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Microlocal Euler classes and Hochschild homology

Masaki Kashiwara* and Pierre Schapira

May 1, 2013

Abstract

We define the notion of a trace kernel on a manifold $M$. Roughly speaking, it is a sheaf on $M \times M$ for which the formalism of Hochschild homology applies. We associate a microlocal Euler class to such a kernel, a cohomology class with values in the relative dualizing complex of the cotangent bundle $T^*M$ over $M$ and we prove that this class is functorial with respect to the composition of kernels.

This generalizes, unifies and simplifies various results of (relative) index theorems for constructible sheaves, $\mathcal{D}$-modules and elliptic pairs.

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1 Introduction

Our constructions mainly concern real manifolds, but in order to introduce the subject we first consider a complex manifold \((X, \mathcal{O}_X)\). Denote by \(\omega_X^{\text{hol}}\) the dualizing complex in the category of \(\mathcal{O}_X\)-modules, that is, \(\omega_X^{\text{hol}} = \Omega_X[d_X]\), where \(d_X\) is the complex dimension of \(X\) and \(\Omega_X\) is the sheaf of holomorphic forms of degree \(d_X\). Denote by \(\mathcal{O}_X^\Delta\) and \(\omega_X^{\text{hol}}\) the direct images of \(\mathcal{O}_X\) and \(\omega_X^{\text{hol}}\) respectively by the diagonal embedding \(\delta: X \hookrightarrow X \times X\). It is well-known (see in particular [Ca05, CaW07]) that the Hochschild homology of \(\mathcal{O}_X\) may be defined by using the isomorphism

\[
\delta_* \mathcal{H} \mathcal{H}(\mathcal{O}_X) \simeq R \mathcal{H} \text{om}_{\mathcal{O}_X \times X}(\mathcal{O}_X^\Delta, \omega_X^{\text{hol}}).
\]

Moreover, if \(\mathcal{F}\) is a coherent \(\mathcal{O}_X\)-module and \(D_{\mathcal{O}}\mathcal{F} := R \mathcal{H} \text{om}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^{\text{hol}})\) denotes its dual, there are natural morphisms

\[
\mathcal{O}_X^\Delta \rightarrow \mathcal{F} \boxtimes D_{\mathcal{O}}\mathcal{F} \rightarrow \omega_X^{\text{hol}},
\]

whose composition defines the Hochschild class of \(\mathcal{F}\):

\[
\text{hh}_{\mathcal{O}}(\mathcal{F}) \in H^0_{\text{Supp}(\mathcal{F})}(X; \mathcal{H} \mathcal{H}(\mathcal{O}_X)).
\]

These constructions have been extended when replacing \(\mathcal{O}_X\) with a so-called DQ-algebroid stack \(\mathscr{A}\) in [KS12] (DQ stands for “deformation-quantization”). One of the main results of loc. cit. is that Hochschild classes are functorial with respect to the composition of kernels, a kind of (relative) index theorem for coherent DQ-modules.

On the other hand, the notion of Lagrangian cycles of constructible sheaves on real analytic manifolds has been introduced by the first named author (see [Ka85]) in order to prove an index theorem for such sheaves, after they first appeared in the complex case (see [Ka73] and [McP74]). We refer to [KS90, Chap. 9] for a systematic study of Lagrangian cycles and for historical comments. Let us briefly recall the construction.
Consider a real analytic manifold $M$ and let $k$ be a unital commutative ring with finite global dimension. Denote by $\omega_M$ the (topological) dualizing complex of $M$, that is, $\omega_M = \text{or}_M \lceil \dim M \rceil$ where $\text{or}_M$ is the orientation sheaf of $M$ and $\dim M$ is the dimension. Finally, denote by $\pi_M : T^* M \to M$ the cotangent bundle to $M$. Let $\Lambda$ be a conic subanalytic Lagrangian subset of $T^* M$. The group of Lagrangian cycles supported by $\Lambda$ is given by $H^0_\Lambda(T^* M; \pi_M^{-1} \omega_M)$. Denote by $D_{\bR, c}(k_M)$ the bounded derived category of $\bR$-constructible sheaves on $M$. To an object $F$ of this category, one associates a Lagrangian cycle supported by $\text{SS}(F)$, the microsupport of $F$. This cycle is called the characteristic cycle, or the Lagrangian cycle or else the microlocal Euler class of $F$ and is denoted here by $\mu_{eu}^M(F)$.

In fact, it is possible to treat the microlocal Euler classes of $\bR$-constructible sheaves on real manifolds similarly as the Hochschild class of coherent sheaves on complex manifolds. Denote as above by $k_{\Delta M}$ and $\omega_{\Delta M}$ the direct image of $k_M$ and $\omega_M$ by the diagonal embedding $\delta_M : M \hookrightarrow M \times M$. Then we have an isomorphism

\[
(1.3) \quad H^0_\Lambda(T^* M; \pi_M^{-1} \omega_M) \simeq H^0_\Lambda(T^* M; \mu_{hom}(k_{\Delta M}, \omega_{\Delta M})),
\]

where $\mu_{hom}$ is the microlocalization of the functor $R \mathcal{H}om$. Then $\mu_{eu}^M(F)$ is obtained as follows. Denote by $D_M F := R \mathcal{H}om(F, \omega_M)$ the dual of $F$. There are natural morphisms

\[
(1.4) \quad k_{\Delta M} \to F \boxtimes D_M F \to \omega_{\Delta M},
\]

whose composition gives the microlocal Euler class of $F$.

In this paper, we construct the microlocal Euler class for a wide class of sheaves, including of course the constructible sheaves but also the sheaves of holomorphic solutions of coherent $\mathscr{D}$-modules and, more generally, of elliptic pairs in the sense of $[\text{ScSn94}]$. To treat such situations, we are led to introduce the notion of a trace kernel.

On a real manifold $M$ (say of class $C^\infty$), a trace kernel is the data of a triplet $(K, u, v)$ where $K$ is an object of the derived category of sheaves $D^b(k_{M \times M})$ and $u, v$ are morphisms

\[
(1.5) \quad u : k_{\Delta M} \to K, \quad v : K \to \omega_{\Delta M}.
\]

One then naturally defines the microlocal Euler class $\mu_{eu}^M(K, u, v)$ of such a kernel, an element of $H^0_\Lambda(T^* M; \mu_{hom}(k_{\Delta M}, \omega_{\Delta M}))$ where $\Lambda = \text{SS}(K) \cap T^*_{\Delta M}(M \times M)$. By (1.4), a constructible sheaf gives rise to a trace kernel.
If $X$ is a complex manifold and $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, we construct natural morphisms (over the base ring $k = \mathbb{C}$)

$$C_{\Delta X} \to \bigoplus_{x \in X} \bigotimes_{\mathcal{D}_{\mathcal{O}_{X,x}}} (\mathcal{M} \boxtimes \mathcal{D}_{\mathcal{M}}) \to \omega_{\Delta X},$$

where $\mathcal{D}_{\mathcal{D}_{\mathcal{M}}}$ denotes the dual of $\mathcal{M}$ as a $\mathcal{D}$-module. In other words, one naturally associates a trace kernel on $X$ to a coherent $\mathcal{D}_X$-module. Moreover, we prove that under suitable microlocal conditions, the tensor product of two trace kernels is again a trace kernel, and it follows that one can associate a trace kernel to an elliptic pair.

We study trace kernels and their microlocal Euler classes, showing that some proofs of [KS12] can be easily adapted to this situation. One of our main results is the functoriality of the microlocal Euler classes: the microlocal Euler class of the composition $K_1 \circ K_2$ of two trace kernels is the composition of the microlocal Euler classes of $K_1$ and $K_2$ (see Theorem 6.3 for a precise statement). Another essential result (which is far from obvious) is that the composition of classes coincides with the composition for $\pi_M^{-1}\omega_M$ constructed in [KS90] via the isomorphism between $\mu hom(k_{\Delta_M}, \omega_{\Delta_M})$ and $\pi_M^{-1}\omega_M$.

As an application, we recover in a single proof the classical results on the index theorem for constructible sheaves (see [KS90], §9.5) as well as the index theorem for elliptic pairs of [ScSn94], that is, sheaves of generalized holomorphic solutions of coherent $\mathcal{D}$-modules. We also briefly explain how to adapt trace kernels to the formalism of the Lefschetz trace formula.

We call here $\mu hom(k_{\Delta_M}, \omega_{\Delta_M})$ the microlocal homology of $M$, and this paper shows that, in some sense, the microlocal homology of real manifolds plays the same role as the Hochschild homology of complex manifolds.

To conclude this introduction, let us make a general remark. The category $D^b_{\text{c}}(k_M)$ of constructible sheaves on a compact real analytic manifold $M$ is “proper” in the sense of Kontsevich (that is, Ext finite) but it does not admit a Serre functor (in the sense of Bondal-Kapranov) and it is not clear whether it is smooth (again in the sense of Kontsevich). However this category naturally appears in Mirror Symmetry (see [FLTZ10]) and it would be a natural question to try to understand its Hochschild homology in the sense of [McC94, Ke99]. We don’t know how to compute it, but the above construction, with the use of $\mu hom(k_{\Delta_M}, \omega_{\Delta_M})$, provides an alternative approach of the Hochschild homology of this category. This result is not totally surprising if one remembers the formula (see [KS90, Prop. 8.4.14]):

$$D_{T^*M}(\mu hom(F,G)) \simeq \mu hom(G,F) \otimes \pi_M^{-1}\omega_M.$$
Hence, in some sense, $\pi_M^{-1} \omega_M$ plays the role of a microlocal Serre functor. Note that thanks to Nadler and Zaslow [NZ09], the category $\mathcal{D}^b_{\mathbb{R}-c}(k_M)$ is equivalent to the Fukaya category of the symplectic manifold $T^*M$, and this is another argument to treat sheaves from a microlocal point of view.

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2 A short review on sheaves

Throughout this paper, a manifold means a real manifold of class $C^\infty$. We shall mainly follow the notations of [KS90] and use some of the main notions introduced there, in particular that of microsupport and the functor $\mu hom$.

Let $M$ be a manifold. We denote by $\pi_M : T^*M \to M$ its cotangent bundle. For a submanifold $N$ of $M$, we denote by $T_N^*M$ the conormal bundle to $N$. In particular, $T^*_M M$ denotes the zero-section. We set $\tilde{T}_M^* := T^*_M \setminus T_N^*M$ and we denote by $\tilde{\pi}_M$ the restriction of $\pi_M$ to $\tilde{T}_M^*$. If there is no risk of confusion, we write simply $\pi$ and $\tilde{\pi}$ instead of $\pi_M$ and $\tilde{\pi}_M$. One denotes by $a : T^*M \to T^*M$ the antipodal map, $(x; \xi) \mapsto (x; -\xi)$ and for a subset $S$ of $T^*_M$, one denotes by $S^a$ its image by this map. A set $A \subset T^*_M$ is conic if it is invariant by the action of $\mathbb{R}^+$ on $T^*_M$.

Let $f : M \to N$ be a morphism of manifolds. To $f$ one associates as usual the maps

$$
\begin{array}{ccc}
T^*_M & \xleftarrow{f_d} & M \times_N T^*N \\
\searrow & & \downarrow \pi \\
\pi_M & \swarrow & f \downarrow \pi_N \\
M & \xrightarrow{f} & N.
\end{array}
$$

(2.1)

(Note that in loc. cit. the map $f_d$ is denoted by $t f^{t-1}$.)

Let $\Lambda$ be a closed conic subset of $T^*N$. One says that $f$ is non-characteristic for $\Lambda$ if the map $f_d$ is proper on $f^{-1}_\pi \Lambda$ or, equivalently, $f^{-1}_\pi \Lambda \cap f_d^{-1}(T^*_M) \subset M \times_N T^*_N$.

Let $k$ be a commutative unital ring with finite global homological dimension. One denotes by $k_M$ the constant sheaf on $M$ with stalk $k$ and by $\mathcal{D}^b(k_M)$ the bounded derived category of sheaves of $k$-modules on $M$. When $M$ is a real analytic manifold, one denotes by $\mathcal{D}^b_{\mathbb{R}-c}(k_M)$ the full triangulated subcategory of $\mathcal{D}^b(k_M)$ consisting of $\mathbb{R}$-constructible objects.
One denotes by $\omega_M$ the dualizing complex on $M$ and by $\omega_M^{-1}$ its dual, that is, $\omega_M^{-1} = R\mathcal{H}om(\omega_M, k_M)$. More generally, for a morphism $f: M \to N$, one denotes by $\omega_{M/N} := f^! k_N \simeq \omega_M \otimes f^{-1}(\omega_N^{-1})$ the relative dualizing complex. Recall that $\omega_M \simeq \mathcal{o}_M [\dim M]$ where $\mathcal{o}_M$ is the orientation sheaf and $\dim M$ is the dimension of $M$. Also recall the natural morphism of functors

\[ (2.2) \quad \omega_{M/N} \otimes f^{-1} \to f^! . \]

We have the duality functors

\[ D'_M F = R\mathcal{H}om (F, k_M), \quad D_M F = R\mathcal{H}om (F, \omega_M). \]

For $F \in D^b(k_M)$, one denotes by $\text{Supp}(F)$ the support of $F$ and by $\text{SS}(F)$ its microsupport, a closed $\mathbb{R}^+$-conic co-isotropic subset of $T^*M$. For a morphism $f: M \to N$ and $G \in D^b(k_N)$, one says that $f$ is non-characteristic for $G$ if $f$ is non-characteristic for $\text{SS}(G)$.

