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Surjectivity of a Gluing for Stable $T^2$-cones in Special Lagrangian Geometry

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Abstract. We suppose that $M$ is a Calabi–Yau (or more generally almost Calabi–Yau) manifold of dimension 3 over $\mathbb{C}$. We define a moduli space $V$ of special Lagrangian varifolds (or currents) in $M$, using geometric measure theory. We suppose that $X$ is a compact special Lagrangian 3-fold in $M$ with singularity at one point and modelled on a stable $T^2$-cone of multiplicity 1. We regard $X$ as a point of $V$, and determine a neighbourhood of $X$ in $V$.

1 Introduction

Gluing is a method of constructing blowing-up solutions to partial differential equations, and we shall say that the gluing is surjective if all blowing-up solutions may be re-constructed by the gluing. In Yang–Mills gauge theory, for example, Taubes [25] constructs some instantons by the method of gluing, and Donaldson [6] proves its surjectivity; for the details we refer also to Freed and Uhlenbeck [8]. In special Lagrangian geometry, by a theory of Joyce [13, 14, 15, 16, 17], we can smooth some isolated conical singularities by the method of gluing, and we wish to prove its surjectivity.

We can formulate the surjectivity problems in terms of moduli spaces. In the situation of Donaldson’s result [6] indeed one has the moduli space of Yang–Mills instantons modulo gauge equivalence; using a result of Uhlenbeck [26] one can compactify it by adding some singular instantons; one defines the boundary of the moduli space as the set of all the singular instantons added for the compactification; using the surjectivity result one can determine a neighbourhood of the boundary in the compactified space.

We wish to do a similar thing in special Lagrangian geometry. We suppose that $M$ is a Calabi–Yau (or more generally almost Calabi–Yau) manifold of dimension $m$ over $\mathbb{C}$. We take $a \in H_m(M; \mathbb{Z})$. We denote by $N$ the set of all compact special Lagrangian submanifolds of $M$ with homology class $a$. We give $N$ the $C^\infty$-topology. Mclean [19, Theorem 3.6] proves that $N$ is a manifold of dimension finite.

For compactifying $N$ we shall use varifolds and currents in geometric measure theory. As special Lagrangian varifolds are calibrated in the sense of Harvey and Lawson [9] they may be oriented and regarded as currents. By a fundamental theorem of Harvey and Lawson special Lagrangian currents are homologically...
area-minimizing, and so their area will not be cancelled; for the details we refer e.g. to Federer [7, 5.4.2] or Simon [24, Theorem 34.5]. In effect we may use varifolds in place of currents and vice versa. We shall mainly use varifolds rather than currents.

We denote by $\mathcal{V}$ the set of all special Lagrangian integral varifolds (or currents) in $M$ with support compact, boundary 0 and homology class $a$. We give $\mathcal{V}$ the weak topology of varifolds. From an integral compactness theorem of Alldard [1, Theorem 6.4] we see that $\mathcal{V}$ is compact. We may regard $\mathcal{N}$ as a subset of $\mathcal{V}$. From a regularity theorem of Allard [1, Theorem 8.19] we see that $\mathcal{N}$ is an open subset of $\mathcal{V}$, and the $C^\infty$-topology on $\mathcal{N}$ is the same as the one induced from $\mathcal{V}$. We denote by $N$ the closure of $\mathcal{N}$ in $\mathcal{V}$. We put $\partial N = N \setminus \mathcal{N}$.

In the situation of Donaldson’s result [6] the boundary of the moduli space consists of instantons with isolated singularities, to which one can apply the gluing method of Taubes. In our situation, however, $\partial N$ consists of special Lagrangian varifolds whose singularities may be non-isolated or branched, and it will be difficult therefore to determine a neighbourhood of $\partial N$ in $\mathcal{V}$. What we shall do is to take a certain point $X$ of $\partial N$ and determine a neighbourhood of $X$ in $\mathcal{V}$.

We have mainly two things to do: the analysis of bubbling-off and the classification of local models. In the proof of Donaldson’s surjectivity theorem one uses a technique of Uhlenbeck [26] for the analysis of bubbling-off, and a result of Atiyah, Hitchin and Singer [3] for the classification of local models.

We suppose that $X$ has singularity only at one point and modelled on Jacobi-integrable smooth cones of multiplicity 1. We can then analyse the bubbling-off near $X$, using a technique of Allard–Almgren [2], Simon [22, 23] and the author [12]. On the other hand it seems difficult in general to classify local models for smoothing isolated conical singularities.

For stable $T^2$-cones in the sense of Joyce [14, Definition 3.6], however, we can classify the local models. We can describe the stable $T^2$-cones as follows. We put

$$C = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{0\} : |z_1| = |z_2| = |z_3|, z_1 z_2 z_3 \in (0, \infty)\}. \quad (1)$$

Harvey and Lawson [9, Chapter III.3.A, Theorem 3.1] prove that $C$ is a special Lagrangian submanifold of $\mathbb{C}^3 \setminus \{0\}$. We put $\Sigma = C \cap S^5$. It is easy to see that $\Sigma$ is diffeomorphic to $T^2$. Joyce proves that $C$ is a stable cone. Haskins [10] proves that every stable $T^2$-cone may be expressed as $a(C)$ for some $a \in SU_3$.

We suppose now that $M$ is of dimension 3 over $\mathbb{C}$. We suppose also that $X$ has singularity only at one point and modelled on $C$ with multiplicity 1. We prove that we have a local model $L$ compatible with $X$ in the situation of Joyce [17, Theorem 10.4]: $L$ is a special Lagrangian submanifold of $\mathbb{C}^3$ asymptotic at infinity to $C$ with multiplicity 1. For each $t > 0$ we can define $tL$ by rescaling $L$ by $t$. We can find a real number $\tau > 0$ such that for each $t \in (0, \tau)$ we can glue $tL$ to $X$ by the method of gluing. Hence we get a smooth embedding $G : (0, \tau) \to \mathcal{N}$ with $G(t)$ tending to $X$ as $t \to +0$.

