Stable $\mathcal{H}^\infty$ Controller Design for Infinite-dimensional Systems via Interpolation-based Approach

By

Masashi Wakaiki

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Abstract

$\mathcal{H}^\infty$ control is well appreciated as a powerful design methodology against system uncertainty. It has been playing an important role in the field of robust control. For infinite-dimensional systems such as time-delay systems, $\mathcal{H}^\infty$ control problems have been under extensive study since the mid-1980s. On the other hand, stable controller design known as strong stabilization is also an important issue of robust control from a practical point of view. This thesis presents a new solution to the $\mathcal{H}^\infty$ control problems for infinite-dimensional systems within the framework of strong stabilization.

First, we study the problem of strong stabilization with sensitivity reduction for multi-input multi-output plants having *infinitely many unstable poles*. The $\mathcal{H}^\infty$ control problem can be reduced to an interpolation problem with a unimodular matrix whose $\mathcal{H}^\infty$-norm is less than one. In conjunction with the Nevanlinna-Pick interpolation theory, this equivalence leads to a computation method of upper and lower bounds on the minimum sensitivity achievable by a stable controller. We also give a design procedure of stable controllers attaining the upper bound.

Second, we design stable controllers providing robust stability for single-input single-output plants with *infinitely many unstable poles*. We transform this robust control problem to an interpolation-minimization problem for a unit element in $\mathcal{H}^\infty$. By using the modified Nevanlinna-Pick interpolation, we obtain upper and lower bounds on the maximum perturbation under which the plant can be stabilized by a stable controller.

Third, strong stabilization with mixed sensitivity reduction is addressed. The plants we consider are allowed to have *pure delays* and *infinitely many unstable zeros*. To overcome the infinite dimensionality, the proposed method gives a new solution rooted in an operator-theoretic approach to interpolation. We introduce a new two-block problem for the design of stable $\mathcal{H}^\infty$ controllers, and then convert the problem to a one-block problem that has been solved by the operator-theoretic approach. As a result, the proposed method offers a direct design procedure. This yields the advantage that the desired controller is constructed with only linear computation as in other interpolation-based methods.

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Notation

$j := \sqrt{-1}$, the imaginary unit.
$\mathbb{R}$: the real line.
$\mathbb{R}^n$: n-dimensional Euclidean space.
$\mathbb{C}$: the complex plane.
$\mathbb{C}_+$: the open right half-plane $\{s \in \mathbb{C} : \ \mathrm{Re}\, s > 0\}$.
$\bar{\mathbb{C}}_+$: the closed right half-plane $\{s \in \mathbb{C} : \ \mathrm{Re}\, s \geq 0\}$.
$\bar{\mathbb{C}}_e$: the extended right half-plane $\bar{\mathbb{C}}_+ \cup \{\infty\}$.
$\mathbb{C}_{-\varepsilon} := \{s \in \mathbb{C} : \ \mathrm{Re}\, s > -\varepsilon\}$.
$j\mathbb{R}$: the imaginary axis $\{s \in \mathbb{C} : \ \mathrm{Re}\, s = 0\}$.
$j\mathbb{R}_e$: the extended imaginary axis $\{j\omega : \ \omega \in \mathbb{R} \cup \{\infty\}\}$.
$\mathbb{D}$: the open unit disc $\{z \in \mathbb{C} : \ |z| < 1\}$.
$\mathbb{T}$: the unit circle $\{\zeta \in \mathbb{C} : \ |\zeta| = 1\}$.
$\bar{s}$: the complex conjugate of $s \in \mathbb{C}$.
$\mathrm{Re}\, s$: the real part of $s \in \mathbb{C}$.
$\mathrm{Arg}\, s$: the argument function of $s \in \mathbb{C} \setminus \{0\}$ whose range is $(-\pi, \pi]$.
$\mathrm{Log}\, s := \ln s + \mathrm{Arg}\, s$: the principal value of the complex logarithm for $s \in \mathbb{C} \setminus \{0\}$.
$\mathbf{M}(R)$: the set of matrices with elements in a commutative ring R, of whatever order.
$R^{p\times q}$: the set of $p \times q$ matrices with entries in a commutative ring R. When it is necessary to show explicitly the size of a matrix, we use this notation.
A^*: the conjugate transpose of $A \in \mathbf{M}(\mathbb{C})$.
$A^{1/2}$: the Hermitian square root of $A \geq 0$.
$A^{-1/2}$: the inverse of the Hermitian square root of $A > 0$.
$\|v\| := (v^*v)^{1/2}$, the Euclidean norm of $v \in \mathbb{C}^p$.
$\|A\| := \sup\{\|Av\| : \ v \in \mathbb{C}^q, \ \|v\| = 1\}$, the Euclidean induced norm of $A \in \mathbb{C}^{p\times q}$.
$\mathcal{B}$: the set of matrices whose norm is less than one: $\{A \in \mathbb{C}^{p\times q} : \ \|A\| < 1\}$.
$\mathcal{H}^\infty$: the space of all bounded holomorphic functions in $\mathbb{C}_+$.
$\mathcal{RH}^\infty$: the subspace of $\mathcal{H}^\infty$ consisting of all real-rational functions. We say that a function is *real-rational* if it is expressible is defined as the ratio of two real polynomials with a nonzero denominator.
$\mathcal{F}^\infty$: the field of fractions of $\mathcal{H}^\infty$.
$\mathcal{H}^2$: the space of all holomorphic function f in $\mathbb{C}_+$ satisfing

$$\sup_{\xi>0} \left(\int_{-\infty}^{\infty} \|f(\xi + j\omega)\|^2 d\omega \right) < \infty.$$

$\det G$: the determinant of $G \in (\mathcal{F}^\infty)^{p\times p}$.
G': the derivative of G.
$G^\sim(s) := G(-\bar{s})^*$, the para-Hermitian conjugate of $G \in \mathbf{M}(\mathcal{H}^\infty)$.
$\|G\|_\infty := \sup_{s\in\mathbb{C}_+} \|G(s)\|$, the $\mathcal{H}^\infty$-norm of $G \in \mathbf{M}(\mathcal{H}^\infty)$.

$(s_i, A_i)_{i=1}^n$: the matrix-valued interpolation data of $G(s_i) = A_i$ $(i = 1, \ldots, n)$ for a matrix-valued function G.

$(s_i, [\xi_i, \eta_i])_{i=1}^n$: the tangential interpolation data of $\xi_i^* G(s_i) = \eta_i^*$ $(i = 1, \ldots, n)$ for a matrix-valued function G.

Chapter 1

Introduction

1.1 Infinite-dimensional systems

Most works on control analysis and synthesis deal only with systems modeled by ordinary differential equations. However, many types of systems in industry do not fall into this category. For example, control systems via networks include time-delays. In repetitive control, a delayed feedback loop is required for asymptotic tracking to periodic reference commands [121]. Moreover, physical phenomena such as heat conduction and structural vibration depend on both position and time. Such a system is called an *infinite-dimensional system* because its state space is infinite-dimensional. In contrast, a system represented by ordinary differential equations has a finite-dimensional state space, so it is called a *finite-dimensional system*.

In this thesis, we employ a frequency domain approach. In other words, instead of the state space representation, we take the transfer function as a model of a system. The transfer function of a finite-dimensional system is a rational function, whereas that of an infinite-dimensional system is an irrational function. This leads to the difficulty of dealing with essential singularities as well as infinitely many poles/zeros See [20] and references therein for examples of transfer functions of infinite-dimensional systems.

A controller design method for infinite-dimensional systems is to approximate the system by a rational function and then apply techniques for finite-dimensional systems to the approximation. This approach provides easily implementable controllers, but the obtained controller stabilizes only a reduced-order model, and not the original model. Hence it is not always successful due to the so-called spillover effects [2]. *Spillover* refers to the phenomenon that the uncontrolled modes lead to instability. Furthermore, the approach introduces additional parameters for approximation. Such approximation parameters may obscure the effects of the physical parameters of the original system.

For this reason, we construct a controller directly from irrational transfer functions in this thesis. Such a direct design has the disadvantage that the resulting controller is generally infinite-dimensional and hence must be approximated by a finite-dimensional system for implementation. However, it gives theoretical insight

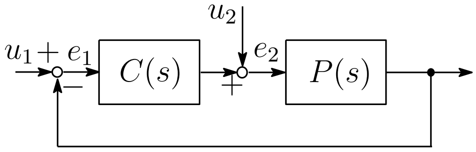


Figure 1.1: Closed-loop system.

into the performance limitation of control systems, which is difficult to obtain by the indirect controller design above.

1.2 Stable $\mathcal{H}^\infty$ controller design

We denote by $\mathcal{H}^\infty$ the space of all bounded holomorphic functions in $\mathbb{C}_+$. The field of fractions of $\mathcal{H}^\infty$ is denoted by $\mathcal{F}^\infty$. For the commutative ring R, $\mathbf{M}(R)$ denotes the set of matrices with entries in R, of whatever order. We say that G is *stable* if $G \in \mathbf{M}(\mathcal{H}^\infty)$. For $G \in \mathbf{M}(\mathcal{H}^\infty)$, the $\mathcal{H}^\infty$-norm is defined as $\|G\|_\infty = \sup_{s\in\mathbb{C}_+} \|G(s)\|$.

In this thesis, we consider the closed-loop system shown in Figure 1.1, where $P \in \mathbf{M}(\mathcal{F}^\infty)$ represents the plant and $C \in \mathbf{M}(\mathcal{F}^\infty)$ is the controller. The plant P is *stabilizable* if there exists $C \in \mathbf{M}(\mathcal{F}^\infty)$ such that the transfer matrix $H(P, C)$ from (u_1, u_2) to (e_1, e_2) satisfies

$$H(P,C) = \begin{bmatrix} (I+PC)^{-1} & -(I+PC)^{-1}P \\ C(I+PC)^{-1} & I - C(I+PC)^{-1}P \end{bmatrix} \in \mathbf{M}(\mathcal{H}^\infty). \tag{1.2.1}$$

For a given P, the set of all $C \in \mathbf{M}(\mathcal{F}^\infty)$ leading to (1.2.1) is denoted by $\mathscr{C}(P)$. P is *strongly stabilizable* if $\mathbf{M}(\mathcal{H}^\infty) \cap \mathscr{C}(P) \neq \emptyset$. We say that C *stabilizes* P if $C \in \mathscr{C}(P)$, and that C *strongly stabilizes* P if $C \in \mathbf{M}(\mathcal{H}^\infty) \cap \mathscr{C}(P)$.

In many situations, it is not enough to achieve only stability of the closed-loop system with nominal model. This is because parameters that are not known exactly can lead to modeling errors. Moreover, the actual system is subject to varied uncertainties such as disturbance and sensor noise. These perturbations can destabilize the closed-loop system and prevent it from achieving the desired performance.

Zames [128] proposed a new approach to robust control theory by introducing a controller design as an optimization problem with the $\mathcal{H}^\infty$-norm of a prespecified transfer function. The $\mathcal{H}^\infty$-norm here gives the maximal gain of the outputs against the inputs, since it is identical to the induced norm of an operator acting on L^2 spaces. Hence the optimization with $\mathcal{H}^\infty$-norm leads to the worst case analysis against system uncertainty. For instance, minimizing of the $\mathcal{H}^\infty$-norm of a closed-loop transfer function called the *sensitivity function* means that the closed-loop system is made less sensitive to disturbance. Also, if we design a controller to reduce the $\mathcal{H}^\infty$-norm of the so-called *complimentary sensitivity function*, then the controller stabilizes the plant having modeling errors. See also the section on motivation in Chapters 3 and 4 for the details of these transfer functions.

Most researches on $\mathcal{H}^\infty$ control theory impose no restriction on the stability of controllers. However, if a sensor fails, an unstable controller can destabilize the closed-loop system even with a stable plant. Moreover, it is sensitive to hard nonlinearities such as the amplitude or rate saturation of actuators. Note that we cannot deal with such nonlinearities by the $\mathcal{H}^\infty$-norm. The following example illustrates the disadvantages of unstable controllers:

Example 1.2.1. Let the plant P and the weighting functions W_1, W_2 be

$$P(s) = \frac{4s^2 - 16s + 3}{s(4s^2 + 12s + 29)}, \quad W_1(s) = \frac{1}{10s + 5.01}, \quad W_2(s) = 0.2(s + 0.6).$$

Let us add the restriction that the real parts of the poles of $H(P, C)$ in (1.2.1) are smaller than -0.5. Here we minimize not

$$\left\| \begin{bmatrix} W_1 S & W_2 T \end{bmatrix} \right\|_\infty, \tag{1.2.2}$$

but the following modified $\mathcal{H}^\infty$ norm:

$$\left\| \begin{bmatrix} W_1 S & W_2 T \end{bmatrix} \right\|_{\infty,-0.5} := \sup_{\mathrm{Re}\, s > -0.5} \left\| \begin{bmatrix} W_1(s)S(s) & W_2(s)T(s) \end{bmatrix} \right\|. \tag{1.2.3}$$

where $S := 1/(1+PC)$ is the sensitivity function and $T := 1-S$ is the complementary sensitivity function.

The $\mathcal{H}^\infty$ optimal controller C_{opt} is given by

$$C_{\mathrm{opt}}(s) = \frac{17.37(s + 0.371)(s^2 - 3.897s + 10.272)}{(s + 0.501)(s^2 - 9.891s + 84.155)}, \tag{1.2.4}$$

which has two unstable poles $p_1, p_2 \approx 4.95 \pm 7.73i$. The unstable controller C_{opt} achieves 3.474 in both norm (1.2.2) and (1.2.3).

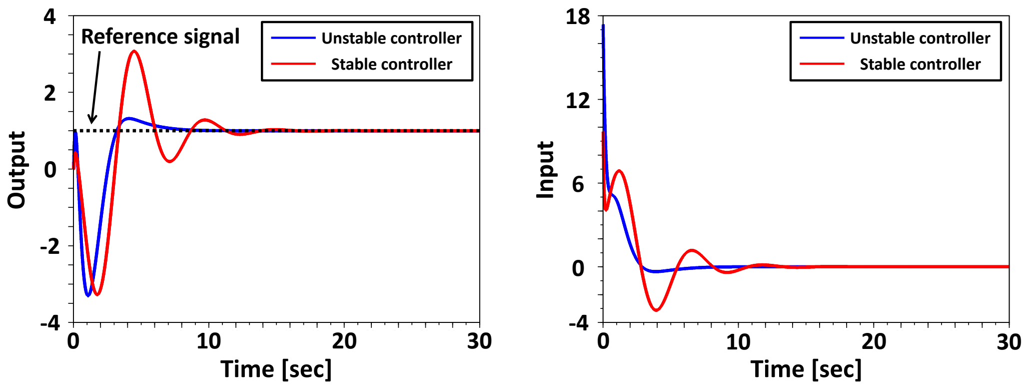
We construct a stable $\mathcal{H}^\infty$ by a MATLAB package HIFOO 3.0 [44]. The resulting controller C_{s} is

$$C_{\mathrm{s}}(s) = \frac{9.683(s + 6.989)(s^2 + 2.587s + 4.926)}{(s + 7.584)(s^2 + 1.921s + 28.196)}. \tag{1.2.5}$$

The modified norm (1.2.3) is 601.3. This is because $W_1 S$ and $W_2 T$ have poles close to $\{-0.5 + j\omega : \ \omega \in \mathbb{R}\}$. However, the stable controller C_{s} attains 3.135 in (1.2.2). We can therefore use C_{s} for performance improvement and robust stabilization.

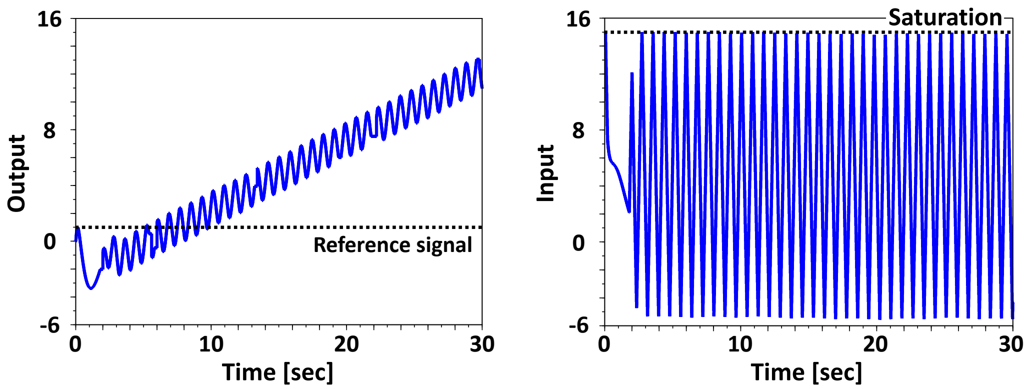
Figure 1.2 shows the step responses in the ideal situation where there are no sensor failures or actuator amplitude or rate saturation. We see from Figure 1.2 (a) that the output with the unstable controller C_{opt} has better tracking performance in terms of both response speed and overshoot than that with the stable controller C_{s}. However, we next show that the unstable controller C_{opt} performs poorly and can lead to instability when sensor failures and actuator amplitude and rate saturation occurs.

Sensor failures/packet losses between sensor and controller: The information on the plant output is not always available to the controller due to sensor failures/packet losses, and hence the controller has access to the output intermittently. Here we assume that the sensor failures/packet losses occur with probability α at every $t = 0.2n$ sec ($n \geq 0$) and that it induces the lack of the information for 0.2 sec.



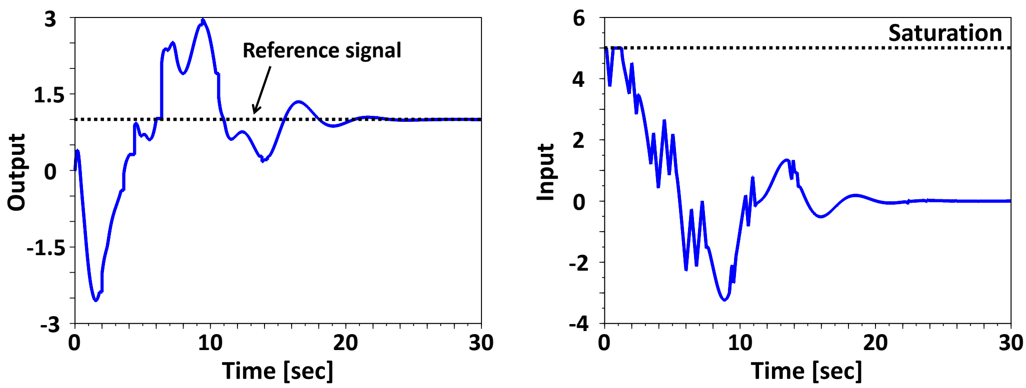
(a) Output of P (b) Input of P

Figure 1.2: Output and input responses without any sensor failures and actuator amplitude and rate saturation.



(a) Output of P (b) Input of P

Figure 1.3: Unstable controller for $\alpha = 0.05$, $L_{\mathrm{a}} = 15$, and $L_{\mathrm{r}} = 50$.



(a) Output of P (b) Input of P

Figure 1.4: Stable controller for $\alpha = 0.1$, $L_{\mathrm{a}} = 5$, and $L_{\mathrm{r}} = 5$.

When sensor failures/packet losses happen, the controller generates its output from the previous successfully transmitted data.

Actuator amplitude and rate saturation: Most actuators have physical constraints that limit the control amplitude and rate. We assume that the input of P and its time derivative are limited to the range $[-L_{\mathrm{a}}, L_{\mathrm{a}}]$ and $[-L_{\mathrm{r}}, L_{\mathrm{r}}]$, respectively.

Figure 1.3 shows the output and input of P with the unstable controller C_{opt} for $\alpha = 0.05$, $L_{\mathrm{a}} = 15$, and $L_{\mathrm{r}} = 50$. We see that the closed-loop system becomes unstable and the input of P oscillates due to the saturation of the actuator amplitude and rate after the first sensor failure at $t \approx 1$ sec.

We also confirm numerically that C_{opt} does not stabilize P for $\alpha = 0$, $L_{\mathrm{a}} = 10$, and $L_{\mathrm{r}} = 45$, that is, the case with no sensor failure. The responses are similar to Figure 1.3, so we omit them.

In contrast, Figure 1.4 shows that the stable controller C_{s} keeps the closed-loop system stable under a more limited situation with $\alpha = 0.1$, $L_{\mathrm{a}} = 5$, and $L_{\mathrm{r}} = 5$.

In [50, 70, 75, 116, 117], further comparisons are made between stable and unstable controllers.

From Example 1.2.1, we see that an unstable controller derived from $\mathcal{H}^\infty$ optimization can lead to instability of the closed-loop system in the presence of sensor failures and actuator amplitude and rate saturation. This implies that an unstable controller is sensitive to such failures and nonlinearities even if the controller is robust in the sense of $\mathcal{H}^\infty$ control theory. Hence stable controller design known as *strong stabilization* is also an important issue in robust control from a practical point of view, and it has been studied since the 1970s. The next section is devoted to a literature review of strong stabilization.

1.3 Literature review

1.3.1 Strong stabilization

We say that a function is *real-rational* if it is expressible as the ratio of two real polynomials with a nonzero denominator. A rational function is said to be *proper* if the degree of the numerator polynomial does not exceed that of the denominator[1]. Let $\mathcal{RH}^\infty$ denote the subspace of $\mathcal{H}^\infty$ consisting of all proper stable real-rational functions.

If the plant and the controller are *real-rational* and proper, then the plant is strongly stabilizable if and only if the plant satisfies the so-called parity interlacing property [127]. For single-input single-output (SISO) systems, this property means that the plant has an even number of real poles between every pair of real zeros in the extended right half-plane $\bar{\mathbb{C}}_e$ (Figure 1.5). This result remains valid for input-delay systems [1]. On the other hand, based on the results in [109], Quadrat [92] shows that *every* stabilizable plant $P \in \mathbf{M}(\mathcal{F}^\infty)$ is strongly stabilizable. However, in this case, the controller C generally belongs to $\mathbf{M}(\mathcal{H}^\infty)$ not to $\mathbf{M}(\mathcal{RH}^\infty)$. This means that $C(s) \in \mathbb{C}$ even for $s \in \mathbb{R}$.

For SISO systems, Vidyasagar [114] obtains a parameterization of all strongly stabilizing controllers by using complex exponential functions. Many interpolation-based methods to construct stable real-rational controllers are developed for SISO systems in [21, 23, 25, 87, 89] and for multi-input multi-output (MIMO) systems in [49, 91, 96, 127], respectively. Furthermore, other various approaches for strong

[1] See also (4.2.11) for the definition of the properness of irrational functions.

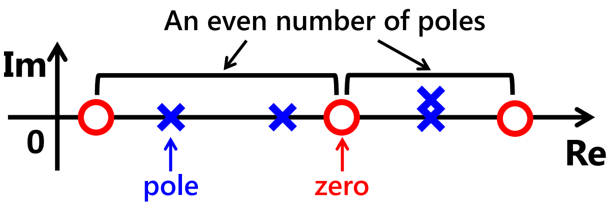


Figure 1.5: Parity interlacing property

stabilization of MIMO systems are proposed: $\mathcal{H}^\infty$ optimization [96], algebraic Riccati equations [130], and linear matrix inequalities [16, 46]. In [74, 86, 88, 102, 107], the order of stable stabilizing controllers is discussed through the investigation on the degree of a rational interpolating function in $\mathcal{H}^\infty$.

For SISO time-delay systems, a design procedure of stable controllers is developed by an interpolation-based method in [104]. Özbay [76] extends the results in [130] to infinite-dimensional MIMO systems. The synthesis of proportional-derivative (PD) controllers (or equivalently stable first-order controllers) is studied for system with input/output-delays [80, 81] and for fractional-order systems [79].

All the results above are for continuous-time, time-invariant systems whose transfer function has a single variable. However, strong stabilization is studied for varied classes of systems. For systems with several variables, sufficient conditions and necessary conditions for strong stabilization are obtained in [68, 123, 124, 126]. The authors of [125] use the results in [123] and derive an algebraic criterion for strong stabilization of time-delay systems by real-rational controllers. A sampled-data system remains strongly stabilizable if so is the original continuous-time system and if the sampling period is sufficiently small [51]. For discrete-time time-varying systems, a sufficient condition for the existence of strongly stabilizing controllers is derived in [29]. On the other hand, for continuous-time time-varying systems, it is shown in [30] that internally stabilizable systems may not be strongly stabilizable unlike in the time-invariant case [92].

Applications of stable stabilizing controllers can be found in high performance robot derives [116], magnetic bearings [99], and two-link planar robots [117].

1.3.2 Stable $\mathcal{H}^\infty$ controller design

The results on the design of stable $\mathcal{H}^\infty$ controllers can be classified in terms of whether they employ the parameterization of all $\mathcal{H}^\infty$ sub-optimal controllers or interpolation with an invertible $\mathcal{H}^\infty$ function.

Let us first review the parameterization-based approach. Most works of this approach study the standard $\mathcal{H}^\infty$ control problem for MIMO systems.

For finite-dimensional systems, the parameterization of all $\mathcal{H}^\infty$ sub-optimal controllers (see, e.g., [131]) offers a large number of design methods of stable $\mathcal{H}^\infty$ controllers by various calculation techniques, e.g., algebraic Riccati equations [6, 7, 15, 65, 66, 90, 129, 130], linear matrix inequalities [16, 46], and bilinear matrix inequalities [13, 14]. For descriptor systems, the authors of [31] provide stable $\mathcal{H}^\infty$ controller design based on the results in [105].

In [110, 111], the results of [14, 66, 130] are extended to systems with multiple input/output-delays via the controller parameterization presented in [72]. For a more general class of SISO time-delay systems, stable controllers for mixed sensitivity reduction are designed in [47], where the parameterization of [108] is used. However, the above results for time-delay systems have computational difficulties due to the infinite dimensionality of the sub-optimal $\mathcal{H}^\infty$ controllers.

We now summarize the results of the interpolation-based approach. Many of them study one-block problems such as sensitivity reduction and construct stable $\mathcal{H}^\infty$ controllers by the Nevanlinna-Pick interpolation; see Chapter 2 and references therein for the details of the Nevanlinna-Pick interpolation.

Strong stabilization with sensitivity reduction for finite-dimensional systems are studied in the SISO case [5, 38, 56] and in the MIMO case [95, 100], respectively. The authors of [53] construct stable and robust controllers for finite-dimensional systems by approximately reducing the robust stabilization problem to a nonlinear min-max optimization problem. In [52], sensitivity improvement by a stable controller is discussed for sampled-data systems via an interpolation-based approach (not with an invertible function).

The technique in [38] is generalized to plants with infinitely many unstable poles in [48, 77]. This has the computational advantage that, by checking the positive definiteness of finitely many Pick matrices, we can obtain the minimum sensitivity achievable by a stable controller even for infinite-dimensional systems.

Using a toolbox HIFOO in MATLAB, we can construct stable $\mathcal{H}^\infty$ controllers with prescribed order [44]. The toolbox is based on nonsmooth and nonconvex optimization.

Stable $\mathcal{H}^\infty$ controllers are used in many applications, e.g., flexible structures [8, 9], DC servo motors [98], and traffic networks [111]. Moreover, the authors in [112] point to the design of stable $\mathcal{H}^\infty/\mu$ controllers for high-precision wafer stage motion as a future work.

1.4 Outline of the thesis

Our design methods of stable $\mathcal{H}^\infty$ controllers are categorized into the latter: the interpolation-based approach. The main contribution of this thesis is to propose computationally attractive solutions to $\mathcal{H}^\infty$ control problems for infinite-dimensional systems within the framework with strong stabilization.

This thesis is organized as follows.

Chapter 2: We study the Nevanlinna-Pick interpolation with boundary conditions in both matrix-valued and tangential cases. First by removing all interior conditions, we reduce this interpolation problem to the problem with boundary conditions only. We next extend the Schur-Nevanlinna algorithm to show that the reduced boundary problem is always solvable.

Chapter 3: Chapter 3 addresses the problem of strong stabilization with sensitivity reduction for MIMO systems. The plants can have infinitely many unstable poles in $\mathbb{C}_+$. We compute lower and upper bounds on the minimum sensitivity achievable by

a stable controller through the Nevanlinna-Pick interpolation. Moreover, we propose a design procedure of stable controllers for sensitivity reduction.
Chapter 4: This chapter aims to construct stable controllers robustly stabilizing SISO systems. The plants we consider may have infinitely many unstable poles as in Chapter 3. We give a computation method for both lower and upper bounds on the largest plant-perturbation permissible by a stable controller. The results are based on the modified Nevanlinna-Pick interpolation proposed in [4].
Chapter 5: In this chapter, we propose the design of stable controllers that simultaneously achieve low sensitivity and robust stability. The plants here are allowed to have pure delays and infinitely many unstable zeros. We introduce a new two-block problem for the design of such stable $\mathcal{H}^\infty$ controllers. The two-block problem can be solved by matrix computation with the help of the skew Toeplitz approach in [33].
Chapter 6: This chapter summarizes the contributions of this thesis and gives some perspectives on future research.

Figure 1.6 outlines the relations between the chapters.

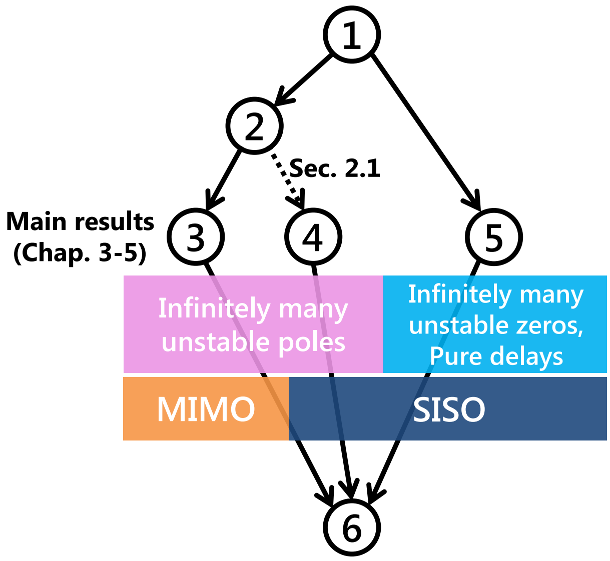


Figure 1.6: Organization of this thesis

Chapter 2

Nevanlinna-Pick Interpolation with Boundary Conditions

The Nevanlinna-Pick interpolation is useful in solving $\mathcal{H}^\infty$ control problems. In this chapter, we study the matrix-valued/tangential Nevanlinna-Pick interpolation with boundary conditions. We use the interpolation problem to solve the problem of strong stabilization with sensitivity reduction for MIMO systems with zeros on the *closed* right half-plane. Using the associated Pick matrix, the authors of [3] have already given a necessary and sufficient condition for such an extended Nevanlinna-Pick interpolation problem. In contrast, here we extend the Schur-Nevanlinna algorithm to propose a new, inductive proof of the theorem. The main contribution of the present chapter is to give a computationally efficient solution. This helps us to construct controllers achieving nearly optimal performance.

2.1 Scalar-valued Nevanlinna-Pick interpolation

In this section, we briefly review the scalar-valued Nevanlinna-Pick interpolation.

Since the results in [3, 33, 114] are developed for the open unit disk $\mathbb{D}$, it is convenient in this and the next sections to map $\mathbb{C}_+$ onto $\mathbb{D}$ via the bilinear transformation

$$s \mapsto \tau(s) := z = \frac{s-1}{s+1}.$$

That is, we consider $\mathcal{H}^\infty(\mathbb{D})$ defined by the set of functions that are bounded and holomorphic in $\mathbb{D}$, and the $\mathcal{H}^\infty$-norm is defined by $\|G\|_\infty := \sup_{z\in\mathbb{D}} \|G(z)\|$ for $G \in \mathbf{M}(\mathcal{H}^\infty(\mathbb{D}))$. We denote the closed unit disk by $\bar{\mathbb{D}}$. Note that the $f \in \mathcal{H}^\infty$ if and only if $f \circ \tau \in \mathcal{H}^\infty(\mathbb{D})$ and also that $\|f\|_\infty = \|f \circ \tau\|_\infty$.

The Nevanlinna-Pick interpolation problem is stated as follows:

Problem 2.1.1 ([3, 25, 33, 120])**.** *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ and $\mu_1, \ldots, \mu_n \in \bar{\mathbb{D}}$. Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct. Find $\phi \in \mathcal{H}^\infty(\mathbb{D})$ satisfying $\|\phi\|_\infty \le 1$ and*

$$\phi(\lambda_i) = \mu_i, \quad i = 1, \ldots, n.$$

In what follows, we use the notation of the form $(\lambda_i, \mu_i)_{i=1}^n$ to denote the interpolation data as above, i.e., associating values μ_i at λ_i.

The following theorem gives a necessary and sufficient condition for the solvability of Problem 2.1.1:

Theorem 2.1.2 ([3, 25, 33, 120]). *Consider Problem 2.1.1. Define the Pick matrix P by*

$$P := \begin{bmatrix} \frac{1-\mu_1\bar{\mu}_1}{1-\lambda_1\bar{\lambda}_1} & \cdots & \frac{1-\mu_1\bar{\mu}_n}{1-\lambda_1\bar{\lambda}_n} \\ \vdots & & \vdots \\ \frac{1-\mu_n\bar{\mu}_1}{1-\lambda_n\bar{\lambda}_1} & \cdots & \frac{1-\mu_n\bar{\mu}_n}{1-\lambda_n\bar{\lambda}_n} \end{bmatrix}. \tag{2.1.1}$$

Problem 2.1.1 is solvable if and only if P is positive semi-definite.

We obtain solutions to Problem 2.1.1 by the Schur-Nevanlinna algorithm [25]. Moreover, we can parameterize all solutions to Problem 2.1.1 by an analytic function.

Theorem 2.1.3 ([3, 33]). *Assume that the Pick matrix P defined in (2.1.1) is positive definite. If necessary, by using an appropriate one-to-one mapping of $\mathbb{D}$ onto $\mathbb{D}$, for example,*

$$s \mapsto \frac{s-\alpha}{1-\bar{\alpha}s}, \qquad \text{where } \alpha \in \mathbb{D} \text{ and } \alpha \neq \lambda_i \quad (i = 1, \ldots, n),$$

we may assume without loss of generality that $\lambda_i \neq 0$ for $i = 1, \ldots, n$.

Define

$$B(z) := \prod_{i=1}^n \frac{\lambda_i - z}{1 - \overline{\lambda_i} z} \cdot \frac{\lambda_i}{|\lambda_i|}$$

$$v_i := \overline{B(0)}/\overline{\lambda_i}, \quad v := \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}^\top$$

$$u := P^{-1}v, \quad \begin{bmatrix} u_1 & u_2 & \ldots & u_n \end{bmatrix} := x^\top.$$

Also set

$$X(z) := \overline{B(0)}B(z) - \sum_{i=1}^n \frac{B(z)}{z-\lambda_i} u_i, \quad Y(z) := -z \left(\sum_{i=1}^n \frac{\overline{\mu_i}}{1-\overline{\lambda_i} z} \overline{u_i} \right)$$

$$\tilde{X}(z) := B(z)\overline{X(1/\bar{z})}, \quad \tilde{Y}(z) := B(z)\overline{Y(1/\bar{z})}.$$

Then all solutions to the Nevanlinna-Pick interpolation problem 2.1.1 are given by

$$\phi = \frac{\tilde{X}f + \tilde{Y}}{X + Yf}, \tag{2.1.2}$$

where $f : \mathbb{D} \to \overline{\mathbb{D}}$ is an arbitrary analytic function.