We shall use systematically the functor $\mu hom$, a variant of Sato’s microlocalization functor. Recall that for a closed submanifold $N$ of $M$, there is a functor $\mu_N: D^b(k_M) \to D^b(k_{T^*_N M})$ constructed by Sato (see [SKK73]) and for $F_1, F_2 \in D^b(k_M)$, one defines in [KS90] the functor

\[ \mu hom: D^b(k_M)^{\text{op}} \times D^b(k_M) \to D^b(k_{T^*_M}), \]

\[ \mu hom(F_1, F_2) := \mu_\Delta R\mathcal{H}om (q_2^{-1} F_1, q_1^! F_2) \]

where $q_1$ and $q_2$ are the first and second projection defined on $M \times M$ and $\Delta$ is the diagonal. This sheaf is supported by $T^*_\Delta(M \times M)$ that we identify with $T^*M$ by the first projection $T^*(M \times M) \simeq T^*M \times T^*M \to T^*M$. Note that

\[ (2.3) \quad \text{Supp}(\mu hom(F_1, F_2)) \subset \text{SS}(F_1) \cap \text{SS}(F_2) \]

and we have Sato’s distinguished triangle, functorial in $F_1$ and $F_2$:

\[ (2.4) R\pi_! \mu hom(F_1, F_2) \to R\pi_* \mu hom(F_1, F_2) \to R\tilde{\pi}_* (\mu hom(F_1, F_2)|_{T^*_M}) \overset{+1}{\to} . \]

Moreover, we have the isomorphism

\[ (2.5) \quad R\pi_* \mu hom(F_1, F_2) \simeq R\mathcal{H}om (F_1, F_2), \]
and, assuming that $M$ is real analytic and $F_1$ is $\mathbb{R}$-constructible, the isomorphism

$$\text{R} \pi_1 \mu\text{hom}(F_1, F_2) \simeq D'_M F_1 \otimes F_2. \quad (2.6)$$

In particular, assuming that $F_1$ is $\mathbb{R}$-constructible and $\text{SS}(F_1) \cap \text{SS}(F_2) \subset T^*_M M$, we have the natural isomorphism (see [KS90, Cor 6.4.3])

$$D'_MF_1 \otimes F_2 \leadsto \text{R} \mathcal{H}\text{om}(F_1, F_2). \quad (2.7)$$

As recalled in the Introduction, assuming that $M$ is real analytic, we have the formula (see [KS90, Prop. 8.4.14]):

$$D_{T^*M}(\mu\text{hom}(F_1, F_2)) \simeq \mu\text{hom}(F_2, F_1) \otimes \pi^{-1}_M \omega_M \quad \text{for } F_1, F_2 \in D^b_{\mathbb{R}c}(k_M). \quad (2.8)$$

## 3 Compositions of kernels

**Notation 3.1.**

(i) For a manifold $M$, let $\delta_M: M \to M \times M$ denote the diagonal embedding, and $\Delta_M$ the diagonal set of $M \times M$.

(ii) Let $M_i$ ($i = 1, 2, 3$) be manifolds. For short, we write $M_{ij} := M_i \times M_j$ ($1 \leq i, j \leq 3$), $M_{123} = M_1 \times M_2 \times M_3$, $M_{1223} = M_1 \times M_2 \times M_2 \times M_3$, etc.

(iii) We will often write for short $k_i$ instead of $k_{M_i}$ and $k_{\Delta_i}$ instead of $k_{\Delta_{M_i}}$ and similarly with $\omega_{M_i}$, etc., and with the index $i$ replaced with several indices $ij$, etc.

(iv) We denote by $\pi_i$, $\pi_{ij}$, etc. the projection $T^*M_i \to M_i$, $T^*M_{ij} \to M_{ij}$, etc.

(v) We denote by $q_i$ the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by $q_{ij}$ the projection $M_{123} \to M_{ij}$. Similarly, we denote by $p_i$ the projection $T^*M_{ij} \to T^*M_i$ or the projection $T^*M_{123} \to T^*M_i$ and by $p_{ij}$ the projection $T^*M_{123} \to T^*M_{ij}$.

(vi) We also need to introduce the maps $p_j$ or $p_{ij}$, the composition of $p_j$ or $p_{ij}$ and the antipodal map on $T^*M_j$. For example,

$$p_{12^a}((x_1, x_2, x_3; \xi_1, \xi_2, \xi_3)) = (x_1, x_2; \xi_1, -\xi_2).$$
(vii) We let $\delta_2 : M_{123} \to M_{1223}$ be the natural diagonal embedding.

We consider the operation of composition of kernels:

$$\circ_2 : \text{D}^b(kM_{12}) \times \text{D}^b(kM_{23}) \to \text{D}^b(kM_{13})$$

$$ (K_1, K_2) \mapsto K_1 \circ_2 K_2 := Rq_{13!}(q_{12}^{-1} K_1 \otimes q_{23}^{-1} K_2) \simeq Rq_{13!} \delta_2^{-1}(K_1 \boxtimes K_2).$$

We will use a variant of $\circ$:

$$\ast_2 : \text{D}^b(kM_{12}) \times \text{D}^b(kM_{23}) \to \text{D}^b(kM_{13})$$

$$ (K_1, K_2) \mapsto K_1 \ast_2 K_2 := Rq_{13*}(q_2^{-1} \omega_2 \otimes \delta_2^!(K_1 \boxtimes K_2)).$$

We also have $\omega_{M_{123}/M_{1223}} \simeq q_2^{-1} \omega_{M_2} \otimes \delta_2^!$ and we deduce from (2.2) a morphism $\delta_2^{-1} \to q_2^{-1} \omega_{M_2} \otimes \delta_2^!$. Using the morphism $R p_{13!} \to R p_{13*}$ we obtain a natural morphism for $K_1 \in \text{D}^b(kM_{12})$ and $K_2 \in \text{D}^b(kM_{23})$:

$$K_1 \circ K_2 \to K_1 \ast K_2.$$  

It is an isomorphism if $p_{12}^{-1} \text{SS}(K_1) \cap p_{23}^{-1} \text{SS}(K_2) \to T^*M_{13}$ is proper.

We define the composition of kernels on cotangent bundles (see [KS90, Prop. 4.4.11])

$$\circ_2 : \text{D}^b(kT^*M_{12}) \times \text{D}^b(kT^*M_{23}) \to \text{D}^b(kT^*M_{13})$$

$$ (K_1, K_2) \mapsto K_1 \circ_2 K_2 := Rp_{13!}(p_{12}^{-1} K_1 \otimes p_{23}^{-1} K_2) \simeq Rp_{13*!}(p_{12}^{-1} K_1 \otimes p_{23}^{-1} K_2).$$

We also define the corresponding operations for subsets of cotangent bundles.

Let $A \subset T^*M_{12}$ and $B \subset T^*M_{23}$. We set

$$A \times_B = p_{12}^{-1}(A) \cap p_{23}^{-1}(B),$$

$$A \circ_2 B = p_{13}(A \times_B)$$

$$= \left\{ (x_1, x_3; \xi_1, \xi_3) \in T^*M_{13} : \text{there exists } (x_2; \xi_2) \in T^*M_2 \text{ such that } (x_1, x_2; \xi_1, -\xi_2) \in A, (x_2, x_3; \xi_2, \xi_3) \in B \right\}.$$
Proposition 3.2. For $G_1, F_1 \in \text{D}^b(k_{M_{12}})$ and $G_2, F_2 \in \text{D}^b(k_{M_{23}})$ there exists a canonical morphism (whose construction is similar to that of [KS90, Prop. 4.4.11] ):

$$\mu_{\text{hom}}(G_1, F_1) \circ \frac{a}{2} \mu_{\text{hom}}(G_2, F_2) \to \mu_{\text{hom}}(G_1 \ast G_2, F_1 \circ F_2).$$

Proof. In Proposition 4.4.8 (i) of loc. cit., one may replace $F_2 \boxtimes_S G_2$ with $j_1^L(F_2 \boxtimes G_2) \otimes \omega_{X \times S Y / X \times Y}^{\otimes -1}$. Then the proof goes exactly as that of Proposition 4.4.11 in loc. cit. Q.E.D.

Let $\Lambda_{ij} \subset T^* M_{ij}$ ($i = 1, 2, j = i + 1$) be closed conic subsets and consider the condition:

(3.6) the projection $p_{13}: \Lambda_{12} \times \Lambda_{23} \longrightarrow T^* M_{13}$ is proper.

We set

(3.7) $\Lambda_{13} = \Lambda_{12} \circ \frac{a}{2} \Lambda_{23}$.

Corollary 3.3. Assume that $\Lambda_{ij} (i = 1, 2, j = i + 1)$ satisfy (3.6). We have a composition morphism

$$\text{R} \Gamma_{\Lambda_{13}} \mu_{\text{hom}}(G_1, F_1) \circ \frac{a}{2} \text{R} \Gamma_{\Lambda_{23}} \mu_{\text{hom}}(G_2, F_2) \to \text{R} \Gamma_{\Lambda_{13}} \mu_{\text{hom}}(G_1 \ast G_2, F_1 \circ F_2).$$

Convention 3.4. In (3.1), we have introduced the composition $\circ \frac{a}{2}$ of kernels $K_1 \in \text{D}^b(k_{M_{12}})$ and $K_2 \in \text{D}^b(k_{M_{23}})$. However we shall also use the notation $M_{22} = M_2 \times M_2$ and consider for example kernels $L_1 \in \text{D}^b(k_{M_{122}})$ and $L_2 \in \text{D}^b(k_{M_{223}})$. Then when writing $L_1 \circ L_2$ we mean that the composition is taken with respect to the last variable of $M_{22}$ for $L_1$ and the first variable for $L_2$. In other words, set $M_4 = M_2$ and consider $L_1$ and $L_2$ as objects of $\text{D}^b(k_{M_{142}})$ and $\text{D}^b(k_{M_{423}})$ respectively, in which case the composition $L_1 \circ L_2$ is unambiguously defined.

4 Microlocal homology

Let $M$ be a real manifold. Recall that $\delta_M : M \hookrightarrow M \times M$ denotes the diagonal embedding. We shall identify $M$ with the diagonal $\Delta_M$ of $M \times M$ and we
sometimes write $\Delta$ instead of $\Delta_M$ if there is no risk of confusion. We shall identify $T^*M$ with $T^*_a(M \times M)$ by the map

$$\delta^a_{T^*M}: T^*M \to T^*(M \times M), \quad (x; \xi) \mapsto (x, x; \xi, -\xi).$$

We denote by $k_{\Delta_M}, \omega_{\Delta_M}$ and $\omega_{\Delta_M}^{-1}$ the direct image by $\delta_M$ of $k_M, \omega_M$ and $\omega_M^{-1} := R\mathcal{H}om(\omega_M, k_M)$, respectively.

The next definition is inspired by that of Hochschild homology on complex manifolds (see Introduction).

**Definition 4.1.** Let $\Lambda$ be a closed conic subset of $T^*M$. We set

$$\mathcal{MH}_\Lambda(k_M) := R\Gamma(\delta^a_{T^*M}; \mathcal{MH}_\Lambda(k_M)),$$

$$\mathcal{MH}_\Lambda^k(k_M) := H^k(\mathcal{MH}_\Lambda(k_M)) = H^k(T^*M; \mathcal{MH}_\Lambda(k_M)).$$

We call $\mathcal{MH}_\Lambda(k_M)$ the **microlocal homology** of $M$ with support in $\Lambda$.

We also write $\mathcal{MH}(k_M)$ instead of $\mathcal{MH}_{T^*M}(k_M)$.

**Remark 4.2.** (i) We have $\mu_{hom}(k_{\Delta_M}; \omega_{\Delta_M}) \simeq (\delta^a_{T^*M})_*\pi^{-1}_M \omega_M$. In particular, we have $\mathcal{MH}_\Lambda(k_M) \simeq R\Gamma(T^*M; \mathcal{MH}_\Lambda(k_M))$ and $\mathcal{MH}_\Lambda^k(k_M) \simeq R\Gamma(M; \omega_M)$. Assuming that $M$ is real analytic and $\Lambda$ is a closed conic subanalytic Lagrangian subset of $T^*M$, we recover the space of Lagrangian cycles with support in $\Lambda$ as defined in [KS90, §9.3].

(ii) The support of $\mu_{hom}(k_{\Delta_M}; \omega_{\Delta_M})$ is $T^*_{\Delta_M}(M \times M)$. Hence, we have $R\Gamma(\delta^a_{T^*M}; \mu_{hom}(k_{\Delta_M}; \omega_{\Delta_M})) \simeq (\delta^a_{T^*M})_*\mathcal{MH}_\Lambda(k_M)$.

(iii) If $M$ is real analytic and $\Lambda$ is a Lagrangian subanalytic closed conic subset, then we have $H^k(\mathcal{MH}_\Lambda(k_M)) = 0$ for $k < 0$ (see [KS90, Prop. 9.2.2]).

In the sequel, we denote by $\Delta_i$ (resp. $\Delta_{ij}$) the diagonal subset $\Delta_M \subset M$ (resp. $\Delta_{Mij} \subset M$).

**Lemma 4.3.** We have natural morphisms:

(i) $\omega_{\Delta_{12}} \circ (k_{\Delta_2} \boxtimes \omega_{\Delta_3}) \to \omega_{\Delta_{13}},$

(ii) $k_{\Delta_{13}} \to k_{\Delta_{12}} \boxtimes (\omega_{\Delta_2}^{-1} \boxtimes k_{\Delta_3}).$
Proof. Denote by $\delta_{22}$ the diagonal embedding $M_{11233} \hookrightarrow M_{11222233}$.

