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Cosmological entropy production and viscous processes in the (1 + 3 + 6)-dimensional space-times

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Cosmological entropy production is studied in the $(1+3+6)$-dimensional space-times consisting of the outer space (the 3-dimensional expanding section) and the inner space (the 6-dimensional section). The inner space expands initially and contracts later. First it is shown how the production of the 3-dimensional entropy S_3 within the horizon is strengthened by the dissipation due to viscous processes between the two spaces, in which we consider the viscosity caused by the gravitational-wave transport. Next it is shown under what conditions we can have the critical epoch when S_3 reaches the value 10^{88} in the Guth level and at the same time the outer space is decoupled from the inner space. Moreover, the total entropy S_9 in the 9-dimensional space at the primeval expanding stage is also shown corresponding to S_3.

Subject Index E75, E80, E84

1. Introduction

Our observed Universe consists of 4-dimensional space-time, and its 3-dimensional spatial section is very isotropic, homogeneous, and flat. According to the super-string theories, on the other hand, the space-time originally has 10 $(=1+9)$ dimensions and our Universe is considered as a partial section of the total space-time after its evolution. In order that the section may be our observed Universe, it must satisfy the famous cosmological condition that it has the vast entropy $\sim 10^{88}$ within the horizon-size region [1].

Recently, the evolution of the space-time was analyzed by Kim et al. [2,3] in a matrix model of super-string theory and it was shown that owing to the dimensional symmetry-breaking the total 9-dimensional space is separated into the outer space (the 3-dimensional expanding section) and the inner space (the 6-dimensional expanding section), and that the expansion rate of the inner space is smaller than that of the outer space. As suggested by them, this may show the beginning of the separation of our $(1+3)$-dimensional Universe from the other section. In order that our Universe may form in this direction, the evolution of the inner space must tend from the expanding phase to the contracting phase, collapse, and finally decouple from the expanding outer space, while the outer space inflates and tends to the Friedman phase. A similar scenario of such a dynamic evolution of anisotropic multi-dimensional space-times was studied in the form of Kaluza–Klein models in 1980–1990 [4–7]. The cosmological entropy problem was also discussed in many papers [8–14] in the framework of classical relativity. Kolb et al. [12] paid attention to the freeze-out epoch and found it is impossible, if the dimension of the inner space is less than 16, that at that epoch we obtain the

3-dimensional entropy within the horizon in the Guth level and the outer space is decoupled from the inner space. Abbott et al. [9,10] showed that it is possible, if the dimension of the inner space is $\sim$40.

In this paper we consider the entropy production which is obtained at an epoch different from the freeze-out epoch, so as to avoid Kolb et al.'s and Abbott et al.'s above results. For this purpose we show in Sect. 2 our previous treatment [14] for the dynamics of multi-dimensional space-times. As for the initial condition, the multi-dimensional universe is assumed to start from the state of nearly isotropic expansion, in the same way as Kolb et al.'s and Abbott et al.'s treatments, but in a different way from that in our previous one [14], in which we treated only highly anisotropic cases. We include imperfect fluid with viscosity caused by the transport of gravitational waves, as well as the perfect fluid. In Sect. 3 we describe the difference between the freeze-out epoch (t_*), the decoupling (or stabilization) epoch, and the epoch ($t_\dagger$) when the 3-dimensional entropy S_3 within the horizon reaches the critical value 10^{88}. In Sect. 4 we study first the behaviors of S_3 and the total entropy S, and compare them in two cases with non-viscous and viscous fluids. The role of viscosity is found to be so strong as to bring vast entropies at the final stage in the collapse of the compact inner space and the inflation of the outer space. Next, using approximate power solutions (at the final stage), the condition (A) for $S_3 \sim 10^{88}$ is derived, together with the condition (B) that the outer space should be decoupled from the inner space, and the compatibility of the two conditions A and B at epoch $t_\dagger$ (different from t_*) is shown.

In Sect. 5, moreover, we solve numerically the differential evolution equations for scale factors from the initial singular epoch to the final epoch, taking account of viscosity due to the gravitational-wave transport, and derive the model parameters satisfying the above two conditions A and B, by comparing their solutions (at the later stage) with the asymptotic power solutions. In Sect. 6 we derive the (9-dimensional) primeval total entropy S_9, and it is found that the primeval entropy in the viscous case is much smaller than that in the non-viscous case. In Sect. 7, the epoch which may cause the dimensional symmetry-breaking is discussed from the viewpoint of energy conservation, and the possible epoch of symmetry-breaking is estimated. In Sect. 8 concluding remarks are given. In Appendix A the treatment of imperfect fluids is shown, in Appendix B the Planck length in the outer space is derived in connection with the sizes of these two spaces, and in Appendix C the relations between S_3 and S_9 are derived.

2. Multi-dimensional space-times with viscous fluid

First we assume that the fluid consists of massless particles with the energy density ϵ and the pressure p expressed as

$$\epsilon = \mathcal{N} a_n T^{4+n} \quad \text{and} \quad p = \epsilon/(3+n), \tag{1}$$

where T is the temperature, $n(=6)$ is the dimension of the inner space, a_n is the $(4+n)$-dimensional Stefan–Boltzmann constant defined by

$$a_n \equiv (3+n)\, \Gamma((4+n)/2)\, \zeta(4+n)/\pi^{(4+n)/2}, \tag{2}$$

and $\mathcal{N}$ is the number of particle species. The units $c = \hbar = k$ (the Boltzmann constant) $= 1$ are used.

Since the total entropy S within a comoving volume V is given from the second law of thermodynamics by

$$T\mathrm{d}S = V[\mathrm{d}\epsilon + (\epsilon + p)\mathrm{d}V/V], \tag{3}$$

S is expressed as

$$S/V = [(4+n)/(2+n)]\mathcal{N} a_n (\epsilon/\mathcal{N} a_n)^{(3+n)/(4+n)}. \tag{4}$$

As viscous quantities, we have the shear viscosity η and the bulk viscosity ζ, but ζ vanishes for the fluid of massless particles. In imperfect fluids, propagating gravitational waves are absorbed in multi-dimensional universes as well as in 4-dimensional universes. The corresponding η is expressed as

$$\kappa\eta = \eta_0\sqrt{\kappa\epsilon}, \tag{5}$$

where

$$\eta_0 \equiv \left[\frac{4+n}{2(3+n)(5+n)}\frac{\mathcal{N}_r}{\mathcal{N}}\right]^{1/2}, \tag{6}$$

$G = \kappa/8\pi$ is a $(4+n)$-dimensional gravitational constant, and $\mathcal{N}_r$ is the number of radiative particle species absorbing gravitational waves. We consider states so hot that there are many interactions between particles, and so we assume $\mathcal{N}_r/\mathcal{N} = 1$ for simplicity. The definition of η and ζ and the derivation of Eq. (6) are shown in Appendix A.

The space-times are described by the line element

$$\begin{aligned} ds^2 &= g_{MN}dx^M dx^N \\ &= -c^2 dt^2 + f(t)^2 g_{ij}dx^i dx^j + h(t)^2 g_{\alpha\beta}dx^\alpha dx^\beta, \end{aligned} \tag{7}$$

where $M, N = 0, \ldots, (3+n)$, $i, j = 1, 2, 3$, and $\alpha, \beta = 4, \ldots, (3+n)$. The spaces with metrics g_{ij} and $g_{\alpha\beta}$ are the spaces with constant curvatures k_f and k_h ($= 0$ or ± 1). The Einstein equations are

$$R_{MN} - \frac{1}{2}Rg_{MN} = \kappa T_{MN}, \tag{8}$$

where $G = \kappa/8\pi$ is a $(4+n)$-dimensional gravitational constant. The energy–momentum tensor T_{MN} for fluids with energy density ϵ, pressure p, and viscosity η is defined in Appendix A, and their components are expressed under the comoving condition ($u^L = \delta^L_0$) as

$$\begin{aligned} T^0_0 &= -\epsilon, \quad T^0_M = 0 \quad (M \neq 0), \\ T^M_N &= p'\delta^M_N - \eta\kappa^M_N \quad (M, N \neq 0), \end{aligned} \tag{9}$$

where $\kappa^i_j = 2(\dot f/f)\delta^i_j$, $\kappa^\alpha_\beta = 2(\dot h/h)\delta^\alpha_\beta$ (an overdot denoting $\partial/\partial t$), and

$$p' \equiv p - \left[\zeta - \frac{2}{3+n}\eta\right]\dot V/V, \tag{10}$$

where $\dot V/V = 3\dot f/f + n\dot h/h$. Then the above Einstein equations are expressed as

$$\begin{aligned} \ddot f/f &= -2(\dot f^2 + k_f)/f^2 - n\dot f\dot h/(fh) - 2\kappa\eta\dot f/f + \kappa A/(2+n), \\ \ddot h/h &= -(n-1)(\dot h^2 + k_h)/h^2 - 3\dot f\dot h/(fh) - 2\kappa\eta\dot h/h + \kappa A/(2+n), \end{aligned} \tag{11}$$

where

$$A \equiv \epsilon - p + \left[\zeta + \frac{2(2+n)}{3+n}\eta\right]\dot V/V, \tag{12}$$

and

$$\kappa\epsilon = 3(\dot f^2 + k_f)/f^2 + \frac{1}{2}n(n-1)(\dot h^2 + k_h)/h^2 + 3n\dot f\dot h/(fh). \tag{13}$$

From the above equations we can get

$$\dot{\epsilon} = -(\epsilon + p)\dot{V}/V + \left[\zeta + \frac{2(2+n)}{3+n}\eta\right](\dot{V}/V)^2 - 2\eta\{3(\dot{f}/f)(2\dot{f}/f + n\dot{h}/h) + n(\dot{h}/h)[3\dot{f}/f + (n-1)\dot{h}/h]\}. \tag{14}$$

For the change in the total entropy S, we obtain from Eqs. (4) and (14)

$$(\epsilon + p)\dot{S}/S = \zeta(\dot{V}/V)^2 + \frac{6n}{(3+n)}\eta(\dot{f}/f - \dot{h}/h)^2 > 0. \tag{15}$$

Evidently $S =$ const for the non-viscous case ($\eta = \zeta = 0$).