Notice that the parameterization (2.1.2) of all solutions has the form of a linear fractional transformation with free parameter f and with X, Y, $\tilde{X}$, and $\tilde{Y}$ determined by the interpolation data $(\lambda_i, \mu_i)_{i=1}^n$.

2.2 Matrix-valued Nevanlinna-Pick interpolation

Our objective in this section is to show that the matrix-valued Nevanlinna-Pick interpolation problem with boundary constraints is solvable if and only if the Pick matrix consisting of the interior constraints is positive definite. We also extend the Schur-Nevanlinna algorithm [114] for the construction of solutions.

2.2.1 Interpolating interior conditions

Let us first introduce an interpolation problem with interior conditions only.

We state *the matrix-valued Nevanlinna-Pick interpolation problem* as follows:

Problem 2.2.1 ([3, 22, 114]). *Given distinct complex numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ and complex matrices $F_1, \ldots, F_n$ satisfying $\|F_i\| < 1$ for every i, find $\Phi \in \mathbf{M}(\mathcal{H}^\infty(\mathbb{D}))$ satisfying $\|\Phi\|_\infty < 1$ and*

$$\Phi(\lambda_i) = F_i, \quad i = 1, \ldots, n.$$

Problem 2.2.1 is solvable if and only if the associated Pick matrix is positive definite:

Theorem 2.2.2 ([3, 22, 114]). *Consider the matrix-valued Nevanlinna-Pick interpolation problem 2.2.1. Define the block matrix*

$$P := \begin{bmatrix} P_{1,1} & \cdots & P_{1,n} \\ \vdots & & \vdots \\ P_{n,1} & \cdots & P_{n,n} \end{bmatrix}, \tag{2.2.1}$$

where

$$P_{k,l} := \frac{1}{1 - \bar{\lambda}_k \lambda_l}(I - F_k^* F_l), \quad k,\ l = 1, \ldots, n.$$

Problem 2.2.1 is solvable if and only if P is positive definite.

Let $\mathcal{B} := \{M \in \mathbb{C}^{p\times q} :\ \|M\| < 1\}$. We need the following lemma when we develop an algorithm for the construction of solutions to the interpolation problem and when we consider the problem with boundary conditions.

Lemma 2.2.3 ([22, 114]). *Let $E \in \mathcal{B}$. Define*

$$\begin{aligned} A &:= (I - EE^*)^{-1/2}, \quad B := -(I - EE^*)^{-1/2}E \\ C &:= -(I - E^*E)^{-1/2}E^*, \quad D := (I - E^*E)^{-1/2}, \end{aligned} \tag{2.2.2}$$

where $M^{-1/2}$ denotes the inverse of the Hermitian square root of $M > 0$. Then the mapping

$$T_E : \mathcal{B} \to \mathcal{B} : X \mapsto (AX + B)(CX + D)^{-1} \tag{2.2.3}$$

is well-defined and bijective. The inverse of T_E is given by

$$T_E^{-1}(Y) = (A - YC)^{-1}(YD - B).$$

We obtain a solution to Problem 2.2.1 by iteratively reducing the number of interpolation conditions.

Theorem 2.2.4 ([22, 114]). *Consider the matrix-valued Nevanlinna-Pick interpolation problem 2.2.1. Define*

$$y(z) := \frac{|\lambda_1|(z-\lambda_1)}{\lambda_1(1-\bar{\lambda}_1 z)} \tag{2.2.4}$$

$$\hat{F}_i := \frac{1}{y(\lambda_i)} T_{F_1}(F_i), \quad i = 2, \ldots, n. \tag{2.2.5}$$

Then the original problem with n interpolation data $(\lambda_i, F_i)_{i=1}^n$ is solvable if and only if the Nevanlinna-Pick interpolation problem with $n-1$ data $(\lambda_i, \hat{F}_i)_{i=2}^n$ is solvable. Furthermore, there exist a solution Φ_n to the original problem with n interpolation conditions and a solution Φ_{n-1} to the problem with $n-1$ interpolation conditions such that

$$\Phi_n(z) = T_{F_1}^{-1}(y(z)\Phi_{n-1}(z)). \tag{2.2.6}$$

For computing solutions to Problem 2.2.1, Theorem 2.2.4 suggests an iterative algorithm called *the Schur-Nevanlinna algorithm.* Moreover, it follows from Theorem 2.2.4 that there exist solutions whose entries are rational whenever the problem is solvable.

2.2.2 Interpolating interior and boundary conditions

We study a matrix-valued interpolation problem that has not only interior conditions but also boundary conditions. We first transform it to an interpolation problem with boundary conditions only, and then show that the boundary interpolation problem is always solvable.

Let $\mathbb{T}$ be the boundary of the unit disc $\mathbb{D}$. *The matrix-valued Nevanlinna-Pick interpolation problem with boundary conditions* is stated as follows:

Problem 2.2.5. *Given distinct complex numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$, $r_1, \ldots, r_m \in \mathbb{T}$ and complex matrices $F_1, \ldots, F_n$, $G_1, \ldots, G_m$ such that $\|F_i\| < 1$, $\|G_k\| < 1$ for all i, k. Find a rational matrix function $\Phi \in \mathbf{M}(\mathcal{H}^\infty(\mathbb{D}))$ satisfying $\|\Phi\|_\infty < 1$ and*

$$\Phi(\lambda_i) = F_i, \quad \Phi(r_k) = G_k, \qquad i = 1, \ldots, n, \quad k = 1, \ldots, m.$$

The scalar version of Problem 2.2.5 is studied in [3, Chap. 21], [33, Chap. 2], and [69]. The approach of [3, Chap. 21] and [33, Chap. 2] is based on the corresponding Pick matrix. On the other hand, the method in [69] uses the Schur-Nevanlinna algorithm. Here we extend the method in [69] to the matrix-valued case.

This subsection aims to prove the next theorem. The theorem means that the boundary conditions $(r_k, G_k)_{k=1}^m$ do not affect the solvability of Problem 2.2.5.

Theorem 2.2.6. *Problem 2.2.5 is solvable if and only if the matrix-valued Nevanlinna-Pick interpolation problem 2.2.1 with interpolation data $(\lambda_i, F_i)_{i=1}^n$ is solvable.*

To prove Theorem 2.2.6, we need to reduce Problem 2.2.5 to the following problem:

Problem 2.2.7. *Given distinct complex numbers $r_1, \ldots, r_m \in \mathbb{T}$ and complex matrices $G_1, \ldots, G_m$ satisfying $\|G_k\| < 1$ for every k. Find a rational matrix function $\Psi \in \mathbf{M}(\mathcal{H}^\infty(\mathbb{D}))$ satisfying $\|\Psi\|_\infty < 1$ and*

$$\Psi(r_k) = G_k, \quad k = 1, \ldots, m.$$

The problem above is called *the boundary matrix-valued Nevanlinna-Pick interpolation problem*. It is obvious that the conditions $\|G_k\| < 1$ are necessary for the existence of solutions to Problem 2.2.7. The following lemma suggests that the conditions are also sufficient.

Lemma 2.2.8 ([3, 103])**.** *The boundary matrix-valued Nevanlinna-Pick interpolation problem 2.2.7 is always solvable.*

We can prove Lemma 2.2.8 in the same way as in [3, 33, 103] based on the associated Pick-matrix; see Section 2.4 for details. By contrast, here we extend the Schur-Nevanlinna algorithm. This gives a proof leading to the more computationally efficient construction of solutions.

Proof of Lemma 2.2.8. It suffices to show that there is always a boundary Nevanlinna-Pick interpolation problem 2.2.7 with $m-1$ interpolation conditions in such a way that if the problem with $m-1$ conditions is solvable, then the original problem with m data $(r_k, G_k)_{k=1}^m$ is also solvable.

Let $\epsilon > 0$. We define

$$y_\epsilon(z) := \frac{1}{r_1} \frac{z - r_1}{(1+\epsilon) - \bar{r}_1 z} \tag{2.2.7}$$

$$\hat{G}_k := \frac{1}{y_\epsilon(r_k)} T_{G_1}(G_k), \qquad k = 2, \ldots, m. \tag{2.2.8}$$

Let us first show that the interpolation data $(r_k, \hat{G}_k)_{k=2}^m$ are well-defined. To see this, we prove that there exists $\epsilon > 0$ such that

$$\|\hat{G}_k\| < 1, \quad k = 2, \ldots, m. \tag{2.2.9}$$

By definition, we have

$$\begin{aligned} \|\hat{G}_k\| &= \left\| \frac{1}{y_\epsilon(r_k)} T_{G_1}(G_k) \right\| \\ &= \left| 1 - \frac{\epsilon r_1}{r_k - r_1} \right| \cdot \|T_{G_1}(G_k)\| \\ &\leq \left(1 + \frac{\epsilon}{|r_k - r_1|} \right) \cdot \|T_{G_1}(G_k)\|. \end{aligned} \tag{2.2.10}$$

Since $G_k \in \mathcal{B}$, it follows that $\|T_{G_1}(G_k)\| < 1$ by Lemma 2.2.3. Hence there exists ϵ such that

$$0 < \epsilon < \min_{k=2,\ldots,m} \left(|r_k - r_1| \cdot \left(\frac{1}{\|T_{G_1}(G_k)\|} - 1 \right) \right). \tag{2.2.11}$$

If ϵ satisfies (2.2.11), then

$$\left(1 + \frac{\epsilon}{|r_k - r_1|}\right) \cdot \|T_{G_1}(G_k)\| < 1.$$

Combining this with (2.2.10), we obtain the desired inequality (2.2.9).

Assume that there exists a solution $\Psi_{m-1} \in \mathbf{M}(\mathcal{H}^\infty(\mathbb{D}))$ to the boundary interpolation problem 2.2.7 with $m-1$ interpolation data $(r_k, \hat{G}_k)_{k=2}^m$. We now show that $\Psi_m(z) := T_{G_1}^{-1}(y_\epsilon(z)\Psi_{m-1}(z))$ is a solution to the original problem with m interpolation data $(r_k, G_k)_{k=1}^m$.

Since the domain of $T_{G_1}^{-1}$ is $\mathcal{B}$, to begin with, we need to show

$$y_\epsilon(z)\Psi_{m-1}(z) \in \mathcal{B}, \quad z \in \bar{\mathbb{D}}. \tag{2.2.12}$$

By definition, $\|y_\epsilon\|_\infty < 1$ and $\|\Psi_{m-1}\|_\infty < 1$, and hence $\|y_\epsilon \Psi_{m-1}\|_\infty < 1$. This is equivalent to (2.2.12).

Clearly, Ψ_m is rational and belongs to $\mathbf{M}(\mathcal{H}^\infty(\mathbb{D}))$. Also, (2.2.12) and Lemma 2.2.3 lead to $\|\Psi_m\|_\infty < 1$.

Now we confirm that Ψ_m satisfies the interpolation conditions. When $k = 1$, it follows from the definition (2.2.7) of y_ϵ that

$$\Psi_m(r_1) = T_{G_1}^{-1}(y_\epsilon(r_1)\Psi_{m-1}(r_1)) = T_{G_1}^{-1}(0) = G_1.$$

For $k = 2, \dots, m$, the definition (2.2.8) of $\hat{G}_k$ gives

$$\begin{aligned}
\Psi_m(r_k) &= T_{G_1}^{-1}(y_\epsilon(r_k)\Psi_{m-1}(r_k)) \\
&= T_{G_1}^{-1}(y_\epsilon(r_k)\hat{G}_k) \\
&= T_{G_1}^{-1}(T_{G_1}(G_k)) \\
&= G_k.
\end{aligned}$$

Thus Φ_m is a solution to the original problem with m interpolation conditions.

We have shown that the boundary interpolation problem 2.2.7 with given interpolation data can be reduced to the same problem 2.2.7 with one interpolation data fewer. Continuing this way, we arrive at Problem 2.2.7 with only one interpolation condition, which always admits a solution. Thus Problem 2.2.7 is always solvable. □

Finally, we prove Theorem 2.2.6 by using Theorem 2.2.4 and Lemma 2.2.8.

Proof of Theorem 2.2.6. The necessity is straightforward.

We show the sufficiency as follows. Suppose that the matrix-valued Nevanlinna-Pick interpolation problem 2.2.1 with interpolation data $(\lambda_i, F_i)_{i=1}^n$ is solvable. Theorem 2.2.4 implies the existence of a function satisfying $n-1$ interior and m boundary constraints derived from (2.2.5). Define the new interpolating values $\check{G}_k$ on the boundary $\mathbb{T}$ by

$$\check{G}_k := \frac{1}{y(r_k)} T_{F_1}(G_k).$$

Since y defined by (2.2.4) satisfies $|y(r_k)| = 1$ for every k and since $\|T_{F_1}(G_k)\| < 1$ by Lemma 2.2.3, we have $\|\check{G}_k\| < 1$. Continuing this way, we can finally transform Problem 2.2.5 to the boundary interpolation problem 2.2.7. Moreover, Lemma 2.2.8 shows that the boundary interpolation problem 2.2.7 has always solutions, and hence Problem 2.2.5 is solvable. $\square$

Combining Theorem 2.2.2 with Theorem 2.2.6, we obtain the following corollary:

Corollary 2.2.9. *Consider Problem 2.2.5. Define the Pick matrix P by* (2.2.1) *with interior conditions $(\lambda_i, F_i)_{i=1}^n$. Problem 2.2.5 is solvable if and only if P is positive definite.*

The proofs of Lemma 2.2.8 and Theorem 2.2.6 suggest that we can compute a solution to Problem 2.2.5 by an iterative algorithm, which is an extension of the Schur-Nevanlinna algorithm.

Example 2.2.10. We compute $\Phi \in \mathbf{M}(\mathcal{H}^\infty(\mathbb{D}))$ satisfying $\|\Phi\|_\infty < 1$ and the following interpolation conditions:

$$\Phi(1) = G_1 := \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \Phi(-1) = G_2 := \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Phi(1/3) = F := \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}. \tag{2.2.13}$$

Since G_1, G_2, $F \in \mathcal{B}$ and we have only one interior condition, there exist solutions to the problem.

We first reduce this problem to the boundary interpolation problem 2.2.7. New interpolation values on the boundary $\mathbb{T}$ are

$$\hat{G}_1 \approx \begin{bmatrix} 0.4701 & -0.0094 \\ 0.1374 & 0.1428 \end{bmatrix}, \quad \hat{G}_2 \approx \begin{bmatrix} -0.3856 & 0.3171 \\ 0.1863 & -0.0836 \end{bmatrix}.$$

We calculate $\Psi \in \mathbf{M}(\mathcal{H}^\infty(\mathbb{D}))$ satisfying $\|\Psi\|_\infty < 1$ and boundary conditions $\Psi(1) = \hat{G}_1$, $\Psi(-1) = \hat{G}_2$. A solution Ψ is given by

$$\Psi(z) \approx \begin{bmatrix} \frac{-0.45(z-1.10)(z-0.84)}{(z-1.156)(z-1.104)} & \frac{0.34(z-1.11)(z-1.00)}{(z-1.156)(z-1.104)} \\ \frac{0.19(z^2-2.22z+1.23)}{(z-1.156)(z-1.104)} & \frac{-0.10(z-1.15)(z-0.84)}{(z-1.156)(z-1.104)} \end{bmatrix}.$$

Finally we define y by (2.2.4), that is,

$$y(z) := \frac{3z-1}{3-z},$$

and then a solution to the original interpolation problem is

$$\begin{aligned} \Phi(z) &= T_F^{-1}(y(z)\Psi(z)) \\ &\approx \begin{bmatrix} \frac{1.63(z-3.09)(z-1.16)(z^2-1.40+0.51)}{(z-3.45)(z-3.04)(z-1.16)(z-1.11)} & \frac{-0.78(z-3.07)(z-1.13)(z-1.05)(z+1)}{(z-3.45)(z-3.04)(z-1.16)(z-1.11)} \\ \frac{-0.47(z-3.17)(z-1.17)(z-0.16)(z+1)}{(z-3.45)(z-3.04)(z-1.16)(z-1.11)} & \frac{0.77(z-3.11)(z-1.69)(z-1.24)(z-1.17)}{(z-3.45)(z-3.04)(z-1.16)(z-1.11)} \end{bmatrix}. \end{aligned}$$

We see that Φ satisfies the interpolation conditions (2.2.13) and $\|\Phi\|_\infty \approx 0.7506 < 1$.

2.3 Tangential Nevanlinna-Pick interpolation

In this section, we consider the Nevanlinna-Pick interpolation problem with tangential interpolation conditions $\xi_i^* \Phi(\alpha_i) = \eta_i^*$. As in the previous section, we give a necessary and sufficient condition for the Nevanlinna-Pick interpolation with boundary conditions. Also, we construct solutions to the problem by extending the Schur-Nevanlinna algorithm. Note that, in this section, we study matrix-valued functions on $\mathbb{C}_+$, not on $\mathbb{D}$.

2.3.1 Interpolating interior conditions

Let us first introduce an interpolation problem with interior conditions only. The problem is called *the tangential Nevanlinna-Pick interpolation problem*. It is formally formulated as follows:

Problem 2.3.1 ([3, 27, 62, 67]). *Given distinct complex numbers $\alpha_1, \dots \alpha_n \in \mathbb{C}_+$ and vector pairs*

$$(\xi_i, \eta_i) \in \mathbb{C}^p \times \mathbb{C}^q, \quad i = 1, \dots, n$$

satisfying

$$\|\xi_i\| - \|\eta_i\| > 0, \quad i = 1, \dots, n. \tag{2.3.1}$$

Find $\Phi \in (\mathcal{H}^\infty)^{p \times q}$ satisfying $\|\Phi\|_\infty < 1$ and

$$\xi_i^* \Phi(\alpha_i) = \eta_i^*, \quad i = 1, \dots, n. \tag{2.3.2}$$

We denote the interpolation data of (2.3.2) by $(\alpha_i, [\xi_i, \eta_i])_{i=1}^n$.

There is a solution to Problem 2.3.1 if and only if the associated Pick matrix is positive definite:

Theorem 2.3.2 ([3, 62, 67]). *Consider the tangential Nevanlinna-Pick interpolation problem 2.3.1. Define the Pick matrix*

$$P := \begin{bmatrix} P_{1,1} & \cdots & P_{1,n} \\ \vdots & & \vdots \\ P_{n,1} & \cdots & P_{n,n} \end{bmatrix}, \tag{2.3.3}$$

where

$$P_{k,l} := \frac{\xi_k^* \xi_l - \eta_k^* \eta_l}{\alpha_k + \bar{\alpha}_l}, \quad k,\ l = 1, \dots, n.$$

Problem 2.3.1 is solvable if and only if P is positive definite.

To calculate a solution to Problem 2.3.1 in an iterative way, we use Lemma 2.3.3 below. This lemma gives a transformation that preserves the condition (2.3.1) on the vector pair (ξ_i, η_i).

Lemma 2.3.3 ([62]). *Let $E \in \mathcal{B}$. Set*

$$\begin{aligned} A &:= (I - EE^*)^{-1/2}, \quad B := (I - EE^*)^{-1/2} E \\ C &:= (I - E^*E)^{-1/2} E^*, \quad D := (I - E^*E)^{-1/2}. \end{aligned} \tag{2.3.4}$$

Define T_{AB} and T_{CD} by

$$\begin{aligned} T_{AB} &: \mathbb{C}^p \times \mathbb{C}^q \to \mathbb{C}^p : (\xi, \eta) \mapsto A\xi - B\eta \\ T_{CD} &: \mathbb{C}^p \times \mathbb{C}^q \to \mathbb{C}^q : (\xi, \eta) \mapsto -C\xi + D\eta. \end{aligned} \tag{2.3.5}$$

Then we have

$$\|\xi\|^2 - \|\eta\|^2 = \|T_{AB}(\xi, \eta)\|^2 - \|T_{CD}(\xi, \eta)\|^2.$$

Theorem 2.3.4 ([62, 67]). *Consider the tangential Nevanlinna-Pick interpolation problem 2.3.1. Define*

$$E := \frac{\xi_1 \cdot \eta_1^*}{\|\xi_1\|^2},$$

and matrices A, B, C, and D by (2.3.4). *Define also*

$$\nu := T_{AB}(\xi_1, \eta_1), \quad \kappa(s) := \frac{s - \alpha_1}{s + \bar{\alpha}_1}, \tag{2.3.6}$$

$$X := I + (\kappa - 1)\frac{\nu \cdot \nu^*}{\|\nu\|^2}. \tag{2.3.7}$$

Then the original problem with n interpolation data $(\alpha_i, [\xi_i, \eta_i])_{i=1}^n$ is solvable if and only if the tangential Nevanlinna-Pick interpolation problem with $n-1$ data

$$(\alpha_i, [X(\alpha_i)^* T_{AB}(\xi_i, \eta_i), T_{CD}(\xi_i, \eta_i)])_{i=2}^n$$

is solvable. Moreover, there exist a solution Φ_n to the original problem with n interpolation conditions and a solution Φ_{n-1} to the problem with $n-1$ interpolation conditions such that

$$\Phi_n(s) = T_E\left(X(s)\Phi_{n-1}(s)\right), \tag{2.3.8}$$

where T_E is defined by (2.2.3).

Similarly to Theorem 2.2.4, an iterative algorithm derived from Theorem 2.3.4 below is called *the Schur-Nevanlinna algorithm.* Theorem 2.3.4 also shows that if the problem is solvable, then there exist always solutions whose elements are rational functions.

Remark 2.3.5.

1. In (2.3.4), we have the same definitions of A and D as in (2.2.2). However, note that the definitions of B and C in (2.2.2) have a minus sign, whereas those in (2.3.4) do not. Moreover, we use the inverse of T_{F_1} in the matrix-valued case (2.2.6) but T_E itself in the tangential case (2.3.8) when we construct Φ_n from Φ_{n-1}.
2. Note that ν in (2.3.6) is nonzero. In fact, since $\|\xi_1\| > \|\eta_1\|$, we have

$$A^{-1}\nu = \xi_1 - E\eta_1 = \xi_1 - \xi_1 \cdot \frac{\|\eta_1\|^2}{\|\xi_1\|^2} \neq 0,$$

and hence $\nu \neq 0$.

2.3.2 Interpolating interior and boundary conditions

In this subsection, we study the tangential Nevanlinna-Pick interpolation problem that has interpolation conditions on the extended imaginary axis $j\mathbb{R}_e := \{j\omega : \omega \in \mathbb{R} \cup \{\infty\}\}$. To solve this problem, we reduce it to an interpolation problem with boundary conditions only, and show that the boundary interpolation problem is always solvable.

The tangential Nevanlinna-Pick interpolation problem with boundary conditions is stated as follows:

Problem 2.3.6 ([3, 67])**.** *Suppose $\alpha_1, \ldots \alpha_n \in \mathbb{C}_+$ and $j\omega_1, \ldots, j\omega_m \in j\mathbb{R}_e$ are distinct. Let vector pairs (ξ_i, η_i) and (x_k, y_k) in $\mathbb{C}^p \times \mathbb{C}^q$ satisfy*

$$\|\xi_i\| - \|\eta_i\| > 0, \quad \|x_k\| - \|y_k\| > 0, \quad i = 1, \ldots, n, \; k = 1, \ldots, m.$$

Find a rational matrix function $\Phi \in (\mathcal{H}^\infty)^{p\times q}$ such that $\|\Phi\|_\infty < 1$ and

$$\xi_i^* \Phi(\alpha_i) = \eta_i^*, \quad x_k^* \Phi(j\omega_k) = y_k^*, \quad i = 1, \ldots, n, \; k = 1, \ldots, m.$$

Remark 2.3.7. In Problem 2.3.6, we may have an interpolation condition at ∞. Since we consider only proper rational functions f in this subsection, $f(\infty)$ is well-defined and finite.

We say that a rational function f is *strictly proper* if $f(\infty) = 0$, that is, the degree of the numerator polynomial is less than that of the denominator[1]. In control theory, we need to deal with interpolation at ∞ if the plant is strictly proper.

Our objective of this subsection is Theorem 2.3.8. The theorem means that the solvability of Problem 2.3.6 is dependent on only its interpolation data in $\mathbb{C}_+$.

Theorem 2.3.8. *Problem 2.3.6 is solvable if and only if the tangential Nevanlinna-Pick interpolation problem 2.3.1 with interpolation data $(\alpha_i, [\xi_i, \eta_i])_{i=1}^n$ is solvable.*

To solve Problem 2.3.6, we transform it to the following problem with boundary conditions only:

Problem 2.3.9 ([3])**.** *Given distinct imaginary numbers $j\omega_1, \ldots, j\omega_m \in j\mathbb{R}_e$ and vector pairs*

$$(x_k, y_k) \in \mathbb{C}^p \times \mathbb{C}^q, \quad k = 1, \ldots, m.$$

satisfying

$$\|x_k\| - \|y_k\| > 0, \quad k = 1, \ldots, m. \tag{2.3.9}$$

Find a rational matrix function $\Psi \in (\mathcal{H}^\infty)^{p\times q}$ satisfying $\|\Psi\|_\infty < 1$ and

$$x_k^* \Psi(j\omega_k) = y_k^*, \quad k = 1, \ldots, m.$$

This problem is called *the boundary tangential Nevanlinna-Pick interpolation problem*. Clearly, the condition (2.3.9) is necessary for the solvability for Problem 2.3.9. The lemma below shows that the condition is also sufficient. We prove it by extending the Schur-Nevanlinna algorithm.

[1] See also (4.2.12) for the definition of the strict properness of irrational functions.

Lemma 2.3.10 ([3])**.** *The boundary tangential Nevanlinna-Pick interpolation problem 2.3.9 is always solvable.*

Proof. It suffices to show that there always exists a boundary Nevanlinna-Pick interpolation problem such that the problem has $m-1$ interpolation conditions and if the problem has a solution, the original problem with m interpolation data $(\omega_k, [x_k, y_k])_{k=1}^m$ is solvable.

Let ω_1 be finite, i.e., not ∞. We can remove the interpolation condition at $j\omega_1$ as follows. As in Theorem 2.3.4, define E by

$$E := \frac{x_1 \cdot y_1^*}{\|x_1\|^2} \tag{2.3.10}$$

and A, B, C, and D by (2.3.4). Fix $\epsilon > 0$ and set

$$\nu := T_{AB}(x_1, y_1), \quad \kappa_\epsilon(s) := \frac{s - j\omega_1}{s - j\omega_1 + \epsilon} \tag{2.3.11}$$

$$X_\epsilon := I + (\kappa_\epsilon - 1)\frac{\nu \cdot \nu^*}{\|\nu\|^2}. \tag{2.3.12}$$

First we show that there exists $\epsilon > 0$ such that

$$\|(X_\epsilon(j\omega_k)^* T_{AB}(x_k, y_k)\| - \|T_{CD}(x_k, y_k))\| > 0, \quad k = 2, \dots, m, \tag{2.3.13}$$

which means that the data

$$(j\omega_k, [X_\epsilon(j\omega_k)^* T_{AB}(x_k, y_k), T_{CD}(x_k, y_k)])_{k=2}^m \tag{2.3.14}$$

lead to the well-defined interpolation conditions.

Since

$$X_\epsilon \nu = \kappa_\epsilon \nu$$

by the definition (2.3.12) of X_ϵ, a routine calculation shows that

$$I - X_\epsilon(j\omega_k)X_\epsilon(j\omega_k)^* = \frac{\epsilon^2}{\epsilon^2 + (\omega_k - \omega_1)^2} \cdot \frac{\nu \cdot \nu^*}{\|\nu\|^2}. \tag{2.3.15}$$

In conjunction with (2.3.3), (2.3.15) shows that

$$\begin{aligned}\big(\|x_k\|^2 - \|y_k\|^2\big) - \big(\|(X_\epsilon(j\omega_k)^* T_{AB}(x_k, y_k)\|^2 - \|T_{CD}(x_k, y_k))\|^2\big) \\ = T_{AB}(x_k, y_k)^*(I - X_\epsilon(j\omega_k)X_\epsilon(j\omega_k)^*)T_{AB}(x_k, y_k) \\ = \frac{\epsilon^2}{\epsilon^2 + (\omega_k - \omega_1)^2} \cdot \frac{|\nu^* \cdot T_{AB}(x_k, y_k)|^2}{\|\nu\|^2}.\end{aligned} \tag{2.3.16}$$

If $\nu^* \cdot T_{AB}(x_k, y_k) = 0$ or $\omega_k = \infty$, (2.3.9) and (2.3.16) lead to the desired inequality (2.3.13). Hence it suffices to consider the case

$$\nu^* \cdot T_{AB}(x_k, y_k) \neq 0 \;\text{ and }\; \omega_k \neq \infty, \quad k = 2, \dots, m.$$

Note that

$$\frac{\epsilon^2}{\epsilon^2+(\omega_k-\omega_1)^2}\cdot\frac{|\nu^*\cdot T_{AB}(x_k,y_k)|^2}{\|\nu\|^2}<\frac{\epsilon^2}{(\omega_k-\omega_1)^2}\cdot\frac{|\nu^*\cdot T_{AB}(x_k,y_k)|^2}{\|\nu\|^2}. \tag{2.3.17}$$

Since $\omega_1\neq\omega_k$ for $k=2,\ldots,m$ and since $\nu\neq 0$ by Remark 2.3.5.2, there exists ϵ such that

$$0<\epsilon<\min_{2\le k\le m}\left(\frac{\|\nu\|\cdot|\omega_k-\omega_1|}{|\nu^*\cdot T_{AB}(x_k,y_k)|}\cdot\sqrt{\|x_k\|^2-\|y_k\|^2}\right) \tag{2.3.18}$$

For every ϵ in (2.3.18), we have

$$\frac{\epsilon^2}{(\omega_k-\omega_1)^2}\cdot\frac{|\nu^*\cdot T_{AB}(x_k,y_k)|^2}{\|\nu\|^2}<\|x_k\|^2-\|y_k\|^2.$$

Thus (2.3.16) and (2.3.17) lead to the desired inequality (2.3.13)

Let Ψ_{m-1} be a solution to a boundary Nevanlinna-Pick interpolation problem with $m-1$ interpolation data (2.3.14). We now prove that

$$\Psi_m(s):=T_E(X_\epsilon(s)\Psi_{m-1}(s))$$

is a solution to the original problem with m interpolation data $(\omega_k,[x_k,y_k])_{k=1}^m$.

Let us denote by $\bar{\mathbb{C}}_e$ the extended right half-plane $\bar{\mathbb{C}}_+\cup\{\infty\}$. First of all, we have to prove $X_\epsilon(s)\Psi_{m-1}(s)\in\mathcal{B}$, that is,

$$\|X_\epsilon(s)\Psi_{m-1}(s)\|<1,\quad s\in\bar{\mathbb{C}}_e, \tag{2.3.19}$$

because the domain of T_E is $\mathcal{B}$. For all $s\in\bar{\mathbb{C}}_e$, we have $|\kappa_\epsilon(s)|\le 1$. Hence

$$I-X_\epsilon^*(s)X_\epsilon(s)=(1-|\kappa_\epsilon(s)|^2)\cdot\frac{\nu\cdot\nu^*}{\|\nu\|^2}\ge 0,\quad s\in\bar{\mathbb{C}}_e,$$

which is equivalent to $\|X_\epsilon\|_\infty\le 1$. In conjunction with $\|\Phi_{m-1}\|_\infty<1$, this leads to (2.3.19).

To prove that Ψ_m is a solution, we should show that $\Psi_m\in(\mathcal{H}^\infty)^{p\times q}$, $\|\Psi_m\|_\infty<1$, and

$$x_k^*\Psi_m(j\omega_k)=y_k^*,\quad k=1,\ldots,m. \tag{2.3.20}$$

Obviously, Ψ_m is a rational matrix function in $(\mathcal{H}^\infty)^{p\times q}$, and (2.3.19) and Lemma 2.2.3 show $\|\Psi_m\|_\infty<1$.

We can prove (2.3.20) as follows. By the definition (2.2.3) of T_E, (2.3.20) is equivalent to

$$\begin{aligned}0&=x_k^*\left(AX_\epsilon(j\omega_k)\Psi_{m-1}(j\omega_k)+B\right)-y_k^*\left(CX_\epsilon(j\omega_k)\Psi_{m-1}(j\omega_k)+D\right)\\&=(x_k^*A-y_k^*C)X_\epsilon(j\omega_k)\Psi_{m-1}(j\omega_k)+(x_k^*B-y_k^*D)\end{aligned} \tag{2.3.21}$$

for $k=1,\ldots,m$. Since a solution $\Psi_{m-1}\in(\mathcal{H}^\infty)^{p\times q}$ does not have an interpolation condition at $j\omega_1$, we split the proof of (2.3.21) into two cases: $k=1$ and $k=2,\ldots,m$.

When $k=1$, (2.3.21) follows from

$$(x_1^*A-y_1^*C)X_\epsilon(j\omega_1)=0,\qquad x_1^*B-y_1^*D=0.$$

In fact, since $B = C^* = ED$, we see from the definition (2.3.11) of κ_ϵ that

$$\begin{aligned}(x_1^* A - y_1^* C) X_\epsilon(j\omega_1) &= (Ax_1 - By_1)^* X_\epsilon(j\omega_1) \\ &= \nu^* X_\epsilon(j\omega_1) \\ &= \kappa_\epsilon(j\omega_1)\nu^* \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}x_1^* B - y_1^* D &= (x_1^* E - y_1^*) D \\ &= \left(x_1^* \cdot \frac{x_1 \cdot y_1^*}{\|x_1\|^2} - y_1^* \right) D \\ &= 0.\end{aligned}$$

Let us consider the case $k = 2, \ldots, m$. Since Ψ_{m-1} satisfies the interpolation conditions

$$(X_\epsilon(j\omega_k)^* T_{AB}(x_k, y_k))^* \Psi_{m-1}(j\omega_k) = T_{CD}(x_k, y_k)^*, \quad k = 2, \ldots, m,$$

it follows that

$$\begin{aligned}&(x_k^* A - y_k^* C) X_\epsilon(j\omega_k) \Psi_{m-1}(j\omega_k) + (x_k^* B - y_k^* D) \\ &\quad = (Ax_k - By_k)^* X_\epsilon(j\omega_k) \Psi_{m-1}(j\omega_k) - (-Cx_k + Dy_k)^* \\ &\quad = (X_\epsilon(j\omega_k)^* T_{AB}(x_k, y_k))^* \Psi_{m-1}(j\omega_k) - T_{CD}(x_k, y_k)^* \\ &\quad = 0.\end{aligned}$$

Thus (2.3.21) holds also for $k = 2, \ldots, n$.