(i) We have the morphisms

$$\omega_{\Delta_{12}} \circ_{22} (k_{\Delta_2} \otimes \omega_{\Delta_3}) = Rq_{11333} \delta_{22}^{-1} (\omega_{\Delta_12} \otimes k_{\Delta_2} \otimes \omega_{\Delta_3})$$

$$\simeq Rq_{11333} \omega_{\Delta_{123}}$$

$$\rightarrow \omega_{\Delta_{13}}.$$  

(ii) The isomorphism

$$\delta_{22} (k_{\Delta_2} \otimes \omega_{\Delta_2}) \simeq k_{\Delta_2}$$

gives rise to the isomorphisms

$$k_{\Delta_{12}} \star_{22} (\omega_{\Delta_2}^{-1} \otimes \omega_{\Delta_2}) = Rq_{11333} (q_{11333}^{-1} \omega_{\Delta_2} \otimes \delta_{22} (k_{\Delta_12} \otimes \omega_{\Delta_2}^{-1} \otimes k_{\Delta_3}))$$

$$\simeq Rq_{11333} \delta_{22} (k_{\Delta_1} \otimes \omega_{\Delta_2} \otimes \omega_{\Delta_23})$$

$$\simeq Rq_{11333} k_{\Delta_{123}}$$

and the result follows by adjunction from the morphism

$$q_{11333}^{-1} k_{\Delta_{13}} \simeq k_{\Delta_1} \otimes k_{\Delta_2} \otimes k_{\Delta_3} \rightarrow k_{\Delta_1} \otimes k_{\Delta_2} \otimes k_{\Delta_3} = k_{\Delta_{123}}.$$

Q.E.D.

Proposition 4.4. Let $M_i$ ($i = 1, 2, 3$) be manifolds. We have a natural composition morphism (whose constructions will be given in the course of the proof):

$$\mu hom(k_{\Delta_{12}}, \omega_{\Delta_{12}}) \circ_{22} \mu hom(k_{\Delta_{23}}, \omega_{\Delta_{23}}) \rightarrow \mu hom(k_{\Delta_{13}}, \omega_{\Delta_{13}}).$$ (4.2)

In particular, let $\Lambda_{ij}$ be a closed conic subset of $T^*M_{ij}$ ($ij = 12, 13, 23$). If $\Lambda_{12} \circ_{23} \Lambda_{23} \subset \Lambda_{13}$, then we have a morphism

$$\mu hom(\omega_{\Delta_2}^{-1}, \omega_{\Delta_2}^{-1}) \circ_{22} \mu hom(k_{\Delta_{23}}, \omega_{\Delta_{23}}) \rightarrow \mu hom(\omega_{\Delta_2}^{-1} \circ k_{\Delta_{23}}, \omega_{\Delta_2}^{-1} \circ \omega_{\Delta_{23}})$$ (4.3)

Proof. Consider the morphism (see Proposition 3.2 and Convention 3.4)

$$\mu hom(\omega_{\Delta_2}^{-1}, \omega_{\Delta_2}^{-1}) \circ_{22} \mu hom(k_{\Delta_{23}}, \omega_{\Delta_{23}}) \rightarrow \mu hom(\omega_{\Delta_2}^{-1} \circ k_{\Delta_{23}}, \omega_{\Delta_2}^{-1} \circ \omega_{\Delta_{23}})$$

$$\simeq \mu hom(\omega_{\Delta_2}^{-1} \otimes k_{\Delta_3}, k_{\Delta_2} \otimes \omega_{\Delta_3}).$$
It induces an isomorphism

\[(4.4) \quad \mu_{\text{hom}}(k_{\Delta_{23}}, \omega_{\Delta_{23}}) \simeq \mu_{\text{hom}}(\omega_{\Delta_{23}}^{-1} \boxtimes k_{\Delta_{3}}, k_{\Delta_{2}} \boxtimes \omega_{\Delta_{3}}).\]

Note that this isomorphism is also obtained by

\[
\mu_{\text{hom}}(k_{\Delta_{23}}, \omega_{\Delta_{23}}) \simeq \mu_{\text{hom}}((\omega_{\Delta_{23}}^{-1} \boxtimes k_{\Delta_{3}}) \boxtimes k_{\Delta_{2}}, (\omega_{\Delta_{23}}^{2} \boxtimes k_{\Delta_{3}}) \boxtimes \omega_{\Delta_{23}})
\]

\[
\simeq \mu_{\text{hom}}(\omega_{\Delta_{23}}^{-1} \boxtimes k_{\Delta_{3}}, k_{\Delta_{2}} \boxtimes \omega_{\Delta_{3}}).
\]

Applying Proposition 3.2, we get a morphism:

\[(4.5) \quad \mu_{\text{hom}}(k_{\Delta_{12}}, \omega_{\Delta_{12}}) \circ \mu_{\text{hom}}(k_{\Delta_{23}}, \omega_{\Delta_{23}}) \rightarrow \mu_{\text{hom}}(k_{\Delta_{12}} \times (\omega_{\Delta_{23}}^{-1} \boxtimes k_{\Delta_{3}}), \omega_{\Delta_{12}} \circ k_{\Delta_{2}} \boxtimes \omega_{\Delta_{3}})).\]

It remains to apply Lemma 4.3. Q.E.D.

**Corollary 4.5.** Let $\Lambda_{ij}$ ($i = 1, 2, j = i + 1$) satisfying (3.6) and let $\Lambda_{13} = \Lambda_{12} \circ \Lambda_{23}$. The composition of kernels in (4.3) induces a morphism

\[(4.6) \quad \circ \overrightarrow{\Lambda_{12}} : \text{MH}_{\Lambda_{12}}(k_{12}) \overset{L}{\otimes} \text{MH}_{\Lambda_{23}}(k_{23}) \rightarrow \text{MH}_{\Lambda_{13}}(k_{13}).\]

In particular, each $\lambda \in \text{MH}_{\Lambda_{12}}(k_{12})$ defines a morphism

\[(4.7) \quad \lambda \circ \overrightarrow{\Lambda_{12}} : \text{MH}_{\Lambda_{23}}(k_{23}) \rightarrow \text{MH}_{\Lambda_{13}}(k_{13}).\]

**Proof.** These morphisms follow from (4.3). The second assertion follows from the isomorphism $H^0(X) \simeq \text{Hom}_{D^b(k)(k, X)}$ in the category $D^b(k)$. Q.E.D.

**Theorem 4.6.** (i) We have the isomorphisms

\[
\mu_{\text{hom}}(k_{\Delta_{M}}, \omega_{\Delta_{M}}) \simeq (\delta_{T_{M}}^{a})_{*} \pi_{M}^{-1} R \mathcal{H}om(k_{M}, \omega_{M})
\]

\[
\simeq (\delta_{T_{M}}^{a})_{*} \pi_{M}^{-1} \omega_{M}.
\]

(ii) We have a commutative diagram

\[(4.8) \quad \mu_{\text{hom}}(k_{\Delta_{12}}, \omega_{\Delta_{12}}) \circ \mu_{\text{hom}}(k_{\Delta_{23}}, \omega_{\Delta_{23}}) \rightarrow \mu_{\text{hom}}(k_{\Delta_{13}}, \omega_{\Delta_{13}})
\]

\[
(\delta_{T_{M_{12}}}^{a})_{*}(\pi_{M_{12}}^{-1} \omega_{M_{12}} \circ \pi_{M_{23}}^{-1} \omega_{M_{23}}) \rightarrow (\delta_{T_{M_{13}}}^{a})_{*} \pi_{M_{13}}^{-1} \omega_{M_{13}}.
\]
Here the top horizontal arrow of (4.8) is given in Proposition 4.4, and the bottom horizontal arrow is induced by
\[
p_{12}^{-1} \pi_{M_12}^{-1} \omega_{M_{12}} \otimes p_{23}^{-1} \pi_{M_{23}}^{-1} \omega_{M_{23}} \simeq \pi_{M_1}^{-1} \omega_{M_1} \boxtimes \pi_{M_2}^{-1} (\omega_{M_2} \otimes \omega_{M_2}) \boxtimes \pi_{M_3}^{-1} \omega_{M_3},
\]
\[
\pi_{M_2}^{-1} (\omega_{M_2} \otimes \omega_{M_2}) \simeq \omega_{T^*M_2},
\]
\[
\text{R}p_{13}!(\pi_{M_1}^{-1} \omega_{M_1} \boxtimes \omega_{T^*M_2} \boxtimes \pi_{M_3}^{-1} \omega_{M_3}) \longrightarrow \pi_{M_1}^{-1} \omega_{M_1} \boxtimes \pi_{M_3}^{-1} \omega_{M_3}.\]

Proof. (i) is obvious.

(ii)–(a) By [KS90, Prop. 4.4.8], we have natural morphisms for \((i, j) = (1, 2)\) or \((i, j) = (2, 3)\):
\[
\mu_{\text{hom}}(k_{\Delta_i}, \omega_{\Delta_i}) \boxtimes \mu_{\text{hom}}(k_{\Delta_j}, \omega_{\Delta_j}) \to \mu_{\text{hom}}(k_{\Delta_{ij}}, \omega_{\Delta_{ij}})
\]
and it follows from (i) that these morphisms are isomorphisms. These isomorphisms give rise to the isomorphism
\[
\mu_{\text{hom}}(k_{\Delta_{12}}, \omega_{\Delta_{12}}) \circ_{22} \mu_{\text{hom}}(k_{\Delta_{23}}, \omega_{\Delta_{23}}) \simeq
\]
\[
\mu_{\text{hom}}(k_{\Delta_1}, \omega_{\Delta_1}) \boxtimes (\mu_{\text{hom}}(k_{\Delta_2}, \omega_{\Delta_2}) \circ_{22} \mu_{\text{hom}}(k_{\Delta_2}, \omega_{\Delta_2})) \boxtimes \mu_{\text{hom}}(k_{\Delta_3}, \omega_{\Delta_3})
\]
Similarly, we have an isomorphism
\[
\pi_{M_12}^{-1} \omega_{M_{12}} \circ_2 \pi_{M_{23}}^{-1} \omega_{M_{23}} \simeq \pi_{M_1}^{-1} \omega_{M_1} \boxtimes (\pi_{M_2}^{-1} \omega_{M_2} \circ_2 \pi_{M_2}^{-1} \omega_{M_2}) \boxtimes \pi_{M_3}^{-1} \omega_{M_3}.
\]
Hence, we are reduced to the case where \(M_1 = M_3 = pt\), which we shall assume now.

(ii)–(b) We change our notations and we set:
\[M := M_2, \quad Y := M \times M,\]
\[\delta_M : M \hookrightarrow Y \text{ the diagonal embedding, } \Delta_M = \delta_M(M),\]
\[j : Y \hookrightarrow Y \times Y \text{ the diagonal embedding, } \Delta_Y = \delta_Y(Y),\]
\[\delta^*_{T^*M} : T^*M \hookrightarrow T^*Y, (x; \xi) \mapsto (x, x; \xi, -\xi),\]
\[\delta^*_{T^*Y} : T^*Y \hookrightarrow T^*Y \times T^*Y,\]
\[p : T^*Y \rightarrow pt \text{ the projection,}\]
\[a_Y : Y \rightarrow pt \text{ the projection.}\]
With these new notations, the composition \( a \circ_{22} \) will be denoted by \( \circ_{T^*Y} \).

Consider the diagram 4.9 similar to Diagram (4.4.15) of [KS90]

\[
\begin{array}{ccc}
T^*M \times T^*M & \xrightarrow{i} & T^*Y \times T^*Y \\
\downarrow j_\pi & & \sim p_! \\
T^*Y \times T^*Y & \xrightarrow{\sim \pi_Y} & T^*_\Delta (Y \times Y) \\
\downarrow j_d & & \downarrow p \\
T^*Y & \xrightarrow{s} & Y \xrightarrow{a_Y} pt.
\end{array}
\]

\((4.9)\)

Here, \( i \) is the canonical embedding induced by \( \delta^a_{T^*M} \), \( p_1 \) is induced by the first projection \( T^*Y \times T^*Y \to T^*Y \), \( s \) : \( Y \hookrightarrow T^*Y \) is the zero-section embedding and \( \tilde{s} \) is the natural embedding. Note that the square labelled by \( \square \) is Cartesian. We have

\[
Rp_! \circ (\delta^a_{T^*Y})^{-1} \simeq Ra_! \circ R\pi_Y! \circ p_1^{-1} \circ (\delta^a_{T^*Y})^{-1} \\
\simeq Ra_! \circ R\pi_Y! \circ \tilde{s}^{-1} \circ j_\pi^{-1} \\
\simeq Ra_! \circ s^{-1} \circ Rj_d! \circ j_\pi^{-1}.
\]

Therefore,

\[
\mu hom(k_{\Delta_M}, \omega_{\Delta_M}) \circ_{T^*Y} \mu hom(k_{\Delta_M}, \omega_{\Delta_M}) \\
\simeq Rp_! (\delta^a_{T^*Y})^{-1} (\mu hom(k_{\Delta_M}, \omega_{\Delta_M})) \boxtimes \mu hom(k_{\Delta_M}, \omega_{\Delta_M}) \\
\simeq Ra_! s^{-1} Rj_d! j_\pi^{-1} \mu hom(k_{\Delta_M} \boxtimes k_{\Delta_M}, \omega_{\Delta_M} \boxtimes \omega_{\Delta_M}).
\]

Hence, by adjunction, to give a morphism

\[
\mu hom(k_{\Delta_M}, \omega_{\Delta_M}) \circ_{T^*Y} \mu hom(k_{\Delta_M}, \omega_{\Delta_M}) \to k
\]

is equivalent to giving a morphism in \( D^b(k_Y) \)

\[
s^{-1} Rj_d! j_\pi^{-1} \mu hom(k_{\Delta_M} \boxtimes k_{\Delta_M}, \omega_{\Delta_M} \boxtimes \omega_{\Delta_M}) \to a_Y^! k_{pt}.
\]

\((4.10)\)

Note that the left hand side of \((4.10)\) is supported on \( \Delta_M \). Hence in order to give a morphism \((4.10)\), it is necessary and sufficient to give a morphism in \( D^b(k_M) \)

\[
\delta_M^{-1} s^{-1} Rj_d! j_\pi^{-1} \mu hom(k_{\Delta_M} \boxtimes k_{\Delta_M}, \omega_{\Delta_M} \boxtimes \omega_{\Delta_M}) \to \delta_M^! a_Y^! k_{pt}.
\]