We shall parametrize $G : (0, \tau) \to \mathcal{N}$ by a family of special Lagrangian 3-folds with isolated conical singularities modelled on $C$. We denote by $\mathcal{X}$ the set
of all elements of $V$ with supported connected, singularity only at one point and modelled on $C$ with multiplicity 1. Joyce [14, Definition 5.6] defines a topology on $X$. We prove that it is actually the same as the one induced from $V$. By a result of Joyce [14, Corollary 6.11] we can take a neighbourhood $Y$ of $X$ in $X$ which has a structure of manifold. Joyce proves [17, Proposition 10.3] also $\dim Y = \dim N - 1$. Making $\tau > 0$ and $Y$ smaller if necessary we can construct a one-to-one local diffeomorphism $G : (0, \tau) \times Y \to N$ with $G(t,Y)$ tending to $Y$ as $t \to +0$ for each $Y \in \mathcal{Y}$.

We can extend $G$ continuously to $[0, \tau) \times Y$ by setting $G(0,Y) = Y$ for each $Y \in \mathcal{Y}$. We note that $G : [0, \tau) \times Y \to N$ is a one-to-one open mapping. Our surjectivity result is:

**Theorem 1.** $G$ maps $[0, \tau) \times Y$ onto a neighbourhood of $X$ in $V$.

We begin in §2 with a review of special Lagrangian geometry and geometric measure theory. In §3 we analyse the bubbling-off of special Lagrangian varifolds near Jacobi-integrable smooth cones of multiplicity 1 (which need not be stable nor $T^2$-cones). In §4 we classify the local models for smoothing $C$. In §5 we prove Theorem 1.

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## 2 Review of SL Geometry and GMT

In this section we shall give a review of special Lagrangian geometry and geometric measure theory. First of all we shall define almost Calabi–Yau manifolds and their special Lagrangian submanifolds.

Let $(M, \omega)$ be a symplectic manifold of dimension $2m$, and let $J$ be a complex structure on $M$ such that if we put $\hat{g}(v,w) = \omega(v,Jw)$ then $\hat{g}$ will be a Kähler metric on $(M,J)$. Let $\Omega$ be a holomorphic $(m,0)$-form on $(M,J)$ with $\Omega|_x \neq 0$ for every $x \in M$. Then we shall call $(M,\omega,J,\Omega)$ an almost Calabi–Yau manifold, and $(\omega,J,\Omega)$ an almost Calabi–Yau structure on $M$. We can define a smooth function $\psi : M \to (0, \infty)$ such that

$$\frac{\psi^{2m}}{m!}\omega^m = (-1)^s \left(\frac{i}{2}\right)^m \Omega \wedge \overline{\Omega} \text{ where } s = \frac{m(m-1)}{2}.$$  

We put $g = \psi^2 \hat{g}$. We shall call $g$ the almost Calabi–Yau metric on $(M,\omega,J,\Omega)$. Here $g$ need not be a Kähler metric on $(M,J)$. If we have $\psi(x) = 1$ for every $x \in M$ then $g$ will be a Kähler metric of Ricci curvature 0. In that case we shall call $(\omega,J,\Omega)$ a Calabi–Yau structure on $M$, and $g$ the Calabi–Yau metric on $(M,\omega,J,\Omega)$.

We define a Calabi–Yau structure on $\mathbb{R}^{2m} = \mathbb{C}^m$ as follows. Let $(z_1, \cdots, z^m)$ be the co-ordinates of $\mathbb{C}^m$. Let $\omega_0 = \frac{i}{2}(dz_1 \wedge d\overline{z}_1 + \cdots + dz^m \wedge d\overline{z}^m)$, $g_0 =$
Let $(M,\omega,J,\Omega)$ be an almost Calabi–Yau manifold, and let $g$ be the almost Calabi–Yau metric on $(M,\omega,J)$. Then $\text{Re}\,\Omega$ will be a calibration on $(M,g)$ in the sense of Harvey and Lawson \cite{HarLaw}, and so we can define $\text{Re}\,\Omega$-submanifolds of $(M,g)$, which we shall call special Lagrangian submanifolds of $(M,\omega,J,\Omega)$. Special Lagrangian submanifolds of $(M,\omega,J,\Omega)$ will be Lagrangian submanifolds of $(M,\omega)$.

We can also define $\text{Re}\,\Omega$-varifolds and currents in $(M,g)$, which we shall call special Lagrangian varifolds and currents in $(M,\omega,J,\Omega)$, respectively. In the remainder of this section we shall consider calibrated geometry not limited to special Lagrangian geometry.

We suppose that $M$ is a manifold. For each $x \in M$ we denote by $G_p(T_xM)$ the Grassmann manifold of all vector subspaces of $T_xM$ of dimension $p$. We put $G_p(TM) = \bigcup_{x \in M} G_p(T_xM)$. By a varifold of dimension $p$ in $M$ we shall mean a Radon measure on $G_p(TM)$.

We suppose that $g$ is a Riemannian metric on $M$, and $\phi$ is a calibration of degree $p$ on $M$. For each $x \in M$ we put $G_\phi(T_xM) = \{S \in G_p(T_xM) : |\phi|_g = 1\}$. We also put $G_\phi(TM) = \bigcup_{x \in M} G_\phi(T_xM)$. By a $\phi$-varifold in $M$ we shall mean a Radon measure on $G_\phi(TM)$.

For each $x \in M$ and $S \in G_\phi(T_xM)$ we define $\vec{S} \in \bigwedge^p T_xM$ as follows. We take an orthonormal basis $(e_1,\cdots,e_p)$ for $T_xM$ with respect to $g|_x$ such that $\langle \phi|_x,e_1 \wedge \cdots \wedge e_p \rangle = 1$. We set $\vec{S} = e_1 \wedge \cdots \wedge e_p$. It is easy to see that $\vec{S}$ is independent of the choice of $(e_1,\cdots,e_p)$, and so well-defined.

