SU(2) COHERENT STATE PATH INTEGRALS LABELED BY A FULL SET OF EULER ANGLES: BASIC FORMULATION

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We develop a basic formulation of the spin (SU(2)) coherent state path integrals based not on the conventional highest or lowest weight vectors but on arbitrary fiducial vectors. The coherent states, being defined on a 3-sphere, are specified by a full set of Euler angles. They are generally considered as states without classical analogues. The overcompleteness relation holds for the states, by which we obtain the time evolution of general systems in terms of the path integral representation; the resultant Lagrangian in the action has a monopole-type term *à la* Balachandran *et al.* as well as some additional terms, both of which depend on fiuducial vectors in a simple way. The process of the discrete path integrals to the continuous ones is clarified. Complex variable forms of the states and path integrals are also obtained. During the course of all steps, we emphasize the analogies and correspondences to the general canonical coherent states and path integrals that we proposed some time ago. In this paper we concentrate on the basic formulation. The physical applications as well as criteria in choosing fiducial vectors for real Lagrangians, in relation to fictitious monopoles and geometric phases, will be treated in subsequent papers separately.

Keywords: Coherent state; path integral; fiducial vector; monopole; geometric phase; displaced number state; rotated spin number state; nonclassical quantum state.

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1. Introduction

It has been approximately four decades since the "coherent state (CS)" for the Heisenberg–Weyl group, i.e., the "canonical CS (CCS)" was extended to wider classes. [1−4] During the period, the broader CS, together with the original one, have had a great influence on almost every branch of modern physics. [5−9]

Since basic properties of CS are that they are continuous functions labeled by some parameters and that they compose "overcomplete" sets, [5] they provide a nat-

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ural way to perform path integrations. Such "coherent state path integrals (CSPI) ", i.e., path integrals (PI) via CS, have highly enriched the methods of PI with their physical applications. [5,8] (In what follows each of the words "CS" and "CSPI" is used as a plural as well as a singular.)

As stated at the beginning, among all the CS the CCS is the original and the best-known CS that was introduced by Schrödinger. [10] The CCS is, in the light of quantum optics, generated by displacing, or driving, the vacuum, i.e., the zero photon state. [11–13] From the viewpoint of CS in terms of unitary irreducible representations of Lie groups *à la* Perelomov, [2,6,7] the unitary operator is a displacing operator and a "fiducial vector (FV)" [a] is the ground state or vacuum.

Some time ago we opened up the CSPI in terms of CCS evolving from an arbitrary FV and investigated the associated geometric phases with an application to quantum optics. [14] Let us look back the results from the physical viewpoint concisely: First, we *set the generic CCS by displacing, or driving, not a usual vacuum, but an arbitrary superposition of photon number states.* If we take a single photon number state as a FV, we find that the CCS reduce to "displaced number states (DNS)". [16–18] So we may state that *the CCS with a generic FV is an arbitrary superposition of the DNS.* Second, using the general CCS we have performed CSPI which give a completely general propagator including geometric phase terms corresponding to a quantum optical state that has no classical analogue; *the resultant action in the Lagrangian includes extra terms reflecting the entanglements between the coefficients of FV.* Third, particularly we investigated the geometric phase for DNS; and we found that the condition for the experimental detection of the phase had changed according to the n-dependence of DNS. Such CCS with a general FV may propose a clue to a universal language for quantum optics especially when combined with SU(1,1) case. [b]

Now, we realize that we have another CS which is of practical importance in a large variety of physical systems: that is the spin or SU(2) CS. It is one of the extensions of CS addressed at the beginning. [c] Hence, following the achievements for the CCS above mentioned,[14] we here endeavor to liberate spin CS from the conventional choice of FV, $|\Psi_0\rangle = |s, s\rangle$ or $|s, -s\rangle$, and perform the path integration via the CS based on arbitrary FV; the SU(2) CS is, in this case, labeled by a full set of three Euler angles (ϕ, θ, ψ); it is defined on a 3-sphere S^3.

We have several concrete reasons for doing such an extension. First, we know that DNS turned out to be non-classical quantum states that had various interest-

[a] In Refs. 14 and 15 we adopted the term "starting vector" which can be found in, e.g. p 14 of Ref. 6. The term seems well fit for the situation. However, we use "fiducial vector" in the present paper since it appears to be more employed in literature. See e.g., Ref. 5. We use also the word "FV" as a plural as well as a singular.

[b] "Displaced squeezed number states" have been discussed in literature; see Refs. 16–18 and references therein.

[c] These two CS are intriguing also from the mathematical viewpoint since each of them is a typical example of CS for nilpotent Lie groups and semisimple ones respectively.[2]

ing properties. [16−18] So the analogous general spin CS may serve new non-classical quantum states that have not been known. And the PI will provide examples of propagators for such states. Second, the usual CCS have been employed as logical gates in quantum computation (QC) .[19] And moreover, the superpositions of DNS, which fall within the CCS with generic FV before mentioned, [14] have already appeared in the context of QC. [20] Since spin systems as well as optical systems are probable candidates for exemplifying QC, spin CS with general FV may be also used in QC. Third, describing geometric phases, which has been one of the crucial topics in fundamental physics[21] in terms of CSPI requires the extension. Let us put it more concretely: once elsewhere we investigated the geometric phases for a spin-s particle under a magnetic field in the formalism of SU(2)CSPI with the conventional FV, i.e., $|s,-s\rangle$.[22] In consequence the results give the geometric phase of a monopole-type that corresponds merely to the adiabatic phase for the lowest eigenstate. However, it has been known that in the adiabatic phase for the same physical setting the strength of a fictitious monopole is proportional to the quantum number m ($m = -s, -s+1, \cdots, s$) of the adiabatic state .[23] We cannot treat the corresponding case by the conventional SU(2)CSPI. Therefore the usual SU(2)CSPI is clearly unsatisfactory; and we had better let CS and CSPI prepare room also for the general cases which are reduced to any mth eigenstate in the adiabatic limit. Thus physics actually needs some extension of spin CSPI. Fourth, now geometric phases have been employed in QC [24]; and, as stated in the third reason, geometric phases are closely related to CSPI. Appreciating both areas it seems that we had better prepare wider CSPI and FV also for QC. Fifth, apart from the third reason, such a formalism of spin CSPI involving arbitrary FV may consequently shed a new light in understanding monopoles themselves in turn. For we have already known that the conventional spin CSPI provide a mathematical description of monopoles naturally. [22] And the description is common to real and fictitious monopoles. Hence it is possible that considering general FV in spin CSPI helps us to understand monopoles deeper. Finally, since the usual spin CS tends to the usual CCS in the high spin limit,[1] we are naturally led to seek the spin CS and CSPI that are contracted to the CCS and CSPI with arbitrary FV described in Ref. 14. For the above several reasons we take a general FV in this paper.

We may grasp the main results by three theorems. The first one shows that we have the overcompleteness relation, or the resolution of unity, for the spin CS with an arbitrary FV. Using the result we obtain the second one as follows: the form of the generic Lagrangian for the SU(2)CSPI is (32). As the Lagrangian for the usual CSPI, it consists of two parts: the topological term related to geometric phases and the dynamical one originating from a Hamiltonian. However, the contents are quite different; in the present case the former has a monopole-type term *à la* Balachandran *et al.* (hereafter called BMS^2), [25,26] whose strength or charge is proportional to the expectation value of the quantum number m in the state of a FV, $|\Psi_0\rangle$, having $(2s+1)$-components; and besides the topological term contains additional terms that reflect the effect of interweaving components of a FV with their next ones. This is also the case for the latter; such interweaving components of a FV appear in the dynamical term as well. In the previous version we merely showed the above results from the formal CSPI.[27] It can be, however, established from the discrete CSPI. In the third theorem we prove that the general spin CS and CSPI contract to the general CCS and CCSPI in Ref. 14.

The plan of the paper is the following. Before going into the spin CS case, we concisely review the CCS and PI evolving from an arbitrary FV[14] in Sec. 2. Next, we describe the spin CS based on arbitrary FV as well as their various properties (Sec. 3) and employ them to perform path integration (Sec. 4). Specifically, there we investigate the process of going from the discrete PI to the continuous ones. Next, we discuss problems related to the Lagrangians: the problems of topological terms, the fictitious gauge potentials and semiclassical equations. Complex variable form of the CS and CSPI are obtained in Sec. 5. The results are applied to the demonstration that the spin CSPI there contract to the CCSPI in Sec. 2. Section 6 gives the summary and prospects. Mathematical tools necessary in the article are enumerated concisely in Appendix A.

2. General Canonical Coherent States and the Path Integrals

In this section we briefly revisit the results of the CCS with a generic FV and the related PI described in Ref. 14. We may compare the expressions in this section with those in the following Secs. 3–5. Notice that α, in this article, denotes a parameter specifying the CCS; not an element of Euler angles.

2.1. *General CCS*

We proceed physically as far as possible.

2.1.1. *Definition of the CCS*

First, we *set the generic CCS, $|\alpha\rangle$, by displacing, or driving, not a usual vacuum, i.e., the zero photon state, but an arbitrary superposition of photon number states*:

$$|\alpha\rangle = \hat{D}(\alpha)|\Psi_0\rangle, \tag{1}$$

where

$$\hat{D}(\alpha) \equiv \exp(\alpha\hat{a}^+ - \alpha^*\hat{a}) = \exp\Big(-(1/2)|\alpha|^2\Big)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a}) \tag{2}$$

and

$$|\Psi_0\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \qquad \text{with} \qquad \sum_{n=0}^{\infty}|c_n|^2 = 1. \tag{3}$$

Here $|n\rangle$ is the photon number state. From (1)–(3), the general CCS, $|\alpha\rangle$, can be put into the form:

$$|\alpha\rangle \equiv \sum_{n}^{\infty} c_n |\alpha, n\rangle \tag{4}$$

with

$$\begin{aligned} |\alpha, n\rangle &\equiv \hat{D}(\alpha)|n\rangle = \sum_{m=0}^{\infty} \Big\langle m|\hat{D}(\alpha)|n\Big\rangle |m\rangle \\ &= \exp\Big(-(1/2)|\alpha|^2\Big) \Big[\sum_{m=0}^{n} \left(\frac{m!}{n!}\right)^{1/2} (-\alpha^*)^{n-m}\ L_m^{(n-m)}(|\alpha|^2)|m\rangle \\ &\quad + \sum_{m=n+1}^{\infty} \left(\frac{n!}{m!}\right)^{1/2} \alpha^{m-n}\ \ L_n^{(m-n)}(|\alpha|^2)|m\rangle \Big], \end{aligned} \tag{5}$$

where $L_k^{(\ell)}(x)$ is the Laguerre polynomials.