\((4.11)\)
Hence, it is enough to check the commutativity of the upper square in the following diagram in $D^b(k_M)$

$$
\begin{array}{cccccccccc}
\delta_M^{-1} s^{-1} & R j_{dt} j_{\pi}^{-1} & \mu hom(k_{\Delta_M} \Leftrightarrow k_{\Delta_M}, \omega_{\Delta_M} \Leftrightarrow \omega_{\Delta_M}) & \rightarrow & \delta_M^{-1} a_Y^1 & k_{pt} \\
\sim & & & & \downarrow & id \\
\delta_M^{-1} s^{-1} R j_{dt} j_{\pi}^{-1} i_* (\pi^{-1}_M \omega_M \Leftrightarrow \pi^{-1}_M \omega_M) & \rightarrow & \delta_M^{-1} a_Y^1 & k_{pt} \\
\sim & & & & \downarrow & id \\
\omega_M & \rightarrow & \omega_M.
\end{array}
$$

(4.12)

The top horizontal arrow is constructed by the chain of morphisms (see [KS90, § 4.4]):

$$
R j_{dt} j_{\pi}^{-1} \mu hom(k_{\Delta_M} \Leftrightarrow k_{\Delta_M}, \omega_{\Delta_M} \Leftrightarrow \omega_{\Delta_M})
\rightarrow \mu hom(j^1 (k_{\Delta_M} \Leftrightarrow k_{\Delta_M}) \otimes \omega_Y, j^{-1} (\omega_{\Delta_M} \Leftrightarrow \omega_{\Delta_M}))
\simeq \mu hom(\omega_{\Delta_M}, \omega_M \Leftrightarrow \omega_{\Delta_M} \Leftrightarrow \omega_M)
\simeq (\delta^{T^*_M}_M \Leftrightarrow \pi^{-1}_M \omega_M)
$$

and

$$
\delta_M^{-1} s^{-1} R j_{dt} j_{\pi}^{-1} \mu hom(k_{\Delta_M} \Leftrightarrow k_{\Delta_M}, \omega_{\Delta_M} \Leftrightarrow \omega_{\Delta_M})
\rightarrow \delta_M^{-1} s^{-1} (\delta^{T^*_M}_M \Leftrightarrow \pi^{-1}_M \omega_M) \simeq \omega_M.
$$

(4.13)

Hence, the commutativity of the diagram (4.12) is reduced to the commutativity of the diagram below:

$$
\begin{array}{cccccccccc}
\delta_M^{-1} s^{-1} & R j_{dt} j_{\pi}^{-1} & \mu hom(k_{\Delta_M} \Leftrightarrow k_{\Delta_M}, \omega_{\Delta_M} \Leftrightarrow \omega_{\Delta_M}) & \rightarrow & \omega_M \\
\sim & & & & \downarrow & \lambda \\
\delta_M^{-1} s^{-1} R j_{dt} j_{\pi}^{-1} i_* (\pi^{-1}_M \omega_M \Leftrightarrow \pi^{-1}_M \omega_M) & \rightarrow & \omega_M.
\end{array}
$$

(4.14)

where the morphism $\lambda$ is given by the morphisms in (4.13). All terms of (4.14) are concentrated at the degree $- \dim M$. Hence the commutativity of (4.14) is a local problem in $M$ and we can assume that $M$ is a Euclidean space. We can checked directly in this case. Q.E.D.
Remark 4.7. Theorem 4.6 may be applied as follows. Let $\Lambda_{ij}$ be a closed conic subset of $T^*M_{ij}$ $(i = 1, 2, j = i + 1)$. Assume (3.6), that is, the projection $p_{13}: \Lambda_{12} \overset{\alpha}{\times} \Lambda_{23} \longrightarrow T^*M_{13}$ is proper and set $\Lambda_{13} = \Lambda_{12} \overset{\alpha}{\times} \Lambda_{23}$. Let $\lambda_{ij} \in \text{MH}_0^0(\Lambda_{ij} \cap \text{MH}_{ij}) \cong H^0_0(\Lambda_{ij}(T^*M_{ij}; \pi^{-1}\omega_{ij}))$. Then
\begin{equation}
\lambda_{12} \overset{\alpha}{\times} \lambda_{23} = \int_{T^*M_2} \lambda_{12} \cup \lambda_{23}
\end{equation}
where the right hand-side is obtained as follows. Set $\Lambda := \Lambda_{12} \overset{\alpha}{\times} \Lambda_{23}$ and consider the morphisms
\begin{align*}
H^0_{\Lambda_{12}}(T^*M_{12}; \pi^{-1}\omega_{12}) & \times H^0_{\Lambda_{23}}(T^*M_{23}; \pi^{-1}\omega_{23}) \\
\rightarrow H^0_{\Lambda}(T^*M_{123}; \pi^{-1}\omega_1 \boxtimes \omega_{T^*M_2} \boxtimes \pi^{-1}\omega_3) \\
\rightarrow H^0_{\Lambda_{13}}(T^*M_{13}; \pi^{-1}\omega_{13}).
\end{align*}
The first morphism is the cup product and the second one is the integration morphism with respect to $T^*M_2$.

5 Microlocal Euler classes of trace kernels

In this section, we often write $\Delta$ instead of $\Delta_M$.

Definition 5.1. A trace kernel $(K, u, v)$ on $M$ is the data of $K \in \mathcal{D}(kM \times M)$ together with morphisms
\begin{equation}
k_{\Delta} \xrightarrow{u} K \quad \text{and} \quad K \xrightarrow{v} \omega_{\Delta}.
\end{equation}

In the sequel, as far as there is no risk of confusion, we simply write $K$ instead of $(K, u, v)$.

For a trace kernel $K$ as above, we set
\begin{equation}
\text{SS}_{\Delta}(K) := \text{SS}(K) \cap T^*_\Delta(M \times M) = (\delta_{T^*M})^{-1}\text{SS}(K).
\end{equation}
(Recall that one often identifies $T^*M$ and $T^*_\Delta(M \times M)$ by $\delta_{T^*M}: T^*M \hookrightarrow T^*M \times T^*M$.)

Definition 5.2. Let $(K, u, v)$ be a trace kernel.
(a) The morphism $u$ defines an element $\tilde{u}$ in $H^0_{SS\Delta}(T^*M; \mu_{hom}(k_\Delta, K))$ and the microlocal Euler class $\mu_{eu_M}(K)$ of $K$ is the image of $\tilde{u}$ by the morphism $\mu_{hom}(k_\Delta, K) \to \mu_{hom}(k_\Delta, \omega_\Delta)$ associated with the morphism $v$.

(b) Let $\Lambda$ be a closed conic subset of $T^*M$ containing $SS\Delta(K)$. One denotes by $\mu_{eu_\Lambda}(K)$ the image of $\tilde{u}$ in $H^0_\Lambda(T^*M; \mu_{hom}(k_\Delta, \omega_\Delta))$.

Hence,

\[
\mu_{eu}(K) \in MH^0_{\Delta}(k_M) \simeq H^0_\Lambda(T^*M; \pi^{-1}\omega_M).
\]

Let $\tilde{v}$ be the element of $H^0_{SS\Delta}(K)(T^*M; \mu_{hom}(K, \omega_\Delta))$ induced by $v$. Then the microlocal Euler class $\mu_{eu_M}(K)$ of $K$ coincides with the image of $\tilde{v}$ by the morphism $\mu_{hom}(K, \omega_\Delta) \to \mu_{hom}(k_\Delta, \omega_\Delta)$ associated with the morphism $u$, which can be easily seen by the commutative diagram:

\[
\begin{array}{ccc}
(\delta^a_{T^*M})^{-1}\mu_{hom}(K, K) & \xrightarrow{u} & (\delta^a_{T^*M})^{-1}\mu_{hom}(K, \omega_\Delta) \\
\downarrow u & & \downarrow u \\
(\delta^a_{T^*M})^{-1}\mu_{hom}(k_\Delta, K) & \xrightarrow{v} & (\delta^a_{T^*M})^{-1}\mu_{hom}(k_\Delta, \omega_\Delta).
\end{array}
\]

One denotes by $eu(K)$ the restriction of $\mu_{eu}(K)$ to the zero-section $M$ of $T^*M$ and calls it the Euler class of $K$. Hence

\[
eu_K(M) \in H^0_{\text{Supp}(K)\cap \Delta}(M; \omega_M).
\]

It is nothing but the class induced by the composition $k_{\Delta_M} \to K \to \omega_{\Delta_M}$.

We say that $L \in D^b(k_M)$ is invertible if $L$ is locally isomorphic to $k_M[d]$ for some $d \in \mathbb{Z}$. Then, $L^{-1} := R\mathcal{H}om(L, k_M)$ is also invertible and $L \otimes L^{-1} \simeq k_M$.

**Proposition 5.3.** Let $L$ be an invertible object in $D^b(k_M)$ and $K$ a trace kernel. Then $K \otimes (L \otimes L^{-1})$ is a trace kernel and $\mu_{eu}(K \otimes (L \otimes L^{-1})) = \mu_{eu}(K)$.

**Proof.** $L \otimes L^{-1}$ is canonically isomorphic to $k_{M \times M}$ on a neighborhood of the diagonal set $\Delta_M$ of $M \times M$. Q.E.D.
Remark 5.4. Of course, we could also have defined a trace kernel as a sequence of morphisms

\[(5.5) \quad \omega_{\Delta M}^{-1} \to \widetilde{K} \to k_{\Delta M} \cdot \]

When treating sheaves, both definitions would give the same microlocal Euler class by taking \( K = \widetilde{K} \otimes (k_M \otimes \omega_M) \). However, when working with \( \mathcal{O} \)-modules or with DQ-modules as in [KS12], the two constructions give different classes. Note that we have chosen an analogue of (5.5) in [KS12].

Trace kernels for constructible sheaves

Let us denote by \( D^b_{cc}(k_M) \) the full triangulated subcategory of \( D^b(k_M) \) consisting of cohomologically constructible sheaves (see [KS90, § 3.4]).

Lemma 5.5. Let \( F \in D^b_{cc}(k_M) \). There are natural morphisms in \( D^b_{cc}(k_{M \times M}) \):

\[(5.6) \quad k_{\Delta M} \to F \overset{L}{\boxtimes} D_M F, \]
\[(5.7) \quad F \overset{L}{\boxtimes} D_M F \to \omega_{\Delta M} \cdot \]

In other words, an object \( F \in D^b_{cc}(k_M) \) defines naturally a trace kernel on \( M \).

Proof. (i) We have

\[ k_M \to R\mathcal{H}om(F, F) \simeq \delta'(F \overset{L}{\boxtimes} D_M F). \]

Hence, the result follows by adjunction.

(ii) The morphism (5.7) may be deduced from (5.6) by duality, or by adjunction from the morphism

\[ \delta^{-1}(F \overset{L}{\boxtimes} D_M F) \to \omega_M. \]

Q.E.D.

Notation 5.6. We shall denote by \( TK(F) \) the trace kernel associated with \( F \in D^b_{cc}(k_M) \), that is the data of \( F \overset{L}{\boxtimes} D_M F \) and the morphisms (5.6), (5.7). Note that we have always \( SS_{\Delta}(TK(F)) \subset SS(F) \) and the equality holds if \( M \) is real analytic and \( F \) is \( \mathbb{R} \)-constructible.
We have the chain of morphisms
\[
\mu_{\text{hom}}(F, F) \simeq (\delta_{T^*M}^a)^{-1}\mu_{\text{hom}}(k_{\Delta}, F \boxtimes DF) \\
\rightarrow (\delta_{T^*M}^a)^{-1}\mu_{\text{hom}}(k_{\Delta}, \omega_\Delta).
\]

We deduce the map
\[(5.8) \quad H^0_{\text{SS}(F)}(T^*M; \mu_{\text{hom}}(F, F)) \rightarrow \mathcal{M}_{\text{SS}(F)}(k_M).
\]

**Definition 5.7.** Let \( F \in D^b_{\text{cc}}(k_M) \). The image of \( \text{id}_F \) by the map \( (5.8) \) is called the microlocal Euler class of \( F \) and is denoted by \( \mu_{\text{eu}}M(F) \).

Clearly, one has
\[(5.9) \quad \mu_{\text{eu}}M(F) = \mu_{\text{eu}}M(TK(F)).
\]

Assume \( M \) is real analytic and denote by \( D^b_{\mathbb{R}-c}(k_M) \) the full triangulated subcategory of \( D^b(k_M) \) consisting of \( \mathbb{R} \)-constructible complexes. Of course, \( \mathbb{R} \)-constructible complexes are cohomologically constructible. In \([KS90, \S\ 9.4]\) the microlocal Euler class of an object \( F \in D^b_{\mathbb{R}-c}(k_M) \) is constructed as above and this class is also called the characteristic cycle, or else, the Lagrangian cycle, of \( F \).

**Remark 5.8.** Let \( (K, u, v) \) be a trace kernel on \( M \). Let \( \delta : M \rightarrow M \times M \) be the diagonal embedding. Then \( u \) and \( v \) decompose as
\[
k_{\Delta_M} \rightarrow \delta_*\delta^!K \rightarrow K \rightarrow \delta_*\delta^{-1}K \rightarrow \omega_{\Delta_M}.
\]

Hence \( \delta_*\delta^!K \) and \( \delta_*\delta^{-1}K \) are also trace kernels. We have evidently
\[
\mu_{\text{eu}}M(\delta_*\delta^!K) = \mu_{\text{eu}}M(\delta_*\delta^{-1}K) = \mu_{\text{eu}}M(K) \quad \text{as elements in } \mathcal{M}_{\text{SS}(F)}(k_M).
\]

**Trace kernels over one point**

Let us consider the particular case where \( M \) is a single point, \( M = \text{pt} \), and let us identify a sheaf over \( \text{pt} \) with a \( k \)-module. In this situation, a trace kernel \( (K, u, v) \) is the data of \( K \in D^b(k) \) together with linear maps
\[
k \xrightarrow{u} K \xrightarrow{v} k.
\]
The (microlocal) Euler class $\text{eu}_{\text{pt}}(K)$ of this kernel is the image of $1 \in \mathbf{k}$ by $v \circ u$.