Let $V$ be a $\phi$-varifold in $M$. Then we can define a $p$-current $\vec{V}$ in $M$ by setting

$$\vec{V}(\chi) = \int_{G_\phi(TM)} \langle \chi|_x, \vec{S} \rangle dV(x, S)$$

for every $p$-form $\chi$ on $M$ with support compact.

Harvey and Lawson \cite[Chapter II.1, Definition 1.4]{HarLaw} define positive $\phi$-currents in $M$, which we shall explain next. First of all we recall a definition of Harvey and Lawson \cite[Chapter II.A, Definition A.1]{HarLaw}: by a $\phi$-non-negative $p$-form on $M$ we shall mean a $p$-form $\chi$ on $M$ with $\langle \chi|_x, \vec{S} \rangle \geq 0$ for every $x \in M$ and $S \in G_\phi(T_xM)$. Harvey and Lawson \cite[Chapter II.A, Proposition A.2]{HarLaw} prove that a $p$-current $T$ in $M$ is a positive $\phi$-current if (and only if) we have $T(\chi) \geq 0$ for every $\phi$-non-negative $p$-form on $M$ with support compact. We have:

**Theorem.** Let $V$ be a $\phi$-varifold in $M$. Then $\vec{V}$ is a positive $\phi$-current in $M$ in the sense of Harvey and Lawson.

**Proof.** We have only to prove that if $\chi$ is a $\phi$-non-negative $p$-form on $M$ with support compact then we have $\vec{V}(\chi) \geq 0$. By the definition of $\phi$-non-negative $p$-forms we have $\langle \chi|_x, \vec{S} \rangle \geq 0$ for every $x \in M$ and $S \in G_\phi(T_xM)$. Hence we get $\vec{V}(\chi) = \int (\chi|_x, \vec{S}) dV(x, S) \geq 0$, completing the proof. \[\square\]
We suppose now that $M$ is compact. We take $a \in H_p(M; \mathbb{R})$. We denote by $\mathcal{V}$ the set of all integral $\phi$-varifolds $V$ in $M$ with support compact, $\partial V = 0$ and $[V] = a$. We give $\mathcal{V}$ the weak topology in the sense of Allard [1, Definition 2.6(2)], i.e. the topology of the Radon measures on $G_p(TM)$. We have then:

**Theorem.** $\mathcal{V}$ is compact.

**Proof.** It is easy to see that $\mathcal{V}$ is a metrizable space. It suffices therefore to prove that if $V_1, V_2, V_3, \cdots \in \mathcal{V}$ then there exists a subsequence of $(V_n)_{n=1}^{\infty}$ converging in $\mathcal{V}$. Since $M$ is compact it is clear that $\phi$ is compactly-supported, and for each $n = 1, 2, 3, \cdots$ therefore we have area $V_n = V_n(\phi) = a \cdot [\phi]$ where $[\phi]$ denotes the de Rham cohomology class of $\phi$. This implies that area $V_n$ is bounded with respect to $n$. By an integral compactness theorem of Allard [1, Theorem 6.4], therefore, we can find a subsequence of $(V_n)_{n=1}^{\infty}$ converging as Radon measures on $G_p(TM)$. We may identify $(V_n)_{n=1}^{\infty}$ with the subsequence. We denote its limit by $V$. It suffices then to prove $V \in \mathcal{V}$. Since $V_n$ tends to $V$ as varifolds we see that $V_n$ tends to $V$ as $p$-currents in $M$. As $\partial V_n = 0$ for every $n = 1, 2, 3, \cdots$ so we have $\partial V = 0$. As $[V_n] = a$ for every $n = 1, 2, 3, \cdots$ so we have $[V] = a$. We have therefore $V \in \mathcal{V}$ as we want. \hfill \Box

Let $V$ be a varifold of dimension $p$ in $M$. Then we shall denote by $\|V\|$ the Radon measure on $M$ defined by setting $\|V||(f) = V(\pi \circ f)$ for every $f \in C_c(M; \mathbb{R})$, where $\pi$ denotes the projection of $G_p(TM)$ onto $M$, and $C_c(M; \mathbb{R})$ denotes the set of all continuous functions on $M$ with support compact.

We denote by $\mathcal{R}$ the set of all Radon measures on $M$ which may be expressed as $\|V\|$ for some $V \in \mathcal{V}$. We give $\mathcal{R}$ the topology of the Radon measures on $M$.

We have then:

**Theorem.** The mapping $V \mapsto \|V\|$ is a homeomorphism of $\mathcal{V}$ onto $\mathcal{R}$.

**Proof.** From the definition of $\mathcal{R}$ it is clear that $V \mapsto \|V\|$ maps $\mathcal{V}$ onto $\mathcal{R}$. Notice also that each $V \in \mathcal{V}$ is rectifiable by a result of Allard [1, Theorem 5.5]. Then we see that $V \mapsto \|V\|$ is one-to-one. From the definition of $\|V\|$, moreover, we see that $V \mapsto \|V\|$ is continuous. We have thus proved that $V \mapsto \|V\|$ is a continuous bijection of $\mathcal{V}$ onto $\mathcal{R}$. Notice also that $\mathcal{V}$ is compact and $\mathcal{R}$ is Hausdorff. Then we see that $V \mapsto \|V\|$ is a homeomorphism of $\mathcal{V}$ onto $\mathcal{R}$. \hfill \Box

We denote by $\mathcal{T}$ the set of all positive $\phi$-currents in $M$ which may be expressed as $\vec{V}$ for some $V \in \mathcal{V}$. We give $\mathcal{T}$ the topology of $p$-currents in $M$. We have then:

**Theorem.** The mapping $V \mapsto \vec{V}$ is a homeomorphism of $\mathcal{V}$ into $\mathcal{T}$.

**Proof.** From the definition of $\mathcal{T}$ it is clear that $V \mapsto \vec{V}$ maps $\mathcal{V}$ onto $\mathcal{T}$. Notice also that $V \mapsto \|V\|$ is one-to-one by the previous proposition. Then we see that $V \mapsto \vec{V}$ is one-to-one. It is also easy to see that $V \mapsto \vec{V}$ is continuous. We
have thus proved that $V \mapsto \overline{V}$ is a continuous bijection of $V$ onto $T$. Notice also that $V$ is compact and $T$ is Hausdorff. Then we see that $V \mapsto \overline{V}$ is a homeomorphism of $V$ onto $T$.