We see that $|\alpha, n\rangle$ in (5) is a DNS. [d] So we may say that *CCS with a general FV is an arbitrary superposition of DNS*. However, we will not use the explicit form of DNS in the present paper.

2.1.2. *Resolution of unity*

For CCS $|\alpha\rangle$ evolving from an *arbitrary FV*, we have the "overcompleteness relation" or "resolution of unity":

$$\frac{1}{\pi}\int |\alpha\rangle d^2\alpha \langle\alpha| = \mathbf{1} \quad \text{with} \quad d^2\alpha \equiv d(\mathrm{Re}\,\alpha)\, d(\mathrm{Im}\,\alpha). \tag{6}$$

2.1.3. *CCS as eigenvectors*

It turns out that the CCS $|\alpha\rangle$ is a "generalized eigenvector" [28] of an annihilation operator:

$$(\hat{a} - \alpha)^{N+1}|\alpha\rangle = 0 \qquad (N = \max n). \tag{7}$$

In the case of DNS $|\alpha, n\rangle$ we also have:

$$(\hat{a}^+ - \alpha^*)(\hat{a} - \alpha)\,|\alpha, n\rangle = n\,|\alpha, n\rangle\,. \tag{8}$$

Concerning (8), we apologize for sign errors in the original expressions in Eqs. (20) and (B1) in Ref. 14.

[d] We made no mention of DNS in Ref. 14; since our central concern was CSPI, we were not aware of it. We appreciate those who contribute to DNS including Refs. 16–18 and 20.

2.2. General CCS path integrals

2.2.1. *CCS path integrals*

Invoking the resolution of unity for CCS (6), we can obtain PI expression for the CCS. We put the results below.

Let us define

$$A(\dot{\alpha}, \dot{\alpha}^*; \{c_n\}) \equiv 2 \sum_{n=1}^{\infty} n^{1/2} \Big(\dot{\alpha}\, c_n^* c_{n-1} - \dot{\alpha}^*\, c_n c_{n-1}^* \Big) \tag{9}$$

and

$$H(\alpha^*, \alpha, t) \equiv \langle \alpha | \hat{H} | \alpha \rangle. \tag{10}$$

Then we find the propagator $K(\alpha_f, t_f; \alpha_i, t_i)$ which starts from $|\alpha_i\rangle$ at $t = t_i$, evolves under the effect of the Hamiltonian $\hat{H}(\hat{a}^+, \hat{a}; t)$ which is assumed to be a suitably-ordered function of $\hat{a}^+$ and $\hat{a}$, and ends up with $|\alpha_f\rangle$ at $t = t_f$ is:

$$K(\alpha_f, t_f; \alpha_i, t_i) = \int \mathcal{D}[\alpha(t)] \; \exp\{(i/\hbar) S[\alpha(t)]\}, \tag{11}$$

where

$$S[\alpha(t)] \equiv \int_{t_i}^{t_f} L\, dt \tag{12}$$

with

$$L \equiv \frac{i\hbar}{2} \Big[(\alpha^* \dot{\alpha} - \dot{\alpha}^* \alpha) + A(\dot{\alpha}, \dot{\alpha}^*; \{c_n\}) \Big] - H(\alpha^*, \alpha, t) \tag{13}$$

and we symbolized

$$\mathcal{D}[\alpha(t)] \equiv \lim_{N \to \infty} \left(\frac{1}{\pi} \right)^N \prod_{j=1}^{N} d^2 \alpha_j. \tag{14}$$

As we see in (9), the extra A-term in the Lagrangian (13) represents the entanglements of the coefficients of FV with their next ones. However, even if the A-term vanishes, since we take a general FV, $H(\alpha^*, \alpha, t)$ in (10) is also different from that for the usual CCS as we showed in the evaluation of geometric phases for DNS in Ref.14.

2.2.2. *Canonical equations*

In the semiclassical limit, i.e., $\hbar \to 0$, the Lagrangian (13) yields the Euler-Lagrange equation:

$$i\hbar\, \dot{\alpha} = \frac{\partial H}{\partial \alpha^*}, \; -i\hbar\, \dot{\alpha}^* = \frac{\partial H}{\partial \alpha}, \tag{15}$$

which is the generalized canonical equations. Since the A-term (9) is expressed as a total derivative, it is not involved in (15); and thus Eq. (15) is the same as that for the usual CCS formally. However, as mentioned above, the meaning of α and $H(\alpha^*, \alpha, t)$ are different.

3. SU(2) Coherent State with General Fiducial Vectors

In this section we investigate the explicit form of the SU(2) CS based on arbitrary FV. And their properties are studied to such extent as we need later. It means that we will consider the spin states analogous to the CCS with a generic FV in Sec. 2.1. The results in Secs. 3–5 include those for the conventional SU(2)CS [1,3,6] and their CSPI [30,31]; the latter follow from the former when we put $c_s = 1$ and $c_m = 0$ $(m \neq s)$, or $c_{-s} = 1$ and $c_m = 0$ $(m \neq -s)$ in later expressions.

3.1. *Construction of the general SU(2) coherent state*

The SU(2) or spin CS are constructed from the Lie algebra satisfying $\mathbf{S} \times \mathbf{S} = i\,\mathbf{S}$, where $\mathbf{S} \equiv (\hat{S}_1, \hat{S}_2, \hat{S}_3)$ is a matrix vector composed of the spin operators. The operators in $\mathbf{S}$ are also the infinitesimal operators of the irreducible representation $R^{(s)}(g)$ of $SO(3)$. Since $SU(2) \simeq SO(3)$ locally, we can also use $SO(3)$ to construct the SU(2) CS. Somewhat similar to the CCS, the SU(2)CS is defined by operating a rotation operator $\hat{R}(\mathbf{\Omega})$ with Euler angles $\mathbf{\Omega} \equiv (\phi, \theta, \psi)$, [e] which is the operator of $R^{(s)}(g)$, on a fixed vector $|\Psi_0\rangle$ in the Hilbert space of $R^{(s)}(g)$ [1−3]:

$$|\mathbf{\Omega}\rangle \equiv |\phi, \theta, \psi\rangle = \hat{R}(\mathbf{\Omega})|\Psi_0\rangle = \exp(-i\phi\hat{S}_3)\exp(-i\theta\hat{S}_2)\exp(-i\psi\hat{S}_3)|\Psi_0\rangle. \tag{16}$$

The vector $|\Psi_0\rangle$, called a FV, is taken as $|s, -s\rangle$ or $|s, s\rangle$ in the conventional choice. [1,3] CS with such FV are closest to the classical states and have various useful properties.[6] We appreciate them truly. According to the general theory of the CS, however, we have much wider possibilities in choosing a FV; and in fact it permits any normalized fixed vector in the Hilbert space.[2,5,6] Thus we can take $|\Psi_0\rangle$ as

$$|\Psi_0\rangle = \sum_{m=-s}^{s} c_m|m\rangle \qquad \text{with} \qquad \sum_{m=-s}^{s} |c_m|^2 = 1. \tag{17}$$

Hereafter $|m\rangle$ stands for $|s, m\rangle$. The FV will bring us all the information in later sections as far as the general theory goes. Looking at the problem in the light of physical applications, we need to take an appropriate $|\Psi_0\rangle$, i.e., $\{c_m\}$, for each system being considered. *We may consider CS with such FV as quantum states which have no classical analogues.* Exploring them surely enrich the understanding of the physical world. Eqs. (16)–(17) correspond to (1)–(3) for CCS. From Appendix A (ii) one can see $|\mathbf{\Omega}\rangle$ is defined on a 3-sphere S^3; it is specified by three real parameters, for which we take $\mathbf{\Omega} = (\phi, \theta, \psi)$. Notice that the reduction of the number of Euler angles is not always possible for an arbitrary $|\Psi_0\rangle$; for any s, $|\Psi_0\rangle$ is not necessarily reached from $|m\rangle$ via $\hat{R}(\mathbf{\Omega})$. Hence we use a full set of three Euler angles and proceed with it in what follows, which seems suitable for later discussions. When $|\Psi_0\rangle = |m\rangle$, we can eliminate ψ from $\mathbf{\Omega}$, thus yielding the spin CS with the phase space of a 2-sphere S^2, the Bloch sphere, labeled by two real parameters (θ, ϕ).

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[e] Hereafter we adapt the abbreviation $\mathbf{\Omega} \equiv (\phi, \theta, \psi)$ from Radcliffe[1] to describe a set of Euler angles which specifies the spin CS.

Having written $|\Psi_0\rangle$ in the form of (17), the SU(2)CS is represented by a linear combination of a set of the vectors $\{|m\rangle\}$ as:

$$|\mathbf{\Omega}\rangle = \sum_{m=-s}^{s} c_m |\mathbf{\Omega}, m\rangle \tag{18}$$

with

$$\begin{aligned} |\mathbf{\Omega}, m\rangle \equiv \hat{R}(\mathbf{\Omega})|m\rangle &= \sum_{m'=-s}^{s} \mathsf{R}^{(s)}_{m'm}(\mathbf{\Omega})\ |m'\rangle \\ &= \sum_{m'=-s}^{s} \exp[-i(m'\phi + m\psi)]\ \mathsf{r}^{(s)}_{m'm}(\theta)\ |m'\rangle. \end{aligned} \tag{19}$$

See Appendix A (i) for the definitions of $\mathsf{R}^{(s)}_{m'm}$ and $\mathsf{r}^{(s)}_{m'm}$. The form of (18)–(19) is valuable for later arguments. One may see that Eqs. (18)–(19) are analogues of Eqs. (4)–(5) for CCS; the ket $|\mathbf{\Omega}, m\rangle$, which corresponds to DNS $|\alpha, n\rangle$, may be called the "rotated spin number state". [f] We once treated $|\mathbf{\Omega}, m\rangle$ and its CSPI in Ref. 33.

The state $|\mathbf{\Omega}\rangle$ may be named the "extended spin CS", yet we will call it just "the CS" or the "general spin CS" in this paper since there have been some arguments about the choice of such a FV [5,6] and the CSPI.[34] [g] We take a simple strategy for the SU(2)CSPI evolving from arbitrary FV here and we will give their explicit forms.

3.2. *Resolution of unity*

The most important property that the CS enjoy is the "overcompleteness relation" or "resolution of unity" which plays a central role in performing the path integration. We have the relation (6) for the CCS. In the present spin CS case, it is expressed as follows.

