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An abstract for
“On WKB theoretic transformations for Painlevé transcendents on degenerate Stokes segments”
Kohei Iwaki

This article is an abstract for the author’s paper entitled “On WKB theoretic transformations for Painlevé transcendents on degenerate Stokes segments”. In the series of papers ([6], [1] and [7]) Aoki, Kawai and Takei discuss the WKB analysis of Painlevé transcendents (P_j) with a large parameter \( \eta \), and give a new approach for asymptotic analysis of Painlevé transcendents. (Here the list of Painlevé equations (P_j) with a large parameter \( \eta \) is given in Table 1 below.) In the paper we construct new WKB theoretic transformations of the Painlevé transcendents (P_j). That is, when the configuration of P-Stokes curves of (P_j) degenerates and contains a P-Stokes curve connecting two P-turning points (we call such a special P-Stokes curve a “P-Stokes segment”), we construct a formal transformation which reduces a 2-parameter solution of (P_j) to a 2-parameter solution of the second Painlevé equation (P_{1\Pi}) or the third Painlevé equation (P_{1\Pi}(D_7)) of type \( D_7 \) (cf. [8]) near the P-Stokes segment.

\[
(P_1) \ : \ \frac{d^2 \lambda}{dt^2} = \eta^2 (6\lambda^2 + t),
\]
\[
(P_{1\Pi}) \ : \ \frac{d^2 \lambda}{dt^2} = \eta^2 (2\lambda^3 + t\lambda + c),
\]
\[
(P_{1\Pi}(D_7)) \ : \ \frac{d^2 \lambda}{dt^2} = \frac{1}{\lambda} \left( \frac{d\lambda}{dt} \right)^2 - \frac{1}{t} \frac{d\lambda}{dt} + \eta^2 \left[ \frac{\lambda^3}{t^2} - \frac{c_{\infty}\lambda^2}{t} + \frac{c_0}{t - \lambda} \right],
\]
\[
(P_{1\Pi}(D_7)) \ : \ \frac{d^2 \lambda}{dt^2} = \frac{1}{\lambda} \left( \frac{d\lambda}{dt} \right)^2 - \frac{1}{t} \frac{d\lambda}{dt} + \eta^2 \left[ \frac{\lambda^2}{t^2} - \frac{1}{t} \right],
\]
\[
(P_{1\Pi}(D_7)) \ : \ \frac{d^2 \lambda}{dt^2} = \frac{1}{2\lambda} \left( \frac{d\lambda}{dt} \right)^2 + \eta^2 \left[ \frac{3}{2} \lambda^3 + 4t\lambda + (2t^2 - 2c_\infty)\lambda - \frac{2c_0^2}{\lambda} \right],
\]
\[
(P_{1\Pi}) \ : \ \frac{d^2 \lambda}{dt^2} = \left( \frac{1}{2\lambda} - \frac{1}{\lambda - 1} \right) \left( \frac{d\lambda}{dt} \right)^2 - \frac{1}{t} \frac{d\lambda}{dt}
+ \eta^2 \frac{2\lambda(\lambda - 1)^2}{t^2} \left[ \frac{c_{\infty}^2}{4\lambda^2} - \frac{c_0^2}{4\lambda^2} - \frac{c_1 t}{(\lambda - 1)^2} - \frac{t^2 \lambda + 1}{4(\lambda - 1)^3} \right],
\]
\[
(P_{1\Pi}) \ : \ \frac{d^2 \lambda}{dt^2} = \frac{1}{2} \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t} \right) \left( \frac{d\lambda}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t} \right) \frac{d\lambda}{dt}
+ \frac{\lambda(\lambda - 1)}{2(t - 1)(\lambda - t)} + \eta^2 \frac{2\lambda(\lambda - 1)(\lambda - t)}{t^2(t - 1)^2} \left[ \frac{c_{\infty}^2}{4\lambda^2} - \frac{c_0^2}{4\lambda^2} \frac{t}{(\lambda - 1)^2} + \frac{c_1^2}{4} \frac{t - 1}{(\lambda - 1)^2} - \frac{c_2^2}{4} \frac{t(t - 1)}{(\lambda - t)^2} \right].
\]