3 Analysis of Bubbling-off

In this section we shall analyse bubbling-off near special Lagrangian Jacobi-integrable smooth cones of multiplicity 1. We can summarize our results as follows.

We take a special Lagrangian varifold $X$ with singularity only at one point $x$ and asymptotic at $x$ to a multiplicity 1 special Lagrangian Jacobi-integrable smooth cone $C$. We consider special Lagrangian varifolds $V$ close to $X$. We define an energy functional for $V$. We prove that if the energy of $V$ is small then $V$ has singularity only at one point $y$ and asymptotic at $y$ to a multiplicity 1 smooth cone close to $C$, and if the energy of $V$ is large then $V$ bubbles off.

We begin in §3.1 with the definition of an energy functional and a discussion about related results. In §3.2 we discuss an important property of special Lagrangian Jacobi-integrable cones. In §3.3 we analyse the bubbling-off.

3.1 Energy Functional

We define an energy functional as follows. For each varifold $V$ of dimension $m$ in $\mathbb{R}^n$ we put

$$\mathcal{E}(V) = \int_{\mathbb{R}^n \times G_m(\mathbb{R}^n)} \frac{|S^\perp y|^2}{|y|^{m+2}} dV(y, S) \in [0, \infty]$$

where $S^\perp$ denotes the orthogonal complement to $S$ in $\mathbb{R}^n$, and $S^\perp y$ denotes the projection of $y$ onto $S^\perp$.

We shall recall a monotonicity formula for stationary varifolds. We put $B_\rho = \{ y \in \mathbb{R}^n : |y| < \rho \}$ for each $\rho > 0$ and $A_{\sigma, \rho} = B_\rho \setminus B_\sigma$ for each $\rho > \sigma > 0$. For each $Z \subset \mathbb{R}^n$ we put $\tilde{Z} = Z \times G_m(\mathbb{R}^n)$. If $V$ is a stationary varifold of dimension $m$ in $(B_1, g_0)$ and if $0 < \sigma < \rho < 1$ then we have

$$\frac{\|V\|({B}_\rho)}{\rho^m} - \frac{\|V\|({B}_\sigma)}{\sigma^m} = \mathcal{E}(V, \tilde{A}_{\sigma, \rho}); \quad (3)$$

for the proof we refer e.g. to Allard [1, Theorem 5.1(1)] or Simon [24, Equation 17.4]. It is easy to extend (3) to Riemannian metrics in place of $g_0$:

**Proposition 1.** Let $g$ be a Riemannian metric on $B_1$ with $\epsilon = \|g - g_0\|_{C^1(B_1)} < \infty$. Then there exists $k > 0$ such that if $V$ is a stationary varifold of dimension $m$ in $(B_1, g)$ and if $0 < \sigma < \rho < 1$ then we have

$$e^{k \epsilon \rho} \frac{\|V\|({B}_\rho)}{\rho^m} - e^{k \epsilon \sigma} \frac{\|V\|({B}_\sigma)}{\sigma^m} \geq \mathcal{E}(V, \tilde{A}_{\sigma, \rho}).$$

6
We suppose now that $n = 2m$ and $(\omega, J, \Omega)$ is an almost Calabi–Yau structure on $B_1$. We denote by $V_0$ the space of all special Lagrangian integral varifolds $V$ with $\partial V = 0$ in $(B_1, \omega, J, \Omega)$.

We define $r : \mathbb{R}^{2m} \to [0, \infty)$ by setting $r(y) = |y|$ for each $y \in \mathbb{R}^{2m}$. We put $g_{cyl} = r^{-2}g_0$, which we shall call the cylindrical metric on $\mathbb{R}^{2m} \setminus \{0\}$.

We regard $(0, \infty)$ as a multiplicative group acting upon $\mathbb{R}^{2m}$ as re-scaling. By a smooth cone in $\mathbb{R}^{2m}$ we shall mean a closed submanifold of $\mathbb{R}^{2m} \setminus \{0\}$ invariant under the re-scaling by $(0, \infty)$. We denote by $C$ the set of all special Lagrangian smooth cones in $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$.

We suppose $C \in C$. We denote by $NC$ the normal bundle to $C$ in $\mathbb{R}^{2m}$ with respect to $g_0$. We get the same bundle even if we use $g_{cyl}$ in place of $g_0$.

For each $u \in C^k(C; NC)$ and $\rho < \sigma$ we define $|u|^{k, cyl}_{[\rho, \sigma]}$ as follows. We put $t = -\log r$ and $\vartheta_k = \frac{\partial}{\partial t}$. We put $\Sigma = C \cap S^{2m-1}$. By the definition of smooth cones $\Sigma$ is a compact submanifold of $S^{2m-1}$. We denote by $\nabla_{\Sigma}$ the Levi-Civita connexion over $\Sigma$ induced from $g_0$. We put

$$|u|^{k, cyl}_{[\rho, \sigma]} = \sup_{C \cap A_{\rho, \sigma}} \sum_{i,j \geq 0, |i+j| \leq k} |\partial_i \nabla^j_{\Sigma} u|.$$

If $|u|^{k, cyl}_{[\rho, \sigma]}$ is sufficient small, then we can define the exponential map $\exp u : C \cap A_{\rho, \sigma} \to A_{\rho, \sigma}$ with respect to the metric $g_{cyl}$, and the image of $\exp u$ will be a submanifold of $A_{\rho, \sigma}$, which we shall denote by $\text{Graph}_{cyl} u$. We put $|\text{Graph}_{cyl} u| = \mathcal{H}^m$, $\text{Graph}_{cyl} u$ where $\mathcal{H}^m$ denotes the Hausdorff $m$-dimensional measure with respect to the almost Calabi–Yau metric $g$.