Theorem 1. *For $|\mathbf{\Omega}\rangle$ with arbitrary FV, we have the resolution of unity:*

$$\int |\mathbf{\Omega}\rangle d\mu(\mathbf{\Omega}) \langle\mathbf{\Omega}| = \mathbf{1} \tag{20}$$

with

$$d\mu(\mathbf{\Omega}) \equiv \frac{2s+1}{8\pi^2}\, d\mathbf{\Omega} \qquad \text{and} \qquad d\mathbf{\Omega} \equiv \sin\theta\, d\theta d\phi d\psi. \tag{21}$$

[f] We may, instead, call it the "rotated magnetic quantum number state" borrowing the term from spectroscopy. [32] However, we feel that the "rotated spin number state" sounds like a generic term and appropriate for a wide variety of spin systems.

[g] It is reviewed in Ref. 35. The authors of Ref. 34 constructed "universal propagators" for various Lie group cases, being independent of the representations, which yields a different action from ours.

For simplicity, we have neglected the difference between an integer s and a half-integer s, which is not essential. Concerning the proof, there is an abstract way making full use of Schur's lemma. [2,5,6] However, we propose proving it by a slightly concrete method which is a natural extension of that used for the original spin CS. [1,3] For it indicates clearly what is to be changed when we use a general $|\Psi_0\rangle$.

Proof: We see from (18)–(19) $\langle \mathbf{\Omega}| = \sum_{\tilde{m}=-s}^{s} \sum_{m''=-s}^{s} c_{\tilde{m}}^{*} \left(\mathsf{R}^{(s)}_{m''\tilde{m}}(\mathbf{\Omega})\right)^{*} \langle m''|$. Then, with the aid of (A.10) we have:

$$\begin{aligned}
\int |\mathbf{\Omega}\rangle d\mathbf{\Omega} \langle \mathbf{\Omega}| &= \sum_{m=-s}^{s} \sum_{\tilde{m}=-s}^{s} c_m c_{\tilde{m}}^{*} \Big\{ \sum_{m'=-s}^{s} \sum_{m''=-s}^{s} \Big[\int_0^{\pi} d\theta \ \sin\theta \\
&\quad \times \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \left(\mathsf{R}^{(s)}_{m''\tilde{m}}(\mathbf{\Omega})\right)^{*} \ \mathsf{R}^{(s)}_{m'm}(\mathbf{\Omega}) \Big] |m'\rangle\langle m''| \Big\} \\
&= \sum_{m=-s}^{s} \sum_{\tilde{m}=-s}^{s} c_m c_{\tilde{m}}^{*} \Big(\sum_{m'=-s}^{s} \sum_{m''=-s}^{s} \frac{8\pi^2}{2s+1} \ \delta_{m'',m'} \delta_{\tilde{m},m} |m'\rangle\langle m''| \Big) \\
&= \frac{8\pi^2}{2s+1} \sum_{m=-s}^{s} c_m c_m^{*} \Big(\sum_{m'=-s}^{s} |m'\rangle\langle m'| \Big) \\
&= \frac{8\pi^2}{2s+1} \Big(\sum_{m=-s}^{s} |c_m|^2 \Big) \ \mathbf{1} = \frac{8\pi^2}{2s+1} \ \mathbf{1},
\end{aligned} \tag{22}$$

which is exactly what we wanted. □

3.3. *Overlap of two coherent states*

The overlap of two CS $|\mathbf{\Omega}_\ell\rangle \equiv |\phi_\ell, \theta_\ell, \psi_\ell\rangle = \sum_{m_\ell=-s}^{s} c_{m_\ell} |\phi_\ell, \theta_\ell, \psi_\ell; m_\ell\rangle$ ($\ell = 1, 2$) is one of those important quantities which we employ for various calculations in the CS. It can be derived, with the help of (16), (A.1) and (A.9), as:

$$\begin{aligned}
\langle \mathbf{\Omega}_2 | \mathbf{\Omega}_1 \rangle &= \sum_{m_1=-s}^{s} \sum_{m_2=-s}^{s} c_{m_1} c_{m_2}^{*} \langle m_2 | \hat{R}(-\psi_2, -\theta_2, -\phi_2) \hat{R}(\phi_1, \theta_1, \psi_1) | m_1 \rangle \\
&= \sum_{m_1=-s}^{s} \sum_{m_2=-s}^{s} c_{m_1} c_{m_2}^{*} \mathsf{R}^{(s)}_{m_2 m_1}(\mathbf{\Omega}_3) \\
&= \sum_{m_1=-s}^{s} \sum_{m_2=-s}^{s} c_{m_1} c_{m_2}^{*} \exp[-i(m_2\varphi + m_1\chi)] \ \mathsf{r}^{(s)}_{m_2 m_1}(\vartheta),
\end{aligned} \tag{23}$$

where (A.2) gives the form of $\mathsf{r}^{(s)}_{m_2 m_1}(\vartheta)$ and $\mathbf{\Omega}_3 \equiv (\varphi, \vartheta, \chi)$ is determined by (A.11) if we replace $\tilde{\mathbf{\Omega}}$ with $\mathbf{\Omega}_3$. It is easy to see that any state $|\mathbf{\Omega}\rangle$ is normalized to unity, as conforms to our construction of the CS.

3.4. *Typical matrix elements*

Typical matrix elements that we may employ in later are:

$$\begin{cases} \langle \mathbf{\Omega} | \hat{S}_3 | \mathbf{\Omega} \rangle = A_0(\{c_m\}) \cos\theta - A_1(\psi; \{c_m\}) \sin\theta \\ \langle \mathbf{\Omega} | \hat{S}_+ | \mathbf{\Omega} \rangle = A_0(\{c_m\}) \sin\theta \exp(i\phi) + A_2(\mathbf{\Omega}; \{c_m\}) = \langle \mathbf{\Omega} | \hat{S}_- | \mathbf{\Omega} \rangle^*, \end{cases} \tag{24}$$

where $\hat{S}_\pm = \hat{S}_1 \pm i\hat{S}_2$ and

$$\begin{cases} A_0(\{c_m\}) = \sum_{m=-s}^{s} m|c_m|^2 \\ A_1(\psi; \{c_m\}) = \frac{1}{2} \sum_{m=-s+1}^{s} f(s,m)[c_m^* c_{m-1} \exp(i\psi) + c_m c_{m-1}^* \exp(-i\psi)] \\ A_2(\mathbf{\Omega}; \{c_m\}) = \frac{1}{2} \sum_{m=-s+1}^{s} f(s,m) \exp(i\phi)\{(1+\cos\theta)\exp(i\psi) c_m^* c_{m-1} \\ \qquad\qquad -(1-\cos\theta)\exp(-i\psi) c_m c_{m-1}^*\} \\ f(s,m) = [(s+m)(s-m+1)]^{1/2}. \end{cases} \tag{25}$$

By $\{c_m\}$ we mean a set of the coefficients of a FV. We can easily verify (24) by (16) and (A.8).

Generating functions for general matrix elements exist as in the original CS cases. [3] In the normal product form it reads

$$\begin{aligned} X_N(z_+, z_3, z_-) &\equiv \langle \mathbf{\Omega}_2 | \exp(z_+ \hat{S}_+) \exp(z_3 \hat{S}_3) \exp(z_- \hat{S}_-) | \mathbf{\Omega}_1 \rangle \\ &= \langle \Psi_0 | \hat{R}^+(\mathbf{\Omega}_2) \hat{R}(\mathbf{\Omega}) \hat{R}(\mathbf{\Omega}_1) | \Psi_0 \rangle = \langle \Psi_0 | \hat{R}(\mathbf{\Omega}'') | \Psi_0 \rangle, \end{aligned} \tag{26}$$

where $\mathbf{\Omega}$ is related to z_ℓ ($\ell = +, 3, -$) through (A.7); and $\mathbf{\Omega}''$ is determined by (A.12) with suitable changes. In principle any matrix elements can be obtained from (26) via partial differentiations with respect to appropriate variables z_i ($i = \pm, 3$).

4. Path Integral via the General Spin CS

We now give the PI expressions in terms of the general spin CS evolving from an arbitrary FV defined in Sec. 3. In Sec. 4.1 the PI form is given. It is proved in the following Sec. 4.2. Some specific aspects of the Lagrangian are discussed in Secs. 4.3–4.5.

4.1. *Path integrals*

In this section we will give the explicit PI expression of the transition amplitude by means of the CS discussed in Sec. 3. What we need is the propagator $K(\mathbf{\Omega}_f, t_f; \mathbf{\Omega}_i, t_i)$ which starts from $|\mathbf{\Omega}_i\rangle$ at $t = t_i$, evolves under the effect of the Hamiltonian $\hat{H}(\hat{S}_+, \hat{S}_-, \hat{S}_3; t)$ which is assumed to be a function of $\hat{S}_+$, $\hat{S}_-$ and $\hat{S}_3$ with a suitable operator ordering and ends up with $|\mathbf{\Omega}_f\rangle$ at $t = t_f$:

$$K(\mathbf{\Omega}_f, t_f; \mathbf{\Omega}_i, t_i) = \langle \mathbf{\Omega}_f, t_f | \mathbf{\Omega}_i, t_i \rangle = \langle \mathbf{\Omega}_f | \mathrm{T} \exp\Big[-(i/\hbar) \int_{t_i}^{t_f} \hat{H}(t)\, dt\Big] | \mathbf{\Omega}_i \rangle, \tag{27}$$

where T denotes the time-ordered product. The overcompleteness relation (20) affords us the well-known prescription of formal CSPI [5,8] to give:

$$K(\mathbf{\Omega}_f, t_f; \mathbf{\Omega}_i, t_i) = \int \exp\{(i/\hbar) S[\mathbf{\Omega}(t)]\}\, \mathcal{D}[\mathbf{\Omega}(t)], \tag{28}$$

where

$$S[\mathbf{\Omega}(t)] \equiv \int_{t_i}^{t_f} \Big[\, \langle \mathbf{\Omega}| i\hbar \frac{\partial}{\partial t} |\mathbf{\Omega}\rangle - H(\mathbf{\Omega}, t) \, \Big] \, dt \equiv \int_{t_i}^{t_f} L(\mathbf{\Omega}, \dot{\mathbf{\Omega}}, t) \, dt \tag{29}$$

with

$$H(\mathbf{\Omega}, t) \equiv \langle \mathbf{\Omega}|\hat{H}|\mathbf{\Omega}\rangle \tag{30}$$

and we symbolized

$$\mathcal{D}[\mathbf{\Omega}(t)] \equiv \lim_{N\to\infty} \prod_{j=1}^{N} d\mu(\mathbf{\Omega}_{t_j}) \equiv \prod_{t} \frac{8\pi^2}{2s+1} \, [\sin\theta(t) d\theta(t) d\phi(t) d\psi(t)]. \tag{31}$$