It has been proved that we can reduce every boundary Nevanlinna-Pick problem to a boundary Nevanlinna-Pick problem that has one interpolation condition fewer. There always exists a solution to the boundary Nevanlinna-Pick interpolation problem having only one condition. Indeed, we can easily check that a constant function Ψ_0 defined by

$$\Psi_0(s) := \frac{x_0 \cdot y_0^*}{\|x_0\|^2}$$

is a solution to the problem with a single boundary condition $x_0^* \Phi_0(j\omega_0) = y_0^*$. Thus the boundary Nevanlinna-Pick problem 2.3.9 is always solvable. □

Combining Theorem 2.3.4 with Lemma 2.3.10, we obtain a proof of Theorem 2.3.8.

Proof of Theorem 2.3.8. The necessity is straightforward.

We prove the sufficiency as follows. Suppose that the tangential Nevanlinna-Pick interpolation problem 2.3.1 with data $(\alpha_i, [\xi_i, \eta_i])_{i=1}^n$ is solvable. Using Theorem 2.3.4, we can show the existence of a function satisfying $n-1$ interior conditions

$$(\alpha_i, [X(\alpha_i)^* T_{AB}(\xi_i, \eta_i), T_{CD}(\xi_i, \eta_i)])_{i=2}^n$$

and m boundary conditions

$$(j\omega_k, [X(j\omega_k)^* T_{AB}(x_k, y_k), T_{CD}(x_k, y_k)])_{k=1}^m.$$

These boundary interpolation values satisfy

$$\|X(j\omega_k)^* T_{AB}(x_k, y_k)\| - \|T_{CD}(x_k, y_k))\| > 0, \quad k = 1, \ldots, m.$$

In fact, since X defined by (2.3.7) satisfies

$$I - X(j\omega)X(j\omega)^* = 0, \quad \omega \in j\mathbb{R}_e,$$

Lemma 2.3.3 shows that

$$\begin{aligned}
&\|X(j\omega_k)^* T_{AB}(x_k, y_k)\|^2 - \|T_{CD}(x_k, y_k))\|^2 \\
&\qquad = T_{AB}(x_k, y_k)^* X(j\omega_k)X(j\omega_k)^* T_{AB}(x_k, y_k) - \|T_{CD}(x_k, y_k))\|^2 \\
&\qquad = \|T_{AB}(x_k, y_k))\|^2 - \|T_{CD}(x_k, y_k))\|^2 \\
&\qquad = \|x_k\|^2 - \|y_k\|^2 > 0.
\end{aligned}$$

Continuing this way, we can finally reduce Problem 2.3.6 to the boundary interpolation problem 2.3.9, which we have shown is always solvable in Lemma 2.3.10. This completes the proof. □

In conjunction with Theorem 2.3.2, Theorem 2.3.8 shows that the solvability of Problem 2.3.6 is equivalent to the positive definiteness of the Pick matrix in (2.3.3):

Corollary 2.3.11. *Consider Problem 2.3.6. Define the Pick matrix P by* (2.3.3) *with interior conditions $(\alpha_i, [\xi_i, \eta_i])_{i=1}^n$. Problem 2.3.6 is solvable if and only if P is positive definite.*

As in the previous section, we see from the proofs of Lemma 2.3.10 and Theorem 2.3.8 that solutions to Problem 2.3.6 can be calculated by the extended Schur-Nevanlinna algorithm.

Example 2.3.12. We compute $\Phi \in \mathbf{M}(\mathcal{H}^\infty)$ satisfying $\|\Phi\|_\infty < 1$ and $\xi^*\Phi(\alpha) = \eta^*$ and $x_k^*\Phi(j\omega_k) = y_k^*$ for $k = 1, 2$, where $\alpha := 1$, $\omega_1 := 0$, $\omega_2 := \infty$,

$$\xi := \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \eta := \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}, \quad x_1 := \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad y_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_2 := \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad y_2 := \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Since $\|\xi\| - \|\eta\| > 0$ and $\|x_k\| - \|y_k\| > 0$ for $k = 1, 2$ and since we have only one interior condition, there exist solutions to the problem.

Let us first convert this problem to the boundary interpolation problem 2.3.9. New boundary interpolation values are

$$(\hat{x}_1, \hat{y}_1 0) \approx \left(\begin{bmatrix} -1.7068 \\ 0.6466 \end{bmatrix}, \begin{bmatrix} 0.2658 \\ 0.5105 \end{bmatrix} \right), \quad (\hat{x}_2, \hat{y}_2) \approx \left(\begin{bmatrix} 1.4974 \\ 1.7487 \end{bmatrix}, \begin{bmatrix} 0.8343 \\ -0.7771 \end{bmatrix} \right).$$

A solution to the boundary interpolation problem 2.3.9 with data $(j\omega_k, [\hat{x}_k, \hat{y}_k])_{k=1}^2$ is given by

$$\Psi(s) \approx \begin{bmatrix} \frac{0.546s-5.97\times10^{-4}}{s+0.0114} & \frac{-0.270s-4.14\times10^{-3}}{s+0.0114} \\ \frac{9.60\times10^{-3}s+3.10\times10^{-3}}{s+0.0114} & \frac{-0.213s-1.95\times10^{-3}}{s+0.0114} \end{bmatrix}.$$

Finally we obtain a solution Φ to the original interpolation problem,

$$\Phi(s) \approx \begin{bmatrix} \frac{0.781(s+0.0324)(s+0.0971)}{(s+0.0141)(s+0.781)} & \frac{0.0668(s+0.0138)(s+6.70)}{(s+0.0141)(s+0.781)} \\ \frac{0.219(s-0.822)(s-0.0238)}{(s+0.0141)(s+0.781)} & \frac{-0.0668(s-3.397)(s+0.0107)}{(s+0.0141)(s+0.781)} \end{bmatrix}.$$

The matrix-valued function Φ satisfies the three interpolation conditions and $\|\Phi\|_\infty \approx 0.8121 < 1$.

Tangential interpolation conditions are less stringent than matrix-valued ones. We see this from Examples 2.2.10 and 2.3.12. The interpolation problems in these examples have the same number of interpolation conditions, but the degree of the solution in the tangential case is smaller than that in the matrix-valued case. This is because tangential interpolation conditions prescribe not matrix values, but some values in a certain direction only.

2.4 Difference from the results in Ball *et al.*

In Sections 2.2 and 2.3, we have obtained a necessary and sufficient condition for the Nevanlinna-Pick interpolation problem by extending the Schur-Nevanlinna algorithm. The condition has already derived in [3] by the approach rooted in the Pick matrices. In this section, we summarize the approach in [3] briefly and show the advantage of the proposed method.

The approach of [3] based on the Pick matrix is the following:

For simplicity, let us study the scalar-valued case. Let $r_1, \dots, r_m \in \mathbb{T}$ and $w_1, \dots, w_m \in \mathbb{D}$, and suppose that $r_1, \dots, r_m$ are distinct. We then consider the problem of finding a rational function $\phi \in \mathcal{H}^\infty(\mathbb{D})$ such that $\|\phi\|_\infty < 1$ and

$$\phi(r_k) = w_k, \qquad k = 1, \dots, m.$$

If a positive number $a < 1$ is sufficiently close to 1, then the Pick matrix P_a defined by

$$P_a := \begin{bmatrix} \frac{1-w_1\bar{w}_1}{1-a^2 r_1 \bar{r}_1} & \cdots & \frac{1-w_1\bar{w}_m}{1-a^2 r_1 \bar{r}_m} \\ \vdots & & \vdots \\ \frac{1-w_m\bar{w}_1}{1-a^2 r_m \bar{r}_1} & \cdots & \frac{1-w_m\bar{w}_m}{1-a^2 r_m \bar{r}_m} \end{bmatrix} \qquad (2.4.1)$$

is positive definite. In fact, as a tends to 1, the diagonal entries can be made arbitrarily large due to $|w_k| < 1$, while the off-diagonal entries remain bounded. Therefore Theorem 2.1.1 shows that for such a there exists $\phi_a \in \mathcal{H}^\infty$ satisfying $\|\phi\|_\infty < 1$ and

$$\phi_a(ar_k) = w_k, \qquad k = 1, \dots, m.$$

If we set $\phi(z) = \phi_a(az)$, then ϕ is the desired function.

The approach above is more straightforward than that in Sections 2.2 and 2.3, but we do not know how close a need to be 1 for the positive definiteness of P_a. Such a bound is necessary for the construction of solutions, in particular, for the design of controllers achieving nearly optimal performance. In contrast, the counterpart ϵ in our approach has bounds in (2.2.11) and (2.3.18), respectively. Thus the extended Schur-Nevanlinna algorithm constructs solutions efficiently

2.5 Summary

In this chapter, we have studied the Nevanlinna-Pick interpolation problem with boundary conditions. We have shown that the problem is solvable if and only if the associated Pick matrix consisting of the interpolation data at the interior points is positive definite. The necessary and sufficient condition was derived in [3]. While the approach there is rooted in the positive definiteness of the Pick matrix, the proposed method is an extension of the Schur-Nevanlinna algorithm. As a result, we can efficiently compute solutions to the interpolation problem.

Chapter 3

Strong Stabilization with Sensitivity Reduction for MIMO Systems

3.1 Motivation and problem statement

In this chapter, we consider MIMO systems. Let $P,\ C \in \mathbf{M}(\mathcal{F}^\infty)$ be a given plant and a controller, respectively. The sensitivity function $S := (I + PC)^{-1}$ is an important performance function that governs a performance of the closed-loop system. The significance of S can be seen from Figure 3.1, where $W_1,\ W_2 \in \mathbf{M}(\mathcal{H}^\infty)$ are given weighting functions. Here W_1S is the transfer function from the disturbance d to the weighted measured output $\tilde{y}$, and W_2 can be interpreted as the generator of d. For example, if $\|W_2(j\omega)\|$ is large in the frequency range $[0,\ \omega_d]$, then the energy of d is concentrated on the range. For disturbance rejection, we should reduce $\|W_1SW_2\|_\infty$ subject to the constraint that C stabilizes P.

Also, W_1S is the transfer function from the reference input r to the weighted error $\tilde{e}$. Suppose that the energy of r is concentrated on the frequency range $[0,\ \omega_r]$. To improve the tracking performance, $\|W_1(j\omega)S(j\omega)\|$ should be small for $\omega \in [0,\ \omega_r]$. We therefore choose the weighting function W_2 such that $\|W_2(j\omega)\|$ is large in $[0,\ \omega_r]$. For tracking of r, we need to reduce $\|W_1SW_2\|_\infty$ with a stabilizing controller.

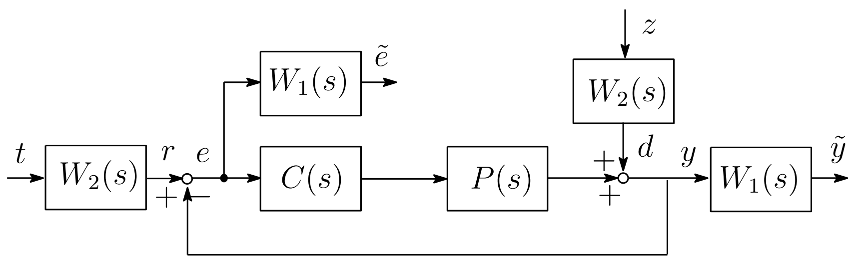


Figure 3.1: Disturbance rejection and tracking.

Then our problem is the following:

Problem 3.1.1. *Given a plant $P \in \mathbf{M}(\mathcal{F}^\infty)$, weighting matrices W_1, $W_2 \in \mathbf{M}(\mathcal{H}^\infty)$, determine whether there exists a controller $C \in \mathbf{M}(\mathcal{H}^\infty) \cap \mathscr{C}(P)$ such that*

$$\|W_1 S W_2\|_\infty < 1, \quad \textit{where} \;\; S := (1 + PC)^{-1}. \tag{3.1.1}$$

Also, if one exists, find such a controller.

The objective of this chapter is to obtain a sufficient condition and also a necessary condition for Problem 3.1.1 that can be checked by matrix computation. Moreover, we propose the design procedure of stable controllers satisfying (3.1.1).

To proceed further, we need to recall the definitions of unimodular functions, multivariable zeros, and coprime factorizations over $\mathcal{H}^\infty$.

Definition 3.1.2 ([114])**.** A matrix $U \in (\mathcal{H}^\infty)^{p\times p}$ is *unimodular* if it has an inverse in $(\mathcal{H}^\infty)^{p\times p}$.

Definition 3.1.3 ([131])**.** Consider a matrix-valued function N whose elements are meromorphic in $\mathbb{C}$. We call $z_0 \in \mathbb{C}$ a *blocking zero* of N if $N(z_0) = 0$. Also, $z_0 \in \mathbb{C}$ is a *transmission zero* of N if $N(z_0)$ is not of full rank.

Definition 3.1.4 ([101])**.** $D,\ N \in \mathbf{M}(\mathcal{H}^\infty)$ are said to be *left coprime* if the Bezout identity

$$NX + DY = I \tag{3.1.2}$$

holds for some $X,\ Y \in \mathbf{M}(\mathcal{H}^\infty)$. $P \in \mathbf{M}(\mathcal{F}^\infty)$ admits a *left coprime factorization* if there exist $D,\ N \in \mathbf{M}(\mathcal{H}^\infty)$ such that $P = D^{-1}N$ and $D,\ N$ are left coprime.

Similarly, $\tilde{D},\ \tilde{N} \in \mathbf{M}(\mathcal{H}^\infty)$ are *right coprime* if the Bezout identity

$$\tilde{X}\tilde{N} + \tilde{Y}\tilde{N} = I \tag{3.1.3}$$

holds for some $\tilde{X},\ \tilde{Y} \in \mathbf{M}(\mathcal{H}^\infty)$. $P \in \mathbf{M}(\mathcal{F}^\infty)$ admits a *right coprime factorization* if there exist $\tilde{D},\ \tilde{N} \in \mathbf{M}(\mathcal{H}^\infty)$ such that $P = \tilde{N}\tilde{D}^{-1}$ and $\tilde{D},\ \tilde{N}$ are right coprime.

If P is a scalar-valued function, we use the expressions *coprime* and *coprime factorization.*

The existences of left and right factorizations are necessary for stabilizablity:

Theorem 3.1.5 ([101])**.** Suppose that $P \in \mathbf{M}(\mathcal{F}^\infty)$ is stabilizable. Then P possesses right and left coprime factorizations over $\mathcal{H}^\infty$.

3.2 Systems with unstable blocking zeros

In this section, we consider the plant having only blocking zeros in $\bar{\mathbb{C}}_+$. First we transform the problem of strong stabilization to that of interpolation by a unimodular matrix in $\mathbf{M}(\mathcal{H}^\infty)$. Next we show that strong stabilization with sensitivity reduction is equivalent to an interpolation problem with a unimodular matrix whose $\mathcal{H}^\infty$-norm is less than one.

3.2.1 Reduction to interpolation with a unimodular matrix of $\mathcal{H}^\infty$-norm less than one

Let us first study strong stabilization only. The following lemma gives a necessary and sufficient condition for strong stabilization:

Lemma 3.2.1. *Suppose that $P \in \mathbf{M}(\mathcal{F}^\infty)$ is stabilizable and has a left coprime factorization $P = D^{-1}N$ with D, $N \in \mathbf{M}(\mathcal{H}^\infty)$. Then C strongly stabilizes P if and only if $C \in \mathbf{M}(\mathcal{H}^\infty)$ and*

$$(D + NC)^{-1} \in \mathbf{M}(\mathcal{H}^\infty). \tag{3.2.1}$$

Proof. Sufficiency. We have

$$(I + PC)^{-1} = (I + D^{-1}NC)^{-1} = (D^{-1}(D + NC))^{-1} = (D + NC)^{-1}D.$$

Moreover,

$$(I + PC)^{-1}P = (D + NC)^{-1}N$$
$$C(I + PC)^{-1} = C(D + NC)^{-1}D$$
$$C(I + PC)^{-1}P = C(D + NC)^{-1}N.$$

Since C, D, N, and $(D + NC)^{-1}$ belong to $\mathbf{M}(\mathcal{H}^\infty)$, we obtain (1.2.1), and hence C strongly stabilizes P.

Necessity. Since P is stabilizable, Theorem 3.1.5 shows that P admits a right coprime factorization:

$$P = \tilde{N}\tilde{D}^{-1}, \quad \tilde{N},\ \tilde{D} \in \mathbf{M}(\mathcal{H}^\infty).$$

Moreover, the Bezout identity (3.1.2) is satisfied for some X, $Y \in \mathbf{M}(\mathcal{H}^\infty)$. It is known that all stabilizing controllers are of the form $(X + \tilde{D}Q)(Y - \tilde{N}Q)^{-1}$ for $Q \in \mathbf{M}(\mathcal{H}^\infty)$ [101]. Since P is strongly stabilizable, there exists $Q_0 \in \mathbf{M}(\mathcal{H}^\infty)$ such that

$$C = (X + \tilde{D}Q_0)(Y - \tilde{N}Q_0)^{-1} \in \mathbf{M}(\mathcal{H}^\infty).$$

In conjunction with the Bezout identity (3.1.2), this leads to

$$\begin{aligned} D + NC &= D + N(X + \tilde{D}Q_0)(Y - \tilde{N}Q_0)^{-1} \\ &= (D(Y - \tilde{N}Q_0) + N(X + \tilde{D}Q_0))(Y - \tilde{N}Q_0)^{-1} \\ &= (Y - \tilde{N}Q_0)^{-1}. \end{aligned}$$

Thus we obtain $(D + NC)^{-1} = Y - \tilde{N}Q_0 \in \mathbf{M}(\mathcal{H}^\infty)$. □

Lemma 3.2.1 suggests the following problem to find stable stabilizing controllers:

Problem 3.2.2. Given D, $N \in \mathbf{M}(\mathcal{H}^\infty)$, find a controller $C \in \mathbf{M}(\mathcal{H}^\infty)$ satisfying (3.2.1).

We can reduce Problem 3.2.2 to an interpolation problem with unimodular matrices in $\mathbf{M}(\mathcal{H}^\infty)$ under the following assumption on D and N:

Assumption 3.2.3. D, $N \in \mathbf{M}(\mathcal{H}^\infty)$ are left coprime, and all elements of $N,\ D,\ X$, and Y in (3.1.2) are meromorphic functions in $\mathbb{C}$. The matrix-valued function N is square and has the form $N = \phi N_o$, where $\phi \in \mathcal{H}^\infty$ and $N_o,\ N_o^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$. Moreover ϕ is a rational function satisfying $\phi(\infty) \neq 0$ and possesses only simple zeros $z_1, \dots, z_n$ in $\bar{\mathbb{C}}_+$.

We will discuss the above conditions on multivariable zeros in $\bar{\mathbb{C}}_+$ in Remark 3.2.14 at the end of this subsection.

Under Assumption 3.2.3, $P := D^{-1}N$ can have only finitely many zeros in $\bar{\mathbb{C}}_+$. Moreover, they are simple and blocking zeros because they arise from the scalar-valued function ϕ.

We prove that Problem 3.2.2 is equivalent to the following problem under Assumption 3.2.3:

Problem 3.2.4. Given $z_1, \dots, z_n \in \bar{\mathbb{C}}_+$ and complex square matrices $A_1, \dots, A_n$, find $U \in \mathbf{M}(\mathcal{H}^\infty)$ satisfying $U^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$ and

$$U(z_i) = A_i, \quad i = 1, \dots, n. \tag{3.2.2}$$

Theorem 3.2.5. *Consider Problem 3.2.2 under Assumption 3.2.3. We restrict the solutions to matrices whose elements are meromorphic functions. Define $A_i := D(z_i)$ for $i = 1, \dots, n$. Then Problem 3.2.2 is equivalent to Problem 3.2.4 with interpolation data $(z_i, A_i)_{i=1}^n$.*

A solution C to Problem 3.2.2 and a solution U to Problem 3.2.4 satisfy

$$C = N^{-1}(U - D), \quad U = D + NC. \tag{3.2.3}$$

Proof. Let C be a solution of Problem 3.2.2. Define U by (3.2.3). Then U satisfies $U,\ U^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$ by (3.2.1). In addition, since $\phi(z_i) = 0$,

$$U(z_i) = D(z_i) + \phi(z_i) N_o(z_i) C(z_i) = D(z_i) = A_i.$$

Hence U is a solution to Problem 3.2.4.

Conversely, suppose that U is a solution to Problem 3.2.4 with interpolation data $(z_i, A_i)_{i=1}^n$. Define C by (3.2.3), that is,

$$C := \frac{1}{\phi} N_o^{-1}(U - D).$$

Then C satisfies $(D + NC)^{-1} = U^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$ and

$$\phi C = N_o^{-1}(U - D) \in \mathbf{M}(\mathcal{H}^\infty). \tag{3.2.4}$$

Assume, to reach a contradiction, that $C \notin \mathbf{M}(\mathcal{H}^\infty)$. Since $\phi C \in \mathbf{M}(\mathcal{H}^\infty)$ by (3.2.4), C has some poles in $\bar{\mathbb{C}}_+$ that are canceled by zeros of ϕ. Let z_k be one of such poles. Since ϕ has only simple zeros in $\bar{\mathbb{C}}_+$, we have $(\phi C)(z_k) \neq 0$. However, (3.2.2) shows that

$$(\phi C)(z_k) = N_o^{-1}(z_k)(U(z_k) - D(z_k)) = N_o^{-1}(z_k)(A_k - A_k) = 0,$$

and we have a contradiction. $\square$

Before proceeding to sensitivity reduction by strongly stabilizing controllers, we recall the definitions of inner, outer, co-inner, and co-outer matrix functions.

A matrix-valued function $L \in \mathbf{M}(\mathcal{H}^\infty)$ is said to be *inner* if $L(j\omega)^* L(j\omega) = I$ almost everywhere. Let $(\mathcal{H}^2)^p$ is the space of all vector-valued functions f that are holomorphic in $\mathbb{C}_+$, take values in $\mathbb{C}^p$, and satisfy

$$\sup_{\xi > 0} \left(\int_{-\infty}^{\infty} \|f(\xi + j\omega)\|^2 d\omega \right) < \infty.$$

The norm of $f \in (\mathcal{H}^2)^p$ is defined by

$$\|f\|_2 := \sup_{\xi > 0} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(\xi + j\omega)\|^2 d\omega \right)^{1/2}.$$

We say that $L \in (\mathcal{H}^\infty)^{p \times q}$ is *outer* if the set $\{Lf : w \in (\mathcal{H}^2)^q\}$ is dense in $(\mathcal{H}^2)^p$.

For $L \in \mathbf{M}(\mathcal{H}^\infty)$, define $\grave{L}(s) := L(\bar{s})^*$ and then $\grave{L} \in \mathbf{M}(\mathcal{H}^\infty)$. $L \in \mathbf{M}(\mathcal{H}^\infty)$ is said to be *co-inner* if $\grave{L}$ is inner. Similarly, $L \in \mathbf{M}(\mathcal{H}^\infty)$ is *co-outer* if $\grave{L}$ is outer.

By definition, $\|\grave{L}\|_\infty = \|L\|_\infty$ for $L \in \mathbf{M}(\mathcal{H}^\infty)$, and hence we have the following lemma:

Lemma 3.2.6. *Let L be co-inner. For every $K \in \mathbf{M}(\mathcal{H}^\infty)$,*

$$\|KL\|_\infty = \|K\|_\infty.$$

Proof. Since $\grave{L}$ is inner, we have $\|\grave{L}K\|_\infty = \|K\|_\infty$ for every $K \in \mathbf{M}(\mathcal{H}^\infty)$ by the definition of inner matrices. Hence

$$\|KL\|_\infty = \|\grave{L}\grave{K}\|_\infty = \|\grave{K}\|_\infty = \|K\|_\infty,$$

which is a desired conclusion. □

The following result shows that every function in $\mathbf{M}(\mathcal{H}^\infty)$ admits a unique co-inner-outer factorization:

Theorem 3.2.7 ([32, 36])**.** *Let $K \in (\mathcal{H}^\infty)^{p \times q}$. K admits a co-inner-outer factorization of the form $K = GF$, where $G \in (\mathcal{H}^\infty)^{p \times r}$ is co-outer and $F \in (\mathcal{H}^\infty)^{r \times q}$ is co-inner for some r. F and G are unique to within multiplication by a constant unitary matrix.*

Now we consider strong stabilization with sensitivity reduction. We place the following additional assumption on the weights W_1, W_2 and the denominator $\mathcal{H}^\infty$ function D of the plant:

Assumption 3.2.8. All elements of W_1 and W_2 are meromorphic functions. W_1 is unimodular in $\mathbf{M}(\mathcal{H}^\infty)$. If we factor DW_2 as $DW_2 = (DW_2)_{co} \cdot (DW_2)_{ci}$, where $(DW_2)_{co}$ is co-outer and $(DW_2)_{ci}$ is co-inner, then $(DW_2)_{co}$ is also unimodular in $\mathbf{M}(\mathcal{H}^\infty)$.

Remark 3.2.9. 1. Since D is determined by the plant, Assumption 3.2.8 imposes constraints on the selection of the weighting functions.

2. Let $K \in \mathbf{M}(\mathcal{H}^\infty)$ and define $R(j\omega) := \grave{K}(j\omega)^*\grave{K}(j\omega)$. The co-outer function K_{co} of K is unimodular if and only if $\det R(j\omega) \neq 0$ a.e. on $j\mathbb{R}$ and all elements of R^{-1} are essentially bounded on $j\mathbb{R}$ [36]. Furthermore, if R satisfies the conditions, we can use an explicit formula in [36] to compute a co-inner-outer factorization. This formula involves inverting the semi-infinite Toeplitz matrix determined by R. Hence we cannot exactly compute a co-inner-outer factorization by the formula. However, the formula is still useful in approximately obtaining a co-outer function, which is needed for the construction of stable $\mathcal{H}^\infty$ controllers in Theorem 3.2.11 below.

We can obtain a solution to Problem 3.1.1 from that to the following interpolation problem:

Problem 3.2.10. Suppose that $z_1, \ldots, z_n \in \bar{\mathbb{C}}_+$ are distinct and that $B_1, \ldots, B_n$ are complex square matrices. Find a unimodular matrix function $F \in \mathbf{M}(\mathcal{H}^\infty)$ such that all elements of F are meromorphic in $\mathbb{C}$, $\|F\|_\infty < 1$, and

$$F(z_i) = B_i, \quad i = 1, \ldots, n.$$

Theorem 3.2.11. *Consider Problem 3.1.1. Assume that there exist $D,\ N \in \mathbf{M}(\mathcal{H}^\infty)$ such that $P = D^{-1}N$. Let Assumptions 3.2.3 and 3.2.8 hold. Define*

$$B_i := W_1(z_i)D(z_i)^{-1}(DW_2)_{co}(z_i), \quad i = 1, \ldots, n. \tag{3.2.5}$$

If there exists a solution F to Problem 3.2.10 with interpolation data $(z_i, B_i)_{i=1}^n$, then

$$C = N^{-1}(DW_2)_{co}F^{-1}W_1 - P^{-1} \tag{3.2.6}$$

gives a solution to Problem 3.1.1.

Conversely, if C is a solution to Problem 3.1.1 and if all entries of C are meromorphic in $\mathbb{C}$, then

$$F = W_1(D + NC)^{-1}(DW_2)_{co} \tag{3.2.7}$$

is a solution to Problem 3.2.10 with interpolation data $(z_i, B_i)_{i=1}^n$.

Proof. In order for (3.2.5) to be well-defined, to begin with, we need to show that $D(z_i)$ is invertible for $i = 1, \ldots, n$. Since $\phi(z_i) = 0$, it follows from the Bezout identity (3.1.2) that $D(z_i)Y(z_i) = I$. Hence $D(z_i)^{-1}$ exists and $D(z_i)^{-1} = Y(z_i)$.

Let F be a solution to Problem 3.2.10 with interpolation data $(z_i, B_i)_{i=1}^n$. Then C defined by (3.2.6) satisfies $C \in \mathbf{M}(\mathcal{H}^\infty) \cap \mathscr{C}(P)$ by Lemma 3.2.1 and Theorem 3.2.5. Also, since

$$\begin{aligned} W_1(I + PC)^{-1}W_2 &= W_1(D + NC)^{-1}DW_2 \\ &= W_1(D + NC)^{-1}(DW_2)_{co} \cdot (DW_2)_{ci} \\ &= F(DW_2)_{ci}, \end{aligned}$$

Lemma 3.2.6 shows that

$$\|W_1(I+PC)^{-1}W_2\|_\infty = \|F(DW_2)_{ci}\|_\infty = \|F\|_\infty. \tag{3.2.8}$$

Hence (3.1.1) holds and C in (3.2.6) is a solution to Problem 3.1.1.

Conversely, suppose that C is a solution to Problem 3.1.1. Define F by (3.2.7) and U by $U := D + NC$. Lemma 3.2.1 and Theorem 3.2.5 show that U is a solution to Problem 3.2.4 with interpolation data $(z_i, D(z_i))_{i=1}^n$. Combining this with (3.2.8), we show that $F = W_1U^{-1}(DW_2)_{co}$ is a solution to Problem 3.2.10 with interpolation data $(z_i, B_i)_{i=1}^n$. □

The following corollary gives a necessary condition for the solvability of Problem 3.1.1.

Corollary 3.2.12. *Consider Problem 3.1.1 whose solutions are restricted to meromorphic matrix functions. Under the same hypotheses of Theorem 3.2.11, suppose that Problem 3.1.1 is solvable. Then there exists $F \in \mathbf{M}(\mathcal{H}^\infty)$ such that $\|F\|_\infty < 1$ and $F(z_i) = B_i$ for $i = 1, \dots, n$.*

Proof. Obvious from Theorem 3.2.11. □

Remark 3.2.13. The question may occur here: *Why can we solve the $\mathcal{H}^\infty$ control problem for infinite dimensional systems via the Nevanlinna-Pick interpolation?*

If D is co-inner, then we see from the proof of Lemma 3.2.1 that

$$\|S\|_\infty = \|Y - \tilde{N}Q_0\|_\infty \tag{3.2.9}$$

with $Y, \tilde{N}, Q_0 \in \mathbf{M}(\mathcal{H}^\infty)$. In (3.2.9), Y is generally an infinite-dimensional system. However, the minimization of the norm (3.2.9) is simply a finite-dimensional problem. In fact, (3.2.9) is the same form of norm constraint as in sensitivity reduction for an *infinite-dimensional* weight and a *finite-dimensional* stable plant. Such an $\mathcal{H}^\infty$ problem is solvable by the Nevanlinna-Pick interpolation [45, 78]. As in the finite-dimensional case, we can therefore use the Nevanlinna-Pick interpolation even for infinite-dimensional systems in Assumption 3.2.3.

Let us finally discuss ϕ in Assumption 3.2.3.

Remark 3.2.14.

1. For simplicity, we have assumed that the unstable zeros of ϕ are simple in Assumption 3.2.3. We can generalize the results in this section by introducing interpolation conditions on the derivatives of N and D. See also Remark 3.3.12.2 in the next section.

2. If D is a rational matrix function, then we can allow ϕ to be strictly proper. However, if D is not rational and if ϕ is strictly proper, in the same way as [48], we should replace ϕ with $\phi_\varepsilon(s) = \phi(s)(1+\varepsilon s)^m$, where $\varepsilon > 0$ and m is the relative degree[1] of ϕ. This makes sure that we do not have to deal with interpolation conditions at infinity, but this leads to an improper term like PD controllers in the $\mathcal{H}^\infty$ controller.

[1] Let us denote by $\deg(p)$ the degree of a polynomial p. For a proper rational function $f = n/d$ with polynomials n and d, the difference $\deg(d) - \deg(n)$ is called the *relative degree* of f.

3. We assume that ϕ is a scalar-valued function, and then we reduce strong stabilization with sensitivity reduction to *matrix-valued* interpolation. However, this assumption of ϕ could be weakened at the cost of going to *tangential* interpolation. In the next section, we shall address the modifications arising from it.

3.2.2 Design of strongly stabilizing controllers attaining low sensitivity

In this section, we develop a design method of strongly stabilizing controllers for sensitivity reduction. Here we extend the technique of [56] to MIMO infinite-dimensional systems.

The design method uses the following lemma for the construction on a unimodular matrix:

Lemma 3.2.15. *Suppose that $G \in \mathbf{M}(\mathcal{H}^\infty)$ is square and that $\|G\|_\infty < 1$. For every complex number $\lambda \neq 0$,*

$$F := \frac{\lambda}{2}(G + I) \tag{3.2.10}$$

satisfies F, $F^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$ and $\|F\|_\infty < |\lambda|$.

Proof. $F \in \mathbf{M}(\mathcal{H}^\infty)$ is evident. Since G satisfies $\|G\|_\infty < 1$, it follows from the small gain theorem [113, 131] that $(G + I)^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$. Hence we have

$$F^{-1} = \frac{2}{\lambda}(G + I)^{-1} \in \mathbf{M}(\mathcal{H}^\infty).$$

Moreover, from the triangle inequality and $\|G\|_\infty < 1$,

$$\|F\|_\infty = \frac{|\lambda|}{2} \cdot \|G + I\|_\infty \leq \frac{|\lambda|}{2} \cdot (\|G\|_\infty + \|I\|_\infty) < \frac{|\lambda|}{2} \cdot 2 = |\lambda|$$

is obtained. □

We derive the following result from Lemma 3.2.15.

Theorem 3.2.16. *Consider Problem 3.2.10. Let λ be a complex number of absolute value 1. If $G \in \mathbf{M}(\mathcal{H}^\infty)$ satisfies $\|G\|_\infty < 1$ and*

$$G(z_i) = \frac{2}{\lambda}B_i - I, \quad i = 1, \ldots, n,$$

then F defined by (3.2.10) *is a solution of Problem 3.2.10.*

Proof. This follows directly from Theorem 3.2.11 and Lemma 3.2.15. □

The problem of finding G in Theorem 3.2.16 and that of finding F in Corollary 3.2.12 are matrix-valued Nevanlinna-Pick interpolation problems 2.2.5. As we mentioned in Chapter 2, this interpolation problem is solvable if and only if the Pick matrix consisting of the interior conditions is positive definite. Furthermore, solutions is derived from the extended Schur-Nevanlinna algorithm.

The proposed solution to Problem 3.1.1 can be summarized as follows:

Design procedure for stable stabilizing controllers providing low sensitivity for plants with unstable blocking zeros

Step 1: Let $\lambda \in \mathbb{C}$ satisfy $|\lambda| = 1$. For each unstable blocking zero z_i of P, define $\bar{B}_i$ by

$$\bar{B}_i = \frac{2}{\lambda} W_1(z_i) D(z_i)^{-1} (DW_2)_{co}(z_i) - I, \quad i = 1, \ldots, n.$$

Step 2: Solve the matrix-valued Nevanlinna-Pick interpolation problem 2.2.5 with data $(z_i, \bar{B}_i)_{i=1}^n$.