We suppose that $(\omega, J, \Omega)$ is an almost Calabi–Yau structure on $B_1$, and $g$ is the almost Calabi–Yau structure on $B_1$. From the proof of the author [12, Theorem 2.2] we get:

**Theorem 2.** There exists $\varepsilon > 0$ depending only on $m$ and $C$ such that if we have

$$|\Omega - \Omega_0|_{C^0(B_1)} + |g - g_0|_{C^1(B_1)} < \varepsilon, \quad V \in V_0, \quad 0 < \rho < 1, \quad \mathcal{E}(V, \tilde{A}_{\rho, 1}) < \varepsilon$$

and $v \in C^\infty(C \cap A_{1/2, 1}; NC)$, $|v|^{2, cyl}_{[1/2, 1]} < \varepsilon$, $|\text{Graph}_{cyl} v| = ||V||_{m, A_{1/2, 1}}$

then we can extend $v$ to $C \cap A_{\rho, 1}$ so that $||V||_{m, A_{\rho, 1}} = |\text{Graph}_{cyl} v|$ and $|v|^{2, cyl}_{[\rho, 1]} < k\varepsilon^\alpha$ for some $k > 0$ and $\alpha \in (0, 1)$ depending only on $m$ and $C$.

**Remark.** For the proof one has to use a result of Lojasiewicz [18], following Simon [22].

We shall give a corollary to the theorem above. We give $C$ the $C^\infty$-topology. Let $C'$ be a neighbourhood of $C$ in $C$. Then we have:

**Corollary 1.** There exists $\varepsilon > 0$ depending only on $m$, $C$ and $C'$ such that if we have

$$|\Omega - \Omega_0|_{C^0(B_1)} + |g - g_0|_{C^1(B_1)} < \varepsilon, \quad V \in V_0, \quad \mathcal{E}(V) < \varepsilon$$

7
and $v \in C^\infty(C \cap A_{1/2,1}; NC)$, \quad \|v\|_{C^1}^2 < \epsilon, \quad |\text{Graph}_C v| = \|V\|_{L^1} A_{1/2,1}$

then $V$ is singular only at 0 and asymptotic at 0 to some $C' \in C$ with multiplicity 1.

Remark. By a result of Simon [22, Theorem 5] $C'$ will be a unique tangent cone to $V$ at 0.

Proof. Let $C'$ be a tangent cone to $V$ at 0. Then we can take $\delta_1 > \delta_2 > \delta_3 > \cdots$ tending to 0 with $\delta_n^{-1} V$ tending to $C'$ as $n \to \infty$. We can also take $\eta > 0$ such that if $v \in C^\infty(\Sigma; NC)$ and $\|v\|_{C^1(\Sigma)} \leq \eta$ then we have Graph$_C v \in C'$. By Theorem 2 we can find $v_n \in C^\infty(C \cap A_{\delta_n,1}; NC)$ such that $\|V\|_{L^1 A_{\delta_n,1}} = |\text{Graph}_C v_n|$ with $\|v_n\|_{C^1} \leq \eta$, and so we can find a subsequence of $\|\delta_n^{-1} V\|_{L^1 A_{1/2,1}}$ converging to $|\text{Graph}_C v| for some $w \in C^\infty(C \cap A_{1/2,1}; NC)$ with $\|w\|_{C^1} \leq \eta$. This implies that $C'$ is smooth of multiplicity 1 and we have $C' \in C$ as we want.

3.2 Special Lagrangian Jacobi-integrable Cones

We shall discuss an important property of special Lagrangian Jacobi-integrable cones. For a moment we shall consider Jacobi-integrable cones which need not be special Lagrangian.

We denote by $\mathcal{K}$ the set of all smooth cones in $\mathbb{R}^n$. We define a functional $F: \mathcal{K} \to (0, \infty)$ by setting $F(K) = \text{area}(K \cap B_1)$ for each $K \in \mathcal{K}$. We denote by Crit $F$ the set of all critical points of $F$.

Let $K \in \mathcal{K}$. Then we have $T_K \mathcal{K} = C^\infty(K; NK)$. We can also define the hessian of $F$ at $K$ as a linear operator hess$_K F: T_K \mathcal{K} \to T_K \mathcal{K}$. We shall say that $K$ is Jacobi-integrable if for every $v \in \text{Ker} \text{hess}_C F$ there exists a one parameter family $(K_t)_{t \in \mathbb{R}}$ in Crit $F$ with

$$K_0 = K \text{ and } \left. \frac{dK_t}{dt} \right|_{t=0} = v.$$

We suppose that $K$ is Jacobi-integrable. Adams and Simon [4, Lemma 1] prove:

Proposition 2. There exists a neighbourhood of $K$ in $\mathcal{K}$ on which $F$ is constant.

For the proof one has to use the real-analyticity of $F$. Adams and Simon also prove that Crit $F$ is a non-degenerate critical manifold in the sense of Bott [5].

We shall do a similar thing for Lagrangian cones. We denote by $\hat{\mathcal{C}}$ the set of all Lagrangian smooth cones in $(\mathbb{R}^{2m}, \omega_0)$. We have $\hat{\mathcal{C}} \subset \mathcal{K}$. We put $\hat{F} = F|_{\hat{\mathcal{C}}}$.

We denote by Crit $\hat{F}$ the set of all critical points of $\hat{F}$.