The explicit form of the Lagrangian yields

$$L(\mathbf{\Omega}, \dot{\mathbf{\Omega}}, t) = \hbar \Big[A_0(\{c_m\})(\dot{\phi}\cos\theta + \dot{\psi}) + A_3(\mathbf{\Omega}, \dot{\mathbf{\Omega}}; \{c_m\}) \Big] - H(\mathbf{\Omega}, t), \tag{32}$$

where

$$A_3(\mathbf{\Omega}, \dot{\mathbf{\Omega}}; \{c_m\}) \equiv -A_1(\psi; \{c_m\}) \, \dot{\phi} \sin\theta + A_4(\psi; \{c_m\}) \, \dot{\theta} \tag{33}$$

with $A_1(\psi; \{c_m\})$ in (25) and

$$A_4(\psi; \{c_m\}) \equiv \frac{1}{2i} \sum_{m=-s+1}^{s} f(s, m)[c_m^* c_{m-1} \exp(i\psi) - c_m c_{m-1}^* \exp(-i\psi)]. \tag{34}$$

The formal proof of (32) is best carried out by the use of the identity:

$$\begin{aligned} \hat{R}^{+}(\mathbf{\Omega}) \frac{\partial}{\partial t} \hat{R}(\mathbf{\Omega}) &= \hat{R}^{+}(\mathbf{\Omega}) \Big(\, \dot{\phi} \frac{\partial}{\partial \phi} + \dot{\theta} \frac{\partial}{\partial \theta} + \dot{\psi} \frac{\partial}{\partial \psi} \, \Big) \hat{R}(\mathbf{\Omega}) \\ &= -i(\dot{\phi}\cos\theta + \dot{\psi})\hat{S}_3 + \frac{1}{2}(i\dot{\phi}\sin\theta - \dot{\theta}) \exp(i\psi)\hat{S}_+ \\ &\qquad + \frac{1}{2}(i\dot{\phi}\sin\theta + \dot{\theta}) \exp(-i\psi)\hat{S}_- \end{aligned} \tag{35}$$

and (24). Since the relation (35) is independent of s, it can be readily verified by the use of a 2×2 matrix (A.4). The detailed and substantial proof of (32) is given in the following subsection Sec. 4.2.

4.2. *From discrete to continuous path integrals*

We can justify the spin CSPI in Sec. 4.1 by showing the process from the discrete PI to the continuous ones. The proof is a straightforward generalization of that in Ref. 33. The method below may be applied to other CSPI; for example, the SU(1, 1) or SU(3) cases.

Theorem 2. *The quantum evolution of a physical system in terms of the general SU(2) CS is represented by* (28)–(34)*.*

Proof: By dividing the time interval into infinite numbers of an infinitesimal one ϵ in (27) and the successive use of the overcompleteness relation, i.e., Eq. (20) in Theorem 1, we obtain:

$$\begin{aligned} K(\mathbf{\Omega}_f, t_f; \mathbf{\Omega}_i, t_i) &= \langle \mathbf{\Omega}_f | \, \mathrm{T} \exp\Big[-(i/\hbar) \int_{t_i}^{t_f} \hat{H}(t)\, dt\Big] \, |\mathbf{\Omega}_i\rangle \\ &= \lim_{N\to\infty} \int d\mu(\mathbf{\Omega}_1) \cdots \int d\mu(\mathbf{\Omega}_N) \langle \mathbf{\Omega}_f, t_f | \mathbf{\Omega}_N, t_N\rangle \cdots \\ &\qquad \times \langle \mathbf{\Omega}_j, t_j | \mathbf{\Omega}_{j-1}, t_{j-1}\rangle \cdots \langle \mathbf{\Omega}_1, t_1 | \mathbf{\Omega}_i, t_i\rangle, \end{aligned} \tag{36}$$

where $\epsilon = [1/(N+1)](t_f - t_i)$ and $t_j = t_i + j\epsilon$. It is clear that we only have to consider a propagator during an infinitesimal time interval, which gives

$$\begin{aligned} \langle \mathbf{\Omega}_j, t_j | \mathbf{\Omega}_{j-1}, t_{j-1}\rangle &\equiv \langle \mathbf{\Omega}_j | \, \mathrm{T} \exp\Big[-(i/\hbar) \int_{t_{j-1}}^{t_j} \hat{H}(t)\, dt\Big] \, |\mathbf{\Omega}_{j-1}\rangle \\ &\simeq \langle \mathbf{\Omega}_j | \Big(1 - (i/\hbar) \int_{t_{j-1}}^{t_j} dt \; \hat{H}(\hat{S}_+, \hat{S}_-, \hat{S}_3; t) \Big) |\mathbf{\Omega}_{j-1}\rangle \\ &= \langle \mathbf{\Omega}_j | \mathbf{\Omega}_{j-1}\rangle \Big(1 - (i/\hbar)\, \epsilon \, H(\mathbf{\Omega}_j, \mathbf{\Omega}_{j-1}; t_{j-1}) \Big) \\ &\simeq \exp[\ln \langle \mathbf{\Omega}_j | \mathbf{\Omega}_{j-1}\rangle] \cdot \exp\Big[-(i/\hbar)\, \epsilon \, H(\mathbf{\Omega}_j, \mathbf{\Omega}_{j-1}; t_{j-1}) \Big], \end{aligned} \tag{37}$$

where

$$H(\mathbf{\Omega}'', \mathbf{\Omega}'; t) \equiv \frac{\Big\langle \mathbf{\Omega}'' \Big| \hat{H}(\hat{S}_+, \hat{S}_-, \hat{S}_3; t) \Big| \mathbf{\Omega}' \Big\rangle}{\langle \mathbf{\Omega}'' | \mathbf{\Omega}' \rangle}. \tag{38}$$

Our next task is to compute the infinitesimal overlap $\langle \mathbf{\Omega}_j | \mathbf{\Omega}_{j-1}\rangle$ in (37). Representing $|\mathbf{\Omega}_j\rangle = \sum_m c_m \hat{R}(\mathbf{\Omega}_j)|m\rangle$ and $|\mathbf{\Omega}_{j-1}\rangle = \sum_{m'} c_{m'} \hat{R}(\mathbf{\Omega}_{j-1})|m'\rangle$, we see from (23)

$$\begin{aligned} \langle \mathbf{\Omega}_j | \mathbf{\Omega}_{j-1}\rangle &= \sum_{m=-s}^{s} \sum_{m'=-s}^{s} c_m^* c_{m'} \langle m | \hat{R}(\tilde{\mathbf{\Omega}}_j) | m' \rangle \\ &= \sum_{m=-s}^{s} \sum_{m'=-s}^{s} c_m^* c_{m'} \exp[-i(m\tilde{\phi}_j + m'\tilde{\psi}_j)] \, \mathsf{r}_{mm'}(\tilde{\theta}_j), \end{aligned} \tag{39}$$

where $\tilde{\mathbf{\Omega}}_j \equiv (\tilde{\phi}_j, \tilde{\theta}_j, \tilde{\psi}_j)$ satisfies the same relation (23) as $\mathbf{\Omega}_3$ if we put $\mathbf{\Omega}_2 = \mathbf{\Omega}_j = (\phi_j, \theta_j, \psi_j)$ and $\mathbf{\Omega}_1 = \mathbf{\Omega}_{j-1} = (\phi_{j-1}, \theta_{j-1}, \psi_{j-1})$. Searching for concrete relations between $\tilde{\mathbf{\Omega}}_j$, $\mathbf{\Omega}_j$ and $\mathbf{\Omega}_{j-1}$, we are brought to (A.11). Then, by the use of the relation: $\Delta\theta_j \equiv \theta_j - \theta_{j-1} \simeq \dot{\theta}_j \epsilon, \Delta\phi_j \equiv \phi_j - \phi_{j-1} \simeq \dot{\phi}_j \epsilon$ and $\Delta\psi_j \equiv \psi_j - \psi_{j-1} \simeq \dot{\psi}_j \epsilon$ we have, to $\mathrm{O}(\epsilon)$,

$$\begin{cases} \tilde{\theta}_j \simeq \Delta\theta_j, \quad \cos(\frac{1}{2}\tilde{\theta}_j) \simeq 1, \quad \sin(\frac{1}{2}\tilde{\theta}_j) \simeq \frac{1}{2}\sin(\tilde{\theta}_j) \simeq \mathrm{O}(\epsilon), \\ \sin\tilde{\theta}_j \exp(i\tilde{\phi}_j) \simeq -\exp(-i\psi_j)(\Delta\theta_j + i\Delta\phi_j \sin\theta_j), \\ \exp[i(\tilde{\phi}_j + \tilde{\psi}_j)] \simeq 1 - i(\Delta\phi_j \cos\theta_j + \Delta\psi_j). \end{cases} \tag{40}$$

The relation (40) above is the master key to the proof which we invoke implicitly in (42)–(44) below. Now we have from (A.2)

$$\mathsf{r}_{mm'}(\tilde{\theta}_j) = \sum_t N(s,m,m';t) \cdot [\cos(\tilde{\theta}_j/2)]^{(s+m-t)+(s-m'-t)} \times [\sin(\tilde{\theta}_j/2)]^{(t)+(t-m+m')}. \qquad (41)$$

From the restrictions on the factorials in $N(s,m,m';t)$ in (A.3) all the terms in the parentheses in the exponents in (41) have to be zero or more. Then it becomes apparent that we only need to pick up, to $\mathrm{O}(\epsilon)$, the terms fulfilling the combinations $(t, t-m+m') = (0,0), (0,1), (1,0)$, from which the following three cases appear: (i) $t=0, m=m'$, (ii) $t=0, m'=m+1$, (iii) $t=1, m'=m-1$. Evaluating terms in (39), the following expressions for each case are derived by (40), (41) and (A.3). First, consider the case (i). Notice that $N(s,m,m'=m;t=0)=1$ and $\mathsf{r}_{mm}(\tilde{\theta}_j) \simeq 1 + \mathrm{O}\Big([\Delta(\theta_j)]^2\Big)$. The relevant terms are:

$$\sum_{m=-s}^{s} |c_m|^2 \exp[-i\, m(\tilde{\phi}_j + \tilde{\psi}_j)]\, \mathsf{r}_{mm}(\tilde{\theta}_j) \simeq 1 + i\, A_0(\{c_m\})(\Delta\phi_j \cos\theta_j + \Delta\psi_j). \qquad (42)$$