Step 3: Calculate a solution F to Problem 3.2.10 from (3.2.10).

Step 4: Compute a solution C to Problem 3.1.1 from (3.2.6).

Remark 3.2.17. From the point of view of controller implementation, it is important to observe that pole-zero pairs in $\bar{\mathbb{C}}_+$ are cancelled in the controller, that is, the controller has internal *unstable pole-zero cancellations.* Since the controller is infinite-dimensional, exact cancellation is not always possible. Thus we should investigate such cancellations in more detail and study new structures of controllers that can be implemented in a stable way. However, even if the obtained controller is not implementable, the bounds on the optimal value help us understand the performance limitation of stable controllers.

The Proposition 3.2.18 below ensures the set of controllers obtained by the proposed method become smaller as the gains of the weighting functions W_1 and W_2 decreases.

Proposition 3.2.18. *Let λ_1, $\lambda_2 \in \mathbb{C} \setminus \{0\}$ satisfy $\lambda_1 = a\lambda_2$ for some $a \in (0, 1)$. Suppose that $z_1, \ldots, z_n$ are distinct complex numbers in $\bar{\mathbb{C}}_+$ and that $A_1, \ldots, A_n$ are complex square matrices. Suppose also that for $k = 1, 2$, $\mathscr{N}_k$ is the set whose members are the solutions G to the matrix-valued Nevanlinna-Pick interpolation problem 2.2.5 with the following conditions:*

$$G(z_i) = \frac{2}{\lambda_k} A_i - I, \quad i = 1, \ldots, n. \tag{3.2.11}$$

Define

$$\mathscr{M}_k := \left\{ \frac{\lambda_k}{2}(G_k + I) : \; G_k \in \mathscr{N}_k \right\}, \qquad k = 1, 2.$$

Then we have

$$\mathscr{M}_1 \subset \mathscr{M}_2. \tag{3.2.12}$$

Proof. Assume that $F \in \mathscr{M}_1$, and let $G_1 \in \mathscr{N}_1$ satisfy

$$F = \frac{\lambda_1}{2}(G_1 + I). \tag{3.2.13}$$

Define G_2 by

$$G_2 := \frac{\lambda_1}{\lambda_2}(G_1 + I) - I. \tag{3.2.14}$$

We first show that

$$G_2 \in \mathscr{N}_2. \tag{3.2.15}$$

we have

$$G_2(z_i) = \frac{\lambda_1}{\lambda_2}\left(\left(\frac{2}{\lambda_1}A_i - I\right) + I\right) - I = \frac{2}{\lambda_1}A_i - I$$

by (3.2.11). Also, since $\|G_1\|_\infty < 1$ and $\lambda_1 = a\lambda_2$, we see that

$$\begin{aligned} \|G_2\|_\infty &= \left\| \frac{\lambda_1}{\lambda_2}(G_1 + I) - I \right\|_\infty \\ &\leq \frac{|\lambda_1|}{|\lambda_2|} \cdot \|G_1\|_\infty + \frac{|\lambda_1 - \lambda_2|}{|\lambda_2|} \\ &< \frac{a|\lambda_2| + (1-a)|\lambda_2|}{|\lambda_2|} = 1. \end{aligned} \tag{3.2.16}$$

Hence we obtain (3.2.15).

On the other hand, (3.2.13) and (3.2.14) show that

$$F = \frac{\lambda_2}{2}(G_2 + I).$$

Thus $F \in \mathscr{M}_2$ and (3.2.12) is obtained. □

In general, the proposed method produces infinite-dimensional controllers due to the infinite-dimensionality of the plant. To obtain an implementable controller, we must approximate the derived controller by a finite-dimensional controller.

The propositions below suggest that a stable rational controller also stabilizes P and achieves low sensitivity of the closed-loop system if the infinite-dimensional controller is approximated by a rational controller closely in the sense of $\mathcal{H}^\infty$-norm. The following results are extensions of Lemmas 3.1 and 3.2 in [37] to MIMO systems.

Proposition 3.2.19. *Let $P \in \mathbf{M}(\mathcal{F}^\infty)$ and suppose that P has a left coprime factorization $P = D^{-1}N$ for some $D,\ N \in \mathbf{M}(\mathcal{H}^\infty)$. Define*

$$\delta := \frac{1}{\|N\|_\infty \cdot \|(D + NC)^{-1}\|_\infty}.$$

For $C \in \mathbf{M}(\mathcal{H}^\infty) \cap \mathscr{C}(P)$, if $C_a \in \mathbf{M}(\mathcal{RH}^\infty)$ satisfies

$$\|C - C_a\|_\infty < \delta,$$

then C_a also stabilizes P.

Proof. By Lemma 3.2.1 It suffices to prove that $U_a := D + NC_a$ satisfies $U_a^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$.

Defining $U := D + NC$, we have

$$\|U - U_a\|_\infty \le \|N\|_\infty \cdot \|C - C_a\|_\infty < \|N\|_\infty \cdot \delta = 1/\|U^{-1}\|_\infty.$$

It follows that

$$\|I - U^{-1}U_a\|_\infty \le \|U^{-1}\| \cdot \|U - U_a\|_\infty < 1.$$

Moreover, $U^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$ from Lemma 3.2.1, and hence $I - U^{-1}U_a \in \mathbf{M}(\mathcal{H}^\infty)$. If we define $V := I - (I - U^{-1}U_a)$, Lemma 3.2.15 with $\lambda = 2$ and with $G = -(I - U^{-1}U_a)$ shows that both $V = U^{-1}U_a$ and V^{-1} belong to $\mathbf{M}(\mathcal{H}^\infty)$. Thus

$$U_a^{-1} = V^{-1}U \in \mathbf{M}(\mathcal{H}^\infty)$$

is obtained. □

Proposition 3.2.20. *Consider Problem 3.1.1. Suppose that P has a left coprime factorization $P = D^{-1}N$ for some $D,\ N \in \mathbf{M}(\mathcal{H}^\infty)$, and that W_1 is unimodular in $\mathbf{M}(\mathcal{H}^\infty)$. For $C \in \mathbf{M}(\mathcal{H}^\infty) \cap \mathscr{C}(P)$ and $C_a \in \mathbf{M}(\mathcal{RH}^\infty) \cap \mathscr{C}(P)$, we define*

$$\begin{aligned} \delta &:= \left\|W_1(I + PC)^{-1}P\right\|_\infty \cdot \|W_1^{-1}\|_\infty \\ \epsilon &:= \|C - C_a\|_\infty \\ S &:= (I + PC)^{-1} \\ S_a &:= (I + PC_a)^{-1}. \end{aligned} \tag{3.2.17}$$

If $\delta\epsilon < 1$, then

$$\|W_1 S_a W_2\|_\infty \le \frac{\|W_1 S W_2\|_\infty}{1 - \delta\epsilon}. \tag{3.2.18}$$

Proof. Since

$$\begin{aligned} W_1SW_2 - W_1S_aW_2 &= W_1\left((I + PC)^{-1} - (I + PC_a)^{-1}\right)W_2 \\ &= W_1(I + PC)^{-1}\left((I + PC_a) - (I + PC)\right)(I + PC_a)^{-1}W_2 \\ &= W_1(I + PC)^{-1}P(C_a - C)W_1^{-1}(W_1S_aW_2), \end{aligned}$$

we obtain

$$\begin{aligned} \|W_1S_aW_2\|_\infty - \|W_1SW_2\|_\infty &\le \|W_1SW_2 - W_1S_aW_2\|_\infty \\ &= \|W_1(I + PC)^{-1}P(C_a - C)W_1^{-1}(W_1S_aW_2)\|_\infty \\ &\le \delta\epsilon\|W_1S_aW_2\|_\infty. \end{aligned}$$

If $\delta\epsilon < 1$, then we have the desired conclusion (3.2.18). □

Remark 3.2.21. In Proposition 3.2.20, $\|W_1^{-1}\|_\infty$ in (3.2.17) may make the estimate (3.2.18) conservative. Since W_1 is not generally commutative with $(I+PC)^{-1}P(C_a - C)$, we cannot cancel W_1 and W_1^{-1} in (3.2.17). If W_1 is a scalar matrix, i.e., a diagonal matrix whose diagonal elements contain the same scalar function, then we can replace (3.2.17) with $\delta := \|(I + PC)^{-1}P\|_\infty$.

Rational approximations can be obtained from the frequency response data with approximation methods for stable infinite-dimensional systems; see, e.g., [42] and its references. The reader can refer to [20, 83, 85] and references therein for other approximation techniques.

3.2.3 Numerical examples

In this subsection, we present numerical examples to show the efficiency of the results. We apply the proposed method to a repetitive control system [19, 54, 121]. Repetitive control attempts to track or reject arbitrary periodic signals of a fixed period. Such control objectives appear in many applications, e.g., disk drives [73] and industrial manipulators [18].

Example 3.2.22. We consider strong stabilization with sensitivity reduction for the following plant and weighting functions:

$$P(s) = \frac{(s-z_1)(s-z_2)}{(s+1)^2}\begin{bmatrix} \frac{1}{3+4e^{-s}} & e^{-2s} \\ 0 & \frac{6(s+1)^2}{(6s-1)(s-e^{-s}+2)} \end{bmatrix}$$

$$W_1(s) = \frac{s+1}{10s+1}\begin{bmatrix} 1 & 1/10 \\ 0 & 1 \end{bmatrix}, \qquad W_2(s) = I,$$

where $z_1,\ z_2 \in \bar{\mathbb{C}}_+$. Using the factorization method of [48] to each elements of P, we can factor P as $P = \phi D^{-1} N_o$, where

$$\phi(s) := \frac{(s-z_1)(s-z_2)}{(s+1)^2}$$

$$D(s) := \begin{bmatrix} \frac{3+4e^{-s}}{3e^{-s}+4} & 0 \\ 0 & \frac{s-1/6}{s+1/6} \end{bmatrix}$$

$$N_o(s) := \begin{bmatrix} \frac{1}{3e^{-s}+4} & \frac{3+4e^{-s}}{3e^{-s}+4}e^{-2s} \\ 0 & \frac{(s+1)^2}{(s+1/6)(s+e^{-s}+2)} \end{bmatrix}.$$

The zeros of ϕ in $\bar{\mathbb{C}}_+$ are z_1 and z_2. N_o satisfies $N_o^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$. We can easily check whether D and $N := \phi N_o$ are left coprime, In fact, we transform the Bezout identity (3.1.2) to

$$X = N_o^{-1} \cdot \frac{I - DY}{\phi}. \tag{3.2.19}$$

Since X needs to belong to $\mathbf{M}(\mathcal{H}^\infty)$, It follows from (3.2.19) that D and N are left coprime if and only if $I - D(z_i)Y(z_i) = 0$ for $i = 1, 2$, that is, $D(z_1)$ and $D(z_2)$ are nonsingular and there exists $Y \in \mathbf{M}(\mathcal{H}^\infty)$ such that $Y(z_i) = D(z_i)^{-1}$ for $i = 1, 2$. Such a matrix-valued function $Y \in \mathbf{M}(\mathcal{H}^\infty)$ always exists. To see this, we apply the Lagrange interpolation [24] to each element of Y.

Let us first take $z_1 \in (1/6,\ 5]$ and $z_2 = 8$. Define

$$\rho_{\text{inf}} := \inf \big\{ \|W_1(I+PC)^{-1}W_2\|_\infty :\ C \in \mathbf{M}(\mathcal{H}^\infty) \cap \mathscr{C}(P), \\ \text{all entries of } C \text{ are meromorphic} \big\}. \tag{3.2.20}$$

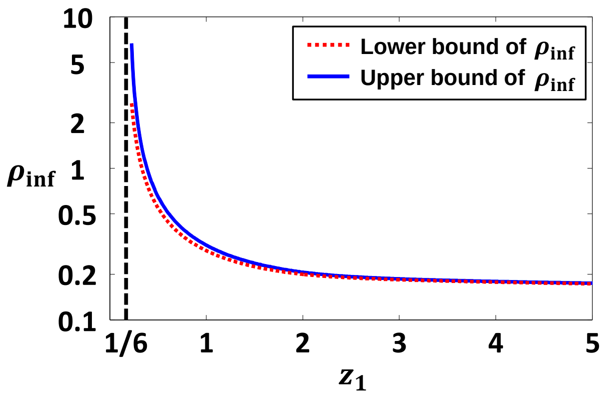


Figure 3.2: Unstable blocking zero z_1 versus minimum sensitivity ρ_{inf}.

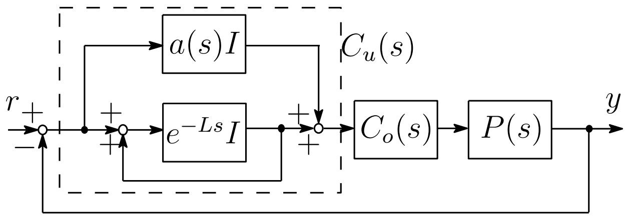


Figure 3.3: Repetitive control system.

Figure 3.2 shows ρ_{inf} dependent on z_1. In Figure 3.2, the solid line indicates a upper bound of ρ_{inf}, which obtained by the proposed method, and the dashed line shows a lower bound of ρ_{inf}. The lower bound is derived from Corollary 3.2.12. Both lines in Figure 3.2 diverge as z_1 becomes closer to 1/6. The reason for this is that an unstable pole-zero cancellation occurs when $z_1 = 1/6$.

When $z_1 = 2j$ and $z_2 = -2j$, the proposed method gives $\rho_{\text{inf}} \le 0.136$ and the stable controller C for $\rho = 0.136$ is

$$C = \frac{1}{\phi} N_o^{-1} F^{-1} W_1 - P^{-1},$$

where

$$F(s) \approx \begin{bmatrix} 0.112 & \frac{0.0093(s^2+3.13\times10^{-7}s+4)^2}{(s^2+4.09\times10^{-7}s+4)^2} \\ 0 & \frac{0.093(s^2+2.92\times10^{-7}s+4)^2}{(s^2+4.09\times10^{-7}s+4)^2} \end{bmatrix}.$$

On the other hand, we obtain $\rho_{\text{inf}} \ge 0.117$ by Corollary 3.2.12.

Example 3.2.23. *(Application to a repetitive control system)*

Consider the repetitive control system [54, 121] given in Figure 3.5. The internal model principle for the class of psedorational impulse response matrices [121] shows that exponential decay of the error signal $r - y$ for any reference signal r with a fixed period L is equivalent to the existence of the internal model $e^{-Ls}/(1 - e^{-Ls})$ under the condition of exponential stability of the closed-loop system. This principle is a precise generalization of the well-known finite-dimensional counterpart [35].

Note that if we use the internal model of the type $e^{-Ls}/(1-e^{-Ls})$, then the closed-loop internal stability cannot be achieved for strictly proper plant [121, Theorem 5.12]. Also, such an internal model leads to a potential loss of *w-stability* [40, Sec. 8] of the closed-loop system. So it is practical to construct modified repetitive controllers [19, 54, 121]. However, such controllers do not accurately track nor reject periodic signals on a high frequency band.

The internal model principle suggests that the controllers we consider can be separated into two part $C = C_u C_o$, where C_u is the part of the internal model and has an infinite number of poles on the imaginary axis, and C_o is the stable part to be designed. For the design of C_o, we can consider the product $C_u P =: P_o$ to be the new plant to be controlled.

To guarantee exponential stability, it is desirable that $H(P,C)$ in (1.2.1) has no poles in the region $\mathbb{C}_{-\varepsilon} := \{s \in \mathbb{C} : \ \operatorname{Re} s > -\varepsilon\}$ for some fixed $\varepsilon > 0$ [119]. Consequently, it is enough to solve the problem of strong stabilization with sensitivity reduction for the following plant and weighting functions:

$$\tilde{P}(s) := P_o(s-\varepsilon) = C_u(s-\varepsilon)P(s-\varepsilon), \tag{3.2.21}$$
$$W_1(s) := \frac{s+1}{10s+1}\begin{bmatrix} 1 & 1/10 \\ 0 & 1 \end{bmatrix}, \quad W_2(s) := I.$$

Once we find a solution $\tilde{C}$ to the problem, we determine the stable part $C_o(s)$ of the controller by $C_o(s) := \tilde{C}(s+\varepsilon)$. Since $\tilde{C} \in \mathbf{M}(\mathcal{H}^\infty)$, it follows that C_o does not have poles in $\mathbb{C}_{-\varepsilon}$.

In Figure 3.3, $L := 3$, $a(s) := s/(s+1)$,

$$P(s) := \begin{bmatrix} \frac{s+1}{s+2} & \frac{e^{-2s}}{s+1} \\ 0 & \frac{s+2}{s-1/15} \end{bmatrix},$$
$$C_u(s) := \left(\frac{e^{-Ls}}{1-e^{-Ls}} + a(s)\right) I = \frac{s+e^{-3s}}{(s+1)(1-e^{-3s})} I.$$

We take $\varepsilon = 0.01$, so $\tilde{P}$ in (3.3.37) has infinitely many unstable poles. However it has only two zeros in $\bar{\mathbb{C}}_+$: $\alpha \approx (0.156+\varepsilon) + 0.607j$, $\beta \approx (0.156+\varepsilon) - 0.607j$, which arise from $C_u(s-\varepsilon)$ and are blocking zeros. Using the factorization method of [48], we can factor $\tilde{P}$ as $\tilde{P}(s) = \phi D^{-1} N_o$, where

$$\phi(s) := \frac{(s-\alpha)(s-\beta)}{(s-\varepsilon+1)^2},$$
$$D(s) := \frac{1-e^{3\varepsilon}e^{-3s}}{e^{-3s}-e^{3\varepsilon}} \begin{bmatrix} 1 & 0 \\ 0 & \frac{s-\varepsilon-1/15}{s+\varepsilon+1/15} \end{bmatrix},$$
$$N_o(s) := \frac{(s-\varepsilon+1)(s-\varepsilon+e^{-3(s-\varepsilon)})}{(e^{-3s}-e^{3\varepsilon})(s-\alpha)(s-\beta)} \begin{bmatrix} \frac{s-\varepsilon+1}{s-\varepsilon+2} & \frac{e^{-2(s-\varepsilon)}}{s-\varepsilon+1} \\ 0 & \frac{s-\varepsilon+2}{s+\varepsilon+1/15} \end{bmatrix}.$$

N_o satisfies $N_o, N_o^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$. We can easily check whether D and $N := \phi N_o$ are left coprime as in Example 3.2.22.

Define ρ_{inf} by (3.2.20). An upper bound of ρ_{inf} derived from the proposed method is $0.578 =: \rho_{\mathrm{u}}$, and the stable controller $\tilde{C}$ for $\rho = \rho_{\mathrm{u}}$ is given as

$$\tilde{C} = \frac{2}{\phi} N_o^{-1}(G+I)^{-1} W_1 - \tilde{P}^{-1},$$

where

$$G(s) \approx \begin{bmatrix} \frac{-0.79(s+0.28)(s-0.073)(s^2+0.46s+0.056)}{(s^2+0.57s+0.081)(s^2+0.51s+0.18)} & \frac{-0.057(s^2+0.49s+0.060)(s^2-0.33s+0.40)}{(s^2+0.57s+0.081)(s^2+0.51s+0.18)} \\ \frac{0.031(s+1.37)(s+0.29)(s^2+0.56s+0.37)}{(s^2+0.57s+0.081)(s^2+0.51s+0.18)} & \frac{-1.00(s-0.27)(s+0.29)}{s^2+0.57s+0.081} \end{bmatrix}$$

On the other hand, by Corollary 3.2.12, we obtain $\rho_{\text{inf}} \geq 0.272$. Combining this lower bound and the above upper bound on ρ_{inf}, we have $0.272 \leq \rho_{\text{inf}} \leq 0.578$. This implies that our proposed method is conservative in this example.

3.3 Systems with unstable transmission zeros

In this section, we place less restrictive constraints on the multivariable zeros of the plant than those in the previous section. We will discuss the difference of the constraints and address the nontrivial modifications arising from it.

Here we prove that *if the plant has transmission zeros in $\bar{\mathbb{C}}_+$, strong stabilization is equivalent to tangential interpolation by a unimodular matrix in $\mathbf{M}(\mathcal{H}^\infty)$.* In conjunction with the tangential Nevanlinna-Pick interpolation, this equivalence enables us to obtain both lower and upper bounds of the minimum sensitivity achievable by a stable controller.

3.3.1 Strong stabilization

We first study strong stabilization only.

Let $\det N$ denote the determinant of $N \in (\mathcal{H}^\infty)^{p\times p}$. Throughout this section, we assume that the following properties holds:

Assumption 3.3.1. *All entries of N, D, X, and Y in* (3.1.2) *are meromorphic in $\mathbb{C}$. Moreover, N is square and* $\det N$ *has the form*

$$\det N = \phi N_o, \quad \text{where } \phi \in \mathcal{RH}^\infty \text{ and } N_o, 1/N_o \in \mathcal{H}^\infty. \tag{3.3.1}$$

The rational function ϕ satisfies $\phi(\infty) \neq 0$ and has only simple zeros $z_1, \dots, z_n$ in $\bar{\mathbb{C}}_+$. For $i = 1, \dots, n$, a left annihilating nonzero vector $v_i \in \mathbb{C}^p$ satisfying

$$v_i^* N(z_i) = 0 \tag{3.3.2}$$

is unique up to multiplication by a constant complex number.

In Remark 3.3.12 at the end on this subsection, we will discuss the two conditions: All functions are meromorphic; $\det N$ has only simple zeros in $\bar{\mathbb{C}}_+$.

In the previous section, we assume that the *matrix*-valued function N can be factored as $N = \phi N_o$, where $\phi \in \mathcal{RH}^\infty$ and $N_o,\ N_o^{-1} \in (\mathcal{H}^\infty)^{p\times p}$. Note that Assumption 3.3.1 requires the factorization (3.3.1) of the *scalar*-valued function $\det N$.

We shall show that Problem 3.2.2 is equivalent to the following problem:

Problem 3.3.2. *Suppose that $s_1, \dots, s_n \in \bar{\mathbb{C}}_+$ are distinct and that $\xi_1, \dots, \xi_n$ and $\eta_1, \dots, \eta_n$ belong to $\mathbb{C}^p$. Find a unimodular matrix $U \in (\mathcal{H}^\infty)^{p\times p}$ such that all elements of U are meromorphic in $\mathbb{C}$ and $\xi_i^* U(s_i) = \eta_i^*$ for $i = 1, \dots, n$.*

Theorem 3.3.3. *Consider Problem 3.2.2 under Assumption 3.3.1. We restrict the solutions to matrices whose entries are meromorphic in $\mathbb{C}$. Then Problem 3.2.2 is equivalent to Problem 3.3.2 with interpolation data $(z_i, [v_i, D(z_i)^* v_i])_{i=1}^n$.*

Furthermore, a solution C to Problem 3.2.2 and a solution U to Problem 3.3.2 satisfy

$$C = N^{-1}(U - D), \quad U = D + NC. \tag{3.3.3}$$

Proof. Let $C \in (\mathcal{H}^\infty)^{p\times p}$ be a meromorphic solution to Problem 3.2.2. Define U by (3.3.3). Then U and U^{-1} belong to $(\mathcal{H}^\infty)^{p\times p}$ by Lemma 3.2.1 and

$$v_i^* U(z_i) = v_i^* D(z_i) + v_i^* N(z_i) C(z_i) = (D(z_i)^* v_i)^*.$$

Hence U is a solution to Problem 3.3.2 with interpolation data $(z_i, [v_i, D(z_i)^* v_i])_{i=1}^n$.

Conversely, let $U \in (\mathcal{H}^\infty)^{p\times p}$ be a solution to Problem 3.3.2 with interpolation data $(z_i, [v_i, D(z_i)^* v_i])_{i=1}^n$. Define C by (3.3.3). Then C satisfies $(D+NC)^{-1} = U^{-1} \in (\mathcal{H}^\infty)^{p\times p}$,

$$NC = U - D \in (\mathcal{H}^\infty)^{p\times p}, \tag{3.3.4}$$

and

$$v_i^*(NC)(z_i) = v_i^*(U(z_i) - D(z_i)) = 0. \tag{3.3.5}$$

We prove $C \in (\mathcal{H}^\infty)^{p\times p}$ by (3.3.4) and (3.3.5) as follows. Define $\Upsilon := NC$. Then $\Upsilon \in (\mathcal{H}^\infty)^{p\times p}$ by (3.3.4) and $v_i^* \Upsilon(z_i) = 0$ by (3.3.5). Let N_c be the transpose of the cofactor matrix of $N \in (\mathcal{H}^\infty)^{p\times p}$. Since we have by Cramer's rule

$$N_c N = N N_c = \det N \cdot I, \tag{3.3.6}$$

it follows from the definition of Υ that

$$\phi C = 1/N_o \cdot N_c \cdot \Upsilon \in (\mathcal{H}^\infty)^{p\times p}. \tag{3.3.7}$$

Also we obtain the following property

$$N_c(z_i)\Upsilon(z_i) = 0, \quad i = 1, \ldots, n. \tag{3.3.8}$$

This is because every row of $N_c(z_i)$ is a constant multiple of v_i^*. To see this, let $(N_c)_m(z_i)$ be the m-th row of $N_c(z_i)$. Since

$$N_c(z_i)N(z_i) = (\phi(z_i)N_o(z_i)) \cdot I = 0$$

by (3.3.6), we have $(N_c)_m(z_i)N(z_i) = 0$ for $m = 1, \ldots, p$. Thus the uniqueness of v_i in Assumption 3.3.1 implies $(N_c)_m(z_i) = k_m v_i^*$ for some $k_m \in \mathbb{C}$.

Since the invertible function N_o has no unstable zero, (3.3.7) and (3.3.8) show

$$(\phi C)(z_i) = 0, \quad i = 1, \ldots, n. \tag{3.3.9}$$

Thus it suffices to prove $C \in (\mathcal{H}^\infty)^{p\times p}$ from the following three conditions: The unstable zeros $z_1, \ldots, z_n$ of ϕ are simple; $\phi C \in (\mathcal{H}^\infty)^{p\times p}$; and (3.3.9) holds.

Suppose $C \notin (\mathcal{H}^\infty)^{p\times p}$. Then, since $\phi C \in (\mathcal{H}^\infty)^{p\times p}$, the unstable poles of C must be zeros of ϕ. Let z_i be one of such poles. Since ϕ has only simple zeros in $\bar{\mathbb{C}}_+$, it follows that $(\phi C)(z_i) \neq 0$. This contradicts (3.3.9), and hence $C \in (\mathcal{H}^\infty)^{p\times p}$. $\square$

Remark 3.3.4. 1. In the previous section, we have considered *matrix-valued* interpolation conditions $U(z_i) = D(z_i)$. This interpolation leads to the stringent assumption that all unstable zeros z_i of the plant be *blocking* zeros, enabling us to handle such multivariable zeros in a way similar to that used for zeros of SISO systems. In contrast, here we address *tangential* interpolation conditions $v_i^* U(z_i) = v_i^* D(z_i)$ so that the plant is allowed to have unstable *transmission* zeros.

2. Prasanth [91] presents a method to find a unimodular matrix in $(\mathcal{RH}^\infty)^{p\times p}$ satisfying finitely many tangential interpolation conditions. Furthermore, a result similar to Theorem 3.3.3 is also developed for finite-dimensional systems in [91].

   However, the argument in [91] makes use of the results of [3] and a *state-space realization* of the plant. Hence it is not applicable to the present situation. The main contribution here is to give a new, straightforward proof in a *transfer-function* approach with Cramer's rule. Moreover, in contrast with [91], we explicitly show the equivalence between strong stabilization and tangential interpolation with a unimodular matrix. From this equivalence, we will derive a necessary and sufficient condition for strong stabilization with sensitivity reduction in the next subsection.

3.3.2 Strong stabilization with sensitivity reduction

We now proceed to the problem of strong stabilization with sensitivity reduction. We further place the same assumption on W_1, W_2, and D as in the case of unstable blocking zeros.

Assumption 3.3.5. *All elements of W_1 and W_2 are meromorphic functions in $\mathbb{C}$. Both W_1 and W_1^{-1} belong to $\mathbf{M}(\mathcal{H}^\infty)$. Let DW_2 have a factorization $DW_2 = (DW_2)_{co} \cdot (DW_2)_{ci}$, where $(DW_2)_{co}$ is co-outer and $(DW_2)_{ci}$ is co-inner. $(DW_2)_{co}$ and $(DW_2)_{co}^{-1}$ also belong to $\mathbf{M}(\mathcal{H}^\infty)$.*

See Remark 3.2.9 for the condition on the co-outer function $(DW_2)_{co}$.

By extending the results of the previous subsection, we will prove that Problem 3.1.1 is equivalent to the following Problem 3.3.6 under Assumptions 3.3.1 and 3.3.5. The only difference between Problems 3.3.2 and 3.3.6 is that the latter problem requires that the $\mathcal{H}^\infty$-norm of a solution be less than one.

Problem 3.3.6. *Suppose $s_1, \dots, s_n \in \bar{\mathbb{C}}_+$ are distinct. Let $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in \mathbb{C}^p$. Find a unimodular matrix $F \in (\mathcal{H}^\infty)^{p\times p}$ such that all elements of F are meromorphic in $\mathbb{C}$, $\|F\|_\infty < 1$, and*

$$\xi_i^* F(s_i) = \eta_i^*, \quad i = 1, \dots, n. \tag{3.3.10}$$

Theorem 3.3.7. *Consider Problem 3.1.1. Suppose there exist $D, N \in \mathbf{M}(\mathcal{H}^\infty)$ such that $P = D^{-1}N$, and let Assumptions 3.3.1 and 3.3.5 hold. Define the vector pairs (ξ_i, η_i) by*

$$\xi_i := (D(z_i)W_1^{-1}(z_i))^* v_i, \quad \eta_i := ((DW_2)_{co}(z_i))^* v_i, \quad i = 1, \dots, n. \tag{3.3.11}$$

If there is a solution F to Problem 3.3.6 with interpolation data $(z_i, [\xi_i, \eta_i])_{i=1}^n$, then a solution to Problem 3.1.1 is given by

$$C = N^{-1}(DW_2)_{co}F^{-1}W_1 - P^{-1}. \tag{3.3.12}$$

Conversely, if there exists a solution C to Problem 3.1.1 and if all entries of C are meromorphic in $\mathbb{C}$, then

$$F = W_1(D + NC)^{-1}(DW_2)_{co} \tag{3.3.13}$$

is a solution to Problem 3.3.6 with interpolation data $(z_i, [\xi_i, \eta_i])_{i=1}^n$.

Proof. Let a unimodular matrix $F \in (\mathcal{H}^\infty)^{p\times p}$ be a solution to Problem 3.3.6 with interpolation data $(z_i, [\xi_i, \eta_i])_{i=1}^n$. Define C by (3.3.12).

To prove $C \in (\mathcal{H}^\infty)^{p\times p} \cap \mathscr{C}(P)$, it suffices to show, by Theorem 3.3.3, that U defined by

$$U := D + NC = (DW_2)_{co}F^{-1}W_1 \tag{3.3.14}$$

satisfies U, $U^{-1} \in (\mathcal{H}^\infty)^{p\times p}$ and $v_i^* U(z_i) = v_i^* D(z_i)$ for $i = 1, \ldots, n$.

Since $(DW_2)_{co}$, F, and W_1 are unimodular, it follows from (3.3.14) that both U and U^{-1} belong to $(\mathcal{H}^\infty)^{p\times p}$. Additionally, since $\xi_i^* = \eta_i^* F(z_i)^{-1}$, we have

$$v_i^* U(z_i) = \eta_i^* F^{-1}(z_i)W_1(z_i) = \xi_i^* W_1(z_i) = v_i^* D(z_i).$$

Hence we obtain $C \in (\mathcal{H}^\infty)^{p\times p} \cap \mathscr{C}(P)$.

Moreover, it follows from Lemma 3.2.6 that

$$\|W_1(1 + PC)^{-1}W_2\|_\infty = \|W_1(D + NC)^{-1}(DW_2)_{co} \cdot (DW_2)_{ci}\|_\infty = \|F\|_\infty. \tag{3.3.15}$$

Thus C is a solution to Problem 3.1.1.

Conversely, suppose C is a solution to Problem 3.1.1 and all the entries are meromorphic. Define F by (3.3.13). Then, since U in (3.3.14) satisfies U, $U^{-1} \in (\mathcal{H}^\infty)^{p\times p}$ by Theorem 3.3.3, it follows that F, $F^{-1} \in (\mathcal{H}^\infty)^{p\times p}$. Also F satisfies $\|F\|_\infty < 1$ by (3.3.15). Since

$$\begin{aligned}
\xi_i^* F(z_i) &= v_i^* D(z_i)(D + NC)^{-1}(z_i)(DW_2)_{co}(z_i) \\
&= v_i^*(D + NC)(z_i) \cdot (D + NC)^{-1}(z_i)(DW_2)_{co}(z_i) \\
&= v_i^*(DW_2)_{co}(z_i) \\
&= \eta_i^*
\end{aligned}$$

by (3.3.11), (3.3.14), and (3.3.2), we obtain (3.3.10). Thus F is a solution to Problem 3.3.6 with interpolation data $(z_i, [\xi_i, \eta_i])_{i=1}^n$. □

See Remark 3.2.13 for the reason why we can reduce the $\mathcal{H}^\infty$ control problem for infinite-dimensional systems to the same interpolation problem as in the finite-dimensional case.

Theorem 3.3.7 suggests that the problem of strong stabilization with sensitivity reduction is equivalent to Problem 3.3.6. A natural question then arises: *Is this interpolation problem solvable?* Since the solution to Problem 3.3.6 must be unimodular, it is difficult to give a necessary and sufficient condition. Here we separately derive a sufficient condition and a necessary condition for Problem 3.3.6 via the tangential Nevanlinna-Pick interpolation.

First we derive a necessary condition. From Theorem 3.3.7, we deduce the next result that provides a lower bound of the minimum sensitivity $\rho_{\inf}$ achievable by a stable controller.

Corollary 3.3.8. *Consider Problem 3.1.1 under the same hypotheses of Theorem 3.3.7. For a given $\rho > 0$, if there exists no $G \in (\mathcal{H}^\infty)^{p\times p}$ such that $\|G\|_\infty < 1$ and $\xi_i^* G(z_i) = \eta_i^*/\rho$ for $i = 1, \ldots, n$, then $\rho_{\inf}$ in* (3.2.20) *satisfies $\rho_{\inf} \geq \rho$.*

Let us next develop a sufficient condition and a design method of stable stabilizing controllers that achieve low sensitivity. The following lemma gives the solution to Problem 3.3.6 via the tangential Nevanlinna-Pick interpolation.