It is assumed that initially the universe expands isotropically, but the expansion of the n-dimensional inner space is slower than that of the 3-dimensional outer space. At the later anisotropic stage the inner space contracts, while the outer space continues to expand. These behaviors correspond to the solutions with $k_h = 1$ and $k_f = 0$ or -1. Their solutions can be derived solving Eq. (11) numerically, but, paying attention to their early isotropic stage and the later anisotropic stage (after the epoch of maximum expansion of the inner space), we can use approximate power solutions which are derived in the following, neglecting the curvature terms. They are expressed as

$$h = h_I(t - t_I)^{\mu}, \quad f = f_I(t - t_I)^{\nu} \tag{16}$$

with $\mu = \nu$ at the isotropic stage, and

$$h = h_A(t_A - t)^{\mu}, \quad f = f_A(t_A - t)^{\nu} \tag{17}$$

with $\mu \neq \nu$ at the anisotropic stage, where h_I, f_I, t_I, h_A, f_A, and t_A are constants. The times t_I and t_A represent the initial epoch of the isotropic stage and the final epoch of the anisotropic stage, respectively. From Eq. (11), we obtain the solutions expressed as

$$\begin{aligned} (n\mu + 3\nu - 1)(n\mu + 3\nu) - H &= 0, \\ (\mu - \nu)(n\mu + 3\nu - 1 - 2\eta_0\sqrt{H}) &= 0, \end{aligned} \tag{18}$$

where $H \equiv 3\nu^2 + \frac{1}{2}n(n-1)\mu^2 + 3n\mu\nu$.

In the isotropic case, the solutions of these equations and the energy density are

$$\mu = \nu = 2/(4+n), \qquad \kappa\epsilon = H(t - t_I)^{-2} \ (> 0). \tag{19}$$

In the anisotropic case, which we consider from now on, they are

$$\begin{aligned} \mu &= (1 + \xi/n)[(3+n)(1 - 4{\eta_0}^2)]^{-1}, \\ \nu &= (1 - \xi/3)[(3+n)(1 - 4{\eta_0}^2)]^{-1} \end{aligned} \tag{20}$$

with

$$\xi^2 \equiv n^2 + 2n(3+n)(1 - 12{\eta_0}^2), \tag{21}$$

and

$$\kappa\epsilon = H(t_A - t)^{-2}, \tag{22}$$

where $H = 4{\eta_0}^2/(1 - 4{\eta_0}^2)$. For viscous fluid and $n = 6$, we have $\mu = 0.345$, $\nu = -0.272$, and $\eta_0 = 0.225$, where Eq. (6) was used. Moreover, $H > 0$ and so Eq. (22) is applicable. For non-viscous fluid ($\eta_0 = 0$), we have $\mu = -\nu = 1/3$, but $H = 0$, so that Eq. (22) is not applicable, and we must take account of higher terms in f and h due to the curvature terms to derive ϵ. For $\eta_0 = 0$, on the other hand, S is constant and so, from Eq. (4),

$$\epsilon \propto V^{-(4+n)/(3+n)} \propto (t_A - t)^{-\alpha}, \tag{23}$$

where $\alpha \equiv (n\mu + 3\nu)(4+n)/(3+n) = 10/9$ for $n = 6$.

The temperature T ($\propto \epsilon^{1/(n+4)}$) is expressed as

$$T \propto (t_A - t)^{-1/5}, \quad (t_A - t)^{-1/9} \tag{24}$$

for $\eta_0 \neq 0, = 0$, respectively. Accordingly, the temperature change for $\eta_0 \neq 0$ is very rapid, compared with that for $\eta_0 = 0$.

Around the epoch when the inner space takes maximum expansion, the above power solution cannot be used and curvature terms must be considered. Here we specify the epoch t_M as the epoch which is comparatively near the epoch of maximum expansion but when the power solutions are approximately applicable. In Sect. 4 we consider the anisotropic stage after t_M for simplicity and use the approximate power solutions. In Sects. 5, 6, and 7 we treat the solutions applicable at all epochs and compare them with the approximate power solutions.

3. Epochs of freeze-out and decoupling of the inner space

While the inner space continues to collapse, the outer space inflates. In order that the expansion of the outer space may change to the Friedman-like slow one, the inner space must be separated from the outer space (see Fig. 1). This epoch is called the decoupling (or stabilization) epoch. At this epoch the size of the inner space is estimated to be comparable with or smaller than the Planck length in the outer space, so that the space-time may be completely quantized on that scale in the outer space.

Around this epoch, we may have an epoch when the physical state in the inner space may change. Kolb et al. [12] suggested that classical treatments of space-time and fluids in the inner space may be questionable around the epoch (t_*) when $hT = 1$, in which h represents the size of the inner space and T is related to ϵ by $\epsilon = T^{4+n}$ ($n = 6$). This is because h is comparable with the mean wavelength of massless particles. This epoch (t_*) is called the freeze-out epoch. Kolb et al. assumed that t_* is near the decoupling epoch, and derived the 3-dimensional entropy S_3 within the horizon. But they

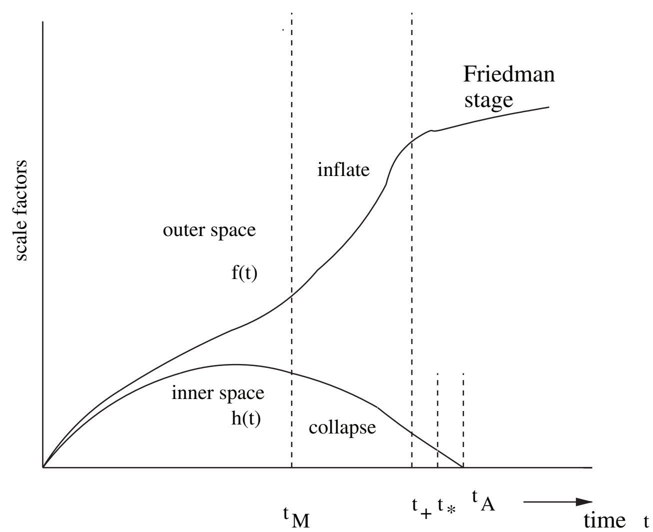

Fig. 1. Scale factors of outer and inner spaces.

found it is impossible for it to reach the expected value $\sim 10^{88}$ in the Guth level [1] if the dimension n of the inner space is less than 16. Abbott et al. [9,10] found that at t_* it reaches the expected value, if n is ~ 40.

In the next section we analyze the entropies at $t_\dagger$ (which is different from t_*), and derive the condition (A) for $S_3 = 10^{88}$ using classical relativity. Moreover, we also consider the Planck length in the outer space (f_{pl}), corresponding to the Newtonian gravitational constant (G_{N}). It is defined in the connection with $G(= \kappa/8\pi)$ by

$$\kappa/8\pi = [h(t_\dagger)]^n f_{\mathrm{pl}}{}^2. \tag{25}$$

Using this relation, we derive the condition (B) that the size of the inner space $h(t)$ is comparable with or smaller than f_{pl}, and examine the compatibility between conditions A and B.

4. Total entropy S and the 3-dimensional entropy S_3

The total entropy S in the multi-dimensional space-time is expressed at the stage of $(t_A - t) \ll t_A$ using Eqs. (4) and (17) as

$$S \propto (t_A - t)^{-\beta}, \tag{26}$$

where $\beta \equiv (3+n)/(4+n) - (n\mu + 3\nu)$ for $\eta_0 \neq 0$. For the non-viscous case, S is constant. The behavior of S was discussed in Ref. [14] using the inflation and collapse factors of the outer and inner spaces. The important quantity which is to be noticed directly from the viewpoint of cosmological observations, however, is not S, but the 3-dimensional entropy S_3 within the horizon l_{h} of 3-dimensional outer space.

Here S_3 is defined by

$$S_3 = s_3 (l_{\mathrm{h}})^3, \tag{27}$$

where the 3-dimensional entropy density s_3 is given by $s_3 = \epsilon_3^{3/4}$, and the 3-dimensional energy density ϵ_3 is

$$\epsilon_3 = h(t)^n \epsilon = h(t)^n T^{n+4}. \tag{28}$$

In contrast to S, S_3 increases with time, not only in the viscous case but also in the non-viscous case, because the common temperature T increases with the decrease of the volume V of the total space in both cases.