Second, for the case (ii) $N(s,m,m'=m+1;t=0)=f(s,m')$; see (25) for the definition of $f(s,m)$. So one finds that the corresponding terms become:

$$\begin{aligned}
&\frac{1}{2}\sum_{m'=-s+1}^{s} c_{m'}c^*_{m'-1} \exp[-i\, m'(\tilde{\phi}_j + \tilde{\psi}_j)] f(s,m') \cdot [\sin\tilde{\theta}_j \exp(i\,\tilde{\phi}_j)] \\
&\simeq \frac{1}{2}\sum_{m=-s+1}^{s} c_m c^*_{m-1}[1 + i\, m(\Delta\phi_j \cos\theta_j + \Delta\psi_j)] \\
&\qquad \times f(s,m)[-\exp(-i\psi_j)(\Delta\theta_j + i\,\Delta\phi_j \sin\theta_j)] \\
&\simeq -\frac{1}{2}\sum_{m=-s+1}^{s} c_m c^*_{m-1} f(s,m) \exp(-i\psi_j)(\Delta\theta_j + i\,\Delta\phi_j \sin\theta_j) + \mathrm{O}(\epsilon^2), \qquad (43)
\end{aligned}$$

where we have renamed m' m. Third, we can deal with the case (iii) just in the same manner as (ii). With $N(s,m,m'=m-1;t=1) = -f(s,m)$ the corresponding terms become:

$$\begin{aligned}
&-\frac{1}{2}\sum_{m=-s+1}^{s} c^*_m c_{m-1} f(s,m) \exp[-i\,(m-1)(\tilde{\phi}_j + \tilde{\psi}_j)] \cdot [\sin\tilde{\theta}_j \exp(-i\,\tilde{\phi}_j)] \\
&\simeq -\frac{1}{2}\sum_{m=-s+1}^{s} c^*_m c_{m-1}[1 + i\,(m-1)(\Delta\phi_j \cos\theta_j + \Delta\psi_j)] \\
&\qquad \times f(s,m)[-\exp(i\psi_j)(\Delta\theta_j - i\,\Delta\phi_j \sin\theta_j)] \\
&\simeq \frac{1}{2}\sum_{m=-s+1}^{s} c^*_m c_{m-1} f(s,m) \exp(i\psi_j)(\Delta\theta_j - i\,\Delta\phi_j \sin\theta_j) + \mathrm{O}(\epsilon^2). \qquad (44)
\end{aligned}$$

Eventually, putting the above results all together, we obtain the infinitesimal overlap:

$$\begin{aligned}\langle \mathbf{\Omega}_j | \mathbf{\Omega}_{j-1} \rangle \simeq 1 &+ i A_0(\{c_m\})(\Delta\phi_j \cos\theta_j + \Delta\psi_j) \\ &- \frac{1}{2} \sum_{m=-s+1}^{s} f(s,m)[c_m c_{m-1}^* \exp(-i\psi_j)(\Delta\theta_j + i\,\Delta\phi_j \sin\theta_j) \\ &- c_m^* c_{m-1} \exp(i\psi_j)(\Delta\theta_j - i\,\Delta\phi_j \sin\theta_j)]. \end{aligned} \tag{45}$$

Substituting (45) into (37) and then (37) into (36), we finally arrive at the expression to $\mathrm{O}(\epsilon)$:

$$K(\mathbf{\Omega}_f, t_f; \mathbf{\Omega}_i, t_i) = \lim_{N\to\infty} \int d\mu(\mathbf{\Omega}_1) \cdots \int d\mu(\mathbf{\Omega}_N) \exp[(i/\hbar)\, S_{1,N+1}] \tag{46}$$

with

$$\begin{aligned} S_{1,N+1} = \sum_{j=1}^{N+1} \Big\{ \hbar \Big[& A_0(\{c_m\})(\Delta\phi_j \cos\theta_j + \Delta\psi_j) \\ & - A_1(\psi_j; \{c_m\})\Delta\phi_j \sin\theta_j + A_4(\psi_j; \{c_m\})\Delta\theta_j \Big] \\ & - \epsilon\, H(\mathbf{\Omega}_j, \mathbf{\Omega}_{j-1}; t_{j-1}) \Big\}. \end{aligned} \tag{47}$$

See (25) and (34) for the definitions of A_1 and A_4 respectively. Hence, it is easy to see that the expressions (46)–(47) agree with those of (28)–(34) in the $\epsilon \to 0$ limit. □

We have thus arrived at *the generic expressions of the PI via the SU(2)CS*, i.e., (28)–(34), which constitute one of the main results of the present paper. They correspond to (9)–(14) for the CCS. The complex variable forms of spin CS and CSPI are presented Sec. 5. There, with the aid of the form, we can easily see the above correspondence by the contraction procedure. The special case when $c_m = 1$ (for a sole m), which includes the conventional SU(2)CSPI, was once treated in Ref. 33.

The transition amplitude between any two states $|i\rangle$ at $t = t_i$ and $|f\rangle$ at $t = t_f$ can be evaluated by:

$$\iint d\mu(\mathbf{\Omega}_f) d\mu(\mathbf{\Omega}_i)\ \langle f | \mathbf{\Omega}_f \rangle (\mathbf{\Omega}_f, t_f; \mathbf{\Omega}_i, t_i) \langle \mathbf{\Omega}_i | i \rangle. \tag{48}$$

Finally, we will make an auxiliary discussion on the derivation of CSPI. We often see a slightly different approach to PI in literature [36–38]; it is essentially after an original one due to Dirac. [39,40] Now let us proceed with it. Define a "moving frame" state vector, [36,37] which represents an intermediate state, as:

$$|\mathbf{\Omega}_j, t_j\rangle \equiv \hat{U}^+(t_j) |\mathbf{\Omega}_j\rangle \qquad \text{with} \qquad \hat{U}(t_j) \equiv \mathrm{T} \exp\Big[-(i/\hbar) \int_{t_i}^{t_j} \hat{H}(t)\, dt \Big]. \tag{49}$$

Then the resolution of unity also holds for $|\boldsymbol{\Omega}_j, t_j\rangle$ whose successive use leads us formally to the same expression as (36). However, this time we have, as a *consequence* of (49),

$$\langle \boldsymbol{\Omega}_j, t_j | \boldsymbol{\Omega}_{j-1}, t_{j-1} \rangle = \langle \boldsymbol{\Omega}_j | \, \mathrm{T} \exp\Big[-(i/\hbar) \int_{t_{j-1}}^{t_j} \hat{H}(t) \, dt \Big] \, | \boldsymbol{\Omega}_{j-1} \rangle. \tag{50}$$

In contrast the same expression is used as a *definition* in (37). Note that in $\hat{U}^{+}(t_j)$ the order of a set of operators $\{\exp[(i/\hbar) \int_{t_{j-1}}^{t_j} \hat{H}(t) dt]\}$ is reversed to that in $\hat{U}(t_j)$; it becomes anti-chronological. The residuary procedure to CSPI is the same as the former.

In the phase space PI "moving frame" vectors $\{|q_j, t_j\rangle\}$ are defined as the eigenstates of the operators $\hat{q}(t_j) \equiv \hat{U}^{+}(t_j) \hat{q} \hat{U}(t_j)$ in the Heisenberg picture, [37,38] from which $|q_j, t_j\rangle = U^{+}(t_j)|q_j\rangle$ results. They are truly the precise descriptions of intermediate states. In the present spin PI case we can do the same thing for a simple FV $|\boldsymbol{\Psi}_0\rangle = |m\rangle$: $|\boldsymbol{\Omega}_j, t_j\rangle$ may be defined as the eigenstate of the operator $[\hat{R}(\boldsymbol{\Omega}) \hat{S}_3 \hat{R}^{+}(\boldsymbol{\Omega})]_{t=t} \equiv \hat{U}^{+}(t) \cdot [\hat{R}(\boldsymbol{\Omega}) \hat{S}_3 \hat{R}^{+}(\boldsymbol{\Omega})]_{t=0} \cdot \hat{U}(t)$ which is in the Heisenberg picture. It corresponds to (8) for the CCS. However, for a generic FV, we are not able to interpret $|\boldsymbol{\Omega}_j, t_j\rangle$ as eigenvectors of some operators no more. And thus we have adopted the former approach which seems more plausible to the generic CSPI in the sense. Of course, *for any FV* CS are clearly defined and the overcompleteness relation holds as (20); indeed it is almost the only relation that CS enjoy. [5] So we can perform PI as we saw it. The relation is the fundamental feature of CS that makes CS such a flexible tool for PI and that makes CSPI so fascinating.

4.3. *The topological term*

The term with the square brackets in the Lagrangian (32),

$$A_0(\{c_m\})(\dot{\phi} \cos\theta + \dot{\psi}) + A_3(\boldsymbol{\Omega}, \dot{\boldsymbol{\Omega}}; \{c_m\}), \tag{51}$$

stemming from $\langle \boldsymbol{\Omega} | (\partial/\partial t) | \boldsymbol{\Omega} \rangle$, may be called the "topological term" that is related to the geometric phases. [h] Here the A_3-term is given by (33). And one can see that the first term in the topological term gives a description of monopoles *à la* BMS^2.[25,26,42] The fictitious gauge potentials corresponding to the whole topological term are also nonsingular as Refs. 25, 26 and 42; see Sec. 4.5 for the point.

In the differential 1-form, the whole topological term κ reads:

$$\kappa = A_0(\{c_m\}) \, (\cos\theta \, d\phi + d\psi) - A_1(\psi; \{c_m\}) \, \sin\theta \, d\phi + A_4(\psi; \{c_m\}) \, d\theta \tag{52}$$

[h] The significance of the term was once recognized by Kuratsuji, who called it the "canonical term", in relation to the semiclassical quantization; note that the geometric phase associated with the term was called the "canonical phase" in Ref. 22 and Ref. 41; see Ref. 41 and references therein. We call them just the geometric phases in the present paper.

and in the 2-form:

$$d\kappa = -\Big[A_0(\{c_m\})\sin\theta + A_1(\psi;\{c_m\})\cos\theta\Big]\, d\theta\wedge d\phi \\ -A_4(\psi;\{c_m\})\,\sin\theta\, d\phi\wedge d\psi + A_1(\psi;\{c_m\})\, d\psi\wedge d\theta. \qquad (53)$$

One may see that *the strength of the well-known monopole-type term depends on* A_0, i.e., the expectation value of the quantum number m in the state of $|\Psi_0\rangle$. *In addition we have other fields with* A_1 *and* A_4*-terms describing the effects of interweaving coefficients of* $|\Psi_0\rangle$ *with their next ones.* We have thus obtained the *general expressions of the topological term in the SU(2)CSPI*, i.e., (52)–(53), which are also one of the main results of the present paper.