Lemma 3.3.9. *Consider Problem 3.3.6. Let $\lambda \in \mathbb{C}$ and define*

$$\zeta_i := \frac{2}{\lambda}\eta_i - \xi_i, \quad i = 1, \ldots, n. \tag{3.3.16}$$

If $G \in (\mathcal{H}^\infty)^{p\times p}$ satisfies $\|G\|_\infty < 1$ and

$$\xi_i^* G(z_i) = \zeta_i^*, \quad i = 1, \ldots, n, \tag{3.3.17}$$

then F defined by

$$F := \frac{\lambda}{2}(G + I) \tag{3.3.18}$$

is unimodular and satisfies $\|F\|_\infty < |\lambda|$ and the interpolation constraints (3.3.10).

Proof. Lemma 3.2.15 shows that $F, F^{-1} \in (\mathcal{H}^\infty)^{p\times p}$ and $\|F\|_\infty < |\lambda|$. By (3.3.16), (3.3.17), and (3.3.18), F also satisfies (3.3.10). □

Combining Theorem 3.3.7 with Lemma 3.3.9, we obtain an upper bound of $\rho_{\inf}$ and a stable controller achieving the bound.

Theorem 3.3.10. *Consider Problem 3.1.1 under the same assumptions and definitions as in Theorem 3.3.7 and Lemma 3.3.9. If there exists $G \in (\mathcal{H}^\infty)^{p\times p}$ such that $\|G\|_\infty < 1$ and* (3.3.17) *holds, then $\rho_{\inf}$ in* (3.2.20) *satisfies $\rho_{\inf} < |\lambda|$ and a solution to Problem 3.1.1 is given by*

$$C = \frac{2}{\lambda} N^{-1} (DW_2)_{co} (G + I)^{-1} W_1 - P^{-1}.$$

Theorem 3.3.10 and Corollary 3.3.8 give upper and lower bounds of the minimum sensitivity $\rho_{\inf}$ by iterative computation of the associated Pick matrices.

We now summarize the design procedure of stable controllers achieving low sensitivity.

Design procedure for stable stabilizing controllers providing low sensitivity for plants with unstable transmission zeros

***Step 1*:** Let $\lambda \in \mathbb{C}$ satisfy $|\lambda| = 1$. For each unstable transmission zero z_i of P, define

$$\xi_i := (D(z_i)W_1^{-1}(z_i))^* v_i, \quad \zeta_i := \frac{2}{\bar{\lambda}}((DW_2)_{co}(z_i))^* v_i - \xi_i, \quad i = 1, \ldots, n.$$

***Step 2*:** Find the solution G to the tangential Nevanlinna-Pick interpolation problem 2.3.6 with data $(z_i, [\xi_i, \zeta_i])_{i=1}^n$.

***Step 3*:** Compute a solution of Problem 3.3.6 from (3.3.18).

***Step 4*:** Calculate a stable controller (3.3.12) achieving a desired sensitivity level.

As in the case of unstable blocking zeros, the controller derived above has unstable pole-zero cancellations. See also Remark 3.2.17.

Let us next investigate the relationship between the gains of W_1, W_2 and the set of controllers derived from the above procedure. The following result is analogous to Proposition 3.2.18:

Proposition 3.3.11. *Let λ_1, $\lambda_2 \in \mathbb{C} \setminus \{0\}$ satisfy $\lambda_1 = a\lambda_2$ for some $a \in (0, 1)$. Suppose that $z_1, \ldots, z_n \in \bar{\mathbb{C}}_+$ are distinct and that $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ are in $\mathbb{C}^p$. Suppose also that for $k = 1, 2$, $\mathscr{N}_k$ is the set of all solutions G to the tangential Nevanlinna-Pick interpolation problem 2.3.6 with the following conditions:*

$$\xi_i^* G(z_i) = \frac{2}{\lambda_k}\eta_i^* - \xi_i^*, \quad i = 1, \ldots, n. \tag{3.3.19}$$

Define

$$\mathscr{M}_k := \left\{ \frac{\lambda_k}{2}(G_k + I) \ : \ G_k \in \mathscr{N}_k \right\}, \qquad k = 1, 2.$$

Then we have

$$\mathscr{M}_1 \subset \mathscr{M}_2. \tag{3.3.20}$$

Proof. Assume that $F \in \mathscr{M}_1$. Then there exists $G_1 \in \mathscr{N}_1$ such that

$$F = \frac{\lambda_1}{2}(G_1 + I). \tag{3.3.21}$$

If we define G_2 by

$$G_2 := \frac{\lambda_1}{\lambda_2}(G_1 + I) - I, \tag{3.3.22}$$

then we have $G_2 \in \mathscr{N}_2$. In fact, by (3.3.19),

$$\begin{aligned}
\xi_i^* G_2(z_i) &= \xi_i^* \left(\frac{\lambda_1}{\lambda_2}(G_1(z_i) + I) - I \right) \\
&= \frac{\lambda_1}{\lambda_2} \left(\frac{2}{\lambda_1} \eta_i^* - \xi_i^* \right) + \frac{\lambda_1 - \lambda_2}{\lambda_2} \xi_i^* \\
&= \frac{2}{\lambda_2} \eta_i^* - \xi_i^*.
\end{aligned}$$

Moreover, since $\|G_1\|_\infty < 1$, it follows from (3.2.16) that $\|G_2\|_\infty < 1$. Thus $G_2 \in \mathscr{N}_2$.

By (3.3.21) and (3.3.22), we also have

$$F = \frac{\lambda_2}{2}(G_2 + I).$$

Since $G_2 \in \mathscr{N}_2$, this leads to $F \in \mathscr{M}_2$. Hence (3.3.20) is obtained. □

We conclude this subsection with two remarks on Assumption 3.3.1.

Remark 3.3.12. 1. In this section, we have assumed that all functions are meromorphic in $\mathbb{C}$ because $\mathcal{H}^\infty$ functions do not necessarily have a finite value on the imaginary axis. If the unstable zeros of $\det N$ are not on the imaginary axis, then we remove the assumption that all elements of the transfer matrices are meromorphic.

2. We have assumed that $\det N$ has only simple zeros in $\bar{\mathbb{C}}_+$, but the results in this section can be generalized to the case in which $\det N$ has unstable zeros of higher order. In this case, we need to introduce interpolation conditions involving derivatives of N and D.

   For example, let z_i be an unstable zero of order 2 of $\det N$, and suppose that v_i and $\bar{v}_i$ are the unique vectors such that

$$v_i^* N(z_i) = 0, \qquad v_i^* N'(z_i) + \bar{v}_i^* N(z_i) = 0, \tag{3.3.23}$$

   where N' denotes the derivative of N. Then the interpolation conditions of U in Theorem 3.3.3 are given by

$$v_i^* U(z_i) = v_i^* D(z_i) =: w_i^*, \tag{3.3.24}$$
$$v_i^* U'(z_i) + \bar{v}_i^* U(z_i) = \bar{v}_i^* D(z_i) + v_i^* D'(z_i) =: \bar{w}_i^*. \tag{3.3.25}$$

   Also, if we assume $(DW_2)_{co} = I$ for simplicity, F in Theorem 3.3.7 must satisfy

$$\xi_i^* F(z_i) = v_i^*, \tag{3.3.26}$$
$$\bar{\xi}_i^* F(z_i) + \xi_i^* F'(z_i) = \bar{v}_i^*, \tag{3.3.27}$$

   where

$$\xi_i := W_1^{-1}(z_i)^* w_i, \tag{3.3.28}$$
$$\bar{\xi}_i := W_1^{-1}(z_i)^* \bar{w}_i + (W_1^{-1})'(z_i)^* w_i. \tag{3.3.29}$$

   Here we give a sketch of a proof for the generalization.

Interpolation Conditions of U

First we obtain the interpolation conditions (3.3.24) and (3.3.25) of U in Theorem 3.3.3.

Suppose $C \in \mathbf{M}(\mathcal{H}^\infty)$ and define $U := D + NC$. Then (3.3.23) shows that U satisfies (3.3.24). Also,

$$\begin{aligned}
v_i^* U'(z_i) &= v_i^* D'(z_i) + v_i^* N(z_i) C'(z_i) + v_i^* N'(z_i) C(z_i) \\
&= v_i^* D'(z_i) - \bar{v}_i^* N(z_i) C(z_i) \\
&= v_i^* D'(z_i) + (\bar{v}_i^* D(z_i) - \bar{v}_i^* U(z_i)),
\end{aligned}$$

and hence we obtain (3.3.25).

Conversely, let $U \in \mathbf{M}(\mathcal{H}^\infty)$ satisfy (3.3.24) and (3.3.25). Define $\Upsilon := NC = U - D$. Then we obtain

$$v_i^* \Upsilon(z_i) = v_i^*(U(z_i) - D(z_i)) = 0$$

and

$$\begin{aligned}
v_i \Upsilon'(z_i) + \bar{v}_i^* \Upsilon(z_i) &= v_i^*(U'(z_i) - D'(z_i)) + \bar{v}_i^*(U(z_i) - D(z_i)) \\
&= (v_i^* U'(z_i) + \bar{v}_i^* U(z_i))) - (v_i^* D'(z_i) + \bar{v}_i^* D(z_i))) \\
&= 0.
\end{aligned}$$

Combining

$$N_c N = \det N, \quad N_c N' + (N_c)' N_c = (\det N)'$$

with the uniqueness of v_i and $\bar{v}_i$, we obtain

$$N_c(z_i)\Upsilon(z_i) = 0 \tag{3.3.30}$$

and

$$(N_c \Upsilon)'(z_i) = N_c(z_i)\Upsilon'(z_i) + (N_c)'(z_i)\Upsilon(z_i) = 0. \tag{3.3.31}$$

Since $\det N \cdot C = N_c \cdot \Upsilon$, (3.3.30) and (3.3.31) show that

$$(\det N \cdot C)(z_i) = 0, \qquad (\det N \cdot C)'(z_i) = 0.$$

These conditions lead to $C \in \mathbf{M}(\mathcal{H}^\infty)$ because the order of z_i is 2.

Interpolation Conditions of F

We now study the interpolation conditions (3.3.26) and (3.3.27) of F in Theorem 3.3.7.

Assume, for simplicity, that $(DW_2)_{co} = I$, and let U satisfy

$$v_i^* U(z_i) = w_i^*, \qquad v_i U'(z_i) + \bar{v}_i^* U(z_i) = \bar{w}_i^*.$$

Define $F := W_1 U^{-1}$ and $L := U^{-1}$. Then

$$v_i^* U(z_i) = w_i^* \quad \Longleftrightarrow \quad w_i^* L(z_i) = v_i^*. \tag{3.3.32}$$

Also, since $(U^{-1})' = -U^{-1}U'U^{-1}$, it follows that

$$\begin{aligned} v_i^* U'(z_i) + \bar{v}_i^* U(z_i) = \bar{w}_i^* \quad &\iff \quad -v_i^* U(z_i) L'(z_i) L^{-1}(z_i) + \bar{v}_i^* L^{-1}(z_i) = \bar{w}_i^* \\ &\iff \quad -w_i^* L'(z_i) + \bar{v}_i^* = \bar{w}_i^* L(z_i) \\ &\iff \quad w_i^* L'(z_i) + \bar{w}_i^* L(z_i) = \bar{v}_i^*. \end{aligned} \tag{3.3.33}$$

Finally, define ξ_i and $\bar{\xi}_i$ by (3.3.28) and (3.3.29), respectively. Since $L = W_1^{-1}F$, (3.3.32) and (3.3.33) show that

$$v_i^* U(z_i) = w_i^* \quad \iff \quad \xi_i^* F(z_i) = v_i^*$$

and

$$\begin{aligned} & v_i^* U'(z_i) + \bar{v}_i^* U(z_i) = \bar{w}_i^* \\ &\iff \quad w_i^* \big((W_1^{-1})'(z_i) F(z_i) + W_1^{-1}(z_i) F'(z_i) \big) + \bar{w}_i^* W_1^{-1}(z_i) F(z_i) = \bar{v}_i^* \\ &\iff \quad \xi_i^* F'(z_i) + \bar{\xi}_i^* F(z_i) = \bar{v}_i, \end{aligned}$$

which are the desired interpolation conditions.

3.3.3 Numerical examples

Here we present a numerical example and also apply the proposed method to a repetitive control system [54, 121]. Furthermore, a coprime factorization technique is developed for MIMO systems with scalar infinite-dimensional part.

Example 3.3.13. We consider strong stabilization with sensitivity reduction for the following infinite-dimensional system and weighting functions:

$$P(s) = \begin{bmatrix} \frac{(s-z_1)(s-z_2)}{(s+1)^2(1+3e^{-2s})} & \frac{e^{-s}}{s+2} \\ 0 & \frac{(s+1)^2}{(s-1)(s-e^{-s}+2)} \end{bmatrix},$$

$$W_1(s) = \frac{s+1}{10s+1} I, \quad W_2(s) = \begin{bmatrix} \frac{s+2}{50s+1} & 0 \\ 0 & \frac{s+1}{200s+1} \end{bmatrix},$$

where $z_1, z_2 \in \mathbb{C}_+$ are distinct.

Let us begin by finding left coprime $D, N \in (\mathcal{H}^\infty)^{2\times 2}$ such that $D^{-1}N = P$. First, applying the factorization method of [43] to each element of P, we can factor P as $P = D^{-1}N$, where

$$D(s) := \begin{bmatrix} \frac{1+3e^{-2s}}{e^{-2s}+3} & 0 \\ 0 & \frac{s-1}{s+1} \end{bmatrix}, \quad N(s) := \begin{bmatrix} \frac{(s-z_1)(s-z_2)}{(s+1)^2(e^{-2s}+3)} & \frac{1+3e^{-2s}}{e^{-2s}+3} \cdot \frac{e^{-s}}{s+2} \\ 0 & \frac{s+1}{s-e^{-s}+2} \end{bmatrix}.$$

The unstable zeros of $\det N$ are z_1 and z_2. The vectors v_i given by

$$v_i = \begin{bmatrix} -\frac{e^{-2z_i}+3}{1+3e^{-2z_i}} \cdot \frac{z_i+2}{e^{-z_i}} & \frac{z_i - e^{-z_i}+2}{z_i+1} \end{bmatrix}^*, \quad i = 1, 2,$$

satisfy $v_i^* N(z_i) = 0$ and they are unique up to multiplication by a constant complex number.

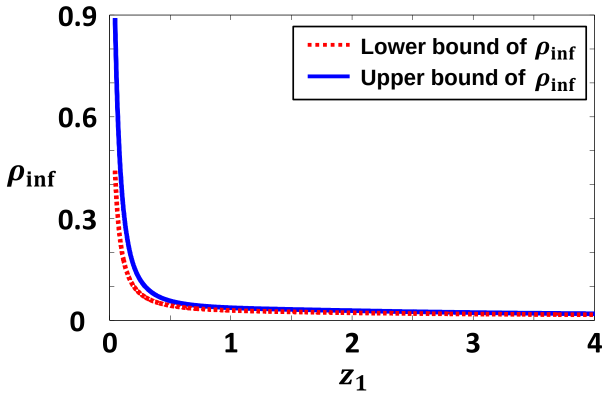


Figure 3.4: Unstable transmission zero z_1 versus minimum sensitivity $\rho_{\rm inf}$.

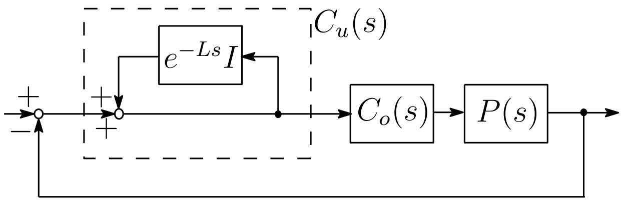


Figure 3.5: Repetitive control system with $a(s) = 1$ in Figure 3.3.

Next, from the same argument leading to $C \in (\mathcal{H}^\infty)^{p\times p}$ in Theorem 3.3.3, we see that D and N are left coprime if and only if there exists $Y \in (\mathcal{H}^\infty)^{2\times 2}$ satisfying the interpolation conditions $v_i^* D(z_i)Y(z_i) = v_i^*$ for $i = 1, 2$. This problem is called the tangential Lagrange interpolation [3, Chap. 16]. We can check the existence of such Y by the tangential Nevanlinna-Pick interpolation with the scaling of the interpolation data.

Let us take $0 < z_1 \leq 4$ and $z_2 = 5$. Figure 3.4 shows the relationship between the unstable transmission zero z_1 and the minimum sensitivity $\rho_{\rm inf}$ in (3.2.20). In Figure 3.4, the solid line indicates an upper bound of $\rho_{\rm inf}$ derived from Theorem 3.3.10. The dashed line shows a lower bound of $\rho_{\rm inf}$ obtained by Corollary 3.3.8. In contrast to Figure 3.2, we see from Figure 3.4 that an unstable pole-zero cancellation at $s = 1$ in $\det P$ does not affect strong stabilization with sensitivity reduction. This is because z_1 is not a *blocking* zero but a simple *transmission* zero. Furthermore, z_1 is not in the same input nor output channel as the pole at $s = 1$.

Example 3.3.14. *(Application to a repetitive control system)*

Consider the repetitive control system in Figure 3.5, where P is a finite-dimensional plant and $C_u(s) = 1/(1 - e^{-Ls}) \cdot I$ is the internal model of any periodic signals with period L. The internal model C_u in Figure 3.5 is the case $a(s) = 1$ in Figure 3.3.

For a given P and C_u, we design C_o to meet performance requirements. Let us here find $C_o \in \mathbf{M}(\mathcal{H}^\infty)$ yielding exponential stability and low sensitivity of the closed-loop system. By the same argument as in the previous section, in order to do this, we study Problem 3.1.1 with $\tilde{P}(s) := C_u(s-\varepsilon)P(s-\varepsilon)$ for some $\varepsilon > 0$. If we find a solution $\tilde{C}$ to the problem, then we design C_o by $C_o(s) = \tilde{C}(s+\varepsilon)$. Since $\tilde{C} \in \mathbf{M}(\mathcal{H}^\infty)$, it follows that C_o is holomorphic and bounded in the region $\{s \in \mathbb{C} : \ \mathrm{Re}\, s > -\varepsilon\}$.

When we apply the proposed method to infinite-dimensional systems, we first raise the following question: *How do we obtain a left coprime factorization of general MIMO infinite-dimensional systems?* If the infinite-dimensional part of the systems is scalar, we can answer this question affirmatively by using a factorization of the finite-dimensional part.

Theorem 3.3.15. *Let Assumption 3.3.1 hold for D, $N \in (\mathcal{RH}^\infty)^{p\times p}$. Suppose $f \in \mathcal{H}^\infty$ is meromorphic in $\mathbb{C}$ and satisfies $f(z_i) \neq 0$ for every i. Then fD and N are left coprime.*

Proof. By the Bezout identity (3.1.2) and the Cramer's rule (3.3.6), we have

$$N_c(I - DY) = \det N \cdot X.$$

The proof of Theorem 3.3.3 shows that m-th row of $N_c(z_i)$ is $k_m v_i^*$ with some $k_m \in \mathbb{C}$ for every m. Hence

$$v_i^*(I - D(z_i)Y(z_i)) = 0 \tag{3.3.34}$$

or $k_1 = \cdots = k_m = 0$, i.e., $N_c(z_i) = 0$.

Let us first prove (3.3.34). Suppose that $N_c(z_i) = 0$. Then there exists $\widehat{N_c} \in (\mathcal{RH}^\infty)^{p\times p}$ such that

$$N_c(s) = (s - z_i)\widehat{N_c}(s).$$

By Assumption 3.3.1, there is also $\widehat{\det N} \in \mathcal{RH}^\infty$ such that

$$\det N(s) = (s - z_i)\widehat{\det N}(s)$$

and $\widehat{\det N}(z_i) \neq 0$. Therefore, since $N\widehat{N_c} = \widehat{\det N} \cdot I$ by Cramer's rule (3.3.6), we have

$$\widehat{\det N}(z_i) \cdot v_i^* = v_i^* N(z_i)\widehat{N_c}(z_i) = 0.$$

This contradicts $\widehat{\det N}(z_i) \neq 0$ and $v_i \neq 0$; hence (3.3.34) always holds.

Next we observe a sufficient condition for the left coprimeness of fD and N. Let $Y_o \in (\mathcal{H}^\infty)^{p\times p}$ satisfy

$$v_i^*\left(I - f(z_i)D(z_i)Y_o(z_i)\right) = 0, \quad i = 1, \dots, n. \tag{3.3.35}$$

Define

$$X_o := N^{-1}(I - fDY_o).$$

Then X_o satisfies the Bezout identity

$$NX_o + fDY_o = I.$$

Moreover, we have $X_o \in (\mathcal{H}^\infty)^{p\times p}$. This proof follows the same line as that of $C \in (\mathcal{H}^\infty)^{p\times p}$ in Theorem 3.3.3, so it is omitted. Thus if there exists Y_o satisfying (3.3.35), then fD and N are left coprime.

The argument given above suggests that, to show the left coprimeness of fD and N, it suffices to prove the following: If there exists $Y \in (\mathcal{RH}^\infty)^{p\times p}$ such that $\xi_i^* Y(z_i) = \eta_i^*$ for $i = 1, \dots, n$, then there also exists $Y_o \in (\mathcal{H}^\infty)^{p\times p}$ such that

$$\xi_i^*(a_i Y_o(z_i)) = \eta_i^*, \quad i = 1, \dots, n, \tag{3.3.36}$$

where $\xi_i := D^*(z_i)v_i$, $\eta_i := v_i$, and $a_i := f(z_i)$.

Since $a_i = f(z_i) \neq 0$, we construct by the Lagrange interpolation [24] a rational function $g \in \mathcal{H}^\infty$ such that $g(z_i) = 1/a_i$ for $i = 1, \ldots, n$. Now define $Y_o := gY \in (\mathcal{H}^\infty)^{p\times p}$. Then, since

$$Y_o(z_i) = \frac{1}{a_i} Y(z_i), \quad i = 1, \ldots, n,$$

it follows that Y_o satisfies (3.3.36). Therefore fD and N are left coprime. □

Theorem 3.3.15 asserts that if there is no unstable pole-zero cancellation in the product $\tilde{P}(s) = C_u(s-\varepsilon)P(s-\varepsilon)$, then $\tilde{P}$ has the following left coprime factorization:

$$\tilde{P}(s) = (C_u(s-\varepsilon)D(s))^{-1} \cdot N(s), \tag{3.3.37}$$

where D, $N \in (\mathcal{RH}^\infty)^{p\times p}$ are left coprime and satisfy $P(s-\varepsilon) = D^{-1}(s)N(s)$.

Finally, taking $\varepsilon = 0.1$, $L = 1$, and

$$P(s) = \begin{bmatrix} \frac{s-1}{s-2} & \frac{1}{s-5} \\ \frac{1}{s+8} & \frac{s}{s-5} \end{bmatrix}, \quad W_1(s) = \frac{s+1}{10s+1} I, \quad W_2 = I, \tag{3.3.38}$$

we solve Problem 3.1.1 for $\tilde{P}$ in (3.3.37), W_1, and W_2. Note that this example is different from Example 3.2.23 in the previous section, where all unstable zeros of P must be *blocking* zeros. The plant P in (3.3.38) has two unstable transmission zeros: 0.846 and 0.291.

By Theorem 3.3.10 and Corollary 3.3.8, we compute both upper and lower bounds of ρ_{inf} in (3.2.20) with $\tilde{P}$; $0.6998 \le \rho_{\text{inf}} \le 0.7176$. A solution $\tilde{C} \in (\mathcal{H}^\infty)^{2\times 2}$ achieving the upper bound $\rho = 0.7176$ is given by $\tilde{C} = N^{-1}D_{co}F^{-1}W_1 - \tilde{P}$, where D_{co} is a co-outer matrix of D and F is a solution to Problem 3.3.6. D_{co}, F, and F^{-1} are given by

$$D_{co}(s) \approx \begin{bmatrix} \frac{0.7071(s+7.862)(s+3.814)(s+1.992)}{(s+3.612)(s+1.698)(s+7.956)} & \frac{-0.0709(s+9.094)(s+7.398)}{(s+3.612)(s+1.698)(s+7.956)} \\ \frac{-0.0709(s-33.59)(s+1.58)}{(s+3.612)(s+1.698)(s+7.956)} & \frac{0.7071(s+7.995)(s+4.984)(s+1.72)}{(s+3.612)(s+1.698)(s+7.956)} \end{bmatrix},$$

$$F(s) \approx \begin{bmatrix} \frac{0.6180(s+0.06924)}{s+0.9034} & \frac{0.2463(s+0.0555)}{s+0.9034} \\ \frac{-0.09049(s-0.9017)}{s+0.9034} & \frac{0.4234(s+1.536)}{s+0.9034} \end{bmatrix},$$

$$F^{-1}(s) \approx \begin{bmatrix} \frac{1.488(s+1.536)}{s+1.078} & \frac{-0.8869(s+0.0555)}{s+1.078} \\ \frac{0.3256(s-0.9017)}{s+1.078} & \frac{2.225(s+0.6924)}{s+1.078} \end{bmatrix}.$$

From this example and Example 3.2.23, we observe that our proposed method is less conservative for plants with unstable *transmission* zeros than for those with unstable *blocking* zeros. This is because transmission zeros lead to tangential interpolation conditions, which are less restrictive than matrix-valued ones arising from blocking zeros. We have already seen similar numerical results in Examples 2.2.10 and 2.3.12 in the Nevanlinna-Pick interpolation problem.

3.4 Summary

We have studied the problem of strong stabilization with sensitivity reduction for MIMO infinite-dimensional systems. The systems possess only finitely many zeros in $\bar{\mathbb{C}}_+$ but they are allowed to have infinitely many poles in $\mathbb{C}_+$. Since we have derived only a sufficient condition and a necessary condition, the problem has not yet been fully solved. However the proposed method gives both upper and lower bounds of the minimum sensitivity via the Nevanlinna-Pick interpolation with boundary conditions. Hence we can obtain these bounds by iterative computation of the associated Pick matrices. We have also proposed the design procedure of stable $\mathcal{H}^\infty$ controllers. A repetitive control system have been discussed as a practical application.

Chapter 4

Robust Stabilization of SISO Systems by Stable Controllers

4.1 Motivation and problem statement

The first step in constructing a controller for a given plant is to obtain a mathematical model for the plant. Since this process always involves some modeling errors, the *actual* plant is generally different from the model used for controller design. Even if we can obtain an exact model, the model often turns out to be too complex for analysis and control of the plant. In such a case, we have to use a simple nominal model whose behavior approximately represents the actual plant.

Here we use the class of multiplicative perturbations as a model of such plant uncertainties. Let us consider SISO systems in this chapter. Suppose that $P_0 \in \mathcal{F}^\infty$ is a given nominal model and that $W \in \mathcal{H}^\infty$ is a given function and represents a frequency-dependent upper bound on the multiplicative perturbations. We cannot measure perturbations accurately, but most of the time, it is possible to find such an upper bound on the perturbations.

Fix $\rho > 0$. In this section, we consider the following set $\mathcal{M}(P_0, W)$ of perturbed plants:

$$\mathcal{M}(P_0, W) := \{P : \ P = (1 + \Delta W)P_0, \ \Delta \in \mathcal{H}^\infty, \ \|\Delta\|_\infty < 1/\rho\}. \tag{4.1.1}$$

Figure 4.1 shows the block diagram of a plant P in $\mathcal{M}(P_0, W)$.

Consider the closed-loop system in Figure 4.2, where the plant $P \in \mathcal{M}(P_0, W)$ and the controller $C \in \mathcal{F}^\infty$. In Figure 4.2, R is the transfer function from δ_{out} to δ_{in}. A simple computation shows that $R = -WT$ with

$$T := \frac{P_0 C}{1 + P_0 C}.$$

If C stabilizes the nominal model P_0 and satisfies $\|R\|_\infty \le \rho$, then C stabilizes $P \in \mathcal{M}(P_0, W)$. In fact, since $\|\Delta\|_\infty < 1/\rho$ and $\|R\|_\infty \le \rho$, it follows that

$$\frac{1}{1 + \Delta R} \in \mathcal{H}^\infty,$$

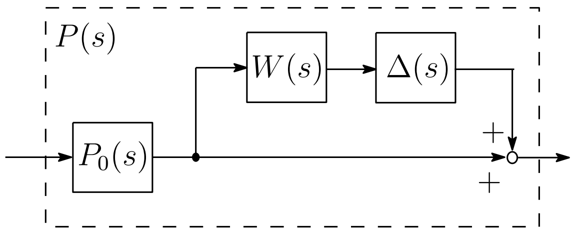


Figure 4.1: Class of multiplicative perturbations.

and hence the sensitivity function S satisfies

$$S = \frac{1}{1 + P_0(1 + \Delta W)C} = \frac{1}{1 + P_0 C} \cdot \frac{1}{1 + \Delta R} \in \mathcal{H}^\infty.$$

Similarly, CS, PS, and $T = 1 - S$ also belong to $\mathcal{H}^\infty$. Thus C stabilizes every $P \in \mathcal{M}(P_0, W)$.

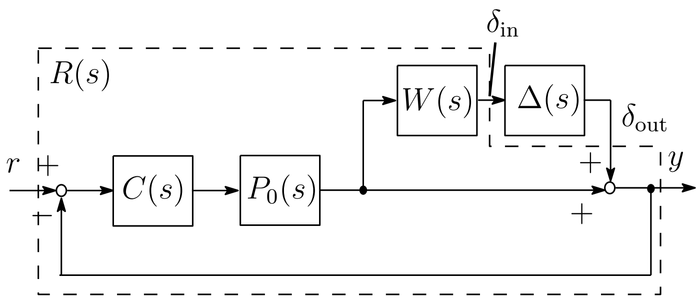


Figure 4.2: Closed-loop system with perturbed plant $P_0(1 + W\Delta)$.

There is a simple way to see the relevance of the condition $\|R\|_\infty \leq \rho$. A routine calculation shows that the closed-loop system in Figure 4.2 is equivalent to that in Figure 4.3. The maximum loop gain in Figure 4.3 equals $\|\Delta R\|_\infty$, which is less than 1 for all allowable Δ if $\|R\|_\infty \leq \rho$.

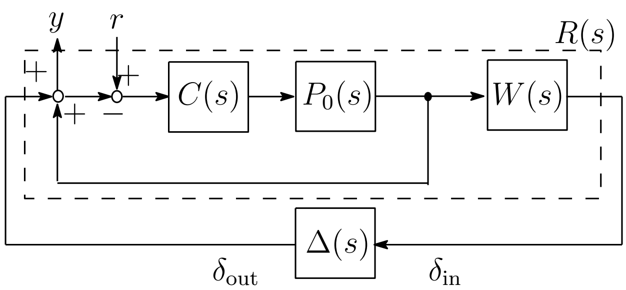


Figure 4.3: Closed-loop system equivalent to that of Figure. 4.2.

Conversely, it is known that the condition $\|R\|_\infty \leq \rho$ is necessary for C to stabilize every $P \in \mathcal{M}(P_0, W)$. See, e.g., [131] for details.

For finite-dimensional systems, it is shown that in [26] that the condition of P in (4.1.1) can be relaxed to the weaker one: The perturbed plant P has the same number of poles in $\mathbb{C}_+$ as the nominal plant P_0 and satisfy

$$\|P(j\omega) - P_0(j\omega)\| < \frac{W(j\omega)}{\rho}.$$

This result can be generalized to several classes of infinite-dimensional systems [11, 33].

We make the following assumption on the plant throughout this chapter:

Assumption 4.1.1. *$P \in \mathcal{F}^\infty$ can be factored as*

$$P = \frac{M_n}{M_d} N_o, \tag{4.1.2}$$

where $M_d \in \mathcal{H}^\infty$, $M_n \in \mathcal{RH}^\infty$ are inner functions and N_o, $1/N_o \in \mathcal{H}^\infty$. We assume that M_n possesses simple zeros $z_1, \dots, z_k$ only and that M_d, M_n are coprime.

Under Assumption 4.1.1, P has only finitely many unstable zeros arising from M_n, but P is allowed to possess infinitely many unstable poles arising from M_d. In [43], it is shown how to factor retarded or neutral time-delay systems into the form (4.1.2) under some mild conditions.

We impose the following assumption on the weighting function:

Assumption 4.1.2. *Both W and $1/W$ belong to $\mathcal{H}^\infty$.*

Our robust stabilization problem by a stable controller can be formulated as follows:

Problem 4.1.3. *Let Assumptions 4.1.1 and 4.1.2 hold. Suppose $\rho > 0$. Determine whether there exists a controller $C \in \mathcal{H}^\infty \cap \mathscr{C}(P)$ such that*

$$\|WT\|_\infty \leq \rho, \quad \text{where } \; T := \frac{PC}{1+PC}. \tag{4.1.3}$$

Also, if one exists, find such a controller C.

We call Problem 4.1.3 *strong and robust stabilization.* The main objective in this chapter is to develop both a sufficient condition and a necessary condition for strong and robust stabilization. These conditions give lower and upper bounds on the largest multiplicative perturbation permissible by a stable controller.

Before proceeding, we recall the definition of a unit element.

Definition 4.1.4 ([114])**.** A function $U \in \mathcal{H}^\infty$ is called *unit* if $1/U \in \mathcal{H}^\infty$.

4.2 Strong and robust stabilization

In this section, we find an interpolation-minimization problem equivalent to Problem 4.1.3. We obtain a sufficient condition as well as a necessary condition for the interpolation-minimization problem via the modified Nevanlinna-Pick interpolation. The modified problem is solvable by computing finitely many Pick matrices as the Nevanlinna-Pick interpolation problem.

The following result shows that Problem 4.1.3 can be reduced to an interpolation-minimization problem by a unit element.

Theorem 4.2.1. *Consider the strong and robust stabilization problem 4.1.3 under Assumptions 4.1.1 and 4.1.2. Problem 4.1.3 is solvable if and only if there exists a function F such that*

$$F,\ 1/F \in \mathcal{H}^\infty \tag{4.2.1}$$

$$\|W - M_d F\|_\infty \leq \rho \tag{4.2.2}$$

$$F(z_i) = \frac{W(z_i)}{M_d(z_i)}, \quad i = 1, \ldots, k. \tag{4.2.3}$$

Furthermore, once such a function F is constructed, the solution of Problem 4.1.3 is given by

$$C = \frac{W - M_d F}{M_n N_o F}. \tag{4.2.4}$$

Proof. Necessity. Let C be a solution to Problem 4.1.3. Define

$$F := \frac{W}{M_d + M_n N_o C}.$$

Since Lemma 3.2.1 shows that

$$\frac{1}{M_d + M_n N_o C} \in \mathcal{H}^\infty,$$

it follows that F satisfies (4.2.1). Also, we have

$$WT = W\left(1 - \frac{M_d F}{W}\right) = W - M_d F, \tag{4.2.5}$$

so F achieves the norm constraint (4.2.2). Moreover, since $M_n(z_i) = 0$ for $i = 1, \ldots, k$,

$$F(z_i) = \frac{W(z_i)}{M_d(z_i) + M_n(z_i) N_o(z_i) C(z_i)} = \frac{W(z_i)}{M_d(z_i)}, \quad i = 1, \ldots, k.$$

Thus F satisfies (4.2.1), (4.2.2), and (4.2.3).