Assume now that $\mathbf{k}$ is a field and denote by $\mathbf{D}_f^b(\mathbf{k})$ the full triangulated subcategory of $\mathbf{D}^b(\mathbf{k})$ consisting of objects with finite-dimensional cohomologies. Let $V \in \mathbf{D}_f^b(\mathbf{k})$ and set $V^* = \text{RHom}(V, \mathbf{k})$. Let $K = TK(V) = V \otimes V^*$, and let $v$ be the trace morphism and $u$ its dual. Then

(a) $\text{eu}_{\text{pt}}(V \otimes V^*) = \text{tr}(\text{id}_V)$, the trace of the identity of $V$.

(b) If $\mathbf{k}$ has characteristic zero, then $\text{eu}_{\text{pt}}(V \otimes V^*) = \chi(V)$, the Euler-Poincaré index of $V$.

(5.10)

Trace kernels for $\mathcal{D}$-modules

In this subsection, we denote by $X$ a complex manifold of complex dimension $d_X$ and the base ring $\mathbf{k}$ is the field $\mathbb{C}$. We denote by $\mathcal{O}_X$ the structure sheaf and by $\Omega_X$ the sheaf of holomorphic forms of maximal degree. We still denote by $\omega_X$ the topological dualizing complex and recall the isomorphism $\omega_X \cong \mathbb{C}_X[2d_X]$.

One denotes by $\mathcal{D}_X$ the sheaf of $\mathbb{C}_X$-algebras of (finite order) holomorphic differential operators on $X$ and refers to [Ka03] for a detailed exposition of the theory of $\mathcal{D}$-modules. We denote by $\text{Mod}(\mathcal{D}_X)$ the category of left $\mathcal{D}_X$-modules and by $\mathbf{D}^b(\mathcal{D}_X)$ its bounded derived category. We also denote by $\text{Mod}_{\text{coh}}(\mathcal{D}_X)$ the abelian category of coherent $\mathcal{D}_X$-modules and by $\mathbf{D}^b_{\text{coh}}(\mathcal{D}_X)$ the full triangulated subcategory of $\mathbf{D}^b(\mathcal{D}_X)$ consisting of objects with coherent cohomologies.

We denote by $\mathbf{D}_{\mathcal{D}} : \mathbf{D}^b(\mathcal{D}_X)^{\text{op}} \to \mathbf{D}^b(\mathcal{D}_X)$ the duality functor for left $\mathcal{D}$-modules:

$$\mathbf{D}_{\mathcal{D}} \mathcal{M} := \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1}[d_X].$$

We denote by $\cdot \boxtimes \cdot$ the external product for $\mathcal{D}$-modules:

$$\mathcal{M} \boxtimes \mathcal{N} := \mathcal{D}_{X \times X} \otimes_{\mathcal{D}_{X \times X} \boxtimes \mathcal{D}_{X \times X}} (\mathcal{M} \boxtimes \mathcal{N}).$$

Let $\Delta$ be the diagonal of $X \times X$. The left $\mathcal{D}_{X \times X}$-module $H^*_\Delta(\mathcal{O}_{X \times X})$ (the algebraic cohomology with support in $\Delta$) is denoted as usual by $\mathcal{B}_\Delta$. Note
that

\[ D_\varnothing B_\Delta \simeq B_\Delta. \]

One shall be aware that here, the dual is taken over \( X \times X \). We also introduce

\[ B_\Delta' := B_\Delta [2d_X]. \]

For \( \mathcal{M} \in D^b_{\text{coh}}(\mathcal{D}_X) \), we have the isomorphism

\[ R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{M}) \simeq R\mathcal{H}om_{\mathcal{D}_X \times X}(B_\Delta, \mathcal{M} \boxtimes D_\varnothing \mathcal{M}) [d_X]. \]

We deduce the morphism in \( D^b(\mathcal{D}_X \times X) \)

\[ (5.11) \quad B_\Delta \rightarrow \mathcal{M} \boxtimes D_\varnothing \mathcal{M} [d_X] \]

and by duality, the morphism in \( D^b(\mathcal{D}_X \times X) \)

\[ (5.12) \quad \mathcal{M} \boxtimes D_\varnothing \mathcal{M} [d_X] \rightarrow B_\Delta'. \]

Denote by \( \mathcal{E}_X \) the sheaf on \( T^*X \) of microdifferential operators of \( \text{SKK73} \). For a coherent \( \mathcal{D}_X \)-module \( \mathcal{M} \) set

\[ E^\mathcal{M} := \mathcal{E}_X \otimes_{\varnothing} \mathcal{M} \pi^{-1} \mathcal{O}_X \]

and recall that, denoting by \( \text{char}(\mathcal{M}) \) the characteristic variety of \( \mathcal{M} \), we have \( \text{char}(\mathcal{M}) = \text{Supp}(E^\mathcal{M}) \). One also sets

\[ C_\Delta := B_\Delta', \quad C_\Delta' := (B_\Delta')^E. \]

We denote by \( D_\varnothing : D^b(\mathcal{E}_X)^{\text{op}} \rightarrow D^b(\mathcal{E}_X) \) the duality functor for left \( \mathcal{E} \)-modules:

\[ D_\varnothing \mathcal{M} := R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X) \otimes_{\varnothing} \pi^{-1} \Omega_{X}^{\varnothing} \]

and we denote by \( \bullet \boxtimes \bullet \) the external product for \( \mathcal{E} \)-modules:

\[ \mathcal{M} \boxtimes \mathcal{N} := \mathcal{E}_X \times \mathcal{E}_X \mathcal{M} \mathcal{N} \]

The morphisms \( (5.11) \) and \( (5.12) \) give rise to the morphisms

\[ (5.13) \quad C_\Delta \rightarrow E^\mathcal{M} \boxtimes D_\varnothing \mathcal{M} [d_X] \rightarrow C_\Delta'. \]
Let $\Lambda$ be a closed conic subset of $T^*X$. One sets
\[
\mathcal{H}\mathcal{H}(\mathcal{E}_X) = (\delta_{T^*X})^{-1} R\mathcal{H}om_{\mathcal{E}_{X\times X}}(\mathcal{E}_\Delta, \mathcal{E}_\Delta),
\]
\[
\mathbb{H}\mathbb{H}_\Lambda(\mathcal{E}_X) = R\Gamma_\Lambda(T^*X; \mathcal{H}\mathcal{H}(\mathcal{E}_X)),
\]
\[
\mathbb{H}\mathbb{H}^k_\Lambda(\mathcal{E}_X) = H^k(\mathbb{H}\mathbb{H}_\Lambda(\mathcal{E}_X)) = H^k_\Lambda(T^*X; \mathcal{H}\mathcal{H}(\mathcal{E}_X)).
\]

We call $\mathcal{H}\mathcal{H}_\Lambda(\mathcal{E}_X)$ the Hochschild homology of $\mathcal{E}_X$ with support in $\Lambda$.

The morphisms in (5.13) define a class
\[
\text{hh}_\mathcal{E}(\mathcal{M}) \in \mathbb{H}\mathbb{H}_{\text{char}(\mathcal{E})}(\mathcal{E}_X)
\]
that we call the Hochschild class of $\mathcal{M}$.

Let $S$ be a closed subset of $X$. By restricting to the zero-section $X$ of $T^*X$ the above construction, we obtain the Hochschild homology of $\mathcal{D}_X$:
\[
\mathcal{H}\mathcal{H}(\mathcal{D}_X) = (\delta_X)^{-1} R\mathcal{H}om_{\mathcal{D}_{X\times X}}(\mathcal{B}_\Delta, \mathcal{B}_\Delta) \simeq \mathcal{H}\mathcal{H}(\mathcal{E}_X)|_X,
\]
\[
\mathbb{H}\mathbb{H}_S(\mathcal{D}_X) = R\Gamma_S(X; \mathcal{H}\mathcal{H}(\mathcal{D}_X)),
\]
\[
\mathbb{H}\mathbb{H}^k_S(\mathcal{D}_X) = H^k(\mathbb{H}\mathbb{H}_S(\mathcal{D}_X)) = H^k_S(X; \mathcal{H}\mathcal{H}(\mathcal{D}_X)).
\]

Then, to $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{D}_X)$ one obtains
\[
\text{hh}_{\mathcal{D}}(\mathcal{M}) := \text{hh}_\mathcal{E}(\mathcal{M})|_X \in \mathbb{H}\mathbb{H}_{\text{Supp}(\mathcal{E})}(\mathcal{D}_X).
\]

We shall make a link between the Hochschild class of $\mathcal{M}$ and the microlocal Euler class of a trace kernel attached to the sheaves of holomorphic solutions of $\mathcal{M}$. We need a lemma.

**Lemma 5.9.** For $\mathcal{N}_1$ and $\mathcal{N}_2$ in $D^b_{\text{coh}}(\mathcal{D}_X)$, there exists a natural morphism
\[
R\mathcal{H}om_\mathcal{E}(\mathcal{N}_1^E, \mathcal{N}_2^E) \to \mu\text{hom}(\Omega_X \otimes \mathcal{D}_X \mathcal{N}_1^L, \Omega_X \otimes \mathcal{D}_X \mathcal{N}_2^L).
\]

Moreover, this morphism is compatible with the composition
\[
R\mathcal{H}om_\mathcal{E}(\mathcal{N}_1^E, \mathcal{N}_2^E) \otimes R\mathcal{H}om_\mathcal{E}(\mathcal{N}_2^E, \mathcal{N}_3^E) \to R\mathcal{H}om_\mathcal{E}(\mathcal{N}_1^E, \mathcal{N}_3^E),
\]
\[
\mu\text{hom}(F_1, F_2) \otimes \mu\text{hom}(F_2, F_3) \to \mu\text{hom}(F_1, F_3).
\]

**Proof.** We have the natural morphism in $D^b(\pi^{-1}\mathcal{D}_X \otimes \pi^{-1}\mathcal{D}_X^\text{op})$ (see [KS85, Prop. 10.6.2])
\[
\mathcal{E}_X \to \mu\text{hom}(\Omega_X, \Omega_X).
\]
This gives rise to the morphisms
\[
\begin{align*}
\mathbf{R}\mathbf{H}\mathbf{o}\mathbf{m}_{\pi^{-1}\mathcal{G}_X}(\pi^{-1}\mathcal{N}_1, \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{G}_X} \pi^{-1}\mathcal{N}_2) \\
&\rightarrow \mathbf{R}\mathbf{H}\mathbf{o}\mathbf{m}_{\pi^{-1}\mathcal{G}_X}(\pi^{-1}\mathcal{N}_1, \mu_{\text{hom}}(\Omega_X, \Omega_X)) \otimes_{\pi^{-1}\mathcal{G}_X} \pi^{-1}\mathcal{N}_2 \\
&\simeq \mu_{\text{hom}}(\Omega_X \otimes_{\mathcal{G}_X} \mathcal{N}_1, \Omega_X \otimes_{\mathcal{G}_X} \mathcal{N}_2).
\end{align*}
\]
Q.E.D.

We have
\[
\begin{align*}
\Omega_{X \times X}[-d_X]^L \otimes_{\mathcal{G}_X \times \mathcal{G}_X} \mathcal{B}_\Delta &\simeq \mathbb{C}_\Delta, \\
\Omega_{X \times X}[-d_X]^L \otimes_{\mathcal{G}_X \times \mathcal{G}_X} \mathcal{B}_\Delta' &\simeq \omega_\Delta.
\end{align*}
\]
Applying Lemma 5.9, one deduces the morphisms
\[
\begin{align*}
\mathbf{R}\mathbf{H}\mathbf{o}\mathbf{m}_{\mathcal{E}_{X \times X}}(\mathcal{C}_\Delta; \mathcal{C}_\Delta') &\rightarrow \mu_{\text{hom}}(\Omega_{X \times X}^L \otimes_{\mathcal{G}_X \times \mathcal{G}_X} \mathcal{B}_\Delta; \Omega_{X \times X}^L \otimes_{\mathcal{G}_X \times \mathcal{G}_X} \mathcal{B}_\Delta') \\
&\simeq \mu_{\text{hom}}(\mathbb{C}_\Delta, \omega_\Delta).
\end{align*}
\]
An easy calculation shows that the first arrow is also an isomorphism. Therefore, we get the isomorphism
\[
(5.16) \quad \mathcal{H}\mathcal{H}(\mathcal{E}_X) \simeq \mathcal{M}\mathcal{H}(\mathbb{C}_X).
\]
Recall that the Hochschild homology of \(\mathcal{E}_X\) has been already calculated in [BG87].

Applying the functor \(\Omega_{X \times X}[-d_X]^L \otimes_{\mathcal{G}_X \times \mathcal{G}_X} \cdot\) to (5.11) and (5.12) we get the morphisms
\[
(5.17) \quad \mathbb{C}_\Delta \rightarrow \Omega_{X \times X}^L \otimes_{\mathcal{G}_X \times \mathcal{G}_X} (\mathcal{M} \otimes \mathcal{D}_\mathcal{G}) \rightarrow \omega_\Delta.
\]

**Notation 5.10.** For \(\mathcal{M} \in \mathbf{D}^b_{\text{coh}}(\mathcal{G}_X)\), we denote by TK(\(\mathcal{M}\)) the trace kernel given by (5.17).