For a FV with $c_m = 1$ (for a sole m), since A_1 and A_4-terms vanish we have

$$\kappa = m\,(\cos\theta\, d\phi + d\psi), \qquad d\kappa = -m\sin\theta\, d\theta\wedge d\phi, \qquad (54)$$

which represents a monopole with the strength m.

4.4. *Semiclassical limit*

In this subsection we will investigate what information the semiclassical limit of CSPI brings. In the situation where $\hbar \ll S[\mathbf{\Omega}(t)]$, the principal contribution in (28) comes from the path that satisfies $\delta S = 0$, which requires the Euler-Lagrange equations for $L(\mathbf{\Omega}, \dot{\mathbf{\Omega}}, t)$. Then we obtain

$$\begin{cases} \hbar\{[A_0(\{c_m\})\sin\theta + A_1(\psi;\{c_m\})\cos\theta]\dot\phi + A_1(\psi;\{c_m\})\dot\psi\} = -(\partial H/\partial\theta) \\ \hbar\{[A_0(\{c_m\})\sin\theta + A_1(\psi;\{c_m\})\cos\theta]\dot\theta - [A_4(\psi;\{c_m\})\sin\theta]\dot\psi\} = \partial H/\partial\phi \\ \hbar\{[A_4(\psi;\{c_m\})\sin\theta]\dot\phi + A_1(\psi;\{c_m\})\dot\theta\} = \partial H/\partial\psi, \end{cases} \qquad (55)$$

where A_0 and A_1 are given by (25) and A_4 is defined by (34).

The expressions in (55) are the variational equations for the spin CS parameters $\mathbf{\Omega}$, which may be compared with (15) in CCSPI.

The special case, i.e., that for SU(2)CS with a FV $|\Psi_0\rangle = |m\rangle$, was once treated in Ref. 33; putting $|\Psi_0\rangle = |-s\rangle$ brings us back to the results for the original case.[30,31]

4.5. *The nature of fictitious gauge potentials*

We now investigate the nature of fictitious gauge potentials corresponding to the whole topological term κ in (52). We follow the strategy in Ref. 42.

Using the orthogonal coordinates:

$$\xi = \phi + \psi, \qquad \eta = \phi - \psi \qquad (56)$$

with (A.5), the metric is:

$$ds^2 = d\mathbf{x}^2 = \frac{1}{4}[d\theta^2 + \cos^2(\theta/2)\, d\xi^2 + \sin^2(\theta/2)\, d\eta^2] \equiv d\mathbf{n}^2, \qquad (57)$$

where $\mathbf{n}$ stands for a unit vector in the (θ, ξ, η)-coordinates. Now let us call the fictitious gauge potential $\tilde{\mathbf{A}} = (\tilde{A}_\theta, \tilde{A}_\xi, \tilde{A}_\eta)$; we use $\tilde{\mathbf{A}}$ so as not to confuse them with A_i $(i = 1, \cdots, 4)$-terms in Secs. 3 –4. Then we have:

$$\kappa = \tilde{\mathbf{A}} \cdot d\mathbf{n} = \tilde{A}_\theta \, \frac{1}{2} d\theta + \tilde{A}_\xi \, \frac{1}{2} \cos(\theta/2) d\xi + \tilde{A}_\eta \, \frac{1}{2} \sin(\theta/2) d\eta. \tag{58}$$

Thus we obtain:

$$\begin{cases} \tilde{A}_\theta = 2A_4\Big(\tfrac{1}{2}(\xi - \eta), \{c_m\}\Big), \\ \tilde{A}_\xi = 2A_0(\{c_m\}) \cos(\tfrac{1}{2}\theta) - 2A_1\Big(\tfrac{1}{2}(\xi - \eta), \{c_m\}\Big) \sin(\tfrac{1}{2}\theta), \\ \tilde{A}_\eta = -2A_0(\{c_m\}) \sin(\tfrac{1}{2}\theta) - 2A_1\Big(\tfrac{1}{2}(\xi - \eta), \{c_m\}\Big) \cos(\tfrac{1}{2}\theta), \end{cases} \tag{59}$$

which are evidently nonsingular.

5. Complex Variable Parametrizations of the Spin CS

The generic spin CS and CSPI in Secs. 3 and 4 can be put into complex variable forms like the conventional ones. [1–3] The number of complex variables is, however, twice. This causes a need for a supplementary condition to recover the proper degrees of freedom of $\mathbf{\Omega}$. These problems are discussed in Sec. 5.1. Next, we employ the complex variable form to illustrate the contraction procedure from the generic spin CS and PI to the corresponding CCS and PI (Sec. 5.2). Besides we add another complex variable form (Sec. 5.3).

5.1. *Complex variable form via Gaussian decomposition*

We can parametrize the spin CS, $|\mathbf{\Omega}\rangle$, by a pair of complex variables $\mathbf{z} \equiv (z_+, z_-)$ and its complex conjugate $\mathbf{z}^* \equiv (z_+^*, z_-^*)$ via the "Gaussian decomposition" of the operator $\hat{R}(\mathbf{\Omega})$, i.e., (A.6)–(A.7):

$$|\mathbf{\Omega}\rangle = |\mathbf{z}\rangle \equiv \hat{R}(\mathbf{z})|\Psi_0\rangle \equiv \hat{R}(z_+, z_3, z_-)|\Psi_0\rangle. \tag{60}$$

We put $\hat{R}(\mathbf{z}) = \hat{R}(z_+, z_3, z_-)$ since from (A.7) z_3 is a function of $\mathbf{z}$:

$$\exp(-z_3/2) = i \, \{ \, z_+^* z_-^* \, / \, [\, |z_+|^2 (1 + |z_+|^2)^2 \,] \, \}^{1/2}. \tag{61}$$

However, we know that the degrees of freedom of the CS, i.e., those of $\mathbf{\Omega}$, are three; and thus it is clear that the representation by $\mathbf{z}$ and $\mathbf{z}^*$ is still redundant. We may remedy the problem by reducing the degrees of freedom with the aid of a subsidiary condition: $|z_+| = |z_-|$ from (A.7).

For the resolution of unity we have:

$$\int |\mathbf{z}\rangle d\nu(\mathbf{z}) \langle \mathbf{z}| = \mathbf{1}, \tag{62}$$

where

$$d\nu(\mathbf{z}) = \frac{2s+1}{2\pi^2} \, \frac{\delta(|z_+| - |z_-|)}{|z_+| \, (1 + |z_+|^2)^2} \, d^2(z_+) d^2(z_-) \tag{63}$$

and $d^2(z_\ell) \equiv d(\mathrm{Re} z_\ell) d(\mathrm{Im} z_\ell)$ $(\ell = +, -)$.

The propagator reads:

$$K(\mathbf{z}_f, t_f; \mathbf{z}_i, t_i) = \int \exp\{(i/\hbar) S[\mathbf{z}(t)]\} \, \mathcal{D}[\mathbf{z}(t)], \tag{64}$$

where

$$S[\mathbf{z}(t)] \equiv \int_{t_i}^{t_f} \Big[\, \langle \mathbf{z} | i\hbar \frac{\partial}{\partial t} | \mathbf{z} \rangle - H(\mathbf{z}, t) \, \Big] \, dt \equiv \int_{t_i}^{t_f} L(\mathbf{z}, \dot{\mathbf{z}}, t) \, dt \tag{65}$$

with

$$H(\mathbf{z}, t) \equiv \langle \mathbf{z} | \hat{H} | \mathbf{z} \rangle \qquad \text{and} \qquad \mathcal{D}[\mathbf{z}(t)] \equiv \lim_{N \to \infty} \prod_{j=1}^{N} d\nu(\mathbf{z}_{t_j}). \tag{66}$$

The explicit form of the Lagrangian yields

$$\begin{aligned} L(\mathbf{z}, \dot{\mathbf{z}}, t) = i\hbar \Big\{ \frac{1}{2|z_+|^2} \Big[A_0(\{c_m\}) \Big(\frac{1 - |z_+|^2}{1 + |z_+|^2} \, (z_+^* \dot{z}_+ - \dot{z}_+^* z_+) + (z_-^* \dot{z}_- - \dot{z}_-^* z_-) \Big) \Big] \\ + A_3(\mathbf{z}, \dot{\mathbf{z}}; \{c_m\}) \Big\} - H(\mathbf{z}, t), \end{aligned} \tag{67}$$

where

$$A_3(\mathbf{z}, \dot{\mathbf{z}}; \{c_m\}) \equiv \frac{1}{|z_+|^2 (1 + |z_+|^2)} \sum_{m=-s+1}^{s} f(s, m) (c_m c_{m-1}^* z_+ \dot{z}_+^* z_- - c.c.). \tag{68}$$

One can obtain the above relations (62)–(68), via (60)–(61) and (A.6)–(A.7), from the Euler angle parametrization forms in Secs. 3 and 4. Or one may confirm them by calculating $\langle \mathbf{z} | (\partial/\partial t) | \mathbf{z} \rangle$ directly. The formulae for the conventional CS, $|z\rangle = (1 + |z|^2)^{-1/2} \exp(z \hat{S}_+) |\Psi_0\rangle$ with $|\Psi_0\rangle = |-s\rangle$, follow by putting $z_- = -z_+^*$ and neglecting or integrating out $|z_-|$-variable and then replacing z_+ with z.