We suppose $\hat{C} \in \hat{\mathcal{C}}$. By a result of Weinstein [27, Corollary 6.2], then $T_{\hat{C}} \hat{\mathcal{C}}$ may be identified with the set of all closed 1-forms on $\hat{C}$. By a result of Oh [21, Theorem 3.5] the hessian of $\hat{F}$ at $\hat{C}$ will be the Laplacian acting upon closed 1-forms on $\hat{C}$. We shall denote the Laplacian by $\Delta$, and say that $\hat{C}$ is Lagrangian.
Jacobi-integrable if for every closed 1-form $\xi$ on $\hat{C}$ with $\Delta \xi = 0$ there exists a one parameter family $(\hat{C}_t)_{t \in \mathbb{R}}$ in Crit $\hat{F}$ with

$$\hat{C}_0 = \hat{C} \text{ and } - \frac{d\hat{C}_t}{dt} \bigg|_{t=0} \omega_0 = \xi.$$  

In a way similar to the proof of Proposition 2 we can prove:

**Proposition 3.** If $\hat{C}$ is Lagrangian Jacobi-integrable then there exists a neighbourhood of $\hat{C}$ in $\hat{C}$ on which $\hat{F}$ is constant.

Joyce [13, Definition 6.7] defines special Lagrangian Jacobi-integrable cones, which we shall recall next. As in §3.1 we denote by $C$ the set of all special Lagrangian smooth cones in $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$. We take $C \in C$. We shall say that $C$ is special Lagrangian Jacobi-integrable if for every closed 1-form $\xi$ on $C$ with $d^\ast \xi = 0$ there exists a one parameter family $(C_t)_{t \in \mathbb{R}}$ in $C$ with

$$C_0 = C \text{ and } - \frac{dC_t}{dt} \bigg|_{t=0} \omega_0 = \xi.$$  

We have:

**Lemma.** If $C$ is special Lagrangian Jacobi-integrable then $C$ is Lagrangian Jacobi-integrable.

**Proof.** Let $\xi$ be a closed 1-form on $C$ with $\Delta \xi = 0$. Then we have $d \ast d \ast \xi = 0$ and so

$$\ast d \ast \xi = a$$  

for some $a \in \mathbb{R}$. We define a map $f^a_t : \mathbb{C}^m \rightarrow \mathbb{C}^m$ by setting $f^a_t(z_1, z_2, \cdots, z_m) = (e^{ia t} z_1, z_2, \cdots, z_m)$. We put $C^a_t = f^a_t(C)$. We have then

$$\Omega_0|_{C^a_t} = (f^a_t \ast \Omega_0)|_{C} = e^{ia t} \Omega_0|_{C} = e^{-ia t} \ast 1.$$  

(5)

We put

$$\xi^a = - \frac{dC^a_t}{dt} \bigg|_{t=0} \omega_0.$$  

Since $C^a_t$ is Lagrangian we see that $\xi^a$ is a closed 1-form on $C$. We may also write $\xi^a = g_0(\tilde{\xi}^a, \bullet)$ for some $\tilde{\xi}^a$. We have then $\frac{dC^a_t}{dt}|_{t=0} = J_0 \tilde{\xi}^a$ and so

$$\frac{d}{dt} \Omega_0|_{C^a_t} \bigg|_{t=0} = d(J_0 \tilde{\xi}^a \Omega_0|_{C}) = d(i \tilde{\xi}^a \Omega_0|_{C}) = d(i \tilde{\xi}^a \ast 1) = i d \ast \xi^a.$$  

This combined with (5) implies that on $C$ we have

$$i d \ast \xi^a = i a \ast 1$$  

and so $d \ast \xi^a = a$. This combined with (4) implies $d \ast (\xi - \xi^a) = 0$. By the special Lagrangian Jacobi-integrability of $C$, therefore, we can find a one-parameter family $(C_t)_{t \in \mathbb{R}}$ in $C$ with

$$C_0 = C \text{ and } - \frac{dC_t}{dt} \bigg|_{t=0} \omega_0 = \xi - \xi^a.$$  

9
Putting $\hat{C}_t = f_t^*(C)$ we get

\[
\hat{C}_t \in \hat{C}, \hat{C}_0 = C \quad \text{and} \quad \frac{d\hat{C}_t}{dt} \bigg|_{t=0} = \omega_0 = \xi,
\]
completing the proof.

By Proposition 3 and the lemma above we have:

**Proposition 4.** Let $C$ be special Lagrangian Jacobi-integrable. Then there exists a neighbourhood $C'$ of $C$ in $\mathcal{C}$ on which the area functional is constant; i.e. for each $C' \in C'$ we have $\text{area}(C' \cap B_1) = \text{area}(C \cap B_1)$.

3.3 Bubbling-off

We suppose that $(\omega, J, \Omega)$ is an almost Calabi–Yau structure on $B_1$ with $J|_0 = J_0$ and $\Omega|_0 = \Omega_0$. We denote by $g$ the almost Calabi–Yau metric on $(B_1, \omega, J, \Omega)$. We write $g = \sum_{i,j=1}^{2m} g_{ij} \, dy^i \, dy^j$ and suppose $g_{ij}(0) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial y^k}(0) = 0$ for each $i, j, k = 1, \cdots, 2m$.

If $W$ is a varifold in $\mathbb{R}^n$ and if $\delta > 0$ then we can define $\delta^{-1}W$ by re-scaling $W$ by $\delta^{-1}$. If $b \in \mathbb{R}^n$ then we can also define $W - b$ by translating $W$ by $-b$.

We take $C \in \mathcal{C}$ and a neighbourhood $C'$ of $C$ in $\mathcal{C}$. $X \in \mathcal{V}_0$. We denote by $X'$ the space of all elements of $\mathcal{V}_0$ with support connected, with singularity only at one point $y$ and asymptotic at $y$ to some element of $C'$ with multiplicity 1. We take $X \in \mathcal{V}_0$ and suppose that 0 is the singular point of $X$ and $C$ is the multiplicity 1 smooth tangent cone to $X$ at 0. We have then:

**Theorem 3.** Let $(X_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{X}'$ converging to $X$. For each $n = 1, 2, 3, \cdots$ let $x_n$ be the singular point of $X_n$, and let $C_n$ be the multiplicity 1 smooth tangent cone to $X_n$ at $x_n$. Then $x_n$ tends to 0 and $C_n$ tends to $C$ as $n \to \infty$.