Since \(\text{char}(\mathcal{M}) = \text{SS}(\mathbf{R}\mathbf{H}\mathbf{o}\mathbf{m}_{\mathcal{G}_X}(\mathcal{M}, \mathcal{E}_X))\) by [KS90, Th. 11.3.3], we get that \(\mu_{\text{eu}}(\text{TK}(\mathcal{M}))\) is supported by \(\text{char}(\mathcal{M})\), the characteristic variety of \(\mathcal{M}\).
Proposition 5.11. After identifying $\mathcal{HH}(\mathcal{E}_X)$ and $\mathcal{MH}(\mathbb{C}_X)$ by the isomorphism (5.16), we have $\text{hh}_e(\mathcal{M}) = \mu \text{eu}_X(\text{TK}(\mathcal{M}))$ in $\mathbb{H}^0_{\text{char}(\mathcal{M})}(\mathbb{C}_X)$.

Proof. This follows from Lemma 5.9 applied to (5.13). Q.E.D.

Note that the class $\mu \text{eu}_X(\text{TK}(\mathcal{M}))$ coincides with the microlocal Euler class of $\mathcal{M}$ already introduced by Schapira-Schneiders in [ScSn94].

6 Operations on microlocal Euler classes I

In this section, we shall adapt to trace kernels the constructions of [KS12, Chap. 4 §3] and we shall show that under natural microlocal conditions of properness, the microlocal Euler class of the composition of two kernels is the composition of the classes.

We use Notations 3.1 and we consider a trace kernel $(K, u, v)$ on $M_{12}$.

Lemma 6.1. Let $K$ be a trace kernel on $M_{12}$. There are natural morphisms in $D^b(k_{M_{11}})$:

\begin{align*}
(6.1) \quad k_{\Delta_{13}} & \to K \ast (\omega_{\Delta_2}^{-1} \boxtimes k_{\Delta_3}), \\
(6.2) \quad K \circ (k_{\Delta_2} \boxtimes \omega_{\Delta_3}) & \to \omega_{\Delta_{13}}.
\end{align*}

Proof. (i) By Lemma 4.3 (ii) we have a morphism $k_{\Delta_{13}} \to k_{\Delta_{12}} \ast (\omega_{\Delta_2}^{-1} \boxtimes k_{\Delta_3})$. By composing this morphism with $k_{\Delta_{12}} \to K$, we get (6.1).

(ii) By Lemma 4.3 (i) we have a morphism $\omega_{\Delta_{12}} \circ (k_{\Delta_2} \boxtimes \omega_{\Delta_3}) \to \omega_{\Delta_{13}}$. By composing this morphism with $K \to \omega_{\Delta_{12}}$ we get (6.2). Q.E.D.

Let $K$ be a trace kernel on $M_{12}$ with microsupport $\text{SS}(K)$ contained in a closed conic subset $\Lambda_{1122}$ of $T^*M_{1122}$ and let $\Lambda_{23}$ a closed conic subset of $T^*M_{23}$. We assume

\begin{equation}
(6.3) \quad \Lambda_{1122} \times_{\Delta_{22}} \cdot_{T^*M_{23}} \Lambda_{23} \text{ is proper over } T^*M_{1133}.
\end{equation}
We set
\[
\begin{align*}
\Lambda_{12} &:= \Lambda_{1122} \cap T_{\Delta_{12}}^* M_{1122}, \\
\Lambda_{133} &:= \Lambda_{1122} \overset{\varphi_{22}}{\circ} \delta_{T^* M_{22}} \Lambda_{23}, \\
\Lambda_{13} &:= \Lambda_{1133} \cap T_{\Delta_{13}}^* M_{1133} = \Lambda_{12} \overset{\varphi_{2}}{\circ} \Lambda_{23}.
\end{align*}
\]

(6.4)

We define a map
\[
\Phi_K : \mathcal{MH}_{\Lambda_{23}}(k_{23}) \rightarrow \mathcal{MH}_{\Lambda_{13}}(k_{13})
\]
by the sequence of morphisms
\[
\begin{align*}
\mathcal{MH}_{\Lambda_{23}}(k_{23}) &\simeq R\Gamma_{\delta_{T^* M_{22}} \Lambda_{23}}(T^* M_{22}; \mu hom(k_{\Delta_{23}}, \omega_{\Delta_{23}})) \\
&\simeq R\Gamma_{\delta_{T^* M_{22}} \Lambda_{23}}(T^* M_{22}; \mu hom(\omega_{\Delta_{2}} \overset{L}{\boxtimes} k_{\Delta_{3}}, k_{\Delta_{2}} \overset{L}{\boxtimes} \omega_{\Delta_{3}})) \\
&\rightarrow R\Gamma_{\Lambda_{1133}}(T^* M_{1133}; \mu hom(K, K) \overset{\varphi_{22}}{\circ} \mu hom(\omega_{\Delta_{2}} \overset{L}{\boxtimes} k_{\Delta_{3}}, k_{\Delta_{2}} \overset{L}{\boxtimes} \omega_{\Delta_{3}})) \\
&\rightarrow R\Gamma_{\Lambda_{1133}}(T^* M_{1133}; \mu hom(K \overset{L}{\boxtimes} k_{\Delta_{13}}, k_{\Delta_{2}} \overset{L}{\boxtimes} \omega_{\Delta_{13}})) \\
&\rightarrow \mu eu_{M_{12}}(K) \overset{\varphi_{12}}{\circ} \mu eu_{M_{12}}(K) \overset{\varphi_{12}}{\circ} \mu eu_{M_{12}}(K) \overset{\varphi_{12}}{\circ} \mu eu_{M_{12}}(K) \overset{\varphi_{12}}{\circ} \mu eu_{M_{12}}(K).
\end{align*}
\]

Here the first arrow is given by id$_K$, the second is given by Proposition 3.2, and the last arrow is induced by the morphisms in Lemma 6.1.

The next result is similar to [KS12, Th. 4.3.5].

**Proposition 6.2.** Let $\Lambda_{1122} \subset T^* M_{1122}$ and $\Lambda_{23} \subset T^* M_{23}$ be closed conic subsets satisfying (6.3) and recall the notation (6.4). Let $K$ be a trace kernel on $M_{12}$ with microsupport contained in $\Lambda_{1122}$. Then the map $\Phi_K$ in (6.5) is the map $\mu eu_{M_{12}}(K) \overset{\varphi_{12}}{\circ} \mu eu_{M_{12}}(K)$ given by Corollary 4.5.

**Proof.** By using the morphism $k_{\Delta_{12}} \rightarrow K$, we find the commutative diagram below:

\[
\begin{array}{ccc}
R\Gamma_{\Lambda_{23}}(T^* M_{22}; \mu hom(k_{\Delta_{21}}, \omega_{\Delta_{21}})) & \rightarrow & R\Gamma_{\Lambda_{13}}(T^* M_{1133}; \mu hom(k_{\Delta_{12}} \overset{22}{\circ} k_{\Delta_{21}}, k_{\Delta_{12}} \overset{22}{\circ} \omega_{\Delta_{21}})) \\
\downarrow & & \downarrow \\
R\Gamma_{\Lambda_{1133}}(T^* M_{1133}; \mu hom(K \overset{22}{\circ} k_{\Delta_{21}}, K \overset{22}{\circ} \omega_{\Delta_{21}})) & \rightarrow & R\Gamma_{\Lambda_{13}}(T^* M_{1133}; \mu hom(k_{\Delta_{12}} \overset{22}{\circ} k_{\Delta_{23}}, K \overset{22}{\circ} \omega_{\Delta_{23}})).
\end{array}
\]
By using the morphism $K \to \omega_{\Delta_{12}}$, we get the commutative diagram

$$
\begin{array}{c}
R\Gamma_{\Delta_{23}}(T^*M_{2233}; \mu hom(k_{\Delta_{12}} \circ \omega_{\Delta_{22}}, k_{\Delta_{23}} \circ \omega_{\Delta_{23}})) \\
\longrightarrow \\
\longrightarrow \\
R\Gamma_{\Delta_{13}}(T^*M_{1133}; \mu hom(k_{\Delta_{12}} \circ \omega_{\Delta_{22}}, K \circ \omega_{\Delta_{23}}))
\end{array}
$$

(6.6)

Recall the morphisms in Lemma 4.3:

$$
\omega_{\Delta_{12}} \circ \omega_{22}(k_{\Delta_2} \otimes \omega_{\Delta_3}) \to \omega_{\Delta_{13}}; \quad k_{\Delta_{13}} \to k_{\Delta_{12}} \ast (\omega_{\Delta_2}^{-1} \otimes \omega_{\Delta_3}).
$$

(6.7)

We get the morphisms

$$
w: R\Gamma_{\delta_{T^*M_{13}; \Lambda_{13}}}(T^*M_{1133}; \mu hom(k_{\Delta_{12}} \circ \omega_{\Delta_{22}}, \omega_{\Delta_{13}}) \circ \omega_{\Delta_{23}})
\simeq R\Gamma_{\delta_{T^*M_{13}; \Lambda_{13}}}(T^*M_{1133}; \mu hom(k_{\Delta_{12}} \ast (\omega_{\Delta_2}^{-1} \otimes k_{\Delta_3}), \omega_{\Delta_{12}} \circ \omega_{\Delta_{23}}(k_{\Delta_2} \otimes \omega_{\Delta_3})))
\to R\Gamma_{\delta_{T^*M_{13}; \Lambda_{13}}}(T^*M_{1133}; \mu hom(k_{\Delta_{13}}, \omega_{\Delta_{13}})).
$$

By its construction, the morphism $\mu eu_{M_{12}}(K) \circ w$ is obtained as the composition with the map $w$ of the top row of the diagram (6.6). Since the composition with $w$ of the two other arrows is the morphism $\Phi_K$, the proof is complete.

Theorem 6.3. Let $K_{ij}$ be a trace kernel on $M_{ij}$ with $SS(K_{ij}) \subset \Lambda_{ijj}$. Assume that

$$
\Lambda_{1122} \circ \Lambda_{2233} \text{ is proper over } T^*M_{1133}.
$$

(6.8)

Set $\Lambda_{1133} = \Lambda_{1122} \circ \Lambda_{2233}$ and $\Lambda_{ij} = \Lambda_{ijj} \cap T^*_{\Delta_j} M_{ijj}$.

Then

(a) $K_{13}$ is a trace kernel on $M_{13},$

(b) $\mu eu_{M_{13}}(K_{13}) = \mu eu_{M_{12}}(K_{12}) \circ \mu eu_{M_{23}}(K_{23})$ as elements of $\mathcal{M}_{\Lambda_{13}}(k_{13}).$
In particular, we have \( \Phi_{K_{12}} \circ \Phi_{K_{23}} \simeq \Phi_{K_{13}} \).

**Proof.** (a) The trace kernel \( K_{23} \) defines morphisms

\[
\omega_{\Delta_2}^{(2)} \boxtimes k_{\Delta_3} \to \tilde{K}_{23} \to k_{\Delta_2} \boxtimes \omega_{\Delta_3}.
\]

Assuming (6.8) and using (6.1) and (6.2), we get that \( K_{13} = K_{12} \circ \tilde{K}_{23} \) is a trace kernel on \( M_{13} \).

(b) We get a commutative diagram in which we set \( \lambda_{23} = \mu \text{eu}_{M_{23}}(K_{23}) \in \mathbb{MH}^0(k_{23}) \simeq \text{Hom}(\omega_{\Delta_2}^{(2)} \boxtimes k_{\Delta_3}, k_{\Delta_2} \boxtimes \omega_{\Delta_3}) \):

\[
\begin{array}{ccc}
k_{\Delta_{13}} & \xrightarrow{K_{12} \ast (\omega_{\Delta_2}^{(2)} \boxtimes k_{\Delta_3})} & K_{12} \circ (k_{\Delta_2} \boxtimes \omega_{\Delta_3}) \xrightarrow{\lambda_{23}} \omega_{\Delta_{13}} \\
& \downarrow K_{12} \ast \tilde{K}_{23} & \downarrow \omega_{\Delta_{23}} \\
& K_{12} \circ \tilde{K}_{23} & \\
\end{array}
\]

The composition of the arrows on the bottom is \( \mu \text{eu}_{M_{13}}(K_{13}) \) and the composition of the arrows on the top is \( \Phi_{K_{12}}(\mu \text{eu}_{M_{23}}(K_{23})) \). Hence, the assertion follows from the commutativity of the diagram by Proposition 6.2.

(c) follows from (b) and Proposition 6.2. Q.E.D.

### 7 Operations on microlocal Euler classes II

We shall combine Theorems 4.6 and 6.3 and make more explicit the operations on microlocal Euler classes for direct or inverse images. In particular, applying our results to the case of constructible sheaves, we shall recover the results of [KS90, Ch. IX §5].

Let \( M \) be a manifold and let \( \iota: N \hookrightarrow M \) be closed embedding of a smooth submanifold \( N \). If there is no risk of confusion, we shall still denote by \( k_N \) and \( \omega_N \) the sheaves \( \iota_* k_N \) and \( \iota_* \omega_N \) on \( M \). Then \( k_N \) is cohomologically constructible and moreover

\[
D_M k_N = R \mathcal{H}om(k_N, \omega_M) \simeq \omega_N.
\]
Hence, $\text{TK}(k_N) = k_N \boxtimes \omega_N$ is a trace kernel on $M$.