Here and in the following we neglect the factors ≈ 1 such as a_0. Then

$$(S_3)^{1/3} = [h(t)T]^{\frac{n+4}{4}} h(t)^{-1} l_{\mathrm{h}}, \tag{29}$$

where the horizon-size l_{h} is

$$l_{\mathrm{h}} = f(t) \int_0^t \mathrm{d}t'/f(t'). \tag{30}$$

4.1. *Entropy in the freeze-out epoch t_**

At epoch t_* with $h(t_*)T_* \equiv 1$, we have

$$(S_3)_* = [h(t_*)^{-1} f(t_*)]^3 \Big[\int_0^{t_*} \mathrm{d}t'/f(t')\Big]^3 \sim [(t_A - t_*)/(t_A - t_M)]^{3(\nu-\mu)} (f/h)_M{}^3, \tag{31}$$

where we assumed that this epoch t_* is at the stage of $(t_A - t) \ll t_A$.

In the non-viscous case, we have $\mu = -\nu = 1/3$, and

$$(S_3)_* \sim [(t_A - t_*)/(t_A - t_M)]^{-2}(f/h)_M{}^3 = [h(t_*)/h(t_M)]^{-6}(f/h)_M{}^3, \tag{32}$$

which is consistent with Kolb et al.'s result—Eq. (5.5) in Ref. [12].

In our viscous case ($n = 6$), we have $\mu = 0.345$, $\nu = -0.272$, and $\eta_0 = 0.225$ [cf. Eq. (6)], and

$$(S_3)_* \sim [(t_A - t_{*v})/(t_A - t_M)]^{-1.83}(f/h)_M{}^3 = [h(t_{*v})/h(t_M)]^{-5.38}(f/h)_M{}^3, \tag{33}$$

where we used the notation t_{*v} to discriminate t_* in the viscous case from t_* in the non-viscous case.

Now let us compare quantities at t_* and t_{*v}. Using Eq. (24), we have

$$\begin{aligned} \frac{h(t_*)T(t_*)}{h(t_M)T(t_M)} &= \left(\frac{t_A - t_*}{t_A - t_M}\right)^{2/9}, \\ \frac{h(t_{*v})T(t_{*v})}{h(t_M)T(t_M)} &= \left(\frac{t_A - t_{*v}}{t_A - t_M}\right)^{0.145} \end{aligned} \tag{34}$$

for $\eta_0 = 0$, $\eta_0 \neq 0$, respectively. Since $h(t_*)T(t_*) = h(t_{*v})T(t_{*v}) = 1$ and $h(t_M)T(t_M)$ is common, we obtain from these equations

$$\left(\frac{t_A - t_*}{t_A - t_M}\right)^{2/9} = \left(\frac{t_A - t_{*v}}{t_A - t_M}\right)^{0.145}, \tag{35}$$

so that S_3 in the non-viscous case is expressed in terms of t_{*v} as

$$(S_3)_* \sim \left(\frac{t_A - t_{*v}}{t_A - t_M}\right)^{0.131}(f/h)_M{}^3 = [h(t_{*v})/h(t_M)]^{-3.78}(f/h)_M{}^3. \tag{36}$$

By comparing this with Eq. (33), therefore, it is found that $(S_3)_*$ in the viscous case is much larger than $(S_3)_*$ in the non-viscous case, since $h(t_{*v})/h(t_M) \ll 1$.

4.2. *Entropy at epoch $t_\dagger$ (different from t_*)*

Now let us derive the conditions A and B suggested in Sect. 3. First we derive S_3 at epoch $t_\dagger$ with

$$h(t_\dagger)T_\dagger \equiv \lambda\ (\neq 1). \tag{37}$$

In this case we have

$$(S_3)^{1/3} = \lambda^{\frac{n+4}{4}}[h(t_\dagger)^{-1}f(t_\dagger)]\int_0^{t_\dagger} dt'/f(t'). \tag{38}$$

4.2.1. The viscous case ($\eta_0 = 0.225$). Using Eq. (24) we obtain

$$\lambda = [h(t_\dagger)/h(t_M)]^{1-\frac{1}{5\mu}}h(t_M)T_M, \tag{39}$$

with $\mu = 0.345$.

For S_3 at $t_\dagger$, we obtain, neglecting the factors of integrals,

$$(S_3)_\dagger \sim [h(t_\dagger)/h(t_M)]^\gamma S_0, \tag{40}$$

with $\gamma \equiv -3(1-\nu/\mu) = -5.362$, where

$$S_0 \equiv \lambda^{3(n+4)/4}(f/h)_M{}^3 \tag{41}$$

and we assumed that the epoch $t_\dagger$ is at the stage of $(t_A - t) \ll t_A$. If we express $(S_3)_\dagger$ in terms of λ, we have, by use of Eq. (39),

$$(S_3)_\dagger \sim \left[\lambda/(hT)_M\right]^\delta S_0 \tag{42}$$

with $\delta \equiv [\gamma/(1-\frac{1}{5\mu})] = -12.77$. Assuming $(S_3)_\dagger = 10^{88}$, we have

$$\begin{aligned} h(t_\dagger)/h(t_M) &= [(S_3)_\dagger/S_0]^{1/\gamma} = 10^{-16.4}(S_0)^{0.186}, \\ \lambda/(hT)_M &= [(S_3)_\dagger/S_0]^{1/\delta} = 10^{-6.89}(S_0)^{0.0783}. \end{aligned} \tag{43}$$

Moreover, we obtain from these relations

$$T_\dagger/T_M = [\lambda/(hT)_M][h(t_\dagger)/h(t_M)]^{-1} = 10^{9.51}(S_0)^{-0.108}. \tag{44}$$

Next we consider the relations of f and h to the Planck length $f_{\rm pl}$ in the outer space (cf. Sect. 3), which are derived in Appendix B. In order that the space-time in the outer space can be treated classically at epoch $t_\dagger$, the condition $f_\dagger/f_{\rm pl} \gg 1$ must be satisfied. This condition gives $\lambda > 1.96$. Moreover, $h_\dagger/f_{\rm pl}$ is smaller than 1 if $\lambda < 1.06 \times 10^4$. It is found, therefore, that, if $\lambda \sim 1.06 \times 10^4$, the outer space at epoch $t_\dagger$ gets $(S_3)_\dagger = 10^{88}$ and is decoupled from the inner space at the same time, because it has the quantized space-time.

For $\lambda = 1.06 \times 10^4$, we obtain, assuming $(f/h)_M = 1.5$,

$$S_0 = 5.22 \times 10^{30} \tag{45}$$

from Eq. (41),

$$\begin{aligned} h(t_\dagger)/h(t_M) &= 2.06 \times 10^{-11}, \\ \lambda/(hT)_M &= 2.28 \times 10^{-5} \end{aligned} \tag{46}$$

from Eq. (43), and

$$T_\dagger/T_M = 1.59 \times 10^6 \tag{47}$$

from Eq. (44). The model-dependent value of $(f/h)_M$ is shown in Sect. 6. Since $f_{\rm pl}$ corresponds to the Planck temperature $T_{\rm pl}$ ($= 10^{19}$ GeV) and $h_\dagger/f_{\rm pl} = 1$, we have $T_\dagger = \lambda T_{\rm pl}$ and $T_M = 6.67 \times 10^{-3} T_{\rm pl}$. From Eq. (46), moreover, we can also obtain

$$(hT)_M = 3.23 \times 10^8. \tag{48}$$

4.2.2. The non-viscous case ($\eta_0 = 0$). Using Eq. (24) we obtain

$$\lambda = [h(t_\dagger)/h(t_M)]^{2/3} h(t_M) T_M. \tag{49}$$

For S_3 at $t_\dagger$, we obtain, neglecting the factors of integrals,

$$(S_3)_\dagger \sim [h(t_\dagger)/h(t_M)]^\gamma S_0, \tag{50}$$

with $\gamma \equiv -3(1-\nu/\mu) = -6$ and S_0 in Eq. (41). If we compare them in terms of λ, we have, by use of Eq. (49),

$$(S_3)_\dagger \sim \left[\lambda/(hT)_M\right]^\delta S_0 \tag{51}$$

with $\delta \equiv (3/2)\gamma = -9$. Assuming $(S_3)_\dagger = 10^{88}$, we have

$$\begin{aligned} h(t_\dagger)/h(t_M) &= [(S_3)_\dagger/S_0]^{1/\gamma} = 10^{-14.7}(S_0)^{0.167}, \\ \lambda/(hT)_M &= [(S_3)_\dagger/S_0]^{1/\delta} = 10^{-9.78}(S_0)^{0.111}. \end{aligned} \tag{52}$$

Moreover, we obtain from these relations

$$T_\dagger/T_M = [\lambda/(hT)_M][h(t_\dagger)/h(t_M)]^{-1} = 10^{4.89}(S_0)^{-0.0556}. \tag{53}$$

Next we consider the relations of f and h to the Planck length $f_{\rm pl}$ in the outer space, which are derived in Appendix B. In order that the space-time in the outer space can be treated classically at epoch $t_\dagger$, the condition $f_\dagger/f_{\rm pl} \gg 1$ must be satisfied. This condition gives gives $\lambda > 10^{-5.88}$. Moreover, $h_\dagger/f_{pl}$ is smaller than 1 if $\lambda < 46.1$. It is found, therefore, that if $\lambda \sim 46.1$, the outer space at epoch $t_\dagger$ gets $(S_3)_\dagger = 10^{88}$ and is decoupled from the inner space at the same time, because it has the quantized space-time.