Sufficiency. Suppose F satisfies (4.2.1), (4.2.2), and (4.2.3), and define C by (4.2.4).

We show $C \in \mathcal{H}^\infty$ as follows. Since $1/N_o,\ 1/F \in \mathcal{H}^\infty$, it follows from (4.2.4) that

$$M_n C = \frac{W - M_d F}{N_o F} \in \mathcal{H}^\infty. \tag{4.2.6}$$

Suppose $C \notin \mathcal{H}^\infty$. Then the unstable poles of C must be the zeros of M_n by (4.2.6). Let z_i be such a pole. Since the zeros of M_n are simple, it follows that $(M_nC)(z_i) \neq 0$. In addition, since the units N_o and F do not have unstable zeros, $N_o(z_i) \neq 0$ and $F(z_i) \neq 0$. Hence

$$W(z_i) - M_d(z_i)F(z_i) = (M_nC)(z_i) \cdot N_o(z_i)F(z_i) \neq 0,$$

which contradicts (4.2.3). Thus C belongs to $\mathcal{H}^\infty$.

Moreover, since

$$\frac{1}{M_d + M_nN_oC} = \frac{W}{F} \in \mathcal{H}^\infty,$$

it follows that C strongly stabilizes P by Lemma 3.2.1. Also, C achieves the norm constraint (4.1.3) by (4.2.2) and (4.2.5). Thus C is a solution of Problem 4.1.3. $\square$

To obtain a sufficient condition as well as a necessary condition for robust stabilizability by a stable controller, we use the following problem:

Problem 4.2.2 ([4, 106]). *Suppose $s_1, \ldots, s_k \in \mathbb{C}_+$ are distinct, and let $\beta_1, \ldots, \beta_k \in \mathbb{C} \setminus \{0\}$. Determine whether there exists a function G such that G, $1/G \in \mathcal{H}^\infty$, $\|G\|_\infty \leq 1$, and $G(s_i) = \beta_i$ for $i = 1, \ldots, k$. Also, if one exists, find such a function G.*

Problem 4.2.2 is called *the modified Nevanlinna-Pick interpolation problem* [106].

The difference between the Nevanlinna-Pick interpolation problem 2.1.1 and the modified problem 4.2.2 is that the latter has the additional condition $1/G \in \mathcal{H}^\infty$. Despite this difference, the solvability of Problem 4.2.2 is also equivalent to the positive semi-definiteness of an associated Pick matrix. To see this, we use the transformation $G = e^{-M}$ proposed in [4]. It follows that $G, 1/G \in \mathcal{H}^\infty$ and $\|G\|_\infty \leq 1$ if and only if M maps $\mathbb{C}_+$ into $\bar{\mathbb{C}}_+$ and $\sup_{s\in\mathbb{C}_+} \operatorname{Re} M(s)$ is finite. Also, G satisfies $G(s_i) = \beta_i$ if and only if M achieves the followng interpolation condition:

$$M(s_i) = -(\operatorname{Log} \beta_i + j2\pi l_i) \qquad \text{for some integer } l_i.$$

Combining this with Theorem 2.1.2, we obtain the following result:

Theorem 4.2.3 ([4, 106]). *Consider the modified Nevanlinna-Pick interpolation problem 4.2.2. Define $\alpha_i := \phi(s_i)$ for every $i = 1, \ldots, k$, where the conformal map ϕ is*

$$\phi \ : \ \mathbb{C}_+ \to \mathbb{D} \ : \ s \mapsto \frac{s-1}{s+1}.$$

Problem 4.2.2 is solvable if and only if there exists an integer set $\{l_1, \ldots, l_k\}$ such that the Pick matrix $\mathbf{P}(\{l_1, \ldots, l_k\})$,

$$\mathbf{P}(\{l_1, \ldots, l_k\}) := \left[\frac{-\operatorname{Log}\beta_p - \operatorname{Log}\overline{\beta_q} + j2\pi(l_q - l_p)}{1 - \alpha_p\overline{\alpha_q}}\right]_{p,q=1}^k \tag{4.2.7}$$

is positive semi-definite.

The next result shows that we can construct a solution to Problem 4.2.2 by the Nevanlinna-Pick interpolation.

Theorem 4.2.4 ([48, 77])**.** *Consider the modified Nevanlinna-Pick interpolation problem 4.2.2. Fix* $\sigma > 0$*. Define* α_i *in the same way as in Theorem 4.2.3 and*

$$\zeta_i := \Psi_\sigma(-\operatorname{Log}\beta_i - j2\pi l_k)$$

for $i = 1, \ldots, k$*, where* $\{l_1, \ldots, l_k\}$ *is an integer set and the conformal map* Ψ_σ *is*

$$\Psi_\sigma \ : \ \{s \in \mathbb{C}_+ \ : \ 0 < \operatorname{Re} s < \sigma\} \to \mathbb{D} \ : \ s \mapsto \frac{je^{-j\pi s/\sigma} - 1}{je^{-j\pi s/\sigma} + 1}.$$

If there exists an analytic function $g : \mathbb{D} \to \mathbb{D}$ *such that* $g(\alpha_i) = \zeta_i$ *for* $i = 1, \ldots, k$*, then*

$$G(s) := \exp\left(-\frac{\sigma}{2} - \frac{j\sigma}{\pi}\operatorname{Log}\left(\frac{1 + g(\phi(s))}{1 - g(\phi(s))}\right)\right) \tag{4.2.8}$$

is a solution to Problem 4.2.2.

Remark 4.2.5.

1. We have an infinite number of Pick matrices $\mathbf{P}(\{l_1, \ldots, l_k\})$ in Theorem 4.2.3. Note, however, that in order that $\mathbf{P}(\{l_1, \ldots, l_k\})$ be positive semi-definite it is necessary that $L_{pq} := l_p - l_q$ be bounded. It turns out that only finitely many distinct $\mathbf{P}(\{l_1, \ldots, l_k\})$ could possibly be positive semi-definite. In fact, for the positive semi-definiteness of $\mathbf{P}(\{l_1, \ldots, l_k\})$, L_{pq} must satisfy the following quadratic inequality:

$$\det \begin{bmatrix} \frac{-\operatorname{Log}\beta_p - \operatorname{Log}\overline{\beta_p}}{1 - \alpha_p\overline{\alpha_p}} & \frac{-\operatorname{Log}\beta_p - \operatorname{Log}\overline{\beta_q} - j2\pi L_{pq}}{1 - \alpha_p\overline{\alpha_q}} \\ \frac{-\operatorname{Log}\beta_q - \operatorname{Log}\overline{\beta_p} + j2\pi L_{pq}}{1 - \alpha_q\overline{\alpha_p}} & \frac{-\operatorname{Log}\beta_q - \operatorname{Log}\overline{\beta_q}}{1 - \alpha_q\overline{\alpha_q}} \end{bmatrix} = aL_{pq}^2 + bL_{pq} + c \geq 0,$$

where $a := -4\pi^2$, $b := 4\pi \operatorname{Re}\,[j(-\operatorname{Log}\beta_p - \operatorname{Log}\overline{\beta_q})]$, and

$$c := \left(\frac{\operatorname{Log}\beta_p + \operatorname{Log}\overline{\beta_p}}{1 - \alpha_p\overline{\alpha_p}} \cdot \frac{\operatorname{Log}\beta_q + \operatorname{Log}\overline{\beta_q}}{1 - \alpha_q\overline{\alpha_q}} - \left|\frac{\operatorname{Log}\beta_p + \operatorname{Log}\overline{\beta_q}}{1 - \alpha_p\overline{\alpha_q}}\right|^2\right) \cdot |1 - \alpha_p\overline{\alpha_q}|^2.$$

Hence $D := b^2 - 4ac \geq 0$ and

$$\frac{b + \sqrt{D}}{2a} \leq L_{pq} \leq \frac{b - \sqrt{D}}{2a}.$$

Thus we can check the solvability of Problem 4.2.2 in a finite number of steps. See [4, 37] for the details.

2. To obtain a *rational* solution to Problem 4.2.2, we can use the parameterization (2.1.2) of all solutions to the Nevanlinna-Pick interpolation problem. In [48], a free function f of (2.1.2) is fixed in the form

$$f(z) = \frac{az + b}{z + c}.$$

The norm constraint $\|f\|_\infty \leq 1$ is satisfied if and only if the parameter pair (a, b, c) belongs to

$$\mathcal{S} := \{(a, b, c) \in \mathbb{R}^3 : \ |c| \leq 1, \ |a + b| \leq |c + 1|, \ |a - b| \leq |c - 1|\}.$$

On the other hand, the solution to the Nevanlinna-Pick interpolation problem is invertible in $\mathcal{H}^\infty(\mathbb{D})$ if there exists $(a, b, c) \in \mathbb{R}^3$ such that

$$(az + b)\tilde{X} + (z + c)\tilde{Y}$$

has no zeros in $\bar{\mathbb{D}}$ for $\tilde{X}, \tilde{Y}$ defined in Theorem 2.1.3. This is equivalent to stabilization of discrete-time systems by first-order controllers. Thus we can construct a rational solution by taking the intersection of $\mathcal{S}$ and the stabilization set.

3. A function f is said to be *real* if $\overline{f(s)} = f(\overline{s})$. A simple calculation shows that $G(s)$ in (4.2.8) is real if $g(z) = j \cdot g_0(z)$, where $g_0(z)$ is real.

The problem of strong stabilization with sensitivity reduction is equivalent to the modified Nevanlinna-Pick interpolation problem 4.2.2. By contrast, the difficulty of strong and robust stabilization is the $\mathcal{H}^\infty$-norm condition (4.2.2) in Theorem 4.2.1.

We now develop both a sufficient and a necessary conditions for (4.2.2). From these conditions, we obtain lower and upper bounds on the perturbation by solving Problem 4.2.2. Theorem 4.2.3 and Remark 4.2.5.1 show that calculations of the finitely many Pick matrices lead to these bounds. Furthermore, we find stable controllers for robust stabilization by Theorem 4.2.4.

Define

$$\rho_{\text{inf}} := \inf_{C \in \mathcal{H}^\infty \cap \mathscr{C}(P)} \|WT\|_\infty.$$

Then $K_{\text{sup}} := 1/\rho_{\text{inf}}$ can be regarded as the largest allowable multiplicative uncertainty bound for robust stabilizability by a stable controller. Theorem 4.2.6 below gives a lower bound of K_{sup} and stable and robust controllers.

Theorem 4.2.6. *Consider the strong and robust stabilization problem 4.1.3 under Assumptions 4.1.1 and 4.1.2. Suppose $\|W\|_\infty < \rho$. Choose W_s satisfying $W_s,\ 1/W_s \in \mathcal{RH}^\infty$ and*

$$|W_s(j\omega)| \leq \rho - |W(j\omega)| \tag{4.2.9}$$

for almost all $\omega \in \mathbb{R}$. Define

$$\beta_i := \frac{W(z_i)}{M_d(z_i)W_s(z_i)}$$

for $i = 1, \ldots, k$. If G is a solution to the modified Nevanlinna-Pick interpolation problem 4.2.2 with interpolation data $(z_i, \beta_i)_{i=1}^n$, then $K_{\text{sup}} \geq 1/\rho$ and

$$C := \frac{W - M_d W_s G}{M_n N_o W_s G} \tag{4.2.10}$$

is a solution to Problem 4.1.3.

Proof. Note that $\beta_i \neq 0$ for each i because the unit W does not have unstable zeros. By Theorem 4.2.1, it suffices to show that there exists F satisfying (4.2.1), (4.2.2), and (4.2.3).

Let us first obtain a sufficient condition for (4.2.2). Since M_d is inner, it follows from (4.2.9) that

$$\begin{aligned} |W(j\omega) - M_d(j\omega)F(j\omega)| &\leq |M_d(j\omega)| \cdot |F(j\omega)| + |W(j\omega)| \\ &= |F(j\omega)| + \rho - (\rho - |W(j\omega)|) \\ &\leq |F(j\omega)| + \rho - |W_{\mathrm{s}}(j\omega)| \end{aligned}$$

for almost all $\omega \in \mathbb{R}$. Moreover,

$$|F(j\omega)| + \rho - |W_{\mathrm{s}}(j\omega)| \leq \rho$$

if and only if

$$|(F/W_{\mathrm{s}})(j\omega)| \leq 1.$$

This shows that if $\|F/W_{\mathrm{s}}\|_\infty \leq 1$, then we have (4.2.2).

Suppose that G is a solution to Problem 4.2.2 with interpolation data $(z_i, \beta_i)_{i=1}^n$. Define $F := W_{\mathrm{s}}G$. By the argument given above, F achieves (4.2.2) because

$$\|F/W_{\mathrm{s}}\|_\infty = \|G\|_\infty \leq 1.$$

Since G and W_{s} are unit elements, F satisfies (4.2.1). Moreover, the interpolation conditions (4.2.3) can be obtained directly from those of G. Thus F satisfies (4.2.1), (4.2.2), and (4.2.3). Substituting $F = W_{\mathrm{s}}G$ into (4.2.4), we can also derive (4.2.10). □

In the same way, an upper bound of K_{sup} can be obtained by the next result:

Theorem 4.2.7. *Consider the strong and robust stabilization problem 4.1.3 under Assumptions 4.1.1 and 4.1.2. Choose W_n satisfying $W_n,\ 1/W_n \in \mathcal{RH}^\infty$ and*

$$|W_n(j\omega)| \geq \rho + |W(j\omega)|$$

for almost all $\omega \in \mathbb{R}$. Define

$$\gamma_i := \frac{W(z_i)}{(M_d(z_i)W_n(s_i)}$$

for $i = 1, \ldots, k$. If Problem 4.2.2 with interpolation data $(z_i, \gamma_i)_{i=1}^n$ is not solvable, then $K_{\mathrm{sup}} \leq 1/\rho$.

Proof. As in the proof of Theorem 4.2.6, we can derive a necessary condition for (4.2.2) from

$$\begin{aligned} |W(j\omega) - M_d(j\omega)F(j\omega)| &\geq |m(j\omega)| \cdot |F(j\omega)| - |W(j\omega)| \\ &= |F(j\omega)| + \rho - (\rho + |W(j\omega)|) \\ &\geq |F(j\omega)| + \rho - |W_{\mathrm{n}}(j\omega)| \end{aligned}$$

for almost all $\omega \in \mathbb{R}$. The rest of the proof follows the same lines as that of Theorem 4.2.6, so it is omitted. □

Remark 4.2.8. 1. Strong stabilization with sensitivity reduction can be transformed to the modified Nevanlinna-Pick interpolation [38, 48, 53, 77]. In contrast, strong and robust stabilization cannot. The reason is that F in Theorem 4.2.1 needs to be invertible in $\mathcal{H}^\infty$. We cannot alter the norm constraint (4.2.2) to a simpler one $\|F\|_\infty < \rho$. We therefore need Theorems 4.2.6 and 4.2.7 as additional procedures. This is different from robust stabilization without restriction on the stability of controllers, which can be reduced the Nevanlinna-Pick interpolation as sensitivity minimization [61].

2. Let $P \in \mathcal{F}^\infty$ has a coprime factorization N/D with $N, D \in \mathcal{H}^\infty$. We say that P is *proper* if

$$\lim_{R\to\infty} \sup_{|s|>R} |N(s)| < \infty. \tag{4.2.11}$$

Also P is *strictly proper* if

$$\lim_{R\to\infty} \sup_{|s|>R} |N(s)| = 0. \tag{4.2.12}$$

The plant P is said to be *biproper* if it is proper but not strictly proper.

In Assumption 4.1.1, we have taken a biproper plant having infinitely many unstable poles as the nominal model. Therefore the condition $\|W\|_\infty < \rho$ in Theorem 4.2.6 implies that the controllers obtained by our proposed method may not robustly stabilize strictly proper plants. In the first place, however, we should pose the question: *Are strictly proper plants with infinitely many unstable poles stabilizable?* The answer is negative; see the next section.

3. By the MATLAB command `fitmagfrd`, we can compute W_{s}, W_{n} in Theorems 4.2.6 and 4.2.7.

4. As in the design procedure of stable controllers in Chapter 2, the controller in (4.2.10) has internal unstable pole-zero cancellations. In general, such cancellations are not exactly achieved due to the infinite-dimensionality of the controller. This may lead to the unstable behavior of the controller in implementation. See also Remark 3.2.17.

Theorem 4.2.6 generally gives an infinite-dimensional controller. A natural question at this stage is the following: *Does a finite-dimensional controller that approximates the derived controller stabilize the plant and satisfy the $\mathcal{H}^\infty$-norm condition (4.1.3)?* The reader can refer to [20, 42, 83, 85] for approximation techniques for stable infinite-dimensional systems.

To ensure that the approximation $C_a \in \mathcal{RH}^\infty$ still stabilizes the plant, we can obtain an error bound on the difference $\|C - C_a\|_\infty$ by Proposition 3.2.19.

Define

$$T_a := \frac{PC_a}{1 + PC_a}. \tag{4.2.13}$$

The following is an analogous result to Proposition 3.2.20 for sensitivity reduction and illustrates that we can also obtain an upper bound of $\|WT_a\|_\infty$ by $\|C - C_a\|_\infty$.

Proposition 4.2.9. *Let $P \in \mathcal{F}^\infty$ and $W \in \mathcal{H}^\infty$. Suppose there exists $C \in \mathcal{H}^\infty \cap \mathscr{C}(P)$ and $C_a \in \mathcal{RH}^\infty \cap \mathscr{C}(P)$. Define*

$$\delta := \left\| \frac{P}{1+PC} \right\|_\infty, \qquad \epsilon := \|C - C_a\|_\infty.$$

If $\delta\epsilon < 1$, then

$$\|WT_a\|_\infty \leq \frac{\delta\epsilon \cdot \|W\|_\infty + \|WT\|_\infty}{1 - \delta\epsilon}, \tag{4.2.14}$$

where T and T_a are defined by (4.1.3) *and* (4.2.13) *respectively.*

Proof. A routine calculation shows that

$$T - T_a = \frac{P}{1+PC}(1 - T_a)(C - C_a).$$

Hence we have

$$\|WT - WT_a\|_\infty \leq \delta\epsilon \cdot \|W(1 - T_a)\|_\infty \leq \delta\epsilon \cdot (\|W\|_\infty + \|WT_a\|_\infty). \tag{4.2.15}$$

Since $\|WT_a\|_\infty - \|WT\|_\infty \leq \|WT - WT_a\|_\infty$, it follows from (4.2.15) that

$$(1 - \delta\epsilon) \cdot \|WT_a\|_\infty \leq \delta\epsilon \cdot \|W\|_\infty + \|WT\|_\infty.$$

Thus we obtain (4.2.14) if $\delta\epsilon < 1$. □

4.3 Stabilizability of strictly proper plants having infinitely many unstable poles

Here we answer the question: *Can a linear time-invariant controller stabilize a strictly proper plant with an infinite number of unstable poles?*

The previous works [45, 60] on $\mathcal{H}^\infty$ control of plants with infinitely many unstable modes assume that the plants are biproper. Moreover, a strictly proper neutral delay system is not stabilizable by a finite-dimensional controller [84]. However the above question is not fully answered. Based on the Bezout identity, the next result shows that more general strictly proper plants with infinitely many unstable poles are not stabilizable in the sense of [101].

Proposition 4.3.1. *Let nonzero $N,\ D \in \mathcal{H}^\infty$ be weakly coprime in the sense of [101], i.e., every greatest common divisor of N and D is a unit element. Suppose that D has infinitely many zeros in $\mathbb{C}_+$, and that the set of these unstable zeros has no limit points on the imaginary axis. If N satisfies* (4.2.12)*, then $P := N/D$ is not stabilizable.*

Proof. Assume, to reach a contradiction, that P is stabilizable. Then we have

$$N(s)X(s) + D(s)Y(s) = 1 \quad \text{for all } s \in \mathbb{C}_+ \tag{4.3.1}$$

for some $X,\ Y \in \mathcal{H}^\infty$ [101], By (4.2.12), for every $\varepsilon > 0$, there exists $R > 0$ such that $|N(s)| \cdot \|X\|_\infty < \varepsilon$ for all $s \in \mathbb{C}_+$ satisfying $|s| > R$. In addition, there exists $z_0 \in \mathbb{C}_+$

such that $D(z_0) = 0$ and $|z_0| > R$. Otherwise the set of the unstable zeros of D has at least one limit point in $\{s \in \mathbb{C}_+ : \ |s| \leq R\}$, which implies that $D(s) = 0$ for all $s \in \mathbb{C}_+$ by Theorem 10.18 in [93]. Let $\varepsilon < 1$. Then

$$|N(z_0)X(z_0) + D(z_0)Y(z_0)| \leq |N(z_0)| \cdot \|X\|_\infty < \varepsilon < 1.$$

This contradicts (4.3.1). Thus P is not stabilizable. $\square$

4.4 Numerical examples

In this section, we first compare the proposed method with the previous work for the design of stable $\mathcal{H}^\infty$ controllers. We also present a numerical example for an infinite-dimensional plant, and apply the proposed method to a repetitive control system [54, 121].

Example 4.4.1. We consider strong and robust stabilization for the following finite-dimensional plant P, weighting function W, and positive function ρ:

$$P(s) = \frac{(s+2)(s-3)(s-4)}{(s-2)(s+3)(s+4)}$$
$$W(s) = K \cdot \frac{s+1}{s+10}, \quad \rho = 1$$

where $K > 0$. The purpose of this example is to compare the proposed method with the MATLAB software HIFOO 3.0 [44]. HIFOO 3.0 computes stable $\mathcal{H}^\infty$ controllers, using a hybrid numerical algorithm for nonsmooth and nonconvex optimization based on quasi-Newton updating and gradient sampling.

Table 4.1: Comparison on Example 4.4.1

K_{sup}	Methods	Stability of Controller
0.1441	HIFOO 3.0 [44]	stable
0.2082	Sufficient condition in Theorem 4.2.6	stable
0.2310	Necessary condition in Theorem 4.2.7	unstable
0.2665	$\mathcal{H}^\infty$ optimal controller	unstable

Results for this example are given in Table 4.1, where K_{sup} indicates the maximum of K achieved by each method. We see that our sufficient condition in Theorem 4.2.6 less conservative than HIFOO 3.0 in this example. Also, an upper bound derived from Theorem 4.2.7 is smaller than K_{sup} achievable by the unstable $\mathcal{H}^\infty$ optimal controller.

Example 4.4.2. Consider Problem 4.1.3 with the following infinite-dimensional system P, weighting function W, and positive constant ρ:

$$P(s) = \frac{(s-\alpha)(s-4e^{-s}+1)}{(s-10)(s-15)(2e^{-s}+1)} \tag{4.4.1}$$
$$W(s) = K \cdot \frac{s+1}{s+10}, \quad \rho = 1, \tag{4.4.2}$$

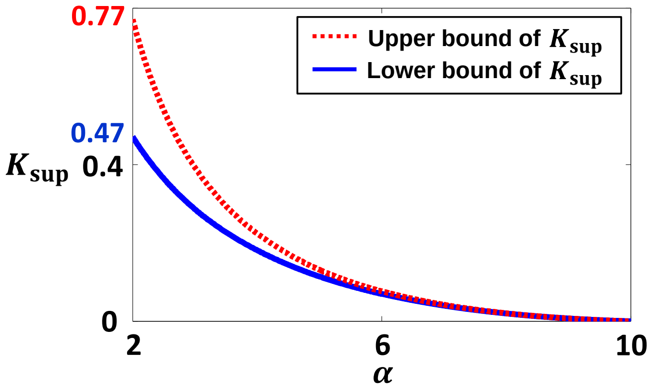


Figure 4.4: Unstable zero α versus supremum gain K_{sup}.

where $2 \leq \alpha < 10$ and $K > 0$. Let p be the only root of $s - 4e^{-s} + 1 = 0$ in $\mathbb{C}_+$ (note that $p \approx 0.7990$). Using the factorization method of [43], P can be factored as $P = M_n N_o / M_d$, where

$$\begin{aligned} M_n(s) &:= \frac{(s-\alpha)(s-p)}{(s+\alpha)(s+p)} \\ M_d(s) &:= \frac{(s-10)(s-15)(2e^{-s}+1)}{(s+10)(s+15)(e^{-s}+2)} \\ N_o(s) &:= \frac{(s+\alpha)(s+p)(s-4e^{-s}+1)}{(s-p)(s+10)(s+15)(e^{-s}+2)}. \end{aligned}$$

Let K_{sup} be the supremum of K such that there exists $C \in \mathcal{H}^\infty \cap \mathscr{C}(P)$ satisfying (4.1.3). Figure 4.4 shows the relationship between α and K_{sup}. In Figure 4.4, the solid line shows a lower bound of K_{sup} obtained by Theorem 4.2.6, and the dashed line indicates an upper bound of K_{sup} derived from Theorem 4.2.7. We compute W_{s} and W_{n} in Theorems 4.2.6, 4.2.7 by the MATLAB function `fitmagfrd`. Both lines in Figure 4.4 decrease to 0 as α becomes closer to 10. The reason for this drop is that an unstable pole-zero cancellation occurs in P when $\alpha = 10$.

Let $\alpha = 2$. Then we obtain the lower bound 0.471 and the upper bound 0.771. We also find a stable controller to achieve robust stability for $K = 0.468$ by Theorem 4.2.4 with $\sigma = 100$. See Fig. 3 of [48] for a discussion on the selection of σ based on a specific numerical example.

When $K = 0.468$, W_{s} in Theorem 4.2.6 and g in Theorem 4.2.4 are given by

$$W_{\text{s}}(s) \approx \frac{0.53(s+10.20)}{(s+5.86)}$$
$$g(z) = j \cdot g_0(z), \quad \text{where } g_0(z) \approx \frac{1.049z+1}{z+1.050}.$$

The above W_{s} is obtained by `fitmagfrd`. The stable controller that provides robust stability is obtained by (4.2.10), where $G(s)$ is defined in (4.2.8) with $g(z)$.

Note that $G(s)$ in (4.2.8) is real by Remark 4.2.5.2. The further investigation of G is conducted through an example in [48].

Example 4.4.3. *(Application to a repetitive control system)*

Consider the repetitive control system given in Figure 3.5, where $L = 1$ and P belongs to the following model set:

$$\mathscr{P} = \left\{ P_\alpha(s) = \frac{(s-6)(s-9)}{(\alpha s+8)(s-5)} \ : \ 0.8 \le a \le 1.2 \right\}.$$

Note that the plant must be biproper for the exponential stability of the closed-loop system [121, Theorem 5.12]; see also Proposition 4.3.1. When the plant is strictly proper, we need a modified repetitive controller [54, 121]. The reader can refer to [122] for the details of robust stabilization of modified repetitive control systems.

The repetitive controller C consists of two parts: C_u and C_o. $C_u = 1/(1-e^{-Ls})$ is the internal model of any periodic signals with period L. On the other hand, C_o is designed for the desired performance, in this example, for robust stabilization. Our goal is to determine whether there exists $C_o \in \mathcal{H}^\infty$ such that $C = C_u C_o$ stabilizes all $P_a \in \mathscr{P}$ and the error $e(t)$ tends exponentially to zero for every $P_\alpha \in \mathscr{P}$.

For $\varepsilon > 0$, let $\mathbb{C}_{-\varepsilon}$ denote $\{s \in \mathbb{C} : \ \operatorname{Re} s > -\varepsilon\}$ and let $\mathcal{H}^\infty(\mathbb{C}_{-\varepsilon})$ denote the set of functions that are bounded and analytic in $\mathbb{C}_{-\varepsilon}$. For exponential stability, it is necessary and sufficient that S, CS, and PS belong to $\mathcal{H}^\infty(\mathbb{C}_{-\varepsilon})$ for some $\varepsilon > 0$ [119, Theorem 3.1]. Moreover, if ε is sufficiently small, then

$$\mathscr{P} \subset \left\{ P_\Delta = (1+W\Delta)P_1 \ : \ \Delta \in \mathcal{H}^\infty(\mathbb{C}_{-\varepsilon}), \ \sup_{s\in\mathbb{C}_{-\varepsilon}} |\Delta(s)| < 1 \right\}, \tag{4.4.3}$$

where

$$\begin{aligned} P_1(s) &:= \frac{(s-6)(s-9)}{(s+8)(s-5)} \\ W(s) &:= \frac{0.25038(s+0.02384)}{s+10}. \end{aligned}$$

Now let us consider the closed-loop system in Figure 4.5. By the preceding discussion, to determine whether there exists $C_o \in \mathcal{H}^\infty$ yielding exponential stability of the closed-loop system for every $P_a \in \mathscr{P}$, we study Problem 4.1.3 with

$$\begin{aligned} \tilde{P}(s) &:= P(s-\varepsilon) = C_u(s-\varepsilon)P_1(s-\varepsilon) \\ \tilde{W}(s) &:= W(s-\varepsilon), \quad \rho := 1. \end{aligned} \tag{4.4.4}$$

Once we find a solution $\tilde{C}$ to this problem, $C_o(s) := \tilde{C}(s+\varepsilon) \in \mathcal{H}^\infty(\mathbb{C}_{-\varepsilon})$ makes the closed-loop system exponentially stable for every $\Delta \in \mathcal{H}^\infty(\mathbb{C}_{-\varepsilon})$ satisfying

$$\sup_{s\in\mathbb{C}_{-\varepsilon}} |\Delta(s)| < 1$$

in Figure 4.5.

Let $\varepsilon = 0.001$, which satisfies (4.4.3). $\tilde{P}$ in (4.4.4) can be factored as $\tilde{P} =$

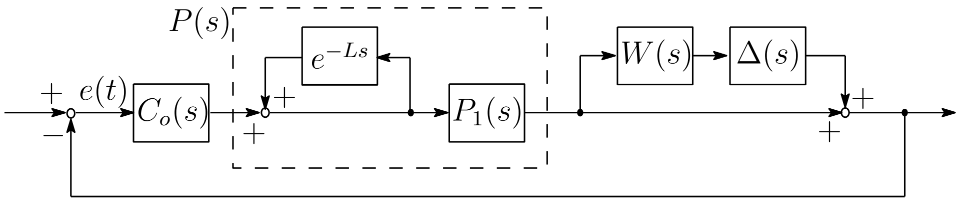


Figure 4.5: Robust stabilization for repetitive control system.

$M_n N_o / M_d$, where

$$M_n(s) := \frac{(s-\varepsilon-6)(s-\varepsilon-9)}{(s+\varepsilon+6)(s+\varepsilon+9)}$$
$$M_d(s) := \frac{(1-e^{\varepsilon}e^{-s})(s-\varepsilon-5)}{(e^{-s}-e^{\varepsilon})(s+\varepsilon+5)}$$
$$N_o(s) := \frac{(s+\varepsilon+6)(s+\varepsilon+9)}{(e^{-s}-e^{\varepsilon})(s+\varepsilon+5)(s-\varepsilon+8)}.$$

Define $\tilde{T} := \tilde{P}\tilde{C}/(1+\tilde{P}\tilde{C})$. It follows from Theorems 4.2.6 and 4.2.7 that

$$0.71 < \inf_{\tilde{C}\in\mathcal{H}^\infty\cap\mathscr{C}(\tilde{P})} \|\tilde{W}\tilde{T}\|_\infty < 0.97.$$

The MATLAB function `fitmagfrd` is used for W_{s} and W_{n} in Theorems 4.2.6, 4.2.7. Since $\inf_{\tilde{C}\in\mathcal{H}^\infty\cap\mathscr{C}(\tilde{P})} \|\tilde{W}\tilde{T}\|_\infty \le \rho = 1$, there exists $C_o \in \mathcal{H}^\infty$ such that the repetitive controller $C = C_u C_o$ stabilizes all $P_a \in \mathscr{P}$ and achieves the exponential decay of $e(t)$ for any $P_a \in \mathscr{P}$.

4.5 Summary

In this chapter, we have studied the strong and robust stabilization problem for SISO infinite-dimensional systems. The plants we consider have only finitely many simple unstable zeros but may possess infinitely many unstable poles. Using the modified Nevanlinna-Pick interpolation, we have obtained both lower and upper bounds on the largest multiplicative perturbation under which a stable controller can stabilize the plant. Hence such bounds can be calculated by checking the positive definiteness of finitely many associated Pick matrices. Moreover, we have found stable controllers to achieve robust stability. A repetitive control system has been discussed as an application of the proposed method.

Chapter 5

Strong Stabilization with Mixed Sensitivity Reduction for SISO Systems

5.1 Motivation and problem statement

In this chapter, we consider SISO systems. We have studied sensitivity reduction in Chapter 3 and robust stabilization in Chapter 4. However, controllers attaining low sensitivity may not be robust against modeling errors, and also robust controllers may not achieve good tracking performance of the closed-loop system. For desired performance and robustness, we therefore need reduction of both the sensitivity function $S = 1/(1 + PC)$ and the complimentary sensitivity function $T = PC/(1 + PC)$, that is, *mixed sensitivity reduction*.

Since $S + T = 1$, the following question arises naturally: *Is it well-posed to reduce S and T simultaneously?* The answer is affirmative in the engineering sense, because in a wide variety of cases of interest in control theory, we can separate the frequency bands where each function should be small. We elaborate on this further from the viewpoints of performance and robustness.

Performance: For good disturbance attenuation and tracking performance, S should be small over a suitable low frequency band. On the other hand, to reduce the effect of measurement noise, we should make T small in a high frequency band.

Robustness: A small value of S in a low frequency band implies good robustness with respect to low frequency plant perturbations caused by parameter uncertainty. In contrast, if T is small in a high frequency band, the closed-loop system is robust against high frequency plant perturbations due to modeling errors and parasitic effects.

The reader can refer to [25, 33, 63, 64] for further theoretical discussions. Mixed sensitivity reduction is one of the basic and practical control objectives. Its successful applications can be found in many engineering fields, such as hot-strip mills [55], flexible structures [57], power flow in transmission systems [115], and vehicle dynamic control systems [10].

We make the following assumptions throughout this chapter:

Assumption 5.1.1. *We consider the following class of plants:*

$$P = \frac{M_n}{M_d} N_1 N_2, \tag{5.1.1}$$

where

- $M_n \in \mathcal{H}^\infty$ *and* $M_d \in \mathcal{RH}^\infty$ *are inner,*
- $N_1,\ 1/N_1 \in \mathcal{H}^\infty$,
- $N_2 \in \mathcal{RH}^\infty$ *is outer.*

We also assume that M_n *has finitely many essential singularities on* $j\mathbb{R}_e$, *and that* $N := M_d N_1 N_2$ *and* $D := M_d$ *are coprime.*

Assumption 5.1.2. *We assume* $W_1,\ W_2N_2,\ 1/(W_2N_2) \in \mathcal{RH}^\infty$.

Note that the plants considered here are different from those in Chapters 3 and 4. The plant in (5.1.1) can have only *finitely many unstable modes* arising from M_d, but it may possess *pure delays* and *infinitely many unstable zeros* arising from M_n.