5.2. *High spin limit: contraction to the canonical CS*

It is well-known that in the high spin limit, i.e., $s \to \infty$, the conventional spin CS with $|\Psi_0\rangle = |s\rangle$ or $|-s\rangle$ approaches to the the usual CCS. [1,3,6] However, as we saw in Sec. 2, the CCS and CCSPI has been extended to an arbitrary FV case; and thus, as we put in Sec. 1, it is natural to ask whether there exits any spin CS and CSPI that tends to the CCS and CCSPI with an arbitrary FV. This has been one of the motivations mentioned in Sec. 1 to construct such general spin CS and CSPI as described in Secs. 3–5.1. The answer is affirmative:

Theorem 3. *The spin CS $|\mathbf{\Omega}\rangle$ and CSPI with a generic FV in Secs. 3–5.1, in the high spin limit, tend to the CCS $|\alpha\rangle$ and CCSPI described in Sec. 2.*

Proof: We adapt the method of Radcliffe[1] and Arecchi *et al.*[3] for a generic FV case. Following the high spin limit of the transformation *à la* Holstein-Primakoff,[43] let us put:

$$\hat{S}_+ \to (2s)^{1/2} \, \hat{a}^+, \quad \hat{S}_- \to (2s)^{1/2} \, \hat{a}, \quad \hat{S}_3 \to -s \, \mathbf{1} + \hat{a}^+ \hat{a}. \tag{69}$$

We also set:

$$z_+ \to \alpha\,(2s)^{-1/2}, \qquad z_- \to -\,z_+^*, \tag{70}$$

which, with (61), gives:

$$z_3 \to |\alpha|^2/(2s). \tag{71}$$

Then the combination of (69)–(71) with (A.6)–(A.7) and (2) produces

$$\hat{R}(\mathbf{\Omega}) = \hat{R}(\mathbf{z}) \longrightarrow \exp(\alpha\hat{a}^+)\exp\Big(-(1/2)|\alpha|^2\Big)\exp(-\alpha^*\hat{a}) = \hat{D}(\alpha). \tag{72}$$

Besides since from (69)

$$\hat{a}^+\hat{a}|n\rangle = n|n\rangle, \qquad \big(n \equiv m+s, |n\rangle \equiv \lim_{s\to\infty}|m\rangle\big), \tag{73}$$

we obtain:

$$|\Psi_0\rangle = \sum_{m=-s}^{s} c_m|m\rangle \longrightarrow \sum_{n=0}^{\infty} c_n|n\rangle, \tag{74}$$

where the numbering of the coefficients has been shifted. From (72) and (74) we obtain that:

$$|\mathbf{\Omega}\rangle = |\mathbf{z}\rangle = \hat{R}(\mathbf{z})|\Psi_0\rangle \longrightarrow \hat{D}(\alpha)\cdot\sum_{n=0}^{\infty} c_n|n\rangle = |\alpha\rangle, \tag{75}$$

which is precisely (1) in Sec. 2.1: the definition of the CCS with a generic FV.

Next, with the aid of (63), (70) and (75), we find that the left side hand of (62) becomes:

$$\begin{aligned}
\int |\mathbf{z}\rangle d\nu(\mathbf{z})\langle\mathbf{z}| &= \frac{2s+1}{2\pi^2}\int |\mathbf{z}\rangle\langle\mathbf{z}|\cdot\frac{\delta(|z_+|-|z_-|)}{|z_+|\,(1+|z_+|^2)^2}\cdot|z_+|\,|z_-| \\
&\qquad \times d(|z_+|)d(\arg z_+)\,d(|z_-|)d(\arg z_-) \\
&\to \frac{2s+1}{2\pi^2}\int |\mathbf{z}\rangle\langle\mathbf{z}|\cdot\frac{|z_+|}{(1+|z_+|^2)^2}\cdot 2\pi\cdot\delta\big(\arg z_- - (\arg z_+ - \pi)\big) \\
&\qquad \times d(|z_+|)d(\arg z_+)\,d(\arg z_-) \\
&= \frac{2s+1}{\pi}\int |z_+\rangle\langle z_+|\cdot\frac{|z_+|}{(1+|z_+|^2)^2}\,d(|z_+|)d(\arg z_+) \\
&\to \frac{1}{\pi}\frac{2s+1}{2s}\int |\alpha\rangle\langle\alpha|\cdot\frac{|\alpha|}{[1+\big(|\alpha|^2/(2s)\big)]^2}\,d(|\alpha|)d(\arg\alpha) \\
&\to \frac{1}{\pi}\int |\alpha\rangle d^2\alpha\langle\alpha|,
\end{aligned} \tag{76}$$

which shows that the resolution of unity for the spin CS, (20) or (62), tends to that for CCS (6). The arguments in the δ-function in (76) should be interpreted as "modulo 2π".

Now that we have both Eqs. (75) and (76), we see that all the results of the spin CS and CSPI here approach to those in Sec. 2, which completes the proof. □

We may also see the results from the complex variable PI expression (64)–(68) with the help of (70) and (71). To this end, notice that we have in $s \to \infty$ limit

$$A_0(\{c_m\}) = \sum_{m=-s}^{m=s} m|c_m|^2 = \sum_{m=-s}^{m=s} (n-s)|c_m|^2 \longrightarrow -s \tag{77}$$

and

$$f(s,m) \longrightarrow n^{1/2}(2s)^{1/2} \ . \tag{78}$$

Then we find that the Lagrangian (67), which is equivalent to (32), for the generic spin CSPI tends to (13) for the CCSPI. And the results in Secs. 3–5.1 are converted to those in Sec. 2. Especially, we see that the A_3-term in (33) and (68) corresponds to the A-term in (9).

The A_3-term is not represented as a total derivative; and thus the A_1- and A_4-terms in the A_3-term take part in variational equations (55) for the spin CS. It is merely in the high spin limit that the A_3-term, approaching to the A-term, becomes a total derivative and its effect disappears in the variational equations. Revisit Secs. 2.2.2 and 4.4 for the point.

5.3. *Another complex variable form*

We have another complex variable representation of the CS.[44] To this end we write $\hat{R}^{(1/2)}(\mathbf{\Omega})$ in (A.4), using a new pair of complex variables $\mathbf{a} = (a_1, a_2)$, in the form of

$$\hat{R}^{(1/2)}(\mathbf{\Omega}) = \hat{R}^{(1/2)}(\mathbf{a}) = \begin{pmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{pmatrix} \qquad \text{with} \qquad |a_1|^2 + |a_2|^2 = 1, \tag{79}$$

which is often used for the SU(2) group. We see from (A.4) and (79)

$$a_1 = \cos(\theta/2)\exp[-i(\phi+\psi)/2], \quad a_2 = \sin(\theta/2)\exp[i(\phi-\psi)/2]. \tag{80}$$

The spin CS, in this case, is specified by

$$|\mathbf{a}\rangle = \hat{R}(\mathbf{a})|\Psi_0\rangle \equiv \hat{R}(\mathbf{\Omega})|\Psi_0\rangle, \tag{81}$$

where $\mathbf{a}$ is related to $\mathbf{\Omega}$ via (80).

The resolution of unity becomes:

$$\int |\mathbf{a}\rangle d\lambda(\mathbf{a})\langle\mathbf{a}| = \mathbf{1}, \tag{82}$$

where

$$d\lambda(\mathbf{a}) = \frac{4(2s+1)}{\pi^2}\,\delta(\mathbf{a}^2 - 1)\,d^2\mathbf{a} \quad \text{and} \quad d^2\mathbf{a} \equiv d^2a_1 d^2a_2. \tag{83}$$

with $d^2(a_\ell) \equiv d(\mathrm{Re}\ a_\ell)\,d(\mathrm{Im}\ a_\ell)$ $(\ell = 1, 2)$. The δ-function leaves the degrees of freedom being three as (63).

The propagator reads:

$$K(\mathbf{a}_f, t_f; \mathbf{a}_i, t_i) = \int \exp\{(i/\hbar) S[\mathbf{a}(t)]\} \, \mathcal{D}[\mathbf{a}(t)], \tag{84}$$

where

$$S[\mathbf{a}(t)] \equiv \int_{t_i}^{t_f} \Big[\, \langle \mathbf{a} | i\hbar \frac{\partial}{\partial t} | \mathbf{a} \rangle - H(\mathbf{a}, t) \, \Big] \, dt \equiv \int_{t_i}^{t_f} L(\mathbf{a}, \dot{\mathbf{a}}, t) \, dt \tag{85}$$

with

$$H(\mathbf{a}, t) \equiv \langle \mathbf{a} | \hat{H} | \mathbf{a} \rangle \qquad \text{and} \qquad \mathcal{D}[\mathbf{a}(t)] \equiv \lim_{N\to\infty} \prod_{j=1}^{N} d\lambda(\mathbf{a}_{t_j}). \tag{86}$$

The explicit form of the Lagrangian yields:

$$\begin{aligned} L(\mathbf{a}, \dot{\mathbf{a}}, t) = i\hbar \Big[& A_0(\{c_m\}) \Big((a_1^* \dot{a}_1 - \dot{a}_1^* a_1) + (a_2^* \dot{a}_2 - \dot{a}_2^* a_2) \Big) \\ & + A_3(\mathbf{a}, \dot{\mathbf{a}}; \{c_m\}) \Big] - H(\mathbf{a}, t), \end{aligned} \tag{87}$$

where

$$A_3(\mathbf{a}, \dot{\mathbf{a}}; \{c_m\}) \equiv \sum_{m=-s+1}^{s} f(s,m) [c_m c_{m-1}^* (a_1 \dot{a}_2 - \dot{a}_1 a_2) - \text{c.c.}]. \tag{88}$$

We may also put the results in a real variable form using $\mathbf{x} \equiv (x_1, \cdots, x_4)$ via (79) and (A.5). In the case the restriction $\mathbf{x}^2 = 1$ keeps the degree of freedom of the CS being three.

6. Summary and Prospects

We have investigated a natural extension of the spin or SU(2)CS and their PI forms by using arbitrary FV, which turns out to be performed successfully.

In the present paper we have worked on the basic formulation. The physical applications, in relation to fictitious monopoles and geometric phases, will be treated in subsequent papers separately. We will discuss criteria in choosing FV for real Lagrangians. The problem has a close link to that of the semiclassical versus full quantum evolutions of CS and FV. It was Stone[45] who first raised the problem commenting on the previous version of our article. [27] He pointed out that an arbitrary FV is not always realized and that there may be restrictions on FV so that quantum evolutions are consistent with the semiclassical ones. The formal CSPI themselves do not give answers to it; and thus, one may ascribe the fault to the formal CSPI. We have resolved, in the present article, the mysteries posed in Ref. 45 to some extent by proving the process from discrete CSPI to the continuous ones as described in Sec. 4.2. The fact that the spin CSPI in Secs. 4–5.1 certainly contract to the CCSPI in Sec. 2 strengthens the validity of the formulation. We will clarify the riddle more deeply the next time around. However, the whole problem seems to have rather profound nature and we will still need much further investigations.