**Proof.** By Allard’s regularity theorem $x_n$ tends to 0 as $n \to \infty$. By Proposition 1 we have

\[
\mathcal{E}(X_n - x_n) \leq e^{k\epsilon \rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^m} - e^{k\epsilon \sigma} \frac{\|X_n - x_n\|(B_\sigma)}{\sigma^m}
\]

where $0 < \sigma < \rho < 1$. Letting $\sigma \to 0$ we get

\[
\mathcal{E}(X_n - x_n) \leq e^{k\epsilon \rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^m} - \text{area}(C_n \cap B_1).
\]

This combined with Proposition 4 implies

\[
\mathcal{E}(X_n - x_n) \leq e^{k\epsilon \rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^m} - \text{area}(C \cap B_1). \tag{6}
\]
Making $\rho$ smaller if necessary we may suppose
\[ e^{ke\rho} \frac{\|X\|(B_\rho)}{\rho^n} - \text{area}(C \cap B_1) < \epsilon. \]

For $n$ sufficiently large, therefore, we have
\[ e^{ke\rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^n} - \text{area}(C \cap B_1) < \epsilon. \]

This combined with (6) implies
\[ \mathcal{E}(X_n - x_n) < \epsilon. \]

By Corollary 1, therefore, $C_n$ tends to $C$ as $n \to \infty$.

We have also:

**Theorem 4.** Let $(V_n)_{n=1}^\infty$ be a sequence in $\mathcal{V}_0 \setminus \mathcal{X}'$ converging to $X$. Then there exists a sequence $(\delta_n)_{n=1}^\infty$ of positive real numbers converging to 0, a sequence $(y_n)_{n=1}^\infty$ in $B_1$ converging to 0, and a subsequence of $(\delta_n^{-1}(V_n - y_n))_{n=1}^\infty$ converging to some varifold $W$ in $\mathbb{R}^{2m}$ asymptotic at infinity to some element of $\mathcal{C}'$ with multiplicity 1 and satisfying $\mathcal{E}(W - b) > 0$ for every $b \in \mathbb{R}^{2m}$.

**Remark.** $W$ will automatically be a special Lagrangian integral varifold with $\partial W = 0$ in $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$.

**Proof.** We take positive real numbers $\rho$ and $\epsilon$, which we shall make smaller if necessary. By assumption we have $V_n \in \mathcal{V}_0 \setminus \mathcal{X}'$. By Corollary 1 therefore we may suppose $\mathcal{E}(V_n - y) \geq \epsilon$ for each $y \in B_{\rho\epsilon}$; otherwise $V_n$ will be singular only at $y$ and asymptotic at $y$ to some element of $\mathcal{C}'$ with multiplicity 1, which contradicts $V_n \in \mathcal{V}_0 \setminus \mathcal{X}'$. We put
\[ \delta_n(y) = \inf \left\{ \delta \in (0, \rho) : \mathcal{E}(\mathcal{E}(V_n - y), \bar{A}_{\delta,\rho}) = \frac{\epsilon}{2} \right\}. \]
Since $\mathcal{E}(V_n - y) \geq \epsilon$ we see $\delta_n(y) > 0$. It is also easy to see that $y \mapsto \delta_n(y)$ is lower semi-continuous. Hence we can find $y_n \in B_{\rho\epsilon}$ with $\delta_n(y_n) = \inf_{y \in B_{\rho\epsilon}} \delta_n(y)$. We put $\delta_n = \delta_n(y_n) > 0$. We have then:

**Lemma.** $\delta_n$ tends to 0 as $n \to \infty$.

**Proof.** Making $\rho$ smaller if necessary we may suppose $\mathcal{E}(X, \tilde{B}_\rho) < \epsilon/2$. We take $\delta > 0$. We have then $\mathcal{E}(X, \tilde{A}_{\delta,\rho}) < \epsilon/2$. Since $V_n$ tends to $X$ as $n \to \infty$ we see that for $n$ sufficiently large we have $\mathcal{E}(V_n, \tilde{A}_{\delta,\rho}) < \epsilon/2$. This implies $\delta_n(0) < \delta$ and so $\delta_n(0)$ tends to 0 as $n \to \infty$. Since $\delta_n \leq \delta_n(0)$ we see that $\delta_n$ also tends to 0 as $n \to \infty$. 

We have also:
Lemma. \((\delta_n^{-1}(V_n - y_n))^\infty_{n=1}\) has a subsequence converging to some varifold \(W\) in \(\mathbb{R}^{2m}\).

Proof. Let \(R > 0\). Then we have only to prove
\[
\sup_{n=1,2,3,\ldots} \|\delta_n^{-1}(V_n - y_n)\|(B_R) < \infty. \tag{7}
\]
Notice that \(\|\delta_n^{-1}(V_n - y_n)\|(B_R) = \delta_n^{-m}\|V_n - y_n\|(B_{\delta_n R})\). Notice also that by Proposition 1 we have
\[
e^{k\epsilon_n\delta_n R} \frac{\|V_n - y_n\|(B_{\delta_n R})}{(\delta_n R)^m} \leq e^{k\epsilon_n\|V_n - y_n\|(B_1)}
\]
which tends to \(\|X\|(B_1)\) as \(n \to \infty\). Then we get (7).

We take \(W\) as above. From the definition of \(\delta_n\) it is easy to see \(\mathcal{E}(W, \tilde{A}_{1,\infty}) \leq \epsilon/2\). In a way similar to the proof of Corollary 1, therefore, we can prove that \(W\) is asymptotic at infinity to a multiplicity 1 smooth cone \(C'\). Making \(\epsilon\) smaller if necessary we may suppose \(C' \in C'\).