For $\lambda = 46.1$, we obtain, assuming $(f/h)_M = 1.5$,

$$S_0 = 1.01 \times 10^{13} \tag{54}$$

from Eq. (41),

$$\begin{aligned} h(t_\dagger)/h(t_M) &= 2.94 \times 10^{-13}, \\ \lambda/(hT)_M &= 4.57 \times 10^{-9} \end{aligned} \tag{55}$$

from Eq. (52), and

$$T_\dagger/T_M = 1.55 \times 10^4 \tag{56}$$

from Eq. (53). Since $f_{\rm pl}$ corresponds to the Planck temperature $T_{\rm pl}$ ($= 10^{19}$ GeV) and $h_\dagger/f_{\rm pl} = 1$, we have $T_\dagger = \lambda T_{\rm pl}$ and $T_M = 2.97 \times 10^{-3} T_{\rm pl}$. From Eq. (55), moreover, we can also obtain

$$(hT)_M = 1.01 \times 10^{10}. \tag{57}$$

These values of T_M and $(hT)_M$ give the condition on the thermal state that the multi-dimensional universe must satisfy at the epoch near the maximum expansion and so at the stage of the early isotropic expansion.

5. Numerical histories of multi-dimensional universes

In this section let us solve numerically equations of scale factors $f(t)$ and $h(t)$ in the outer and inner spaces and equations of the energy density $\epsilon(t)$—given by Eqs. (11) and (13)—at the total stage and relate their behaviors at the final asymptotic stage (which were treated in the previous section) to the behaviors at the earliest stage. Here we assume that the inner space has positive curvature ($k_h = 1$) and the outer space is flat or has negative curvature ($k_f = 0$ or -1).

Now let us transform variables $(t, f, h, \epsilon, p, \eta)$ to $(\bar{t}, \bar{f}, \bar{h}, \bar{\epsilon}, \bar{p}, \bar{\eta})$ as follows, for the convenience of numerical calculations:

$$t = \zeta \bar{t},\ \ f = \zeta \bar{f},\ \ h = \zeta \bar{h},\ \ \epsilon = \zeta^{-2}\bar{\epsilon},\ \ p = \zeta^{-2}\bar{p},\ \ \eta = \zeta^{-1}\bar{\eta}, \tag{58}$$

where ζ is a positive constant and η_0, k_f, k_h are assumed to be invariant. For this transformation, the forms of Eqs. (11), (12), and (13) are invariant. In this section these equations for the new variables are solved, where ζ is determined so as to be consistent with the initial condition in the following.

To determine the initial condition at $t = t_i$ for solving these equations, we consider the approximate solutions $f(t)$ and $h(t)$ around the isotropic solution $t^{1/5}$ (in the limit of small t) as

$$\begin{aligned} f &= f_0 t^{1/5}[1 + f_1(t)], \\ h &= h_0 t^{1/5}[1 + h_1(t)] \end{aligned} \tag{59}$$

with constants h_0 and f_0 $(= h_0)$, where we assume $f_1 \ll 1$, $h_1 \ll 1$. Here the isotropic solution is equal to Eqs. (16) and (19) with $T_I = 0$, and t_i is a small positive time. Then we obtain, from Eq. (11),

$$\begin{aligned} \ddot{f}_1 &= -\frac{28 + 24\eta_0}{15t}\dot{f}_1 - \frac{2 - 24\eta_0}{15t}\dot{h}_1 + \frac{5}{3}(1 - k_f)\frac{t^{-2/5}}{{h_0}^2}, \\ \ddot{h}_1 &= -\frac{1 - 12\eta_0}{15t}\dot{f}_1 - \frac{29 + 12\eta_0}{15t}\dot{h}_1 - \frac{(10 - k_f)}{3}\frac{t^{-2/5}}{{h_0}^2}, \end{aligned} \tag{60}$$

where the last terms come from the positive curvature $(k_h = 1)$ in the inner space and the curvature k_f $(= 0$ or $1)$ in the outer space, and, from Eq. (13),

$$\kappa\epsilon = \frac{36}{25t^2}\left[1 + \frac{10}{3}t(\dot{f}_1 + 2\dot{h}_1) + \frac{25}{12}(5 + k_f)\frac{t^{8/5}}{{h_0}^2}\right]. \tag{61}$$

Solving these equations, we obtain the following approximate solutions:

$$\begin{aligned} f &= h_0 t^{1/5}(1 + f_{10}t^{8/5}), \\ h &= h_0 t^{1/5}(1 + h_{10}t^{8/5}), \end{aligned} \tag{62}$$

where

$$\begin{aligned} f_{10} &= \frac{125}{288 \times 13}\frac{14 - 12\eta_0 - \frac{4}{5}(16 + 3\eta_0)k_f}{1 + \eta_0}{h_0}^{-2}, \\ h_{10} &= -\frac{125}{288 \times 13}\frac{25 + 12\eta_0 - \frac{2}{5}(7 - 6\eta_0)k_f}{1 + \eta_0}{h_0}^{-2}, \end{aligned} \tag{63}$$

and

$$f_{10} - h_{10} = \frac{125 \times 3}{288}\frac{1 - 2k_f}{1 + \eta_0}{h_0}^{-2}. \tag{64}$$

Next, using these solutions, we make six types of initial conditions $f, h, \dot{f}$, and $\dot{h}$ at $t = t_i$ which are discriminated with η_0, h_0, and t_i as

a. (010): $\eta_0 = 0$, $h_0 = 1$, $t_i = 0.1$,
b. (007): $\eta_0 = 0$, $h_0 = 0.7$, $t_i = 0.064$,
c. (015): $\eta_0 = 0$, $h_0 = 1.5$, $t_i = 0.166$,
d. (210): $\eta_0 = 0.225$, $h_0 = 1$, $t_i = 0.1$,
e. (207): $\eta_0 = 0.225$, $h_0 = 0.7$, $t_i = 0.064$,
f. (215): $\eta_0 = 0.225$, $h_0 = 1.5$, $t_i = 0.166$,

where these parameters satisfy the relation ${h_0}^{-2}(t_i)^{8/5} = 0.025$. After the transformation (58) corresponding to these initial conditions, t is dimensionless, while ζ has a dimension of length.

For these initial values, Eq. (11) was solved numerically using the Runge–Kutta method, and the result is shown by solid curves for $t \geq t_i$ and by short-dashed curves for $t < t_i$ in Figs. 2, 3, 4, 5, 6, and 7 for $k_f = 0$. The behavior of f and h for $k_f = -1$ is similar to that for $k_f = 0$.

In these solutions we estimated the singular epoch $t = t_A$ and derived the power solutions with $f = f_A(t_A - t)^\nu$, $h = h_A(t_A - t)^\mu$, using it where $(\mu, \nu) = (1/3, -1/3)$, $(0.345, -0.272)$ for

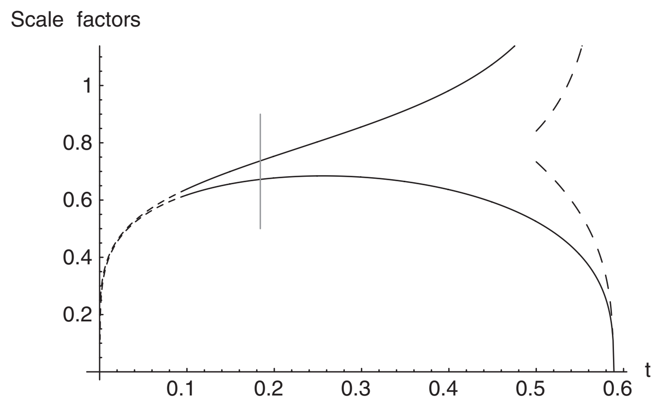

Fig. 2. Scale factors of the outer and inner spaces in a. (010). Solid and short-dashed curves denote the scale factors f and h for $t \geq t_i$ and $< t_i$, respectively. Long-dashed curves denote the power solutions tangent to f and h at epoch t_A. The thin solid line denotes the possible epoch of dimensional symmetry-breaking.

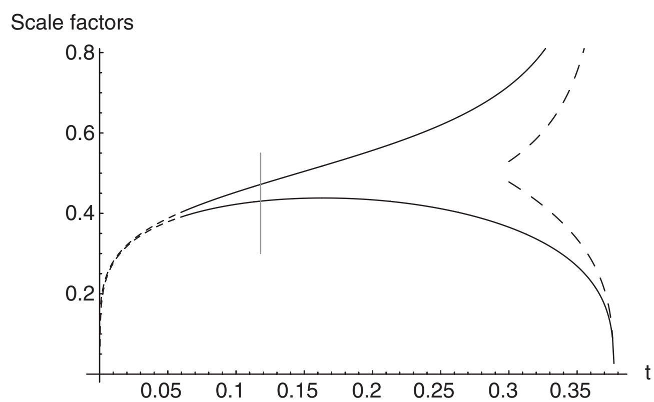

Fig. 3. Scale factors of the outer and inner spaces in b. (007). The notation of the curves is as in Fig. 2.