Next, from a broader viewpoint let us put the prospects of the future below. First, conventional CCS and spin CS have been playing the roles of macroscopic wave functions in vast fields from lasers, superradiance, superfluidity and superconductivitiy to nuclear and particle physics. [5] CS have such potential. And thus we may expect that by choosing appropriate sets of $\{c_m\}$ the CS evolving from arbitrary FV will serve as approximate states or trial wave functions for the collective motions, having higher energies in various macroscopic or mesoscopic quantum phenomena such as spin vortices[46] and domain walls, which may not be treated by the former. We hope that numerous applications of the CS and CSPI will be found in the near future. Second, from the viewpoint of mathematical physics as well as physical applications, it is desirable that the present CS and CSPI formalism is extended to wider classes. The generalization to the SU(1, 1) CS case, which is closely related to squeezed states in lightwave communications and quantum detections, [47,48] is one of the highly probable candidates. We also have another candidate, i.e., the SU(3)CS case. Remembering that the SU(2)CS with a general FV here extends the SU(2) BMS[2] Lagragngian and gives a clear insight into the topological terms, the SU(3)CS case may also shed a new light on the original SU(3) Wess–Zumino term. [49,50] Finally, as we put in Sec. 1, we may regard CS with arbitrary FV as quantum states without classical analogues. We have already known some of such states. [16–18,20] It is true that CS with the conventional FV are closest to classical states and have useful properties.[7] However, since “the physical world is quantum mechanical”, [51] it seems definitely right to search boldly new quantum states whether their classical counterparts exist or not.

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Note Added to the KURENAI Edition

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The following research “Restriction on types of coherent states due to gauge symmetry” has already been published in *Int. J. Mod. Phys. B*, Vol. 28, No. 14 (2014) 1450082; arXiv: 1303.3179 v.2. There we have resolved the problem posed at the beginning of Sec. 6 by imposing restriction on quantum states when a Lagrangian has a sort of gauge symmetry.

Appendix A. Some Formulae for Rotation Matrices

Some basic formulae on the properties of the rotation matrices are enumerated. [53–56] We employ them in Secs. 3–5. We mainly follow the notation and convention of Messiah. [53]

(i) *Matrix elements*

A rotation with Euler angles $\mathbf{\Omega} \equiv (\phi, \theta, \psi)$ of a spin-s particle is specified by an operator $\hat{R}(\mathbf{\Omega}) = \exp(-i\phi\hat{S}_3)\exp(-i\theta\hat{S}_2)\exp(-i\psi\hat{S}_3)$; it has a $(2s+1)\times(2s+1)$ matrix representation whose (m,m')-entry is:

$$\mathsf{R}^{(s)}_{mm'}(\mathbf{\Omega}) \equiv \langle m|\hat{R}(\mathbf{\Omega})|m'\rangle = \exp(-i\phi m)\,\mathsf{r}^{(s)}_{mm'}(\theta)\,\exp(-i\psi m'). \tag{A.1}$$

Here $\mathsf{r}^{(s)}_{mm'}(\theta) \equiv \langle m|\exp(-i\theta\hat{S}_2)|m'\rangle$ is determined by the formula due to Majorana [57] and to Wigner[53]:

$$\mathsf{r}^{(s)}_{mm'}(\theta) = \sum_t N(s,m,m';t)\cdot[\cos(\theta/2)]^{2s+m-m'-2t}\cdot[\sin(\theta/2)]^{2t-m+m'} \tag{A.2}$$

with

$$N(s,m,m';t) \equiv (-1)^t \frac{[\,(s+m)!\,(s-m)!\,(s+m')!\,(s-m')!\,]^{1/2}}{(s+m-t)!\,(s-m'-t)!\,t!\,(t-m+m')!}, \tag{A.3}$$

where the sum runs over any integer t by which all the factorials in (A.3) make sense. In particular, if $s=\frac{1}{2}$, $\mathsf{r}^{(s)}_{mm'}$ is extremely simple to give:

$$\hat{R}^{(1/2)}(\mathbf{\Omega}) = \begin{pmatrix} \cos(\frac{1}{2}\theta)\exp[-\frac{1}{2}i(\phi+\psi)] & -\sin(\frac{1}{2}\theta)\exp[-\frac{1}{2}i(\phi-\psi)] \\ \sin(\frac{1}{2}\theta)\exp[\frac{1}{2}i(\phi-\psi)] & \cos(\frac{1}{2}\theta)\exp[\frac{1}{2}i(\phi+\psi)] \end{pmatrix}. \tag{A.4}$$

Most of the following relations, being independent of s, can be readily verified by the use of (A.4). For a higher spin s one can find explicit expressions of $\mathsf{r}^{(s)}_{mm'}$ in Refs. 54 and 55.

(ii) *Group manifold*

Introducing the real variables $\mathbf{x} \equiv (x_1,\cdots,x_4)$ via

$$\hat{R}^{(1/2)}(\mathbf{\Omega}) = \begin{pmatrix} x_1 + i\,x_2 & -x_3 + i\,x_4 \\ x_3 + i\,x_4 & x_1 - i\,x_2 \end{pmatrix} \equiv \hat{R}^{(1/2)}(\mathbf{x}), \tag{A.5}$$

we see that $\mathbf{x}^2 = x_1^2+x_2^2+x_3^2+x_4^2 = 1$ results; thus the $SO(3)$ manifold is isomorphic to a 3-sphere S^3.

(iii) *Gaussian decomposition*[3,6,7,58]

The rotation matrix $\hat{R}(\mathbf{\Omega})$ can be put into the normal or anti-normal ordering form in which $\hat{R}$ is specified by a set of complex variables:

$$\begin{aligned}\hat{R}(\mathbf{\Omega}) = \hat{R}(z_+,z_3,z_-) &\equiv \exp(z_+\hat{S}_+)\exp(z_3\hat{S}_3)\exp(z_-\hat{S}_-) \\ &= \exp(z_-\hat{S}_-)\exp(-z_3\hat{S}_3)\exp(z_+\hat{S}_+)\end{aligned} \tag{A.6}$$

The relation between the Euler angles and the complex parameters is given by:

$$\begin{cases} z_+ = -\tan(\frac{1}{2}\theta)\exp(-i\phi) \\ z_3 = -2\ln\{\cos(\frac{1}{2}\theta)\exp[\frac{1}{2}i(\phi+\psi)]\} \\ z_- = \tan(\frac{1}{2}\theta)\exp(-i\psi). \end{cases} \tag{A.7}$$

(iv) *Combinations with* $\mathbf{S}$[53]

$$\begin{cases} \hat{R}^{+}(\mathbf{\Omega})\hat{S}_3\hat{R}(\mathbf{\Omega}) = \cos\theta\hat{S}_3 - \frac{1}{2}\sin\theta[\exp(i\psi)\hat{S}_+ + \exp(-i\psi)\hat{S}_-] \\ \hat{R}^{+}(\mathbf{\Omega})\hat{S}_\pm\hat{R}(\mathbf{\Omega}) = \exp(\pm i\phi)\{\sin\theta\hat{S}_3 + \frac{1}{2}[(\cos\theta\pm 1)\exp(i\psi)\hat{S}_+ \\ \qquad\qquad + (\cos\theta \mp 1)\exp(-i\psi)\hat{S}_-]\}. \end{cases} \tag{A.8}$$

(v) *Inverse*
$\hat{R}(\mathbf{\Omega})$ is unitary and its inverse matrix is given by:

$$\hat{R}^{+}(\phi,\theta,\psi) = \hat{R}^{-1}(\phi,\theta,\psi) = \hat{R}(-\psi,-\theta,-\phi). \tag{A.9}$$

(vi) *Orthogonality relation*
The relation stems from integrating the products of the unitary irreducible representations of a compact group over the element of the group; thus it is a generic relation for the representations. In the present case it reads [54,56]:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \left(\mathsf{R}^{(s)}_{mm'}(\mathbf{\Omega})\right)^* \mathsf{R}^{(s')}_{nn'}(\mathbf{\Omega})\sin\theta\, d\phi d\theta d\psi = \frac{8\pi^2}{2s+1}\,\delta_{m,n}\delta_{m',n'}\delta_{s,s'}. \tag{A.10}$$

(vii) *Two successive rotations*
Two successive rotations specified by Euler angles $\mathbf{\Omega}_\ell \equiv (\phi_\ell,\theta_\ell,\psi_\ell)$ $(\ell = 1,2)$ produce $\hat{R}(\tilde{\mathbf{\Omega}}) \equiv \hat{R}(\mathbf{\Omega}_2)\hat{R}(\mathbf{\Omega}_1)$, where $\tilde{\mathbf{\Omega}} \equiv (\tilde{\phi},\tilde{\theta},\tilde{\psi})$ obeys:

$$\begin{cases} \cos\tilde{\theta} = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos(\phi_1+\psi_2) \\ \sin\tilde{\theta}\exp(i\tilde{\phi}) = \exp(i\phi_2)\Big[\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\cos(\phi_1+\psi_2) \\ \qquad\qquad + i\sin\theta_1\sin(\phi_1+\psi_2)\Big] \\ \cos(\frac{1}{2}\tilde{\theta})\exp[\frac{1}{2}i(\tilde{\phi}+\tilde{\psi})] \\ = \exp[\frac{1}{2}i(\phi_2+\psi_1)]\Big\{\cos(\frac{1}{2}\theta_1)\cos(\frac{1}{2}\theta_2)\exp[\frac{1}{2}i(\phi_1+\psi_2)] \\ \qquad\qquad - \sin(\frac{1}{2}\theta_1)\sin(\frac{1}{2}\theta_2)\exp[-\frac{1}{2}i(\phi_1+\psi_2)]\Big\}. \end{cases} \tag{A.11}$$

(viii) *Three successive rotations*
In a similar manner to that in (vii), the Euler angles made of three successive rotations can be calculated. Assuming that the rotations are specified by Euler angles (ϕ_1,θ_1,ψ_1), (ϕ,θ,ψ) and (ϕ_2,θ_2,ψ_2), which happen in this order, the composed rotation yields $\hat{R}(\mathbf{\Omega}') \equiv \hat{R}(\mathbf{\Omega}_2)\hat{R}(\mathbf{\Omega})\hat{R}(\mathbf{\Omega}_1)$, where $\mathbf{\Omega}' \equiv (\phi',\theta',\psi')$ obeys:

$$\begin{aligned} \cos\theta' = {} & [\cos\theta_1\cos\theta - \sin\theta_1\sin\theta\cos(\phi_1+\psi)]\cos\theta_2 \\ & + \{\sin\theta_1[\sin(\phi_1+\psi)\sin(\phi+\psi_2) - \cos(\phi_1+\psi)\cos\theta\cos(\phi+\psi_2)] \\ & - \cos\theta_1\sin\theta\cos(\phi+\psi_2)\}\sin\theta_2 \end{aligned} \tag{A.12}$$

and two additional equations that we omit here; they describe $\sin\theta'\exp(i\phi')$ and $\cos(\theta'/2)\exp[i(\phi'+\psi')/2]$ in terms of $\mathbf{\Omega}_1$, $\mathbf{\Omega}$ and $\mathbf{\Omega}_2$ as in (A.11).

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