Let $M_i$ be a manifold ($i = 1, 2$), let $K_i$ be a trace kernel on $M_i$ and let $\Lambda_{ii}$ be a closed conic subset of $T^*M_i$ with $\text{SS}(K_i) \subset \Lambda_{ii}$. We set

$$\Lambda_i = \Lambda_{ii} \cap T^*_\Delta M_{ii}.$$  

For a morphism of manifolds $f : M_1 \to M_2$, we denote by $\Gamma_f$ its graph, a smooth closed submanifold of $M_{12}$ and we set for short

$$\Lambda_f := T^*_\Gamma_f(M_{12}), \quad \tilde{f} = (f, f) : M_{11} \to M_{22}.$$  

Recall the diagram (2.1)

$\begin{align*}
T^*M_1 \xleftarrow{f_d} M_1 \times M_2 & \xrightarrow{f_\pi} T^*M_2 \\
\pi_{M_1} & \quad \pi & \quad \pi_{M_2}
\end{align*}$

$\xrightarrow{f} M_1 \xrightarrow{f} M_2.$

Note that

$$\Lambda_{11} \circ \Lambda_{11} = f_\pi f_d^{-1} \Lambda_{11}, \quad \Lambda_f \circ \Lambda_{22} = f_d f_\pi^{-1} \Lambda_{22}.$$  

In the sequel, we shall identify $M_{1212}$ with $M_{1122}$. We take as kernel the sheaf $\text{TK}(k_{\Gamma_f})$. Then

(7.1) 

$$\text{TK}(k_{\Gamma_f}) = k_{\Gamma_f} \boxtimes \omega_{\Gamma_f} \simeq k_{\Gamma_f} \otimes (k_1 \boxtimes \omega_1 \boxtimes k_{22})$$  

$$\simeq \omega_{\Delta_1 \circ \Lambda_{11}} ((\omega_1 \boxtimes \omega_1 \boxtimes k_{22}) \otimes k_{\Gamma_f}).$$

Moreover, we have (see (5.9)):

$$\mu_{\text{eu}_{M_{12}}}(\text{TK}(k_{\Gamma_f})) = \mu_{\text{eu}_{M_{12}}}(k_{\Gamma_f}).$$

Also note that

$$R\tilde{f}_! K_1 \simeq K_1 \circ k_{\Gamma_f}, \quad \tilde{f}^{-1} K_2 \simeq k_{\Gamma_f} \circ K_2.$$
External product

Applying Theorem 4.6 with $M_2 = \text{pt}$ and $M_3$ being here $M_2$, we get the commutative diagram

\[
\begin{array}{ccc}
\mathcal{MH}_{\Lambda_1}(k_{M_1}) \boxtimes \mathcal{MH}_{\Lambda_2}(k_{M_2}) & \longrightarrow & \mathcal{MH}_{\Lambda_1 \times \Lambda_2}(k_{M_{12}}) \\
\sim & & \sim \\
R\Gamma_{\Lambda_1}(\pi_{M_1}^{-1}\omega_{M_1}) \boxtimes R\Gamma_{\Lambda_2}(\pi_{M_2}^{-1}\omega_{M_2}) & \longrightarrow & R\Gamma_{\Lambda_1 \times \Lambda_2}(\pi_{M_{12}}^{-1}\omega_{M_{12}})
\end{array}
\]

and taking the global sections and the 0-th cohomology,

\[
\begin{array}{ccc}
\mathcal{MH}^0_{\Lambda_1}(k_{M_1}) \otimes \mathcal{MH}^0_{\Lambda_2}(k_{M_2}) & \longrightarrow & \mathcal{MH}^0_{\Lambda_1 \times \Lambda_2}(k_{M_{12}}) \\
\sim & & \sim \\
H^0_{\Lambda_1}(T^*M_1; \pi_{M_1}^{-1}\omega_{M_1}) \otimes H^0_{\Lambda_2}(T^*M_2; \pi_{M_2}^{-1}\omega_{M_2}) & \longrightarrow & H^0_{\Lambda_1 \times \Lambda_2}(T^*M_{12}; \pi_{M_{12}}^{-1}\omega_{M_{12}}).
\end{array}
\]

Applying Theorem 6.3, we obtain

**Proposition 7.1.** The object $K_1 \boxtimes K_2$ is a trace kernel on $M_{12}$ and

\[
\mu_{\text{eu}}_{M_{12}}(K_1 \boxtimes K_2) = \mu_{\text{eu}}_{M_1}(K_1) \boxtimes \mu_{\text{eu}}_{M_2}(K_2).
\]

**Direct image**

Let $f: M_1 \to M_2$ and $\Gamma_f$ be as above. Applying Theorem 4.6 with $M_1 = \text{pt}$ and $M_2, M_3$ being the current $M_1, M_2$, we get the commutative diagram

\[
\begin{array}{ccc}
\mathcal{MH}(k_{M_1}) \circlearrowleft_{1} \mathcal{MH}(k_{M_{12}}) & \longrightarrow & \mathcal{MH}(k_{M_2}) \\
\sim & & \sim \\
\pi_{M_1}^{-1}\omega_{M_1} \circlearrowleft_{1} \pi_{M_{12}}^{-1}\omega_{M_{12}} & \longrightarrow & \pi_{M_2}^{-1}\omega_{M_2}.
\end{array}
\]

Now we assume

(7.2) \quad $f$ is proper on $\Lambda_1 \cap T^*_{M_1} M_1$, or, equivalently, $f_\pi$ is proper on $f_d^{-1}\Lambda_1$. 

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We set

\[ f_\mu(\Lambda_1) = \Lambda_1 \circ f = f_\pi(f_d^{-1}(\Lambda_1)). \]

Taking the global sections and the 0-th cohomology of the diagram above, we obtain the commutative diagram:

\[
\begin{array}{ccc}
\mathbb{M} \mathbb{H}^0_{\Lambda_1}(k_{M_1}) & \xrightarrow{o_{\mu \Gamma}(k_{f_\mu})} & \mathbb{M} \mathbb{H}^0_{f_\mu \Lambda_1}(k_{M_2}) \\
\sim & & \sim \\
H^0_{\Lambda_1}(T^*M_1; \pi_{M_1}^{-1}\omega_{M_1}) & \xrightarrow{o_{\mu \Gamma}(k_{f_\mu})} & H^0_{f_\mu \Lambda_1}(T^*M_2; \pi_{M_2}^{-1}\omega_{M_2}).
\end{array}
\]

We have natural morphism and isomorphisms, already constructed in [KS90]:

\[ f_\pi \pi_! f_d^{-1} \pi_{M_1}^{-1}\omega_{M_1} \simeq f_\pi \pi_! f_d^{-1}\pi_{M_1}^{-1}\omega_{M_1} \simeq f_\pi \pi_! \pi_{M_2}^{-1}\omega_{M_2} \]

It induces a morphism:

\[ f_\mu : R\Gamma_{\Lambda_1}(\pi_{M_1}^{-1}\omega_{M_1}) \to R\Gamma_{f_\mu \Lambda_1}(\pi_{M_2}^{-1}\omega_{M_2}). \]

**Lemma 7.2.** Let \( \lambda \in H^0_{\Lambda_1}(T^*M_1; \pi_{M_1}^{-1}\omega_{M_1}) \). Then \( \lambda \circ \mu_{M_1}(k_{\Gamma_f}) = f_\mu(\lambda) \).

**Proposition 7.3.** Assume that \( \tilde{f} \) is proper on \( \Lambda_{11} \cap T^*_{M_{11}}M_{11} \). Then the object \( R\tilde{f}_!K_1 \) is a trace kernel on \( M_2 \) and

\[ \mu_{M_2}(\tilde{R}\tilde{f}_!K_1) = \mu_{M_2}(K_1) \pi_! \mu_{M_2}(k_{\Gamma_f}) = f_\mu(\mu_{M_2}(K_1)). \]

**Proof.** Note that \( \mu_{M_2}(k_{\Gamma_f}) = \mu_{M_2}((\omega_{M_1}^{-1} L \otimes \omega_1 L \otimes k_{22} L) \otimes TK(k_{\Gamma_f})) \) by Proposition 5.3. We have \( R\tilde{f}_!K_1 \simeq K_1 \circ_{11} \left( \omega_{\Lambda_1}^{-1} \circ \left( \omega_{\Lambda_1}^{-1} \otimes \omega_1 \otimes k_{22} \right) \otimes TK(k_{\Gamma_f}) \right) \). It remains to apply Theorem 6.3 in which one replaces \( M_1, M_2, M_3 \) with \( pt, M_1, M_2 \), respectively. Q.E.D.
Inverse image

Let \( f : M_1 \to M_2 \) and \( \Gamma_f \) be as above. Applying Theorem 4.6 with \( M_3 = \text{pt} \), we get the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}(k_{M_1}) & \xrightarrow{\partial} & \mathcal{M}(k_{M_2}) \\
\sim & & \sim \\
\pi^{-1}_{M_2} \omega_{M_2} & \xrightarrow{\partial} & \pi^{-1}_{M_1} \omega_{M_1} \\
\end{array}
\]

Now we assume

\[(7.3) \quad f \text{ is non-characteristic for } \Lambda_2, \text{ or, equivalently, } f_d \text{ is proper on } f^{-1}_\pi \Lambda_2.\]

We set

\[ f^\mu(\Lambda_2) = \Lambda_f \circ \Lambda_1 = f_d(f^{-1}_\pi(\Lambda_2)). \]

Taking the global sections and the 0-th cohomology of the diagram above, we obtain the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M}_0(k_{M_2}) & \xrightarrow{\partial} & \mathcal{M}_0(k_{M_1}) \\
\sim & & \sim \\
H^0_\Lambda_2(T^*M_2; \pi^{-1}_{M_2} \omega_{M_2}) & \xrightarrow{\partial} & H^0_\mu\Lambda_2(T^*M_1; \pi^{-1}_{M_1} \omega_{M_1}). \\
\end{array}
\]

We have a natural morphism constructed in the proof of [KS90, Prop. 9.3.2]:

\[ f^\mu : f_d f^{-1}_\pi \pi^{-1}_{M_2} \omega_{M_2} \to \pi^{-1}_{M_1} \omega_{M_1}. \]

Hence, we get a map:

\[ f^\mu : \text{R} \Gamma_{\Lambda_2}(\pi^{-1}_{M_2} \omega_{M_2}) \to \text{R} \Gamma_{\mu \Lambda_2}(\pi^{-1}_{M_1} \omega_{M_1}). \]

**Lemma 7.4.** Let \( \lambda \in H^0_{\Lambda_1}(T^*M_2; \pi^{-1}_{M_2} \omega_{M_2}) \). Then \( \mu \text{eu}_{M_12}(k_{\Gamma_f}) \circ \lambda = f^\mu(\lambda) \).
Proposition 7.5. Assume that $\tilde{f}$ is non characteristic with respect to $\Lambda_{22}$. Then the object $(k_1 \boxtimes \omega_{M_1/M_2}) \otimes \tilde{f}^{-1}K_2$ is a trace kernel on $M_1$ and

$$
\mu e_{M_1}(\omega_{\Lambda_1} \circ \tilde{f}^{-1}(\omega_{\Lambda_2}^{-1} \circ K_2)) = \mu e_{M_2}(\omega_{\Lambda_1} \circ \tilde{f}^{-1}(\omega_{\Lambda_2}^{-1} \circ K_2)) = f^\mu(\mu e_{M_2}(K_2)).
$$

Proof. Applying Theorem 6.3 with $M_3 = pt$, we get that

$$(k_1 \boxtimes \omega_{M_1/M_2}) \otimes \tilde{f}^{-1}K_2 \simeq TK(k_f) \circ_2 (\omega_{\Lambda_2}^{-1} \circ (k_1 \boxtimes \omega_{M_1/M_2}) \otimes K_2)$$

is a trace kernel. Since $\mu e_{M_2}((\omega_2 \boxtimes \omega_{M_2}^{-1}) \otimes K_2) = \mu e_{M_2}(K_2)$ by Proposition 5.3, we obtain the result. Q.E.D.

Tensor product

Consider now the case where $M_1 = M_2 = M$ and the $\Lambda_i$'s satisfy the transversality condition

$$(7.4) \quad \Lambda_{11} \cap \Lambda_{22} \subset T^*_M(M \times M).$$

Then by composing the external product with the restriction to the diagonal, we get a convolution map

$$(7.5) \quad \star: MH_{\Lambda_1}(k_M) \times MH_{\Lambda_2}(k_M) \to MH_{\Lambda_1+\Lambda_2}(k_M).$$

Applying Propositions 7.1 and 7.5, we get:

Proposition 7.6. Assume (7.4). Then the object $K_1 \otimes (k_M \boxtimes \omega_{M}^{-1}) \otimes K_2$ is a trace kernel on $M$ and

$$
\mu e_M(K_1 \otimes (k_M \boxtimes \omega_{M}^{-1}) \otimes K_2) = \mu e_M(K_1) \star \mu e_M(K_2).
$$

Following [ScSn94, II Cor. 5.6], we shall recall the link between the product $\star$ and the cup product.

Proposition 7.7. Let $\lambda_i \in H^0_{\Lambda_i}(T^*M_i; \pi^{-1}_M\omega_M)$ ($i = 1, 2$), and $\Lambda_1 \cap \Lambda_2 \subset T^*_M(M$. Then

$$(7.6) \quad (\lambda_1 \star \lambda_2)|_M = \int_{\pi_M} (\lambda_1 \cup \lambda_2)$$

as elements of $H^0_{\pi(\Lambda_i \cap \Lambda_2)}(M; \omega_M)$. 