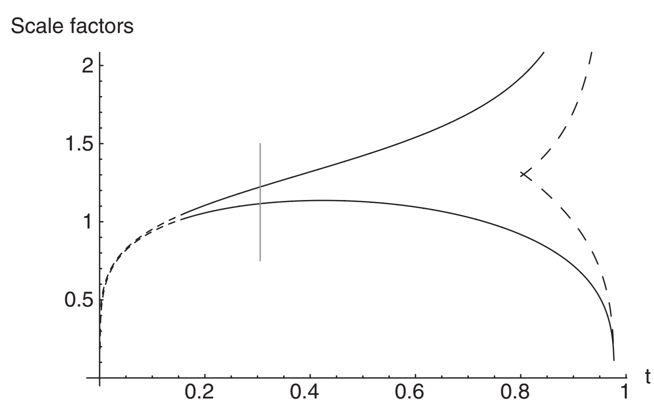

Fig. 4. Scale factors of the outer and inner spaces in c. (015). The notation of the curves is as in Fig. 2.

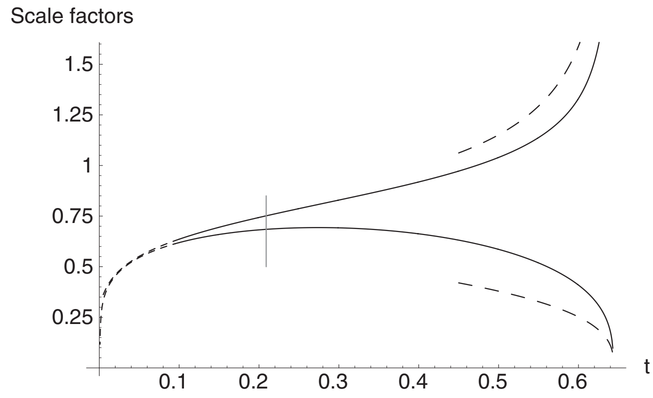

Fig. 5. Scale factors of the outer and inner spaces in d. (210). The notation of the curves is as in Fig. 2.

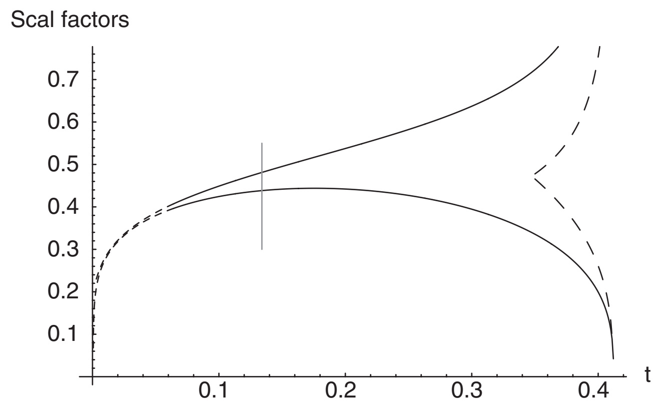

Fig. 6. Scale factors of the outer and inner spaces in e. (207). The notation of the curves is as in Fig. 2.

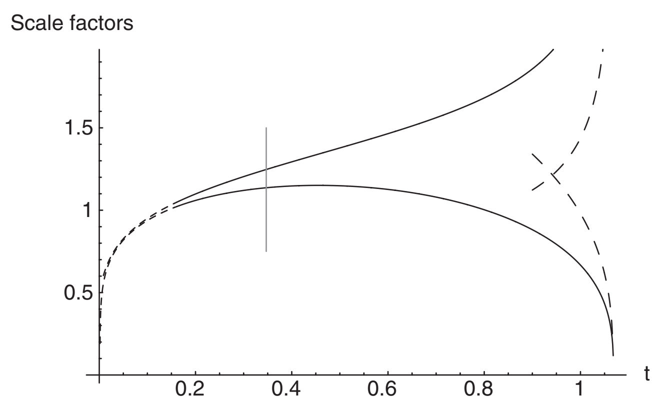

Fig. 7. Scale factors of the outer and inner spaces in f. (215). The notation of the curves is as in Fig. 2.

$\eta_0 = (0,\ 0.225)$, respectively. Constants f_A and h_A are determined so that these power solutions are tangent to the true solutions at epoch t_A. These solutions are shown by long-dashed curves in Figs. 2, 3, 4, 5, 6, and 7.

Moreover, the factors [appearing in Eqs. (B2) and (B5)]

$$\Phi_0(t) \equiv \sqrt{\frac{8\pi}{\kappa\epsilon}}(f/h^2)(h/f)^{0.667},$$
$$\Phi_2(t) \equiv \sqrt{\frac{8\pi}{\kappa\epsilon}}(f/h^2)(h/f)^{0.06} \tag{65}$$

for $\eta_0 = 0$, 0.225, respectively, are shown as functions of t by solid curves in Figs. 8, 9, 10, 11, 12, and 13 for $k_f = 0$. The temperature T [$= (\epsilon/\mathcal{N}a_n)^{1/10}$] is also shown by dashed curves in the same figures. These two quantities $\Phi_0(t)$ and $\Phi_2(t)$ are invariant for the transformation (58) and so they can be regarded as quantities on the ordinary scale of $(t, f, h, \epsilon, p, \eta)$. Moreover, λ, S_0 and the ratios of f, h, and T also, which are determined in the next section using these $\Phi_0(t)$ and $\Phi_2(t)$, can be treated as quantities on the ordinary scale.

6. Physical states at the initial and decoupling epochs and the primeval entropy

Let us use the formulation in Appendix B to derive λ [$\equiv h(t_\dagger)T_\dagger$] and the formulation in Sect. 4.2.2 to derive S_0, h_M [$\equiv h(t_M)$], and T_M [$\equiv T(t_M)$], where t_M is the earliest epoch in the time interval when each solution can be approximated by the corresponding power solutions. For this purpose we first determine the epochs t_A and t_M, and the factors Φ_0 and Φ_2 at epoch t_M, $(f/h)_M$, h_i/h_M,

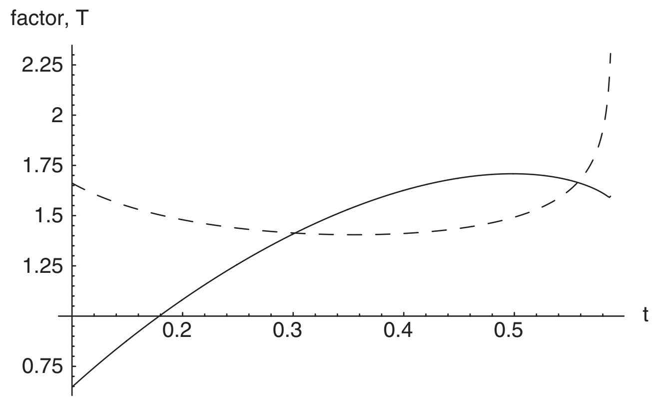

Fig. 8. Factor Φ_0 and temperature in a. (010). The solid and dashed curves denote Φ_0 and T, respectively.

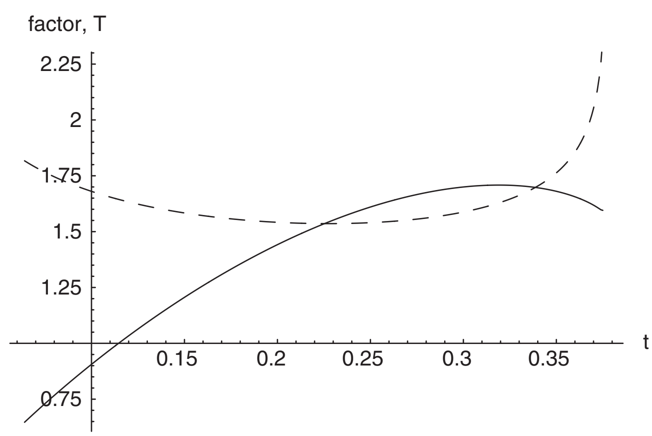

Fig. 9. Factor Φ_0 and temperature in b. (007). The notation of the curves is as in Fig. 8.

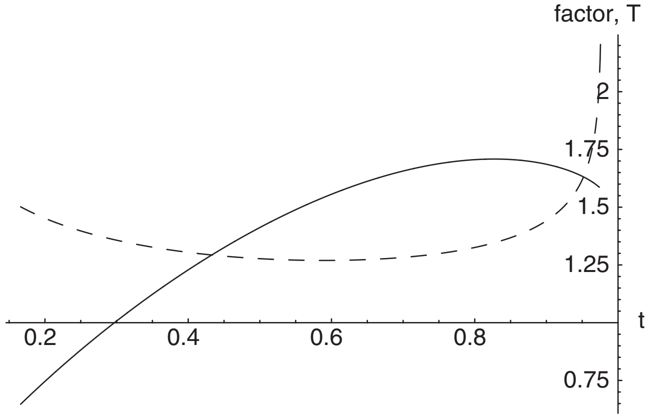

Fig. 10. Factor Φ_0 and temperature in c. (015). The notation of the curves is as in Fig. 8.