Assumption 5.1.1 requires the coprime-inner/outer factorization of P. A calculation method of this factorization for general SISO time-delay systems has been developed in [43].

We aim to find a stable controller that provides both lower sensitivity and robust stability. Then our problem is stated as follows:

Problem 5.1.3. *Let Assumptions 5.1.1 and 5.1.2 hold. Determine whether there exists a controller* $C \in \mathcal{H}^\infty \cap \mathscr{C}(P)$ *such that*

$$\|W_1 S\|_\infty < 1, \quad \textit{where}\ \ S := \frac{1}{1+PC}, \tag{5.1.2}$$

$$\|W_2 T\|_\infty < 1, \quad \textit{where}\ \ T := \frac{PC}{1+PC} = 1 - S. \tag{5.1.3}$$

Also, if one exists, find such a controller.

Problem 5.1.3 is called *strong stabilization with mixed sensitivity reduction.* Our objective of this chapter is to introduce a new two-block problem for the sufficiency of Problem 5.1.3. The two-block problem can be converted to a one-block problem that is solvable by matrix computation only. We also present a design method of such stable $\mathcal{H}^\infty$ controllers.

5.2 $\mathcal{H}^\infty$ control by stable controllers

5.2.1 Strong stabilization with sensitivity reduction

Let us first study the problem of finding a stable controller that stabilizes the plant and achieves low sensitivity (5.1.2) only. Here we obtain a sufficient condition for this problem by extending the results for finite-dimensional systems in [56].

The following result gives two necessary and sufficient conditions for a controller to strongly stabilize the plant:

Lemma 5.2.1. *Suppose N, $D \in \mathcal{H}^\infty$ are coprime. For $P := N/D$, each of the following three conditions implies the other two:*

(i) $C \in \mathscr{C}(P) \cap \mathcal{H}^\infty$.

(ii) $C \in \mathcal{H}^\infty$ *and* $1/(D + NC) \in \mathcal{H}^\infty$.

(iii) $C \in \mathscr{C}(P)$ *and* $D + NC \in \mathcal{H}^\infty$.

Proof. From a simple calculation, we immediately see that (ii) ⇒ (i) and (i) ⇒ (iii). We prove (iii) ⇒ (ii) as follows.

Since $C \in \mathscr{C}(P)$, there exists $Q \in \mathcal{H}^\infty$ such that $Y - NQ \neq 0$ and

$$C = \frac{X + DQ}{Y - NQ}. \tag{5.2.1}$$

Substituting (5.2.1) into $D + NC$, we have

$$D + NC = \frac{1}{Y - NQ}. \tag{5.2.2}$$

Since $D+NC \in \mathcal{H}^\infty$, (5.2.1) and (5.2.2) implies $C \in \mathcal{H}^\infty$. In addition, $1/(D+NC) = Y - NQ \in \mathcal{H}^\infty$ by (5.2.2). Thus (iii) implies (ii). □

The equivalence between (i) and (iii) in Lemma 5.2.1 leads to the following necessary and sufficient condition for a controller to strongly stabilize the plant and attain low sensitivity (5.1.2).

Lemma 5.2.2. *Suppose that N, $D \in \mathcal{H}^\infty$ are coprime, and that X, $Y \in \mathcal{H}^\infty$ satisfy the Bezout equation* (4.3.1). *Define $P := N/D$ and let $W_1 \in \mathcal{RH}^\infty$. Then the following two conditions are equivalent:*

(i) *The controller $C \in \mathcal{H}^\infty \cap \mathscr{C}(P)$ achieves low sensitivity* (5.1.2).

(ii) *The parameter $Q \in \mathcal{H}^\infty$ in* (5.2.1) *satisfies*

$$\|W_1 D(Y - NQ)\|_\infty < 1 \tag{5.2.3}$$

$$\frac{1}{Y - NQ} \in \mathcal{H}^\infty. \tag{5.2.4}$$

Proof. In conjunction with (5.2.2), the equivalence between (i) and (iii) in Lemma 5.2.1 shows that there exists $C \in \mathcal{H}^\infty \cap \mathscr{C}(P)$ if and only if (5.2.4) holds for some $Q \in \mathcal{H}^\infty$. Moreover, since $S = D(Y - NQ)$, it follows that C achieves (5.1.2) if and only if Q satisfies (5.2.3). □

We are now in a position to set a one-block problem for the sufficiency of sensitivity reduction by a stable controller.

Theorem 5.2.3. *Let Assumption 5.1.1 hold. Suppose that* $W_1 \in \mathcal{RH}^\infty$ *and that nonzero* $\lambda \in \mathbb{C}$ *satisfies* $|\lambda| \le 1$. *If there exists* $Q \in \mathcal{H}^\infty$ *such that*

$$\left\| \left(\frac{2}{\lambda}W_1 - D\right) - \frac{2}{\lambda}W_1 N(X+DQ)\right\|_\infty < 1, \tag{5.2.5}$$

then C *in* (5.2.1) *satisfies* $C \in \mathcal{H}^\infty \cap \mathscr{C}(P)$ *and low sensitivity* (5.1.2).

Proof. By Lemma 5.2.2, it suffices to establish (5.2.3) and (5.2.4) for $Q \in \mathcal{H}^\infty$ satisfying (5.2.5).

Define

$$V := \frac{2}{\lambda}W_1(Y - NQ) - 1. \tag{5.2.6}$$

it is clear that $V \in \mathcal{H}^\infty$. Before using Lemma 3.2.15, let us check $\|V\|_\infty < 1$.

Since $D = M_d$ is inner and since $DY = 1 - NX$ by the Bezout equation (4.3.1), it follows that

$$\begin{aligned} \|V\|_\infty &= \|DV\|_\infty \\ &= \left\| \frac{2}{\lambda}W_1(1 - N(X+DQ)) - D\right\|_\infty \\ &= \left\| \left(\frac{2}{\lambda}W_1 - D\right) - \frac{2}{\lambda}W_1 N(X+DQ)\right\|_\infty . \end{aligned} \tag{5.2.7}$$

Thus $\|V\|_\infty < 1$ can be obtained by (5.2.5).

Lemma 3.2.15 shows that U defined by

$$U := \frac{\lambda}{2}(V+1) \tag{5.2.8}$$

satisfies $U,\ 1/U \in \mathcal{H}^\infty$ and $\|U\|_\infty < 1$. We therefore have (5.2.4) because

$$\frac{1}{Y-NQ} = \frac{2}{\lambda}\frac{W_1}{V+1} = \frac{W_1}{U} \in \mathcal{H}^\infty.$$

Moreover,

$$\|W_1 D(Y-NQ)\|_\infty = \|W_1(Y-NQ)\|_\infty = \|U\|_\infty < 1,$$

and hence (5.2.3) is achieved. Thus the controller C in (5.2.1) satisfies $C \in \mathcal{H}^\infty \cap \mathscr{C}(P)$ and low sensitivity (5.1.2) by Lemma 5.2.2. □

The following corollary is for stable plants in Theorem 5.2.3. The restricted result is still interesting because we can directly apply the skew Toeplitz approach [33] to the one-block problem (5.2.9) below.

Corollary 5.2.4. *Let Assumption 5.1.1 with $M_d = N_2 = 1$ hold. Suppose that $W_1,\ 1/W_1 \in \mathcal{RH}^\infty$ and that nonzero $\lambda \in \mathbb{C}$ satisfies $|\lambda| \le 1$. If there exists $Q_s \in \mathcal{H}^\infty$ such that*

$$\left\| \left(\frac{2}{\lambda} W_1 - 1 \right) - M_n Q_s \right\|_\infty < 1 \tag{5.2.9}$$

then the controller C defined by

$$C := \frac{\lambda Q_s}{2W_1 N_1 - \lambda P Q_s} \tag{5.2.10}$$

satisfies $C \in \mathcal{H}^\infty \cap \mathscr{C}(P)$ and (5.1.2).

Proof. We obtain (5.2.9) and (5.2.10) by substituting $N = P = M_n N_1,\ X = 0$, $D = Y = 1$, and $Q_s = \frac{2}{\lambda} W_1 N_1 Q$ into (5.2.5) and (5.2.1). □

In Theorem 5.2.3 and Corollary 5.2.4, we encounter the following question: *As the gain of the weight W_1 is larger, does the set of all controllers given there become smaller?* The next result provides a positive answer. Proposition 5.2.5 shows that the set of all parameters satisfying (5.2.5) become smaller.

Proposition 5.2.5. *Let Assumption 5.1.1 hold, and let $W_1 \in \mathcal{RH}^\infty$. Suppose that nonzero $\lambda_1,\ \lambda_2 \in \mathbb{C}$ satisfy $\lambda_1 = a\lambda_2$ for some $a \in (0,1)$. Define*

$$\mathscr{Q}_i := \left\{ Q_i \in \mathcal{H}^\infty : \ \left\| \left(\frac{2}{\lambda_i} W_1 - D \right) - \frac{2}{\lambda_i} W_1 N (X + DQ_i) \right\|_\infty < 1 \right\}$$

for $i = 1, 2$. Then $\mathscr{Q}_2 \supset \mathscr{Q}_1$.

Proof. First of all, note that

$$\left\| \left(\frac{2}{\lambda_i} W_1 - D \right) - \frac{2}{\lambda_i} W_1 N (X + DQ_i) \right\|_\infty = \left\| \frac{2}{\lambda_i} W_1 (Y - NQ_i) - 1 \right\|_\infty \tag{5.2.11}$$

by (5.2.6) and (5.2.7). Let $Q_1 \in \mathcal{H}^\infty$ satisfy

$$\left\| \frac{2}{\lambda_1} W_1 (Y - NQ_1) - 1 \right\|_\infty < 1. \tag{5.2.12}$$

Define G_1 and G_2 by

$$G_1 := \frac{2}{\lambda_1} W_1 (Y - NQ_1) - 1, \quad G_2 := \frac{\lambda_1}{\lambda_2}(G_1 + 1) - 1.$$

Then by definition

$$G_2 = \frac{2}{\lambda_2} W_1 (Y - NQ_1) - 1. \tag{5.2.13}$$

Since $|\lambda_2| > |\lambda_1|$, we see from (5.2.12) that

$$\begin{aligned} \|G_2\|_\infty &\le \left\| \frac{\lambda_1}{\lambda_2}(G_1 + 1) - 1 \right\|_\infty \\ &\le \left| \frac{\lambda_1}{\lambda_2} \right| \cdot \|G_1\|_\infty + \frac{|\lambda_2| - |\lambda_1|}{|\lambda_2|} < 1. \end{aligned} \tag{5.2.14}$$

By (5.2.13) and (5.2.14), we obtain (5.2.12) with λ_2 in place of λ_1, Hence we deduce from (5.2.11) that every $Q_2 \in \mathscr{Q}_2$ belongs to $\mathscr{Q}_1$. This gives the desired conclusion. □

Remark 5.2.6. 1. A naive approach for sensitivity reduction by a stable controller leads to *two* $\mathcal{H}^\infty$-norm constraints: one is for sensitivity reduction (5.1.2) and the other is for the stabilization of the stabilizing controller through the small gain theorem [131]. However, note that we have derived *one* $\mathcal{H}^\infty$-norm condition (5.2.5) in Theorem 5.2.3.

2. The results in Chapter 3 and the earlier studies [38, 48, 56, 77] are based on the equivalence between (i) and (ii) in Lemma 5.2.1. The key point is that a controller strongly stabilizes the plant if and only if a unit in $\mathcal{H}^\infty$ satisfies only finitely many interpolation conditions. Hence the Nevanlinna-Pick interpolation can be applied to the $\mathcal{H}^\infty$ control problem; see also Remark 3.2.13. However, this approach does not work for the plant in Assumption 5.1.1. In fact, its extension requires that the unit cancel essential singularities and infinitely many poles arising from M_n. One can prove (we shall omit this) that if $N_2 = 1$ in (5.1.1), then the necessary and sufficient condition for strong stabilizability is the existence of a unit $U \in \mathcal{H}^\infty$ achieving the *generalized* interpolation [97],

$$U(\mathbf{T}) = M_d(\mathbf{T}), \tag{5.2.15}$$

where $\mathbf{T}$ is the compressed shift operator on the orthogonal complement of $\{M_n f : f \in \mathcal{H}^2\} \subset \mathcal{H}^2$; see [33, 120] for details. To avoid technical issues, however, we do not proceed further along this line.

In this chapter, we have used the equivalence between (i) and (iii) in Lemma 5.2.1. By this equivalence, we can study Problem 5.1.3 with the aid of the parameterization (5.2.1) of all stabilizing controllers. We will see in the next subsection that this parameterization plays an important role in strong stabilization with mixed sensitivity reduction. Moreover, to address the infinite dimensionality of M_n or equivalently (5.2.15), we employ the results of the operator-theoretic approach in [33].

5.2.2 Strong stabilization with mixed sensitivity reduction

We now consider the problem of strong stabilization with mixed sensitivity reduction. We begin by introducing a two-block problem for this problem.

Theorem 5.2.3 gives the $\mathcal{H}^\infty$-norm constraint (5.2.5) as a sufficient condition for strong stabilization with sensitivity reduction. On the other hand, substituting (5.2.1) into the definition of T, we have $T = N(X + DQ)$. Hence, if there exists a solution $Q \in \mathcal{H}^\infty$ to the two-block problem

$$\left\| \begin{bmatrix} \left(\dfrac{2}{\lambda}W_1 - D\right) - \dfrac{2}{\lambda}W_1 N(X + DQ) \\ W_2 N(X + DQ) \end{bmatrix} \right\|_\infty < 1 \tag{5.2.16}$$

for some nonzero $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$, then C in (5.2.1) is a solution to Problem 5.1.3.

Since $D,\ W_1 \in \mathcal{RH}^\infty$, it follows that W_d and W_λ defined by

$$W_d := \frac{2}{\lambda}W_1 - D, \quad W_\lambda := \frac{2}{\lambda}W_1, \tag{5.2.17}$$

belong to $\mathcal{RH}^\infty$ for $\lambda \in \mathbb{R}$. Therefore we make the next assumption instead of Assumption 5.1.2.

Assumption 5.1.2′. *We assume W_d, W_λ, W_2N_2, $1/(W_2N_2) \in \mathcal{RH}^\infty$.*

The preceding discussion suggests that solutions to Problem 5.1.3 can be constructed by solving the following problem:

Problem 5.2.7. *Let Assumptions 5.1.1 and 5.1.2′ hold. Find $Q \in \mathcal{H}^\infty$ such that*

$$\left\| \begin{bmatrix} W_d - W_\lambda N(X+DQ) \\ W_2 N(X+DQ) \end{bmatrix} \right\|_\infty < 1. \tag{5.2.18}$$

On the other hand, the two-block problem for the mixed sensitivity minimization without consideration of the stability of controllers is given by

$$\left\| \begin{bmatrix} W_1 - W_1 N(X+DQ) \\ W_2 N(X+DQ) \end{bmatrix} \right\|_\infty < 1. \tag{5.2.19}$$

The only difference between (5.2.18) and (5.2.19) is whether W_d and W_λ are equal to W_1. Recall that (5.2.19) can be reduced to a one-block problem solvable by matrix computation [33]. Then a question naturally arises: *Can we also transform the two-block problem* (5.2.18) *to such a one-block problem?* This question is answered affirmatively as follows.

For $G \in \mathcal{H}^\infty$, let $G^\sim(s) := G(-\bar{s})^*$ be its para-Hermitian conjugate. We convert (5.2.18) to the two-block problem (5.2.22) below, which has the parameter $\bar{Q}$ in the first block only.

Theorem 5.2.8. *Consider Problem 5.2.7. Compute G such that G, $1/G \in \mathcal{RH}^\infty$ and*

$$N_2^\sim W_\lambda^\sim W_\lambda N_2 + N_2^\sim W_2^\sim W_2 N_2 = G^\sim G. \tag{5.2.20}$$

Let $E \in \mathcal{RH}^\infty$ satisfy

$$G_1 = \frac{E - XN_1}{M_d} \in \mathcal{H}^\infty, \tag{5.2.21}$$

and let $M_w \in \mathcal{RH}^\infty$ be a finite Blaschke product of minimal degree such that

$$W := M_w \frac{W_d W_\lambda^\sim N_2^\sim}{G^\sim} \in \mathcal{RH}^\infty.$$

Then Problem 5.2.7 is solvable if and only if there exists $\bar{Q} \in \mathcal{H}^\infty$ such that

$$\left\| \begin{bmatrix} W - M\hat{W} - MM_d\bar{Q} \\ G_0 \end{bmatrix} \right\|_\infty < 1, \tag{5.2.22}$$

where

$$G_0 := \frac{W_d W_2 N_2}{G} \in \mathcal{RH}^\infty, \quad \hat{W} := GE \in \mathcal{RH}^\infty,$$
$$M := M_n M_w \in \mathcal{H}^\infty.$$

Furthermore, the solution Q to Problem 5.2.7 is given by

$$Q = \frac{1}{N_1}\left(\frac{\bar{Q}}{G} - G_1\right). \tag{5.2.23}$$

Proof. We can prove this result in a way similar to the transformation of (5.2.19) in [33, Chap. 5].

Define $Q_1 := N_1Q - G_1$. Since $N_1,\ 1/N_1,\ G_1 \in \mathcal{H}^\infty$, it follows that $Q \in \mathcal{H}^\infty$ if and only if $Q_1 \in \mathcal{H}^\infty$. By (5.2.21),

$$\begin{bmatrix} W_dW_\lambda N(X+DQ) \\ W_2N(X+DQ) \end{bmatrix} = \begin{bmatrix} W_d \\ 0 \end{bmatrix} - \begin{bmatrix} W_\lambda N_2 \\ -W_2N_2 \end{bmatrix} M_n(E + M_dQ_1).$$

Define the matrices Θ_1 and Θ_2 by

$$\Theta_1 := \begin{bmatrix} W_\lambda N_2/G & N_2^\sim W_2^\sim/G^\sim \\ -W_2N_2/G & N_2^\sim W_\lambda^\sim/G^\sim \end{bmatrix}, \quad \Theta_2 := \begin{bmatrix} M_w & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $\Theta_1^\sim$ is unitary by (5.2.20) and Θ_2 is inner,

$$\begin{aligned}
&\left\| \begin{bmatrix} W_d \\ 0 \end{bmatrix} - \begin{bmatrix} W_\lambda N_2 \\ -W_2N_2 \end{bmatrix} M_n(E + M_dQ_1) \right\|_\infty \\
&\qquad = \left\| \Theta_2\Theta_1^\sim \cdot \left(\begin{bmatrix} W_d \\ 0 \end{bmatrix} - \begin{bmatrix} W_\lambda N_2 \\ -W_2N_2 \end{bmatrix} M_n(E + M_dQ_1) \right) \right\|_\infty \\
&\qquad = \left\| \Theta_2 \cdot \left(\begin{bmatrix} \frac{W_dW_\lambda^\sim N_2^\sim}{G^\sim} \\ G_0 \end{bmatrix} - \begin{bmatrix} GM_n \\ 0 \end{bmatrix} (E + M_dQ_1) \right) \right\|_\infty \\
&\qquad = \left\| \begin{bmatrix} W - M_1\hat{W} - M_1M_d\bar{Q} \\ G_0 \end{bmatrix} \right\|_\infty,
\end{aligned}$$

where $\bar{Q} := GQ_1$. This means that Problem 5.2.7 is solvable if and only if (5.2.22) holds for some $\bar{Q} \in \mathcal{H}^\infty$. Furthermore, since $\bar{Q} = GQ_1 = G(N_1Q - G_1)$, we obtain (5.2.23). □

Remark 5.2.9. As in (5.2.20), a factorization $F = G^\sim G$ with $G, 1/G \in \mathcal{RH}^\infty$ is called a *spectral factorization* of F and G is a *spectral factor*. If F has the properties: F and $1/F$ are proper real-rational functions without poles on the imaginary axis; $F(\infty) > 0$; $F = F^\sim$, then F has a spectral factorization [34].

We transform (5.2.22) to a one-block problem. The following technique has been used in the literature on $\mathcal{H}^\infty$ control; see, e.g., [33, Chap. 5] and [34, Chap. 8].

Define

$$\sigma := \max\left\{ \|G_0\|_\infty, \quad \left\| \begin{bmatrix} W(j\omega_k) \\ G_0(j\omega_k) \end{bmatrix} \right\| \right\}, \tag{5.2.24}$$

where $j\omega_k \in j\mathbb{R}_e$ runs over the essential singularities of M_n. Note that since M_n has finitely many essential singularities by Assumption 5.1.1, it is trivial to compute σ in (5.2.24). We see from [82, Proposition 1] that if $\sigma \geq 1$, Problem 5.2.7 does not have any solutions. Therefore we assume $\sigma < 1$.

Fix $\gamma = 1 - \varepsilon$, where $\varepsilon > 0$ is so small that $\sigma < \gamma$. Compute a spectral factor F_γ such that $F_\gamma,\ 1/F_\gamma \in \mathcal{RH}^\infty$ and

$$F_\gamma^\sim F_\gamma = \gamma^2 - G_0^\sim G_0.$$

Define the $\mathcal{H}^\infty$ functions L_γ and $\hat{L}_\gamma$ by

$$L_\gamma := \frac{W}{F_\gamma}, \qquad \hat{L}_\gamma := \frac{\hat{W}}{F_\gamma}.$$

Then it follows from Theorem 5.2.8 that Problem 5.2.7 amounts to finding $\bar{Q}_1 \in \mathcal{H}^\infty$ such that

$$\|L_\gamma - M\hat{L}_\gamma - MM_d\bar{Q}_1\|_\infty \leq 1. \tag{5.2.25}$$

This one-block problem can be solved by computing the smallest singular value of a certain matrix; see Appendix A and [33] for details.

Finally, let $\bar{Q}_1$ be a solution to the one-block problem (5.2.25). From (5.2.23) and $\bar{Q} = F_\gamma \bar{Q}_1$, the solution Q to Problem 5.2.7 is given by

$$Q = \frac{1}{N_1}\left(\frac{F_\gamma \bar{Q}_1}{G} - G_1\right). \tag{5.2.26}$$

The results of this section can be summarized as follows:

Design procedure for stable stabilizing controllers achieving low sensitivity (5.1.2) and robust stability (5.1.3)

Step 1: Define W_d and W_λ by (5.2.17).

Step 2: Formulate the two-block problem (5.2.18).

Step 3: Transform the above two-block problem to the one-block problem (5.2.25).

Step 4: Compute a solution $\bar{Q}_1$ to the one-block problem (5.2.25) by the skew Toeplitz approach in Appendix A and [33].

Step 5: Calculate a stable controller from (5.2.1) and (5.2.26).

Remark 5.2.10. Combining the proposed method with the results in [58], we can also construct stable $\mathcal{H}^\infty$ controllers for pseudorational systems [118].

5.3 Numerical examples

To illustrate the results above, we present the following examples in this section:

1. Strong stabilization with *sensitivity reduction* for a *stable* input-delay system.

2. Strong stabilization with *mixed sensitivity reduction* for an *unstable* input-delay system.

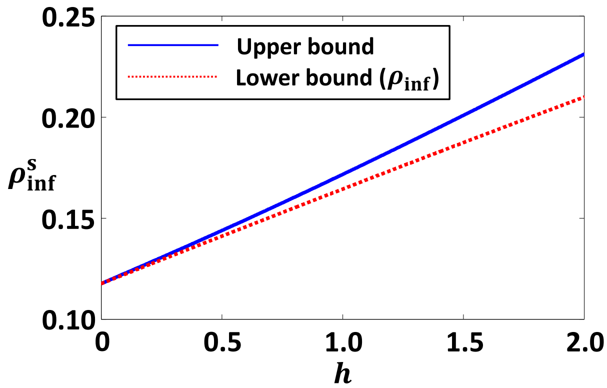


Figure 5.1: Input-delay h versus minimum sensitivity $\rho^{\text{s}}_{\text{inf}}$

Example 5.3.1. We consider the following stable input-delay system and weighting function:

$$P(s) = e^{-hs} \cdot \frac{s-5}{s+5}, \quad W(s) = \frac{s+1}{10s+1}, \quad 0 < h \leq 2.$$

Define $\rho^{\text{s}}_{\text{inf}}$ by

$$\rho^{\text{s}}_{\text{inf}} := \inf_{C \in \mathcal{H}^\infty \cap \mathscr{C}(P)} \|WS\|_\infty.$$

Figure 5.1 shows the relationship between the input-delay h and the sensitivity $\rho^{\text{s}}_{\text{inf}}$. The solid line indicates an upper bound of $\rho^{\text{s}}_{\text{inf}}$ obtained by Corollary 5.2.4. On the other hand, the dashed line denotes

$$\rho_{\text{inf}} := \inf_{C \in \mathscr{C}(P)} \|WS\|_\infty,$$

which can be regarded as a lower bound of $\rho^{\text{s}}_{\text{inf}}$. We computed ρ_{inf} by the skew Toeplitz approach [33].

The $\mathcal{H}^\infty$ optimal controller in this example has poles close to

$$\frac{\log(10\rho_{\text{inf}}) \pm j(2\pi n + 1)}{h} \tag{5.3.1}$$

for sufficiently large n [33]. This means that the controller achieving ρ_{inf} has infinitely many unstable poles for every $h \in (0,\ 2]$.

In Figure 5.1, the gap between the lower and upper bounds increases as the input-delay h becomes longer. One reason is that the $\mathcal{H}^\infty$ optimal controller for the plant with longer input-delay has its infinitely many unstable poles more densely. We see this from (5.3.1).

Example 5.3.2. Consider the unstable input-delay system P, weighting functions W_1, W_2 given by

$$P(s) = \frac{e^{-hs}}{s-1}, \quad W_1(s) = \frac{s+1}{\rho(10s+1)}, \quad W_2(s) = \frac{s+1}{8}, \tag{5.3.2}$$

where $h \geq 0$.

As in Example 5.3.1, let $\rho^{\text{s}}_{\text{inf}}$ be the infimum of ρ such that there exists $C \in \mathcal{H}^\infty \cap \mathscr{C}(P)$ satisfying (5.1.2) and (5.1.3). Also, let $\bar{\rho}^{\text{s}}_{\text{inf}}$ and ρ_{inf} be the infimum of

ρ such that the two block problems (5.2.16) and (5.2.19) are solvable, respectively. Note that $\bar{\rho}^{\mathrm{s}}_{\mathrm{inf}}$ is an upper bound of $\rho^{\mathrm{s}}_{\mathrm{inf}}$ and that ρ_{inf} may not be achieved by a stable controller. Here we draw a comparison between $\bar{\rho}^{\mathrm{s}}_{\mathrm{inf}}$ and ρ_{inf}.

The plant P in (5.3.2) can be factored as $M_nN_1N_2/M_d$, where $M_n(s) := e^{-hs}$, $N_1(s) := 1$,

$$M_d(s) := \frac{s-1}{s+1}, \quad N_2(s) := \frac{1}{s+1}.$$

First we take $h = 0.3$. Matrix computation shows that Problem 5.2.7 with $\rho =$ *0.196* and $\lambda = 1$ is solvable, which leads to $\bar{\rho}^{\mathrm{s}}_{\mathrm{inf}} \leq 0.196$. On the other hand, by using the results in [33, 108], we obtain $\rho^{\mathrm{s}}_{\mathrm{inf}} = 0.203$. These results suggest a question: *Why is the performance by the $\mathcal{H}^\infty$ optimal controller worse than that by a stable controller?*

Of course, this is simply because we use different performance measures. Here, however, we can closely examine the question above by transforming (5.2.18) to the form of (5.2.19). This transformation cannot be always done, but if possible, it leads to a detailed comparison of the performance indices.

Let us convert (5.2.18) to the form of (5.2.19). Set $\rho = 0.196$ and $\lambda = 1$. Since W_d in (5.2.17) is given by

$$W_d(s) \approx \frac{0.204(s+143.7)(s+0.382)}{(s+1)(10s+1)},$$

it follows that W_d, $1/W_d \in \mathcal{H}^\infty$. This invertibility of W_d leads to the reduction of (5.2.18) to the form of (5.2.19).

Define

$$\overline{N}_1 := \frac{W_\lambda}{W_d}N_1, \quad \overline{N} := \frac{\overline{N}_1}{N_1}N, \quad \overline{W}_2 := \frac{W_d}{W_\lambda}W_2.$$

Then a simple calculation shows that (5.2.18) is equivalent to

$$\left\| \begin{bmatrix} W_d - W_d\overline{N}(X+DQ) \\ \overline{W}_2\overline{N}(X+DQ) \end{bmatrix} \right\|_\infty < 1. \tag{5.3.3}$$

Its difference from (5.2.19) are only W_d, $\overline{W}_2$, and $\overline{N}$. Here the $\mathcal{H}^\infty$ performance is not affected by the difference between N and $\overline{N}$, because $\overline{N}_1$ as well as N_1 are invertible in $\mathcal{H}^\infty$. Thus (5.2.18) holds if and only if Q satisfies (5.2.19) with W_d and $\overline{W}_2$ in place of W_1 and W_2. This implies that the weights W_1, W_2, W_d, and $\overline{W}_2$ determine which of the two-block problems is conservative.

Figure 5.2 shows the Bode plots of W_1 and W_2 with $\rho = 0.203$ and those of W_d and $\overline{W}_2$ with $\rho = 0.196$ and $\lambda = 1$. We see from Figure 5.2 that W_d is much smaller than W_1 in the high frequency band. Furthermore, W_2 starts to increase at 10^0 Hz, whereas $\overline{W}_2$ does at 10^2 Hz. These properties of the weighting functions verify the numerical results above.

Figure 5.3 shows the sensitivity $\bar{\rho}^{\mathrm{s}}_{\mathrm{inf}}$ and ρ_{inf} dependent on the input-delay h that is varied between 0 and 0.3.

From this figure, we see that as h become longer, $\bar{\rho}^{\mathrm{s}}_{\mathrm{inf}}$ increases at the almost same rate as ρ_{inf}.

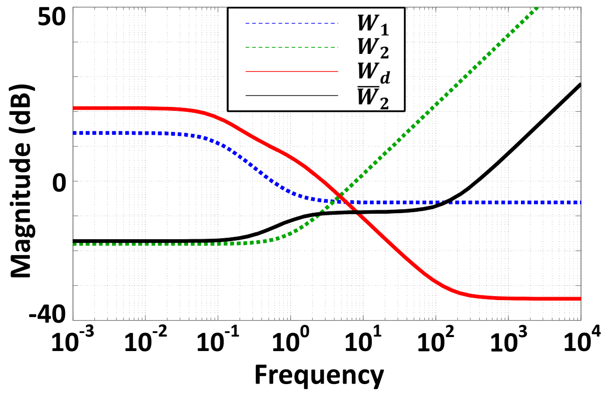


Figure 5.2: Bode plots of W_1, W_2 in (5.2.19) and W_d, $\overline{W}_2$ in (5.3.3)

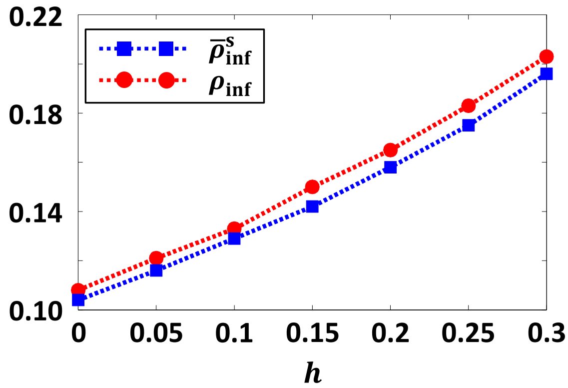


Figure 5.3: Input-delay h versus sensitivities $\bar{\rho}^{\mathrm{s}}_{\mathrm{inf}}$ and ρ_{inf}

5.4 Summary

In this chapter, we have constructed a stable controller that simultaneously achieves low sensitivity and robust stability for SISO time-delay systems. The plants we considered have finitely many unstable poles but they are allowed to possess pure delays and infinitely many unstable zeros. Strong stabilization of the plants in Chapters 3 and 4 is equivalent to the interpolation at finitely many points in $\mathbb{C}_+$, whereas that in this chapter requires the operator-theoretic approach to the interpolation. For a sufficient condition for the stable $\mathcal{H}^\infty$ controller design, we have derived the one-block problem. This problem is solvable by the techniques in [33]. Compared with the parameterization-based approach in [48, 110], the proposed method has the computational advantage that the desired controller is constructed by matrix calculation only.

Chapter 6

Conclusion

In this thesis, we have studied the problems of finding stable $\mathcal{H}^\infty$ controllers for infinite-dimensional systems. The results are based on the fact that the interpolation with an invertible $\mathcal{H}^\infty$ function is a necessary and sufficient condition for strong stabilization. The earlier works rooted in the parameterization of all $\mathcal{H}^\infty$ sub-optimal controllers can be computationally expensive, because such controllers for infinite-dimensional plants are infinite-dimensional. In contrast, the proposed methods lead to the design procedures with matrix computation only.

In Chapter 2, we have shown that the extended Schur-Nevanlinna algorithm gives a necessary and sufficient condition for the Nevanlinna-Pick interpolation problem with boundary conditions. We have reduced the interpolation problem with both interior and boundary conditions to that with boundary conditions only, and then have shown that the reduced boundary interpolation problem is always solvable. Compared with the approach of [3] based on the Pick matrix, the proposed method efficiently constructs solutions to this interpolation problem.

In Chapter 3, we have addressed the problem of finding stable stabilizing controllers that provide low sensitivity for MIMO infinite-dimensional systems. The systems considered in Chapter 3 has finitely many zeros in $\bar{\mathbb{C}}_+$, but are allowed to possess infinitely many poles in $\mathbb{C}_+$. We have derived a sufficient condition and a necessary condition for the $\mathcal{H}^\infty$ control problem. Both of them are in the form of the Nevanlinna-Pick interpolation with boundary conditions. Hence, these conditions give upper and lower bounds of the minimum sensitivity achievable by stable controllers through the computation of the associated Pick matrix. Also, we have constructed stable controllers attaining the upper bound via the extended Schur-Nevanlinna algorithm.

In Chapter 4, we have studied the strong and robust stabilization problem for SISO infinite-dimensional systems. As in Chapter 3, the plants have only finitely many simple unstable zeros but they can possess infinitely many unstable poles. If we do not consider the stability of controllers, then we can transform the problem of robust stabilization to the same one-block problem as that of sensitivity reduction. However we cannot treat the strong and robust stabilization problem in this way. The reason is that the solution to its equivalent one-block problem needs to be invertible in

$\mathcal{H}^\infty$. Through additional procedures for the weighting function, however, the modified Nevanlinna-Pick interpolation leads to both lower and upper bounds on the largest admissible perturbation for which the plant is stabilizable by a stable controller. This means that we can obtain such bounds by computing finitely many Pick matrices.