Table 1: Painlevé equations with a large parameter \( \eta \).
Here let us recall the notions of 2-parameter solution, $P$-turning points and $P$-Stokes curves of $(P_J)$. As is clear from Table 1, each $(P_J)$ has the following form,

$$(P_J) : \frac{d^2 \lambda}{dt^2} = G_J \left( \lambda, \frac{d\lambda}{dt}, t \right) + \eta^2 F_J(\lambda, t),$$  \hspace{1cm} (0.1)$$

where $F_J$ is a rational function in $t$ and $\lambda$, and $G_J$ is a polynomial in $d\lambda/dt$ with degree equal to or at most 2, and rational in $\lambda$ and $t$. Define the set $\text{Sing}_J \subset \mathbb{P}^1$ of singular points of $(P_J)$ by

$$\text{Sing}_I = \{\infty\}, \text{Sing}_{II} = \{0, \infty\},$$

$$\text{Sing}_{IV} = \{0, 1, \infty\}. $$

Let $t_* \in \mathbb{P}^1 \setminus \text{Sing}_J$ be a generic point and fix a holomorphic function $\lambda_0(t)$ defined in a neighborhood $V$ of $t_*$ satisfying $F_J(\lambda_0(t), t) = 0$. The 2-parameter solutions of $(P_J)$ defined on $V$ are formal solutions of $(P_J)$ of the following form:

$$\lambda_J(t, \eta; \alpha, \beta) = \lambda_0(t) + \eta^{-1/2} \sum_{j=0}^{\infty} \eta^{-j/2} \Lambda_{J/2}(t, \eta; \alpha, \beta). \hspace{1cm} (0.2)$$

Here $(\alpha, \beta) = (\sum_{n=0}^{\infty} \eta^{-n} \alpha_n, \sum_{n=0}^{\infty} \eta^{-n} \beta_n)$ is a pair of formal power series whose coefficients $\{(\alpha_n, \beta_n)\}_{n=0}^{\infty}$ parametrize the formal solution, and the functions $\Lambda_{J/2}(t, \eta; \alpha, \beta)$ labeled by half-integers have the following form:

$$\Lambda_{J/2}(t, \eta; \alpha, \beta) = \sum_{j=0}^{j+1} a_{j+1-2m}(t) \exp((j + 1 - 2m) \Phi_J(t, \eta)), \hspace{1cm} (0.3)$$

where $a_{j+1-2m}(t)$ are analytic functions defined on $V$, and

$$\Phi_J(t, \eta) = \eta \phi_J(t) + \alpha_0 \beta_0 \log(\theta_J(t) \eta^2),$$  \hspace{1cm} (0.4)$$

$$\phi_J(t) = \int^t \sqrt{F_J^{(1)}(t)} \ dt, \quad F_J^{(1)}(t) = \frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t),$$  \hspace{1cm} (0.5)$$

and $\theta_J(t)$ is a certain function analytic on $V$ (see [1], [7]).

Next we review the definitions of $P$-turning points and $P$-Stokes curves of $(P_J)$.

**Definition 0.1** ([6, Definition 2.1]). Let $\lambda_J = \lambda_J(t, \eta; \alpha, \beta)$ be a 2-parameter solution of $(P_J)$ and $\lambda_0(t)$ be its top term.

- A point $t = r \notin \text{Sing}_J$ is said to be a $P$-turning point of $\lambda_J$ if

$$F_J^{(1)}(r) = 0,$$  \hspace{1cm} (0.6)$$

where $F_J^{(1)}(t)$ is defined by (0.5).

- A $P$-turning point $t = r$ of $\lambda_J$ is called simple if

$$\frac{\partial^2 F_J}{\partial \lambda^2}(\lambda_0(r), r) \neq 0.$$  \hspace{1cm} (0.7)$$
For a $P$-turning point $t = r$ of $\lambda_J$, a $P$-Stokes curve of $\lambda_J$ (emanating from $t = r$) is an integral curve defined by

$$\text{Im} \int_r^\ell \sqrt{F_j^{(1)}(t)} \, dt = 0.$$  \hspace{1cm} (0.8)

$P$-turning points and $P$-Stokes curves are defined in terms of only the top term $\lambda_0(t)$ of the 2-parameter solution in question. Although they are defined for a fixed branch of the algebraic function $\lambda_0(t)$, we may regard them as objects on the Riemann surface of $\lambda_0(t)$. By “a $P$-turning point (resp., a $P$-Stokes curve)” we may mean “a $P$-turning point (resp., a $P$-Stokes curve) of some 2-parameter solution $\lambda_J$”, simply. By the $P$-Stokes geometry (of $(P_J)$) we mean the configuration of $P$-turning points, singular points and $P$-Stokes curves (of $(P_J)$).