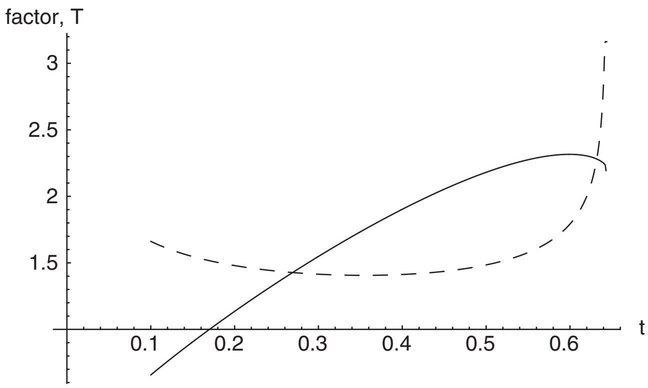

Fig. 11. Factor Φ_2 and temperature in d. (210). The solid and dashed curves denote Φ_2 and T, respectively.

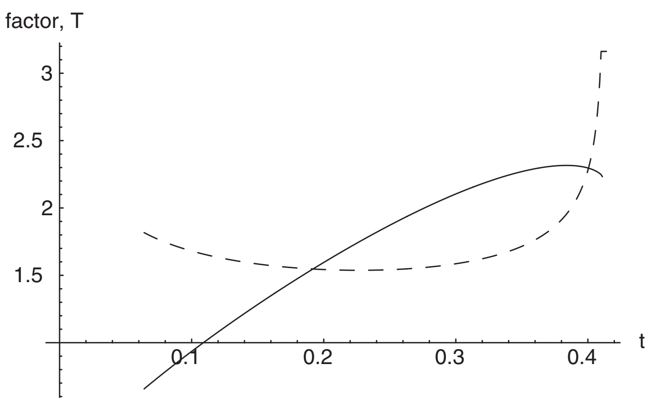

Fig. 12. Factor Φ_2 and temperature in e. (207). The notation of the curves is as in Fig. 11.

and T_i/T_M using the numerical results. The results are shown in Tables 1 and 2 for $k_f = 0$ and -1, respectively.

Next, λ, S_0, T_M, and h_M are obtained using the following formulas (derived in Sect. 4) and the values of Φ_{0M}, Φ_{2M}, and $(f/h)_M$ (in Tables 1 and 2).

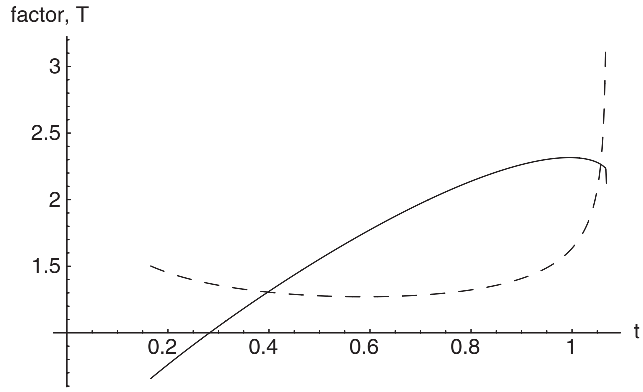

Fig. 13. Factor Φ_2 and temperature in f. (215). The notation of the curves is as in Fig. 11.

Table 1. Model parameters in cases a, b, ..., f ($k_f = 0$). $\Phi_{0M} = \Phi_0(t_M)$ and $\Phi_{2M} = \Phi_2(t_M)$.

model types	t_A	t_M	Φ_{0M}, Φ_{2M}	$(f/h)_M$	h_i/h_M	T_i/T_M
a. (010)	0.589	0.575	1.63	7.30	2.110	0.9092
b. (007)	0.376	0.370	1.62	8.49	2.279	0.8855
c. (015)	0.977	0.970	1.58	15.34	3.074	0.7996
d. (210)	0.644	0.630	2.30	5.74	2.217	0.7500
e. (207)	0.412	0.410	2.25	13.69	3.585	0.5746
f. (215)	1.068	1.060	2.26	10.65	3.090	0.6222

Case of $\eta_0 = 0$:

$$\begin{aligned} \lambda &= 46.13[\Phi_0(t_M)]^{-0.1715}(\mathcal{N}a_n)^{-0.08576}, \\ S_0 &= \lambda^{7.5}(f/h)_M{}^3, \\ T_M/T_{\rm pl} &= 10^{-4.89}\lambda(S_0)^{0.0556}, \\ h_M/f_{\rm pl} &= 10^{14.7}S_0{}^{-0.167}. \end{aligned} \tag{66}$$

Case of $\eta_0 = 0.225$:

$$\begin{aligned} \lambda &= 1.164 \times 10^4[\Phi_2(t_M)]^{-0.1307}(\mathcal{N}a_n)^{-0.06536}, \\ S_0 &= \lambda^{7.5}(f/h)_M{}^3, \\ T_M/T_{\rm pl} &= 10^{-9.51}\lambda(S_0)^{0.108}, \\ h_M/f_{\rm pl} &= 10^{16.4}S_0{}^{-0.186}. \end{aligned} \tag{67}$$

Moreover, T_i and h_i are derived from T_M and h_M using (h_i/h_M) and (T_i/T_M) (given in Tables 1 and 2). In all cases a, ..., f, we show their values in Tables 3 and 4 for $k_f = 0$ and -1, respectively, in the case of $\mathcal{N}a_n = 1$. The values of t_M were determined due to the comparison between the numerical solutions (f and h) and their corresponding power solutions. So, h_M and T_M depend on the epoch t_M, but h_i and T_i do not depend on it.

Here, f_i/h_i and t_i/h_i are derived from the relations in Sect. 5 as

$$\begin{aligned} f_i/h_i &= (1 + f_{10}t_i{}^{8/5})/(1 + h_{10}t_i{}^{8/5}) \simeq 1 + 0.0327(1 - 2k_f)/(1 + \eta_0), \\ h_i/t_i &= h_0 t_i{}^{-4/5}(1 + h_{10}t_i{}^{8/5}) \simeq 0.0251^{-1/2} = 6.325. \end{aligned} \tag{68}$$

Table 2. Model parameters in cases a, b, ..., f ($k_f = -1$). $\Phi_{0M} = \Phi_0(t_M)$ and $\Phi_{2M} = \Phi_2(t_M)$.

model types	t_A	t_M	Φ_{0M}, Φ_{2M}	$(f/h)_M$	h_i/h_M	T_i/T_M
a. (010)	0.582	0.570	2.08	10.05	2.321	0.9155
b. (007)	0.372	0.366	2.08	11.14	2.445	0.8996
c. (015)	0.965	0.958	2.07	19.62	3.251	0.8171
d. (210)	0.643	0.630	3.34	7.92	2.318	0.7568
e. (207)	0.412	0.410	3.34	19.41	3.893	0.5735
f. (215)	1.068	1.060	3.34	14.56	3.299	0.6202

Table 3. Model parameters in cases a, b, ..., f ($k_f = 0$).

	λ	S_0	$10^3\ T_M/T_{\rm pl}$	$10^3\ T_i/T_{\rm pl}$	$h_M/f_{\rm pl}$	$h_i/f_{\rm pl}$
a	42.42	6.266×10^{14}	3.437	3.043	1.694×10^{12}	3.564×10^{12}
b	42.47	9.946×10^{14}	3.532	3.127	1.568×10^{12}	3.574×10^{12}
c	42.65	6.051×10^{15}	3.924	3.138	1.158×10^{12}	3.558×10^{12}
d	1.044×10^4	2.605×10^{32}	9.998	7.498	2.348×10^{10}	5.205×10^{10}
e	1.047×10^4	3.615×10^{33}	13.27	7.626	1.440×10^{10}	5.160×10^{10}
f	1.046×10^4	1.699×10^{33}	12.26	7.629	1.652×10^{10}	5.114×10^{10}

Table 4. Model parameters in cases a, b, ..., f ($k_f = -1$).