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Proof. Denote by \( \delta : \Delta \hookrightarrow M_{12} = M \times M \) the diagonal embedding and let us identify \( M \) with \( \Delta \). Consider the diagram

\[
\begin{array}{ccc}
T^*_{\Delta}M_{12} & \longrightarrow & \Delta \times_{M_{12}} T^*M_{12} \\
\pi \downarrow & & \downarrow \delta_d \\
\Delta & \longrightarrow & T^*\Delta
\end{array}
\]

where \( \pi \) is the projection, \( \delta_d \) is the map associated with \( \delta \), \( s \) is the zero-section embedding and \( f \) is the restriction to \( \Delta \times M T^*M_{12} \) of the embedding \( T^*_{\Delta}M_{12} \hookrightarrow T^*M_{12} \). Since this diagram is Cartesian, we have

\[
s^{-1}\delta_d! \simeq \pi_! f^{-1}.
\]

Now let \( \lambda_1 \times \lambda_2 \in H^0_{\Lambda_1 \times \Lambda_2}(T^*M_{12}; \pi^{-1}\omega_{M_{12}}) \) and denote by \( \lambda_1 \times_M \lambda_2 \) its image by the map

\[
H^0_{\Lambda_1 \times \Lambda_2}(T^*M_{12}; \pi^{-1}\omega_{M_{12}}) \rightarrow H^0_{\Lambda_1 \times_M \Lambda_2}(\Delta \times_{M_{12}} T^*M_{12}; \pi^{-1}\omega_{M_{12}}).
\]

(Here, on the right hand side, we still denote by \( \pi \) the restriction of the projection \( \pi_{M_{12}} \) to \( \Delta \times_{M_{12}} T^*M_{12} \).) Then

\[
\int_\pi (\lambda_1 \cup \lambda_2) = \pi_! f^{-1}(\lambda_1 \times_M \lambda_2),
\]

\[
(\lambda_1 \star \lambda_2)|_M = s^{-1}\delta_d!(\lambda_1 \times_M \lambda_2).
\]

Q.E.D.

Corollary 7.8. Let \( K_1 \) and \( K_2 \) be two trace kernels on \( M \) with \( SS(K_i) \subset \Lambda_{ii} \). Assume (7.4) and assume moreover that \( \text{Supp}(K_1) \cap \text{Supp}(K_2) \) is compact. Then the object \( \text{R}^\Gamma(M \times M; K_1 \otimes (k_M \boxtimes \omega^\otimes -1 \otimes K_2) \otimes K_2) \) is a trace kernel on \( pt \) and

\[
\text{eu}_{pt}(\text{R}^\Gamma(M; K_1 \otimes (k_M \boxtimes \omega^\otimes -1 \otimes K_2) \otimes K_2) = \int_{T^*M} \text{eu}_{M}(K_1) \cup \text{eu}_{M}(K_2).
\]

Remark 7.9. Let \( M \) be a real analytic manifold and let \( F \in \mathbb{D}_{R-c}^0(k_M) \). Recall that one associates to \( F \) the trace kernel \( \text{TK}(F) = F \boxtimes D_MF \) and that \( \text{mu}_{M}(F) = \text{mu}_{M}(\text{TK}(F)) \). Assume now that \( f : M_1 \rightarrow M_2 \) is a morphism of real analytic manifolds.
Let $F_1 \in D_{b,c}^b(k_{M_1})$ and assume that $f$ is proper on $\text{Supp}(F_1)$. Applying Proposition 7.3 and noticing that

\[ R\tilde{f}_1 \text{TK}(F_1) \simeq \text{TK}(Rf_1 F_1), \]

we find that $\mu\text{eu}(Rf_1 F_1) = f_\mu(\mu\text{eu}(F_1))$. It is nothing but [KS90, Prop. 9.4.2].

Let $F_2 \in D_{b,c}^b(k_{M_2})$ and assume that $f$ is non characteristic with respect to $F_2$. Applying Proposition 7.5 and noticing that

\[ \text{TK}(f^{-1} F_2) \simeq (k_1 \boxtimes \omega_{M_1/M_2}) \otimes \tilde{f}^{-1} \text{TK}(F_2), \]

we find that $\mu\text{eu}(f^{-1} F_2) = f^\mu(\mu\text{eu}(F_2))$. Hence, we recover [KS90, Prop. 9.4.3].

8 Applications: $\mathcal{D}$-modules and elliptic pairs

As an application of Theorem 6.3, we shall recover the theorem of [ScSn94] on the index of elliptic pairs. In this section, $X$ is a complex manifold, $k = \mathbb{C}$, $\mathcal{M}$ is an object of $D_{\text{coh}}^b(\mathcal{D}_X)$ and $F$ is an object of $D_{b,c}^b(\mathcal{O}_X)$.

Recall that we have denoted by $\text{TK}(F)$ and $\text{TK}(\mathcal{M})$ (see Notation 5.10) the trace kernels associated with $F$ and with $\mathcal{M}$, respectively:

\[
\text{TK}(F) := F \boxtimes D_X F, \\
\text{TK}(\mathcal{M}) := \Omega_{X \times X} \otimes \omega_{\mathcal{M}/\mathcal{M}} (\mathcal{M} \boxtimes D_{\mathcal{M}}). 
\]

The pair $(\mathcal{M}, F)$ is called an elliptic pair in loc. cit. if $\text{char}(\mathcal{M}) \cap \text{SS}(F) \subset T^*_X X$. From now on, we assume that $(\mathcal{M}, F)$ is an elliptic pair.

It follows from Proposition 7.6 that the tensor product of $\text{TK}(F)$ and $\text{TK}(\mathcal{M})$ shifted by $-2d_X$ is again a trace kernel. We denote it by $\text{TK}(\mathcal{M}, F)$. Hence

\[
\text{TK}(\mathcal{M}, F) \simeq \Omega_{X \times X} \otimes \omega_{X \times X} (\mathcal{M} \boxtimes D_{\mathcal{M}}) \otimes (F \boxtimes D_X F). 
\]

Moreover the same statement gives:

\[
\mu\text{eu}_X (\text{TK}(\mathcal{M}, F)) = \mu\text{eu}_X (\mathcal{M}) \ast \mu\text{eu}_X (F). 
\]
We set

\[ \text{Sol}(\mathcal{M}, F) := \text{RHom}_{\mathcal{O}_X}(\mathcal{M} \otimes F, \mathcal{O}_X), \]

(8.3)

\[ \text{DR}(\mathcal{M}, F) := \Gamma(X; \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \otimes F)[d_X]. \]

(8.4)

As explained in [ScSn94], Theorem [KS90, Th 11.3.3] and isomorphism (2.7) provides a generalization of the classical Petrovsky regularity theorem, namely, the natural isomorphisms

\[ \text{RHom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{D}_X' F \otimes \mathcal{O}_X) \xrightarrow{\sim} \text{RHom}_{\mathcal{O}_X}(\mathcal{M} \otimes F, \mathcal{O}_X). \]

(8.5)

Now assume that \( \text{Supp}(\mathcal{M}) \cap \text{Supp}(F) \) is compact and let us take the global sections of the isomorphism (8.5). We find the isomorphism

\[ \text{RHom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{D}_X' F \otimes \mathcal{O}_X) \xrightarrow{\sim} \text{RHom}_{\mathcal{O}_X}(\mathcal{M} \otimes F, \mathcal{O}_X). \]

(8.6)

It is proved in [ScSn94] \(^1\) that one can represent the left hand side of (8.6) by a complex of topological vector spaces of type DFN and the right hand side of (8.6) by a complex of topological vector spaces of type FN. It follows that the complexes \( \text{Sol}(\mathcal{M}, F) \) and \( \text{DR}(\mathcal{M}, F) \) have finite dimensional cohomology and are dual to each other. More precisely, denoting by \( (\cdot)^* \) the duality functor in \( \mathcal{D}^b_f(C) \), we have

\[ (\text{Sol}(\mathcal{M}, F))^* \simeq \text{DR}(\mathcal{M}, F). \]

It follows from the finiteness of the cohomology of the complexes \( \text{Sol}(\mathcal{M}, F) \) and \( \text{DR}(\mathcal{M}, F) \) that

\[ \Gamma(X \times X; \mathcal{K}(\mathcal{M}, F)) \simeq \text{Sol}(\mathcal{M}, F) \otimes \text{DR}(\mathcal{M}, F). \]

One checks that this isomorphism commutes with the composition of the morphisms \( \mathbb{C} \to \Gamma(X \times X; \mathcal{K}(\mathcal{M}, F)) \to \mathbb{C} \) and \( \mathbb{C} \to \text{Sol}(\mathcal{M}, F) \otimes \text{DR}(\mathcal{M}, F) \to \mathbb{C} \), which implies

\[ \text{eu}_\mathbb{p}_{\text{t}}(\Gamma(X \times X; \mathcal{K}(\mathcal{M}, F))) = \chi(\text{Sol}(\mathcal{M}, F)). \]

(8.7)

Therefore, one recovers the index formula of loc. cit.

\[ \chi(\text{RHom}_{\mathcal{O}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)) = \int_X (\mu \text{eu}_X(\mathcal{M}) \ast \mu \text{eu}_X(F))|_X \]

(8.8)

\[ = \int_{T \times X} \mu \text{eu}_X(\mathcal{M}) \cup \mu \text{eu}_X(F). \]

\(^1\)In fact, the finiteness of the cohomology of this complex is only proved in loc. cit. under the hypothesis that \( \mathcal{M} \) admits a good filtration, but this hypothesis may be removed thanks to the results of [KS96, Appendix].
Remark 8.1. In general the direct image of an elliptic pair is no more an elliptic pair. However, it remains a trace kernel.

Remark 8.2. As already mentioned in [ScSn94], formula (8.8) has many applications, as far as one is able to calculate $\mu_{\text{eu}}(\mathcal{M})$ (see the final remarks below). For example, if $M$ is a compact real analytic manifold and $X$ is a complexification of $M$, one recovers the Atiyah-Singer theorem by choosing $F = D'C_M$. If $X$ is a complex compact manifold, one recovers the Riemann-Roch theorem: one takes $F = \mathcal{C}_X$ and if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, one sets $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}$.

9 The Lefschetz fixed point formula

In this section, we shall briefly show how to adapt the formalism of trace kernels to the Lefschetz trace formula as treated in [KS90, § 9.6]. Here we assume that $k$ is a field.

Assume to be given two maps $f, g: N \to M$ of real analytic manifolds, an object $F \in D^b_{\text{Rel}}(kM)$ and a morphism

$$\varphi: f^{-1}F \to g^!F.$$  

(9.1)

Set

$$h = (g, f): N \times N \to M \times M,$$

$$S = \text{Supp}(F), \quad L = h^{-1}(\Delta_M) = \{(x, y) \in N \times N : g(x) = f(y)\},$$

$$i: L \hookrightarrow N \times N,$$

$$T = f^{-1}(S) \cap g^{-1}(S).$$

One makes the assumption

(9.2) \quad The set $T$ is compact.

Then we have the maps

$$\text{R}\Gamma(M; F) \to \text{R}\Gamma_{f^{-1}S}(N; f^{-1}F) \xrightarrow{\varphi} \text{R}\Gamma_T(N; g^!F) \to \text{R}\Gamma(M; F).$$

The composition gives a map

(9.3) \quad \int \varphi: \text{R}\Gamma(M; F) \to \text{R}\Gamma(M; F),$$

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and this map factorizes through $R\Gamma_T(N; g^1 F)$ which has finite-dimensional cohomologies. Hence, we can define the trace $\text{tr}(\int \varphi)$.

We have the chain of morphisms

$$
\begin{align*}
\mathbf{k}_N &\rightarrow R\mathcal{H}om (g^1 F, g^1 F) \\
&\xrightarrow{\cdot} R\mathcal{H}om (f^{-1} F, g^1 F) \simeq \delta_N^1 (g^1 F \boxtimes D_N f^{-1} F) \\
&\simeq \delta_N^1 (g^1 F \boxtimes f^1 D_M F) \simeq \delta_N^1 h^1 (F \boxtimes D_M F).
\end{align*}
$$

We have thus constructed the morphism

$$
\mathbf{k}_{\Delta_N} \rightarrow h^1 (F \boxtimes D_M F).
$$

By using the morphism $F \boxtimes D_M F \rightarrow \omega_{\Delta_M}$ and the isomorphism $h^1 \omega_{\Delta_M} \simeq i_* \omega_L$, we get the morphisms

$$(9.4) \quad \mathbf{k}_{\Delta_N} \rightarrow h^1 (F \boxtimes D_M F) \rightarrow i_* \omega_L$$

in $D^b(k_{N \times N})$. The support of the composition is contained in $\delta_N(T) \cap L$.

**Theorem 9.1** ([KS90, Proposition 9.6.2]). The trace $\text{tr}(\int \varphi)$ coincides with the image of $1 \in \mathbf{k}$ by the composition of the morphisms

$$
\mathbf{k} \rightarrow R\Gamma(N, \mathbf{k}_N) \rightarrow R\Gamma_c(L, \omega_L) \rightarrow \mathbf{k}.
$$

Here the middle arrow is derived from (9.4).

Although (9.4) is not a trace kernel in the sense of Definition 5.1, it should be possible to adapt the previous constructions to the case of $\mathcal{D}$-modules and to elliptic pairs, then to recover a theorem of [Gu96] but we do not develop this point here (see [RTT12] for related results).

**Final remarks**

The microlocal Euler class of constructible sheaves is easy to compute since it is enough to calculate some multiplicities at generic points. We refer to [KS90] for examples.

On the other hand, there is no direct method to calculate the microlocal Euler class of a coherent $\mathcal{D}$-module $\mathcal{M}$ (except in the holonomic case). In
the authors made a precise conjecture relying \( \mu ev_{\mathcal{X}}(\mathcal{M}) \) and the Chern character of the associated graded module (an \( \mathcal{O}_{T^*X} \)-module), and this conjecture has been proved by Bressler-Nest-Tsygan [BNT02].

Similarly, the Hochschild class of coherent \( \mathcal{O}_X \)-modules is usually calculated through the so-called Hochschild-Kostant-Rosenberg isomorphism, but this isomorphism does not commute with proper direct images, and a precise conjecture (involving the Todd class) has been made by Kashiwara in [Ka91] and this conjecture has recently been proved in the algebraic case by Ramadoss [Ra06] and in the general case by Grivaux [Gr09].

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