In Chapter 5, we have proposed a design method of stable controllers attaining both low sensitivity and robust stability for SISO time-delay systems. Unlike in Chapters 3 and 4, the plants have finitely many unstable poles but may possess pure delays and infinitely many unstable zeros. This infinite dimensionality needs the operator-theoretic approach to the interpolation for strong stabilization. However, as in Chapters 3 and 4, the sufficient condition we obtained can be checked by matrix computation only. Consequently, the proposed design procedure requires less computational efforts than previous methods based on the parameterization of $\mathcal{H}^\infty$ sub-optimal controllers.

We conclude this thesis by indicating some open problems.

A necessary and sufficient condition for the stable $\mathcal{H}^\infty$ control problems: In Chapters 3 and 5, we have used the small gain theorem to construct units in $\mathcal{H}^\infty$. However, this leads to only a sufficient condition. A necessary and sufficient condition may be derived if we use complex exponential/logarithm functions as in the modified Nevanlinna-Pick interpolation. Using such functions, we may avoid the construction of an invertible $\mathcal{H}^\infty$ function. However, this leads to infinitely many steps, because complex logarithm functions are multi-valued functions.

An extension of the Toker-Özbay formula to strong stabilization with mixed sensitivity reuduction: In [108], a necessary and sufficient condition called the Toker-Özbay formula is derived for the two-block problem (5.2.19). This formula not only offers an efficient method to construct the solutions but also provides a comprehensible structure of an $\mathcal{H}^\infty$ sub-optimal controller. Such a structure enables us to construct controllers with additional properties such as controllers of low degree. The two-block problem (5.3.3) introduced in Chapter 5 is a natural generalization of the two-block problem (5.2.19) considered in the Toker-Özbay formula. It is interesting to extend the Toker-Özbay formula to the stable $\mathcal{H}^\infty$ control problem.

The standard $\mathcal{H}^\infty$ control problem with stable controllers for infinite-dimensional plants: In this thesis, we have studied the design of stable controllers for basic $\mathcal{H}^\infty$ control problems. To fulfill various control requirements, we need to construct stable controllers for the standard $\mathcal{H}^\infty$ control problem. However, only systems with multiple input/output delays has been investigated; see [110, 111] for the details. The standard problem with stable controllers still remains open for general infinite-dimensional systems. A parameterization of all solutions to the standard $\mathcal{H}^\infty$ control problem is provided for general infinite-dimensional systems in [59, 60]. Combining this with the parameterization-based approach for stable $\mathcal{H}^\infty$ controllers, we may find a solution to the standard problem with stable controllers.

Stable $\mathcal{H}^\infty$ controller design for fractional-order systems: Fractional-order models and fractional control have received increased attention over the last decade; see, e.g., [12, 17]. Various amount of literature of fractional-order systems is devoted to systems of commensurate order, $G(s) = R(s^r)$ with $R \in \mathcal{RH}^\infty$ and $0 < r < 1$. This is due to the following stability property: Let $R = q/p$ stand for a rational function with p and q coprime polynomials. A transfer function $G(s) = R(s^r)$ is bounded-input bounded-output (BIBO) stable if and only if $|\arg \sigma| > r\pi/2$ for every σ with $p(\sigma) = 0$ [71]. Applications of such fractional-order systems include lead acid batteries with the Warburg impedance [94] and bridge structures having elastomeric bearings [28]. Most controller designs for systems of commensurate order are tuning of classical and fractional-order proportional-integral-derivative (PID) controllers. Such controllers are simple but conservative. From both theoretical and practical points of view, it is important to construct stable (possibly fractional-order) $\mathcal{H}^\infty$ controllers for fractional-order systems.

Appendix A

Skew Toeplitz Approach for Mixed Sensitivity Reduction

Here we review the skew Toeplitz approach for mixed sensitivity reduction. The following result is based on the method of [41]. In what follows, we use the notation in Chapter 5.

In Chapter 5, we have transformed the problem of strong stabilization with mixed sensitivity reduction to the following problem:

Problem A. *For $\gamma < 1$, define $L_\gamma, \hat{L}_\gamma \in \mathcal{RH}^\infty$ as in Chapter 5. Suppose that $M_d \in \mathcal{RH}^\infty$ and $M \in \mathcal{H}^\infty$ are inner functions. Determine whether there exists $\gamma < 1$ such that*

$$\|L_\gamma - M\hat{L}_\gamma - MM_dQ\| \leq 1. \tag{A-1}$$

for some $Q \in \mathcal{H}^\infty$. Also, if one exists, find all $Q \in \mathcal{H}^\infty$ achieving (A-1) *for a fixed $\gamma < 1$.*

Remark A. Define σ by (5.2.24) and $\gamma_{\inf}$ by

$$\gamma_{\inf} := \inf_{\bar{Q}\in\mathcal{H}^\infty} \left\| \begin{bmatrix} W - M\hat{W} - MM_d\bar{Q} \\ G_0 \end{bmatrix} \right\|_\infty .$$

Then $\gamma_{\inf}$ satisfies $\gamma_{\inf} \geq \sigma$ by Proposition 1 in [82], and we can avoid the nongeneric case $\gamma_{opt} = \sigma$ by choosing the weighting functions W_1 and W_2 properly [41]. Hence, *we shall assume that $\gamma_{\inf} > \sigma$ in the appendix.*

Let $L_\gamma = b_\gamma/k_\gamma$ and $M_d = m^\sim/m$, where b_γ, k_γ, and m are polynomials with real coefficients and b_γ, k_γ are coprime. Let n and l be the degrees of k_γ and m, respectively. Suppose that $M_d = C_{M_d}(sI - A_{M_d})^{-1}B_{M_d} + d_{M_d}$ and $L_\gamma = C_L(sI - A_L)^{-1}B_L + d_L$ are minimal realizations. Define $\pi := b_\gamma b_\gamma^\sim - k_\gamma k_\gamma^\sim$ and

$$A_\gamma := \frac{1}{1-d_L^2}\begin{bmatrix} A_L(1-d_L^2) + d_L B_L C_L & B_L B_L^* \\ -C_L^* C_L & -(A_L(1-d_L^2) + d_L B_L C_L)^* \end{bmatrix}$$

$$B_\gamma := -\frac{1}{1-d_L^2}\begin{bmatrix} B_L \\ -d_L C_L^* \end{bmatrix}, \quad C_\gamma := \frac{1}{1-d_L^2}\begin{bmatrix} d_L C_L \\ B_L^* \end{bmatrix}, \quad d_\gamma := \frac{1}{d_L^2 - 1}.$$

Using functions of matrices (see, e.g., [39]), we set

$$\begin{aligned}
\phi_{11}(\gamma) &:= M(A_\gamma)b_\gamma(-A_\gamma)\\
\phi_{12}(\gamma) &:= k_\gamma(A_\gamma)\\
\phi_{21}(\gamma) &:= \pi(A_{M_d}) - k_\gamma(-A_{M_d})\hat{L}_\gamma(-A_{M_d})M(-A_{M_d})b_\gamma(A_{M_d})\\
\phi_{22}(\gamma) &:= -k_\gamma(-A_{M_d})\hat{L}_\gamma(-A_{M_d})k_\gamma(-A_{M_d})\\
\phi_{31}(\gamma) &:= \phi_{22}(\gamma)\\
\phi_{32}(\gamma) &:= \phi_{21}(\gamma)
\end{aligned}$$

$$\begin{aligned}
\Phi_{11}(\gamma) &:= \phi_{11}(\gamma)\mathcal{R}(A_\gamma, B_\gamma, N), & \hat{\Phi}_{11}(\gamma) &:= \phi_{11}(\gamma)\mathcal{R}(A_\gamma, B_\gamma, N+1)\\
\Phi_{12}(\gamma) &:= \phi_{12}(\gamma)\mathcal{R}(A_\gamma, B_\gamma, N), & \hat{\Phi}_{12}(\gamma) &:= \phi_{12}(\gamma)\mathcal{R}(A_\gamma, B_\gamma, N+1)\\
\Phi_{21}(\gamma) &:= \phi_{12}(\gamma)\mathcal{R}(-A_{M_d}, B_{M_d}, N), & \hat{\Phi}_{21}(\gamma) &:= \phi_{12}(\gamma)\mathcal{R}(-A_{M_d}, B_{M_d}, N+1)\\
\Phi_{22}(\gamma) &:= \phi_{22}(\gamma)\mathcal{R}(-A_{M_d}, B_{M_d}, N), & \hat{\Phi}_{22}(\gamma) &:= \phi_{22}(\gamma)\mathcal{R}(-A_{M_d}, B_{M_d}, N+1)\\
\Phi_{31}(\gamma) &:= \phi_{31}(\gamma)\mathcal{R}(A_{M_d}, B_{M_d}, N), & \hat{\Phi}_{31}(\gamma) &:= \phi_{31}(\gamma)\mathcal{R}(A_{M_d}, B_{M_d}, N+1)\\
\Phi_{32}(\gamma) &:= \phi_{32}(\gamma)\mathcal{R}(A_{M_d}, B_{M_d}, N), & \hat{\Phi}_{32}(\gamma) &:= \phi_{32}(\gamma)\mathcal{R}(A_{M_d}, B_{M_d}, N+1),
\end{aligned}$$

where $N := n + l$ and $\mathcal{R}(A, B, r)$ is the controllability matrix of the pair (A, B) with degree r, that is,

$$\mathcal{R}(A, B, r) = \begin{bmatrix} B & AB & \dots & A^{r-1}B \end{bmatrix}.$$

The following theorem gives a necessary and sufficient condition for Problem A and a parameterization of all solutions:

Theorem A ([41])**.** *Assume that M and m are nonzero at the zeros of π. Define σ by* (5.2.24) *and let γ_o be the largest $\gamma > \sigma$ such that*

$$\begin{bmatrix} \Phi_{11}(\gamma) & \Phi_{12}(\gamma) \\ \Phi_{21}(\gamma) & \Phi_{22}(\gamma) \\ \Phi_{31}(\gamma) & \Phi_{32}(\gamma) \end{bmatrix}$$

is singular. Then $\gamma_o = \gamma_{\inf}$ and there exists $\gamma < 1$ such that (A-1) *holds for some $Q \in \mathcal{H}^\infty$ if and only if $\gamma_o < 1$.*

Furthermore, assume that $\gamma_o < \gamma < 1$. Then there always exist v_p and v_q such that

$$\begin{bmatrix} \hat{\Phi}_{11}(\gamma) & \hat{\Phi}_{12}(\gamma) \\ \hat{\Phi}_{21}(\gamma) & \hat{\Phi}_{22}(\gamma) \\ \hat{\Phi}_{31}(\gamma) & \hat{\Phi}_{32}(\gamma) \end{bmatrix} \begin{bmatrix} v_p \\ v_q \end{bmatrix} = 0.$$

Define

$$\begin{aligned}
p(s) &:= \begin{bmatrix} 1 & s & \dots & s^{n+l} \end{bmatrix} v_p\\
q(s) &:= \begin{bmatrix} 1 & s & \dots & s^{n+l} \end{bmatrix} v_q\\
x_0 &:= -\frac{k_\gamma q + M L_\gamma^{\sim} k_\gamma^{\sim} p}{\pi m}\\
y_0 &:= -\left(\frac{k_\gamma^{\sim} q + M^{\sim} L_\gamma k_\gamma^{\sim} q}{\pi m^{\sim}}\right)^{\sim}.
\end{aligned}$$

All solutions $Q \in \mathcal{H}^\infty$ satisfying (A-1) *are given by*

$$Q = (MM_d)^{\tilde{}} L_\gamma - M_d^{\tilde{}} \hat{L}_\gamma - \frac{x_0^{\tilde{}} - y_0^{\tilde{}} \theta}{x_0 \theta - y_0},$$

where θ is an arbitrary $\mathcal{H}^\infty$ function with $\|\theta\|_\infty \leq 1$.

Publications

Chapters 2 and 3

- M. Wakaiki, Y. Yamamoto, and H. Özbay. Sensitivity reduction by strongly stabilizing controllers for MIMO distributed parameter systems. *IEEE Trans. Automat. Control*, 57:2089–2094, 2012.
- M. Wakaiki, Y. Yamamoto, and H. Özbay. Tangential Nevanlinna-Pick interpolation for strong stabilization of MIMO distributed parameter systems. In *Proc. 51st IEEE CDC*, 2012.
- M. Wakaiki, Y. Yamamoto, and H. Özbay. Sensitivity reduction by stable controllers for MIMO infinite dimensional systems via the tangential Nevanlinna-Pick interpolation. To appear in *IEEE Trans. Automat. Control*.

Chapter 4

- M. Wakaiki, Y. Yamamoto, and H. Özbay. Robust stabilization for distributed parameter systems by stable controllers. In *Proc. MTNS 2012*, 2012.
- M. Wakaiki, Y. Yamamoto, and H. Özbay. Stable controllers for robust stabilization of systems with infinitely many unstable poles. *Systems Control Lett.*, 62:511–516, 2013.

Chapter 5

- M. Wakaiki and Y. Yamamoto. Mixed sensitivity reduction for time-delay systems by stable controllers. In *Proc. 52nd IEEE CDC*, 2013.

References

[1] J. L. Abedor and K. Poolla. On the strong stabilization of delay systems. In *Proc. 28th IEEE CDC*, 1989.

[2] M. J. Balas. Active control of flexible systems. *J. Optim. Theory Appl.*, 25:415–436, 1978.

[3] J. A. Ball, I. Gohberg, and L. Rodman. *Interpolation of Rational Matrix Functions*. Birkhäuser, Basel, 1990.

[4] J. A. Ball and J. W. Helton. Interpolation with outer functions and gain equalization in amplifiers. In *Proc. MTNS 1979*, 1979.

[5] A. E. Barabanov. Desing of $\mathcal{H}^\infty$ optimal stable stabilizing compensator. In *Proc. 35th IEEE CDC*, 1996.

[6] D. U. Campos-Delgado and K. Zhou. $\mathcal{H}_\infty$ strong stabilization. *IEEE Trans. Automat. Control*, 46:1968–1972, 2001.

[7] D. U. Campos-Delgado and K. Zhou. A parametric optimization approach to $\mathcal{H}_\infty$ and $\mathcal{H}_2$ strong stabilization. *Automatica*, 39:1205–1211, 2003.

[8] A. Cavallo, C. De Maria, G. Natale, and S. Pirozzi. Classes of strongly stabilizing bandpass controllers for flexible structures. *Adv. Acoust. Vibr.*, Article ID: 249478, 2012.

[9] A. Cavallo, G. De Maria, C. Natale, and S. Pirozzi. Robust control of flexible structures with stable bandpass controllers. *Automatica*, 44:1251–1260, 2008.

[10] V. Cerone, M. Milanese, and D. Regruto. Yaw stability control design through a mixed-sensitivity approach. *IEEE Trans. Control Systems Tech.*, 17:1096–1104, 2009.

[11] M. J. Chen and C. A. Desoer. Necessary and sufficient conditions for robust stability of linear distributed feedback systems. *Int. J. Control*, 29:880–888, 1984.

[12] Y. Q. Chen, I. Petráš, and D. Xue. Fractional order control - a tutorial. In *Proc. ACC 2009*, 2009.

[13] P. Cheng, Y.-Y. Cao, and Y. Sun. Some new results on strong γ_k-γ_{cl} H_∞ stabilization. *IEEE Trans. Automat. Control*, 53:1268–1273, 2008.

[14] P. Cheng, Y.-Y. Cao, and Y Sun. Strong γ_k-γ_{cl} H_∞ stabilization with a new slack vaiable approach. *Automatica*, 45:1861–1867, 2009.

[15] Y. Choi and W. K. Chung. On the stable $\mathcal{H}_\infty$ controller parameterization under sufficient condition. *IEEE Trans. Automat. Control*, 46:1618–1623, 2001.

[16] Y. S. Chou, J. L. Leu, and Y. C. Chu. Stable controller design for MIMO systems: an LMI approach. *IET Control Theory Appl.*, 1:817–829, 2007.

[17] A. M. Concepción, B. M. Chen, Y.-Q. Vinagre, D. Xue, and V. Feliu. *Fractional-order Systems and Controls - Fundamentals and Applications.* Springer, London, 2010.

[18] C. Cosner, G. Anwar, and M. Tomizuka. Plug in repetitive control for industrial robotic manipulators. In *Proc. IEEE ICRA 1990*, 1990.

[19] L. Cuiyan, Z. Dongchun, and Z. Xianyi. A survey of repetitive control. In *Proc. IEEE/RSJ IROS 2004*, 2004.

[20] R. Curtain and K. Morris. Transfer functions of distributed parameter systems: A tutorial. *Automatica*, 45:1101–1116, 2009.

[21] S. Cusumano, K. Poolla, and T. Ting. On robust stabilization synthesis for plants with block structured modeling uncertainty. In *Proc. 26th IEEE CDC*, 1987.

[22] PH. Delsarte, Y. Genin, and Y. Kamp. The Nevanlinna-Pick problem for matrix-valued functions. *SIAM J. Appl. Math.*, 36:47–61, 1979.

[23] P. Dorate, H. B. Park, and Y. Li. An algorithm for interpolation with units in H_∞. *Automatica*, 25:427–430, 1989.

[24] P. Dorato. *Analytic Feedback Design: An Interpolation Approach.* Pacific Grove, CA: Brooks/Col, 2000.

[25] J. C. Doyle, B. A. Francis, and A. Tannenbaum. *Feedback Control Theory.* New York: Macmillan, 1992.

[26] J. C. Doyle and G. Stein. Multivariable feedback design: Concepts for a classical/modern synthesis. *IEEE Transactions on Automatic Control*, 26:4–16, 1981.

[27] H. Dym. A basic interpolation problem. In *Holomorphic Space*, pages 381–423. Cambridge University Press, 1998.

[28] C. Farges, L. Fadiga, and J. Sabatier. $\mathcal{H}_\infty$ analysis and control of commensurate fractional order systems. *Mechatronics*, 23:772–780, 2013.

[29] A. Feintuch. On strong stabilization for linear time-varying systems. *Systems Control Lett.*, 54:1091–1095, 2005.

[30] A. Feintuch. On strong stabilization for linear continuous time time-varying systems. *Systems Control Lett.*, 57:691–695, 2008.

[31] Y. Feng, M. Yagoubi, and P. Chevrel. Simultaneous H_∞ control for continuous-time descriptor systems. *IET Control Theory Appl.*, 5:9–18, 2011.

[32] C. Foiaş and A. E. Frazho. *The Commutant Lifting Approach to Interpolation Problems.* Birkhäuser, Basel, 1990.

[33] C. Foiaş, H. Özbay, and A. Tannenbaum. *Robust Control of Infinite Dimensional Systems: Frequency Domain Methods.* Springer, London, 1996.

[34] B. A. Francis. *A Course in H_∞ Control Theory.* Springer, New York, 1987.

[35] B. A. Francis and W. M. Wonham. The internal model principle for linear multivariable regulators. *J. Appl. Math. Optim.*, 2:170–194, 1975.

[36] A. E. Frazho and W. Bhosri. *An Operator Perspective on Signals and Systems.* Birkhäuser, Basel, 2010.

[37] C. Ganesh. *Synthesis of optimal control systems with stable feedback.* PhD thesis, Rice University, http://scholarship.rice.edu/handle/1911/16056, 1987.

[38] C. Ganesh and J. B. Pearson. Design of optimal control systems with stable feedback. In *Proc. ACC 1986*, 1986.

[39] F. R. Gantmacher. *The Theory of Matrices*, volume 1. Chelsea, NewYork, 1953.

[40] T. T. Georgiou and M. C. Smith. Graphs, causality and sstabilizability: linear, shift-invariant systems on $\mathscr{L}_2[0,\infty)$. *Math. Control Signals Systems*, 6:195–223, 1993.

[41] C. Gu, O. Toker, and H. Özbay. On the two-block H^∞ problem or a class of unstable distributed systems. *Linear Algebra Appl.*, 234:227–244, 1996.

[42] G. Gu, P. P. Khargonekar, and E. B. Lee. Approximation of infinite-dimensional systems. *IEEE Trans. Automat. Control*, 34:610–618, 1989.

[43] S. Gümüşsoy. Coprime-inner/outer factorization of SISO time-delay systems and FIR structure of their optimal H-infinity controllers. *Int. J. Robust Nonlinear Control*, 22:981–998, 2012.

[44] S. Gümüşsoy, M. Millstone, and M. L. Overton. $\mathcal{H}^\infty$ strong stabilization via HIFOO, a package for fixed-order controller design. In *Proc. 47th IEEE CDC*, 2008.

[45] S. Gümüşsoy and H. Özbay. On the mixed sensitivity minimization for systems with infinitely many unstable modes. *Systems Control Lett.*, 53:211–216, 2004.

[46] S. Gümüşsoy and H. Özbay. Remarks on strong stabilization and stable $\mathcal{H}^\infty$ controller design. *IEEE Trans. Automat. Control*, 50:2083–2087, 2005.

[47] S. Gümüşsoy and H. Özbay. Stable $\mathcal{H}^\infty$ controller design for time-delay systems. *Int. J. Control*, 81:546–556, 2008.

[48] S. Gümüşsoy and H. Özbay. Sensitivity minimization by strongly stabilizing controllers for a class of unstable time-delay systems. *IEEE Trans. Automat. Control*, 54:590–595, 2009.

[49] A. N. Gündeş and H. Özbay. Strong stabilization of a class of mimo systems. *IEEE Trans. Automat. Control*, 56:1445–1452, 2011.

[50] S. Gunnarsson and M. Östring. On regulator stability in control of flexible mechanical systems. In *Proc. 32nd ISR*, 2001.

[51] T. Hagiwara and M. Araki. On preservation of strong stabilizability under sampling. *IEEE Trans. Automat. Control*, 33:1080–1082, 1988.

[52] S. Hara and H.-K. Sung. Sensitivity imprimprove by a stable controller in SISO digital control systems. *Systems Control Lett.*, 12:123–128, 1989.

[53] S. Hara and M. Vidyasagar. Sensitivity minimization and robust stabilization by stable controller SISO systems. In *Proc. MTNS 1989*, 1989.

[54] S. Hara, Y. Yamamoto, T. Omata, and M. Nakano. Repetitive control system: A new type servo system for periodic exogenous signals. *IEEE Trans. Automat. Control*, 33:659–668, 1988.

[55] H. Imanari, Y. Morimatsu, K. Sekiguchi, H. Ezure, R. Matuoka, A. Tokuda, and H. Otobe. Looper H-infinity control for hot-strip mills. *IEEE Trans. Indust. Appl.*, 33:790–796, 1997.

[56] H. Ito, H. Ohmori, and A. Sano. Design of stable controllers attaining low $\mathscr{H}^\infty$ weighted sensitivity. *IEEE Trans. Automat. Control*, 38:485–488, 1993.

[57] I. N. Kar, K. Seto, and F. Doi. Multimode vibration control of a flexible structure using H_∞-based robust control. *IEEE/ASME Trans. Mechatronics*, 5:23–31, 2000.

[58] K. Kashima, H. Özbay, and Y. Yamamoto. A Hamiltonian-based solution to the mixed sensitivity optimization problem for stable pseudorational plants. *Systems Control Lett.*, 54:1063–1068, 2005.

[59] K. Kashima and Y. Yamamoto. Finite rank criteria for H^∞ control of infinite-dimensional systems. *IEEE Trans. Automat. Control*, 53:881–893, 2008.

[60] K. Kashima and Y. Yamamoto. On standard H^∞ control problems for systems with infinitely many unstable poles. *Systems Control Lett.*, 57:309–314, 2008.

[61] H. Kimura. Robust stabilizability for a class of transfer functions. *IEEE Trans. Automat. Control*, 29:788–793, 1984.

[62] H. Kimura. Directional interpolation approach to H^∞-optimization and robust stabilization. *IEEE Trans. Automat. Control*, 32:1085–1093, 1987.

[63] H. Kwakernaak. Robust control and $\mathscr{H}_\infty$-optimization–tutorial paper. *Automatica*, 29:255–273, 1993.

[64] H. Kwakernaak. Mixed Sensitivity Design. In *Proc. 15th IFAC WC*, 2002.

[65] P. H. Lee and Y. C. Soh. Synthesis of stable H^∞ controller via the chain scattering framework. *Systems Control Lett.*, 47:121–127, 2002.

[66] P. H. Lee and Y. C. Soh. Synthesis of simultaneous stabilizing $\mathcal{H}^\infty$ controller. *Int. J. Control*, 78:1437–1446, 2005.

[67] D. J. N. Limebeer and B. D. O. Anderson. An interpolation theory approach to H^∞ controller degree bounds. *Linear Algebra Appl.*, 98:347–386, 1988.

[68] Z. Lin, J. Q. Ying, and L. Xu. An algebraic approach to strong stabilizability of linear nD MIMO systems. *IEEE Trans. Automat. Control*, 47:1510–1514, 2002.

[69] L. A. Luxemburg and P. R. Brown. The scalar Nevanlinna-Pick interpolation problem with boundary conditions. *J. Comput. Appl. Math.*, 235:2615–2625, 2011.

[70] D. C. MacMartin and J. P. How. Implementation and prevention of unstable optimal compensators. In *Proc. of ACC 1994*, 1994.

[71] D. Matignon. Stability properties for generalized fractional differential systems. *ESAIM Proceedings*, 5:145–158, 1998.

[72] G. Meinsma and L. Mirkin. H^{∞} control of systems with multiple I/O delays via decomposition to adobe problems. *IEEE Trans. Automat. Control*, 30:199–211, 2005.

[73] J.-H. Moon, M.-N. Lee, and M. J. Chung. Repetitive control for the track-following servo system of an optical disk drive. *IEEE Trans. Control Syst. Tech.*, 6:663–670, 1998.

[74] Y. Ohta, H. Maeda, and S. Kodama. Unit interpolation in H_{∞}: Bounds of norm and degree of interpolants. *Systems Control Lett.*, 17:251–256, 1991.

[75] H. Olsson and K. J. Åström. Friction generated limit cycles. *IEEE Trans. Control Systems Tech.*, 9:629–636, 2001.

[76] H. Özbay. On strongly stabilizing controller synthesis for time delay systems. In *Proc. 17th IFAC WC*, 2008.

[77] H. Özbay. Stable $\mathscr{H}^{\infty}$ controller design for systems with time delays. In *Perspectives in Mathematical System Theory, Control, and Signal Processing*, pages 105–113. Springer, Berlin, 2010.

[78] H. Özbay. An operator theoretic approach to robust control of infinite dimensional systems. *TWMS J. Pure Appl. Math.*, 4:3–19, 2013.

[79] H. Özbay, C. Bonnet, and A. R. Fioravanti. PID controller design for fractional-order systems with time delays. *Systems Control Lett.*, 61:18–23, 2012.

[80] H. Özbay and A. N. Gündeş. Resilient PI and PD controller designs for a class of unstable plants with I/O delays. *Appl. Comput. Math.*, 6:18–26, 2007.

[81] H. Özbay, A. N. Gündeş, and A. B. Özgüler. PID controller synthesis for a class of unstable MIMO plants with I/O delays. *Automatica*, 43:135–142, 2007.

[82] H. Özbay, M. C. Smith, and A. Tannenbaum. Mixed-sensitivity optimization for a class of unstable infinite-dimensional systems. *Linear Algebra Appl.*, 178:43–83, 1993.

[83] J. R. Partington. Some frequency-domain approaches to the model reduction of delay systems. *Annu. Rev. Control*, 28:65–73, 2004.

[84] J. R. Partington and C. Bonnet. H_∞ and BIBO stabilization of delay systems of neutral type. *Systems Control Lett.*, 52:283–288, 2004.

[85] J. R. Partington and P. M. Mäkilä. Rational approximation of distributed-delay controllers. *Int. J. Control*, 78:1295–1301, 2005.

[86] V. V. Patel. On least-degree unit interpolation in RH_∞. *Automatica*, 32:1333–1335, 1996.

[87] V. V. Patel and K. B. Datta. A dual look at unit interpolation in H_∞. *Int. J. Control*, 62:813–829, 1995.

[88] V. V. Patel and K. B. Datta. A modification of theorem for least bound of the minimum degree of interpolating unit. *Systems Control Lett.*, 25:235–236, 1995.

[89] B. M. Patre, S. D. Agashe, S. Jayakumar, and B. Bandyopadhyay. On interpolation with units in H_∞. In *Proc. IEEE ISCAS 1997*, 1997.

[90] I. R. Petersen. Robust H^∞ control of an uncertain system via a stable output feedback controller. *IEEE Trans. Automat. Control*, 54:1418–1423, 2009.

[91] R. K. Prasanth. Two-sided tangential interpolation with real rational units in $\mathscr{H}^\infty$. *Automatica*, 34:861–874, 1998.

[92] A. Quadrat. On a general structure of the stabilizing controllers based on stable range. *SIAM J. Control Optim.*, 42:2264–2285, 2004.

[93] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, 1974.

[94] J. Sabatier, M. Aoun, A. Oustaloup, G. Grégoire, and F. Ragot. Fractional system identification for lead acid battery state of charge estimation. *Signal Processing*, 86:2645–2657, 2006.

[95] A.-W. A. Saif. Sensitivity minimization of MIMO systems with a stable controller: An H_∞ interpolation approach. *Systems Anal. Model. Simul.*, 42:1011–1024, 2002.

[96] A.-W. A. Saif, D.-W. Gu, and I. Postlethwaite. Strong stabilization of MIMO systems via unimodular interpolation in H_∞. *Int. J. Control*, 32:111–120, 1998.

[97] D. Sarason. Generalized interpolation in H^∞. *Trans. American Math. Society*, 127:179–203, 1967.

[98] A. Sarjaš, R. Svečko, and A. Chowdhury. Strong stabilization servo controller with optimization of performance criteria. *ISA Trans.*, 50:419–431, 2011.

[99] J. Shi and W. S. Lee. Analytical feedback design via interpolation approach for the strong stabilization of a magnetic bearing system. In *Proc. 21st CCDC*, 2009.

[100] A. Sideris and M. G. Safonov. Infinity-norm optimization with a stable controller. In *Proc. ACC 1985*, 1985.

[101] M. C. Smith. On stabilization and the existence of coprime factorizations. *IEEE Trans. Automat. Control*, 34:1005–1007, 1989.

[102] M. C. Smith and K. P. Sondergeld. On the order of stable compensators. *Automatica*, 22:127–129, 1986.

[103] T. Sugie and S. Hara. H_∞-suboptimal control problem with boundary constraints. *Systems Control Lett.*, 13:93–99, 1989.

[104] K. Suyama. Strong stabilization of systems with time-delays. In *Proc. IECON'91*, 1991.

[105] K. Takaba, N. Morihira, and T. Katayama. H_∞ control of descriptor system: A J-spectral factorization approach. In *Proc. 33rd IEEE CDC*, 1994.

[106] A. Tannenbaum. Modified Nevanlinna-Pick interpolation and feedback stabilization of linear plants with uncertainty in the gain factor. *Int. J. Control*, 36:331–336, 1982.

[107] O. Toker. On the order of simultaneous stabilizing compensators. *IEEE Trans. Automat. Control*, 41:430–433, 1996.

[108] O. Toker and H. Özbay. H^∞ optimal and suboptimal controllers for infinite dimensional SISO plants. *IEEE Trans. Automat. Control*, 40:751–755, 1995.

[109] S. Treil. The stable rank of the algebra H^∞ equals 1. *J. Funct. Anal.*, 109:130–154, 1992.

[110] H. U. Ünal and A. İftar. Stable $\mathcal{H}^\infty$ controller design for systems with multiple input/output time-delays. *Automatica*, 48:563–568, 2012.

[111] H. U. Ünal and A. İftar. Stable $\mathcal{H}^\infty$ flow controller design using approximation of FIR filters. *Trans. Inst. Meas. Control*, 34:3–25, 2012.

[112] M. van de Wal, G. van Baars, F. Sperling, and O. Bosgra. Multivariable $\mathscr{H}_\infty/\mu$ feedback control design for high-precision wafer stage motion. *Control Eng. Pract.*, 10:739–755, 2002.

[113] M. Vidyasagar. *Input-Output Analysis of Large-Scale Interconnected Systems.* Springer, Berlin, 1981.

[114] M. Vidyasagar. *Control System Synthesis: A Factorization Approach.* Cambridge, MA: MIT Press, 1985, Republished in Morgan & Claypool, 2011.

[115] M. Vilathgamuwa, X. Zhu, and S. S. Choi. A robust control method to improve the performance of a unified power flow controller. *Electr. Power Systems Res.*, 55:103–111, 2000.

[116] A. Wallenborg and K. J. Åström. Limit cycle oscillations in high performance robot drives. In *Proc. IEE International Conference: Control 88*, 1988.

[117] X. Xin and Y. Liu. Reduced-order stable controllers for two-link underactuated planar robots. *Automatica*, 49:2176–2183, 2013.

[118] Y. Yamamoto. Pseudo-rational input/output maps and their realizations: A fractional representation approach to infinite-dimensional systems. *SIAM J. Control Optim.*, 26:1415–1430, 1988.

[119] Y. Yamamoto. Equivalence of internal and external stability for a class of distributed systems. *Math. Control Signals Systems*, 4:391–409, 1991.

[120] Y. Yamamoto. *From Vector Spaces to Function Spaces: Introduction to Functional Analysis with Applications.* SIAM Press, Philadelphia, PA, 2012.

[121] Y. Yamamoto and S. Hara. Relationships between internal and external stability for infinite-dimensional systems with applications to a servo problem. *IEEE Trans. Automat. Control*, 33:1044–1052, 1988.

[122] Y. Yamamoto and S. Hara. Internal and external stability and robust stability condition for a class of infinite dimensional systems. *Automatica*, 28:81–93, 1992.

[123] J. Q. Ying. Conditions for the strong stabilizabilities of n-D linear systems. *Multidimens. Systems Signal Process.*, 9:125–148, 1998.

[124] J. Q. Ying. On the strong stabilizability of MIMO n-dimensional linear systems. *SIAM J. Control Optim.*, 38:313–335, 2000.

[125] J. Q. Ying, Z. Lin, and L. Xu. Some algebraic aspects of the strong stabilizability of time-delay linear systems. *IEEE Trans. Automat. Control*, 46:454–457, 2001.

[126] J. Q. Ying and L. Xu. A computation method for determining strong stabilizability of n-D systems. *J. Symbolic Computation*, 27:479–499, 1999.

[127] D. Youla, J. Bongiorno, and C. Lu. Single-loop feedback stabilization of linear multivariable dynamical plants. *Automatica*, 10:159–173, 1974.

[128] G. Zames. Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Trans. Automat. Control*, 26:301–320, 1981.

[129] M. Zeren and H. Özbay. On the synthesis of stable $\mathcal{H}^\infty$ controllers. *IEEE Trans. Automat. Control*, 44:431–435, 1999.

[130] M. Zeren and H. Özbay. On the strong stabilization and stable H^∞-controller design problems for MIMO systems. *Automatica*, 36:1675–1684, 2000.

[131] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice Hall, Upper Saddle River, NJ, 1996.