	λ	S_0	$10^3\ T_M/T_{\rm pl}$	$10^3\ T_i/T_{\rm pl}$	$h_M/f_{\rm pl}$	$h_i/f_{\rm pl}$
a	40.67	1.191×10^{15}	3.369	3.084	1.522×10^{12}	3.533×10^{12}
b	40.68	1.626×10^{15}	3.429	3.085	1.445×10^{12}	3.533×10^{12}
c	40.73	8.968×10^{15}	3.924	3.085	1.086×10^{12}	3.533×10^{12}
d	0.9944×10^4	4.771×10^{32}	10.16	7.693	2.098×10^{10}	4.864×10^{10}
e	0.9941×10^4	6.996×10^{33}	13.57	7.782	1.273×10^{10}	4.966×10^{10}
f	0.9943×10^4	2.958×10^{33}	12.37	7.672	1.474×10^{10}	4.863×10^{10}

Table 5. The primeval total entropies S_9 in the cases of $k_f = 0, -1$. The values of $10^{-77} S_9$ are shown.

model types	a	b	c	d	e	f
$10^{-77} S_9\ (k_f = 0)$	2.29×10^{13}	3.00×10^{13}	2.97×10^{13}	1.26	1.36	1.26
$10^{-77} S_9\ (k_f = -1)$	2.87×10^{13}	2.88×10^{13}	2.88×10^{13}	1.81	2.38	1.77

Now let us consider the total entropy S_9 within the volume $V\ (= f^3 h^6)$ at epoch t_i. From Eqs. (1) and (4), the total entropy S is given by $S = \frac{5}{4}\mathcal{N} a_n V T^9$ for $n = 6$, so that S_9 is defined by

$$S_9 \equiv T_i{}^9 (f_i{}^3 h_i{}^6) = (T_i h_i)^9 (f_i/h_i)^3. \tag{69}$$

The values in cases a, ..., f on the ordinary scale are shown in Table 5 for $k_f = 0$ and -1. In the case of $k_f = 0$ the average values of S_9 for $\eta_0 = 0$ and 0.225 are 2.75×10^{90} and 1.29×10^{77}, respectively, and so $(S_9)_{\eta_0=0}/(S_9)_{\eta_0=0.225} = 2.13 \times 10^{13}$. In the case of $k_f = -1$ the average values of S_9 for $\eta_0 = 0$ and 0.225 are 2.87×10^{90} and 1.99×10^{77}, respectively, and so $(S_9)_{\eta_0=0}/(S_9)_{\eta_0=0.225} = 1.45 \times 10^{13}$.

As shown in Appendix C, S_9 and the 3-dimensional entropy $(S_3)_\dagger$ are closely related, and so these values of S_9 are the primeval total entropies necessary at the starting point of multi-dimensional universes. In the non-viscous case ($\eta_0 = 0$), entropy S_9 equal to $\sim 10^{90}[(S_3)_\dagger/10^{88}]^2$ is needed, while in the viscous case of $\eta_0 = 0.225$, only the entropy S_9 equal to $\sim 10^{77}[(S_3)_\dagger/10^{88}]^{0.71}$ is needed at the starting point, and so most entropy is produced by the dissipation.

We assumed $\mathcal{N}a_n = 1$ in the above calculations. This factor $\mathcal{N}a_n$ appears often in the calculations, as $h_i/f_{\rm pl} \propto (\mathcal{N}a_n)^{0.1074}$ and $T_i/T_{\rm pl} \propto (\mathcal{N}a_n)^{-0.1215}$ for $\eta_0 = 0$, and $h_i/f_{\rm pl} \propto (\mathcal{N}a_n)^{0.09118}$ and $T_i/T_{\rm pl} \propto (\mathcal{N}a_n)^{-0.1182}$ for $\eta_0 = 0.225$. But the product does not much depend on $\mathcal{N}a_n$ as $T_i h_i \propto (\mathcal{N}a_n)^{0.014}$, $(\mathcal{N}a_n)^{0.027}$, respectively.

7. Dimensional symmetry breaking and the primeval state of multi-dimensional universes

Kim et al. [2,3] showed in a matrix model of super-string theory that, due to the symmetry-breaking, the $(1+3+6)$-dimensional universe with isotropic expansion changes to that with anisotropic expansion, in which the 3-dimensional space expands at a larger rate than the 6-dimensional space. In the present treatment due to classical relativity, such a symmetry-breaking cannot be studied accurately. From the viewpoint of energy balance, however, we may examine the behavior of the symmetry-breaking in simplified situations.

7.1. *Case of $k_f = 0$*

Let us assume that at epoch t_{br} the symmetry-breaking occurred from the isotropic state with scale factors $\bar{f} = \bar{h}$ ($k_{\bar{f}} = k_{\bar{h}} = 1$) to the anisotropic state with $f \neq h$ ($k_f = 0$ and $k_h = 1$), without change in $\dot{f}/f$, $\dot{h}/h$, and ϵ. Then, from Eq. (13), we obtain the following condition for consistency:

$$\frac{18}{\bar{h}_{br}^2} = \frac{15}{{h_{br}}^2} \quad \text{or} \quad h_{br}/\bar{h}_{br} = \sqrt{5/6}, \tag{70}$$

so that

$$(\delta h/\bar{h})_{br} \equiv [(h-\bar{h})/\bar{h}]_{br} = \sqrt{5/6} - 1 = -0.087. \tag{71}$$

On the other hand, since we assume that f does not change, i.e. $(\delta f/\bar{f})_{br} = 0$, we have

$$(\delta h/\bar{h} - \delta f/\bar{f})_{br} = -0.087. \tag{72}$$

This difference can be regarded as the difference of f and h from the average at the instant of symmetry-breaking, which is given by Eq. (64) in Sect. 5:

$$(\delta h/\bar{h} - \delta f/\bar{f})_{br} = -\frac{125 \times 3}{288} \frac{1}{1+\eta_0} {h_0}^{-2} (t_{br}/t_i)^{8/5} {t_i}^{8/5}. \tag{73}$$

From Eq. (72) and this relation we obtain

$$t_{br}/t_i = 1.84(1+\eta_0)^{5/8} = 1.84,\ 2.09 \tag{74}$$

for $\eta_0 = 0$, 0.225, respectively. These epochs (t_{br}) are indicated in Figs. 2, ..., 7 by thin solid lines.

7.2. *Case of $k_f = -1$*

We assume also in this case that at epoch t_{br} the symmetry-breaking occurred from the isotropic state with with scale factors $\bar{f} = \bar{h}$ ($k_{\bar{f}} = k_{\bar{h}} = 1$) to the anisotropic state with $f \neq h$ ($k_f = -1$ and $k_h = 1$), without change in $\dot{f}/f$, $\dot{h}/h$, f, and ϵ. Then, similarly to the case of $k_f = 0$, we obtain the following condition for consistency from Eq. (13):

$$\frac{21}{\bar{h}_{br}^2} = \frac{15}{{h_{br}}^2} \quad \text{or} \quad h_{br}/\bar{h}_{br} = \sqrt{5/7}, \tag{75}$$

so that

$$(\delta h/\bar{h})_{br} \equiv [(h - \bar{h})/\bar{h}]_{br} = \sqrt{5/7} - 1 = -0.155. \tag{76}$$

On the other hand, since we assume that f does not change, we have

$$(\delta h/\bar{h} - \delta f/\bar{f})_{br} = -0.155. \tag{77}$$

This difference can be regarded as the difference of f and h from the average at the instant of symmetry-breaking, which is given by Eq. (64):

$$(\delta h/\bar{h} - \delta f/\bar{f})_{br} = -\frac{125 \times 9}{288}\frac{1}{1+\eta_0}{h_0}^{-2}(t_{br}/t_i)^{8/5}{t_i}^{8/5}. \tag{78}$$

From Eq. (77) and this relation we obtain

$$t_{br}/t_i = 1.57(1+\eta_0)^{5/8} = 1.57,\ 1.78 \tag{79}$$

for $\eta_0 = 0, 0.225$, respectively.

The primeval entropy S_9 within the closed 9-dimensional space before the symmetry-breaking is equal to $(S_9)_i \times {(\bar{h}/h)_{br}}^6 = 1.73(S_9)_i$, where $(S_9)_i$ is given by Eq. (69).

8. Concluding remarks

In this paper we first derived the 3-dimensional entropy S_3 within the horizon, and compared it in the viscous and non-viscous cases. It was found that the time evolutions of temperature T and S_3 in the viscous case are much larger than those in the non-viscous case. Such a remarkable change in these quantities is caused by the change in the energy density ϵ and the viscous dissipation. Next, we derived the values of f, h, and T at the epoch ($t_\dagger$) satisfying the condition that the 3-dimensional entropy S_3 within the horizon should be $\sim 10^{88}$. Then, we examined the condition $(f/f_{\rm pl})_\dagger \gg 1$, where $f_{\rm pl}$ is the Planck length in the outer space, and it was found that λ [$\equiv (hT)_\dagger$] must be $\gg 1.96$ and $10^{-5.88}$ in the viscous and non-viscous cases, respectively. It was found, moreover, that, at epoch $t_\dagger$ with $\lambda \sim 1.06 \times 10^4$, 46.1 (different from the freeze-out epoch t_*), the condition $(h/f_{\rm pl}) \sim 1$ is satisfied at the same time, so that the outer space may be decoupled from the inner space.

Moreover, in Sect. 6 we considered the primeval entropy S_9, and found that, for an equal value of S_3, S_9 in the viscous case can be much smaller than that in the non-viscous case. If S_9 in the universe at the primeval stage is comparable with the critical value (such as in Table 5), the outer space does not need any additional 3-dimensional inflation and dissipation after the decoupling epoch. If S_9 in the universe is much smaller than the critical value, on the other hand, the outer space must create an additional entropy by the 3-dimensional inflation and dissipation after the decoupling epoch, so that S_3 may reach $\sim 10^{88}$.

