adjacency matrix whose eigenvalues being with multiplicity one or two. Below we show two examples when $n = 6, 7$.

Let Γ be a 6-cell regular coupled cell network with D_6 symmetry and $\{\mu_i\}_{i=0,\dots,5}$ be eigenvalues of A_Γ. We know that $\mu_1 = \mu_5$, $\mu_2 = \mu_4$. As discussed above we have the following:

$$\begin{cases} \mu_0 = \mu_1 & \Longleftrightarrow \quad b_2 + 3b_3 + 2b_4 = 0 \\ \mu_0 = \mu_2 & \Longleftrightarrow \quad 3b_2 + 3b_3 = 0 \\ \mu_0 = \mu_3 & \Longleftrightarrow \quad 4b_2 + 2b_4 = 0 \\ \mu_1 = \mu_2 & \Longleftrightarrow \quad 2b_2 - 2b_4 = 0 \\ \mu_1 = \mu_3 & \Longleftrightarrow \quad 3b_2 - 3b_3 = 0 \\ \mu_2 = \mu_3 & \Longleftrightarrow \quad b_2 - 3b_3 + 2b_4 = 0 \end{cases}$$

where b_2, b_3, b_4 are shown in Figure 3. Hence we conclude that A_Γ has no eigenvalue with multiplicity more than two if the following holds:

$$b_2 \ne b_3, \quad b_2 \ne b_4, \quad b_2 + 2b_4 \ne 3b_3.$$

Let Γ be a 7-cell regular coupled cell network with D_7 symmetry. Similarly, we conclude that A_Γ has no eigenvalue with multiplicity more than two if the following holds:

$$b_2 \ne b_3, \quad b_2 \ne b_4, \quad b_3 \ne b_4.$$

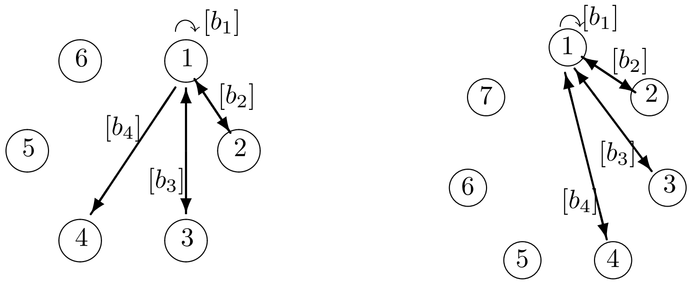

Figure 3: A generator of n-cell regular coupled cell network with D_n symmetry with $n = 6, 7$. $[\cdot]$ denotes the multiplicity of the corresponding arrow.

Furthermore, note that every eigenvalue of the adjacency matrix A_{BR_n} of the n-cell bidirectional ring is with multiplicity one or two for any $n \in \mathbb{N}$.

Now we show the difficulty of the synchrony-breaking steady-state bifurcation problem

$$F(X, \lambda) = 0 \qquad (X \in \mathbb{R}^n)$$

under the condition $F(X, \lambda) \equiv 0$ for all $\lambda \in \mathbb{R}$, where F is a coupled cell system on an n-cell coupled cell network with D_n symmetry and $D_x F(0)$ has a critical eigenvalue with multiplicity m greater than two. It is easy to check that $dim\,\mathrm{Ker} D_x F(0) = m$. By the Lyapunov-Schmidt reduction and Theorem 1.28 in [3], we can obtain a reduced equation $g : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ which is D_n-equivariant and whose bifurcating solutions are in one-to-one correspondence with those of F. Because of its irrelevance we omit the bifurcation parameter and we can identify $\mathbb{R}^m \cong \mathbb{C}^s$ if $m = 2s$ or $\mathbb{R}^m \cong \mathbb{R} \times \mathbb{C}^s$ if $m = 2s + 1$. Let $m = 2s$ and

$$g(z_1, \ldots, z_s)) = \big(g_1(z_1, \ldots, z_s), \ldots, g_s(z_1, \ldots, z_s)\big).$$

Then one can easily obtain the following from the D_n-equivariance of g:

$$\begin{cases} g(e^{i\alpha_1} z_1, \ldots, e^{i\alpha_s} z_s) = \big(e^{i\alpha_1} g_1(z_1, \ldots, z_s), \ldots, e^{i\alpha_s} g_s(z_1, \ldots, z_s)\big) \\ g(\overline{z}_1, \ldots, \overline{z}_s) = \big(\overline{g_1(z_1, \ldots, z_s)}, \ldots, \overline{g_s(z_1, \ldots, z_s)}\big) \end{cases}$$

for some $0 < \alpha_1 < \cdots < \alpha_s < 2\pi$ where $\cos n\alpha_j = 1$ for all $j = 1, \ldots, s$ and for all $(z_1, \ldots, z_s) \in \mathbb{C}^s$. Note that we can obtain a similar representation of an action of the D_n symmetry for the case $m = 2s+1$. When $s > 1$, even if there is explicit representation of an action of the D_n symmetry for the reduced vector field g, it is hard to express a general form or find a normal form of g because of the higher dimension. Therefore it is hard to obtain all solutions of the bifurcation problem and hence we can not define reducibility of the bifurcation.

The second result in this paper shows that the D_n symmetry impose a great influence on the reducibility of synchrony-breaking steady-state bifurcations. Further we would like to find another class of coupled cell networks such that bifurcations in coupled cell systems on those are reducible, e.g., by considering other symmetries. It would be interesting to study reducibility of bifurcations in a coupled cell network in relation to the symmetry of the network in general.

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