We are interested in the degenerate situations of the $P$-Stokes geometry; that is, situations where degenerate $P$-Stokes segments (or $P$-Stokes segments for short) appears. Recall that a $P$-Stokes segment connects $P$-turning points (of a 2-parameter solution $\lambda_J$) of $(P_J)$. Typically there are two types of $P$-Stokes segments appear for the $P$-Stokes geometry of $(P_J)$ in a generic situation: A $P$-Stokes segment of the first type connects two different simple $P$-turning points, while a Stokes segment of the second type (sometimes called a loop-type $P$-Stokes segment) emanates from and returns to the same simple turning point and hence forms a closed loop. Figure 0.1 depicts the $P$-Stokes geometry of $(P_{II})$ when $c = i$, and $(P_{III(D_7)})$ when $c = i$ (described on the Riemann surfaces of $\lambda_0(t)$ after introducing certain new variables). Three $P$-Stokes segments appear in the $P$-Stokes geometry of $(P_{II})$, and a loop-type $P$-Stokes segment appears in the $P$-Stokes geometry of $(P_{III(D_7)})$.

![Diagram of $P$-Stokes geometries](image-url)

The $P$-Stokes geometry of $(P_{II})$ when $c \in i \mathbb{R}_{>0}$.  \hspace{1cm} The $P$-Stokes geometry of $(P_{III(D_7)})$ when $c \in i \mathbb{R}_{>0}$.

Figure 0.1: The $P$-Stokes geometries with $P$-Stokes segments.

Then, our main results are formulated as follows. Since we simultaneously deal with two different Painlevé equations $(P_J)$ and $(P_{II})$ or $(P_{III(D_7)})$, we put symbol $\sim$ over variables or functions relevant to $(P_J)$ in order to avoid confusions. Under some geometric assumptions for the Stokes geometry of linear differential equation associated with $(P_J)$ via the isomonodromic deformation (see [5]), when a $P$-Stokes segment which connects two different simple $P$-turning points (resp., a loop-type $P$-Stokes segment) appears in the $P$-Stokes geometry, then any 2-parameter solution of $(P_J)$ is reduced to a 2-parameter solution of $(P_{II})$ (resp., $(P_{III(D_7)})$) on the $P$-Stokes segment in the following sense.
Theorem 0.1. Assume that \((P_J)\) has a \(P\)-Stokes segment connecting two different simple \(P\)-turning points of \((P_J)\). Then, for any 2-parameter solution \(\lambda_J(t, \eta; \tilde{\alpha}, \tilde{\beta})\) of \((P_J)\), we can find

- formal coordinate transformation series \(x(\tilde{x}, \tilde{t}, \eta)\) and \(t(\tilde{t}, \eta)\) of dependent and independent variables,
- a 2-parameter solution \(\lambda_{II}(t, \eta; \alpha, \beta)\) of \((P_{II})\) with a suitable choice of the constant \(c\) in the equation,

satisfying

\[
x(\lambda_J(\tilde{t}, \eta; \tilde{\alpha}, \tilde{\beta}), \tilde{t}, \eta) = \lambda_{II}(t(\tilde{t}, \eta), \eta; \alpha, \beta)
\]  

(0.9)
in a neighborhood of a point \(\tilde{t} = \tilde{t}_*\) which lies on the \(P\)-Stokes segment.

Theorem 0.2. Assume that \((P_J)\) has a \(P\)-Stokes segment of loop-type. Then, for any 2-parameter solution \(\lambda_J(t, \eta; \tilde{\alpha}, \tilde{\beta})\) of \((P_J)\), we can find

- formal coordinate transformation series \(x(\tilde{x}, \tilde{t}, \eta)\) and \(t(\tilde{t}, \eta)\) of dependent and independent variables,
- a 2-parameter solution \(\lambda_{III(D')}(t, \eta; \alpha, \beta)\) of \((P_{III(D')}\) with a suitable choice of the constant \(c\) in the equation,

satisfying

\[
x(\lambda_J(\tilde{t}, \eta; \tilde{\alpha}, \tilde{\beta}), \tilde{t}, \eta) = \lambda_{III(D')}(t(\tilde{t}, \eta), \eta; \alpha, \beta)
\]  

(0.10)
in a neighborhood of a point \(\tilde{t} = \tilde{t}_*\) which lies on the \(P\)-Stokes segment of loop-type.

These theorems are proved with the aid of a relationship between the geometric properties of \(P\)-Stokes geometry of \((P_J)\) and the Stokes geometry of associated linear equation via the isomonodromic deformation discovered by [6]. Our main results can be considered as non-linear analogues of the transformation theory of [2] (to the Weber equation) and [9] (to the Bessel-type equation). In this sense the equations \((P_{II})\) and \((P_{III(D')}\) give canonical equations of Painlevé equations on a \(P\)-Stokes segment connecting different simple \(P\)-turning points and a loop-type \(P\)-Stokes segment, respectively. We expect that, together with the previous results [3] and [4], our transformation theory plays an important role in the analysis of parametric Stokes phenomena for Painlevé transcendents.

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