Here we considered only the viscosity due to the transport of multi-dimensional gravitational waves. If we consider quantum particle creation [13] and the other transport processes, the viscous entropy production and S_3 may be much larger than those in the present treatment.

Our results were derived using classical relativity and thermodynamics. As the decoupling epoch may be in the world of the quantum super-string, the results may not hold in the original form, but may give a qualitative trend of entropy production.

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Appendix A. Imperfect fluid

The energy–momentum tensor of an imperfect fluid is expressed as

$$\begin{aligned} T^{MN} = \epsilon u^M u^N + \; pH^{MN} \; - \; \eta H^{MK} H^{NL} \left[u_{K;L} + u_{L;K} - \frac{2}{3+n} g_{KL} \Theta \right] \\ - \; \zeta H^{MN} \Theta \; - \; \chi (H^{MK} u^N + H^{NK} u^M)(T_{;K} + T u_{K;L} u^L), \end{aligned} \tag{A1}$$

where the $(4+n)$-dimensional velocity u^M satisfies $g_{MN} u^M u^N = -1$, and H^{MN} is the projection tensor defined by

$$H^{MN} = g^{MN} + u^M u^N, \tag{A2}$$

and Θ denotes $u^L{}_{;L}$. From the equation of relativistic radiative transport, we obtain (see Appendix B of Ref. [14] for details):

$$\begin{aligned} \eta &= \frac{4+n}{(3+n)(5+n)} \epsilon_r \tau, \\ \zeta &= (4+n) \epsilon_r \tau [1/(3+n) - (\partial p / \partial \epsilon)_V]^2, \\ \chi &= \frac{(4+n)}{(3+n)} \epsilon_r \tau, \quad \epsilon_r = \mathcal{N}_r a_n T^{4+n}, \end{aligned} \tag{A3}$$

where η, ζ, and χ are the shear viscosity, the bulk viscosity, and the heat conductivity, respectively, τ is the mean free time, and $\mathcal{N}_r$ is the number of radiative particle species. For the radiative matter, we have $\zeta = 0$ and χ does not contribute to our present treatment.[1]

The gravitational waves are absorbed in imperfect fluids. Their mean free time τ is expressed as

$$\tau = (2\kappa\eta)^{-1}, \tag{A4}$$

as shown in Appendix A of Ref. [14].

Appendix B. The Planck length $f_{\rm pl}$ in the outer space

Let us consider the relation of f and h with the Planck length $f_{\rm pl}$ given by Eq. (25).

B.1. *The viscous case*

From Eq. (25) we get

$$\left(\frac{f}{f_{\rm pl}} \right)^2_\dagger = \left[\frac{8\pi}{\kappa\epsilon} (f^2/h^4) \right]_M \lambda^{4+n} \left[\frac{f_\dagger {h_\dagger}^{-2} (t_A - t_\dagger)}{f_M {h_M}^{-2} (t_A - t_M)} \right]^2 \mathcal{N} a_n, \tag{B1}$$

where $\kappa\epsilon$ is given by Eqs. (13) and (22). Here, we use Eqs. (41) and (43) and the t-dependence of f and h, such as $fh^{-2}(t_A - t) \propto h^{-2+(\nu+1)/\mu}$. Then we obtain, for $n = 6$,

$$\left(\frac{f}{f_{\rm pl}} \right)_\dagger = 10^{-1.804} \left[\sqrt{\frac{8\pi}{\kappa\epsilon}} (f/h^2) \right]_M \lambda^{5.15} (f/h)_M{}^{0.06} (\mathcal{N} a_n)^{1/2} \tag{B2}$$

[1] A misprint was found in the expression for η in the previous paper (Ref. [14]).

and

$$\left(\frac{h}{f_{\rm pl}}\right)_\dagger = \left(\frac{h}{f}\right)_\dagger \left(\frac{f}{f_{\rm pl}}\right)_\dagger = 10^{-29.3}\lambda^{5/2}\left(\frac{f}{f_{\rm pl}}\right)_\dagger . \tag{B3}$$

Here we estimate $\left[\sqrt{\frac{8\pi}{\kappa\epsilon}}(f/h^2)\right]_M (f/h)_M{}^{0.06}$ [$\equiv \Phi_2(t_M)$] to be 2, since t_M is near the epoch of maximum expansion of the inner space. Its model-dependent value (Φ_2) is shown in Sect. 6. Then, if we assume $(f/f_{\rm pl})_\dagger \gg 1$, we get the condition $\lambda \gg 1.96$. But if $\lambda < 1.06 \times 10^4$, $h/f_{\rm pl}$ is smaller than 1. Here the factor $\mathcal{N}a_n$ was neglected, because its contribution is small.

B.2. The non-viscous case

From Eq. (25) we get, similarly,

$$\left(\frac{f}{f_{\rm pl}}\right)_\dagger^2 = \left[\frac{8\pi}{\kappa\epsilon}(f^2/h^4)\right]_M \lambda^{4+n}\left[\frac{f_\dagger h_\dagger{}^{-2}(t_A - t_\dagger)^{\alpha/2}}{f_M h_M{}^{-2}(t_A - t_M)^{\alpha/2}}\right]^2 \mathcal{N}a_n, \tag{B4}$$

where $\kappa\epsilon$ is given by Eqs. (13) and (23), and $\alpha = 10/9$ for $n = 6$. Here we use Eqs. (41) and (52) and the t-dependence of f and h, such as $fh^{-2}(t_A - t)^{\alpha/2} \propto h^{-2+(\nu+\alpha/2)/\mu}$. Then we obtain

$$\left(\frac{f}{f_{\rm pl}}\right)_\dagger = 10^{19.6}\left[\sqrt{\frac{8\pi}{\kappa\epsilon}}(f/h^2)\right]_M \lambda^{3.33}(h/f)_M{}^{0.667}(\mathcal{N}a_n)^{1/2} \tag{B5}$$

and

$$\left(\frac{h}{f_{\rm pl}}\right)_\dagger = \left(\frac{h}{f}\right)_\dagger \left(\frac{f}{f_{\rm pl}}\right)_\dagger = 10^{-29.3}\lambda^{5/2}\left(\frac{f}{f_{\rm pl}}\right)_\dagger . \tag{B6}$$

Here we estimate $\left[\sqrt{\frac{8\pi}{\kappa\epsilon}}(f/h^2)\right]_M (h/f)_M{}^{0.667}$ [$\equiv \Phi_0(t_M)$] to be 1, since t_M is near the epoch of maximum expansion of the inner space. Its model-dependent value (Φ_0) is shown in Sect. 6. Then, if we assume $(f/f_{\rm pl})_\dagger \gg 1$, we get the condition $\lambda \gg 10^{-5.88}$. But if $\lambda < 46.1$, $h/f_{\rm pl}$ is smaller than 1. On the other hand, $h/f_{\rm pl} = 10^{-9.7}$ for $\lambda = 1$ (or at epoch t_*), which is consistent with Eq. (5.7) of Kolb et al. [12].

Thus, if λ [$= (hT)_\dagger$] is comparable with 1, $h/f_{\rm pl}$ is much smaller than 1 in both cases, and for $\eta_0 = (0.225, 0)$, we obtain $(h/f_{\rm pl}) \sim 1$ if $\lambda \sim (1.06 \times 10^4, 46.1)$.

Appendix C. Dependence of $(S_9)_i$ on the 3-dimensional entropy $(S_3)_\dagger$

In the text we treated only the case when $(S_3)_\dagger = 10^{88}$. Here we consider the dependence of $(S_9)_i$ on $(S_3)_\dagger/10^{88}$ ($\equiv R$), under the condition that $h(t_\dagger)$ is equal to the Planck length $f_{\rm pl}$.

1. The viscous case

From Eqs. (43), we obtain

$$\lambda/(hT)_M \propto [(S_3)/S_0]^{1/\delta}, \tag{C1}$$

where $\delta = -12.77$ and $S_0 \equiv \lambda^{15/2}(f/h)_M{}^3$. Here, λ is determined by the above condition (that $h(t_\dagger)$ is equal to the Planck length) which is given in Appendix B, but it is independent of S_3. Therefore we have

$$(hT)_M \propto R^{0.0783}, \tag{C2}$$

and, using the ratios h_i/h_M and T_i/T_M in Tables 1 and 2 and the values of μ and ν (which are independent of S_3), we obtain

$$T_i h_i \propto R^{0.0783}, \tag{C3}$$

so that

$$(S_9)_i = (T_i h_i)^9 \propto R^{0.7047}. \tag{C4}$$

2. The non-viscous case

From Eqs. (52), similarly, we obtain

$$\lambda/(hT)_M \propto [(S_3)/S_0]^{-1/9}, \tag{C5}$$

$$T_i h_i \propto R^{1/9}, \tag{C6}$$

so that

$$(S_9)_i = (T_i h_i)^9 \propto R. \tag{C7}$$

It is found, therefore, that $(S_9)_i$ is proportional to $(S_3)_\dagger/10^{88}$ ($\equiv R$) for $\eta_0 = 0$ and the dependence of $(S_9)_i$ on R is smaller for $\eta_0 = 0.225$ than for $\eta_0 = 0$.

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