It remains to prove \(\mathcal{E}(W - b) > 0\) for every \(b \in \mathbb{R}^{2m}\). We put \(V_n = V_n - y_n - \delta_n b\). By Proposition 1 we have
\[
e^{k\epsilon_n\delta_n} \|V_n - y_n\|(B_{\delta_n R}) \leq e^{k\epsilon_n\|V_n - y_n\|(B_1)} \geq \epsilon/2. \tag{8}
\]
We note that \(V_n\) tends to \(X\) and \(\delta_n^{-1}V_n\) tends to \(W - b\) as \(n \to \infty\). By (8) therefore we have
\[
\frac{\|X\|(B_1)}{\rho^m} - \|W - b\|(B_1) \geq \frac{\epsilon}{2}. \tag{9}
\]
Making \(\rho > 0\) smaller if necessary we may suppose
\[
\frac{\|X\|(B_1)}{\rho^m} \leq \text{area}(C \cap B_1) + \frac{\epsilon}{4}.
\]
This combined with (9) implies
\[
\text{area}(C \cap B_1) - \|W - b\|(B_1) > \frac{\epsilon}{4}.
\]
This combined with Proposition 4 implies
\[
\text{area}(C' \cap B_1) - \|W - b\|(B_1) > \frac{\epsilon}{4}. \tag{10}
\]
On the other hand by (3) we have
\[
\frac{\|W - b\|(B_R)}{R^m} - \|W - b\|(B_1) = \mathcal{E}((W - b)_\lambda, \tilde{A}_{\lambda, R})
\]
for each \(R > 1\). Letting \(R \to \infty\) we get
\[
\text{area}(C' \cap B_1) - \|W - b\|(B_1) = \mathcal{E}((W - b)_\lambda, \tilde{A}_{\lambda, \infty}).
\]
This combined with (10) implies \(\mathcal{E}((W - b)_\lambda, \tilde{A}_{1, \infty}) > \epsilon/4\) and so \(\mathcal{E}(W - b) > 0\), completing the proof. \(\Box\)
4 Classification of Local Models

We define $C$ by (1) in §1. In this section we shall classify local models for smoothing $C$. We can describe the local models as follows. We put

$$L_1 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - 1 = |z_2|^2 = |z_3|^2, z_1 z_2 z_3 \leq 0, \infty\},$$

$$L_2 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 - 1 = |z_3|^2, z_1 z_2 z_3 \leq 0, \infty\},$$

$$L_3 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 = |z_3|^2 - 1, z_1 z_2 z_3 \leq 0, \infty\}. \quad (11)$$

If $L \in \{L_1, L_2, L_3\}$ then $L$ will be a submanifold of $\mathbb{C}^3$ of dimension 3 over $\mathbb{R}$. We note that $L$ is non-singular in spite of the condition $z_1 z_2 z_3 \leq 0, \infty$.

Harvey and Lawson [9, Chapter III.3.A, Theorem 3.1] prove that $L$ is a special Lagrangian submanifold of $\mathbb{C}^3$. It is easy to see that $L$ is asymptotic at infinity to $C$ with multiplicity 1. Our classification result is:

**Theorem 5.** Let $W$ be a special Lagrangian integral varifold with $\partial^\infty_{\mathbb{R}} W = 0$ in $(\mathbb{R}^6, \omega_0, J_0, \Omega_0)$ asymptotic at infinity to $C$ with multiplicity 1. Then we have $\|W\| = |sL + b|$ for some $s > 0$, $L \in \{C, L_1, L_2, L_3\}$ and $b \in \mathbb{R}^6$.

Here $|sL + b|$ denotes the Radon measure on $\mathbb{R}^6$ associated to $sL + b$ with multiplicity 1 with respect to the Euclidean metric $g_0$.

In what follows we give a sketch of the proof of Theorem 5.

First of all we analyse the asymptotic behaviour of $W$ at infinity. From the stability of $C$ we find $b \in \mathbb{R}^6$ such that $W - b$ will decay rapidly to $C$ at infinity. For the sake of simplicity, therefore, we may suppose that $W$ decays rapidly to $C$ at infinity.

We define a $T^2$-action on $\mathbb{R}^6 = \mathbb{C}^3$ by setting

$$(e^{i\theta}, e^{i\phi}) \cdot (z_1, z_2, z_3) = (e^{i\theta} z_1, e^{i\phi} z_2, e^{-i\theta - i\phi} z_3).$$

We note that $\omega_0, J_0, \Omega_0$ and $C$ are invariant under the $T^2$-action. We take a moment map $\mu : \mathbb{C}^3 \to \mathbb{R}^2$. We prove that $\mu$ is constant on $\text{Spt} \|W\|$. The idea of the proof is as follows.

We suppose for a moment that $W$ is a submanifold of $\mathbb{C}^3$. By the $T^2$-action, then, we can deform $W$ as a special Lagrangian submanifold of $\mathbb{C}^3$. The infinitesimal deformation may be identified with $d\mu|_W$, and so $d\mu|_W$ will be a harmonic 1-form on $W$, by the theory of Mclean. Therefore $\mu|_W$ will be a harmonic function on $W$. Since $W$ decays rapidly to $C$ at infinity we shall see that $\mu|_W$ decays rapidly at infinity to some constant. By the maximum principle, therefore, $\mu|_W$ will be a constant function.

Actually $W$ need not be a submanifold of $\mathbb{R}^6$. Using some basic results on varifolds, however, we can modify the argument above. Thus we see that $\mu$ is constant on $\text{Spt} \|W\|$. Harvey and Lawson [9, Chapter III.3.A, Theorem 3.1] construct a special Lagrangian fibration $F : \mathbb{C}^3 \to \mathbb{R}^3$ invariant under the $T^2$-action. From the proof of Harvey and Lawson we see that every $T^2$-invariant special Lagrangian submanifold of $\mathbb{C}^3$ is contained in a fibre of $F$. 

13
We suppose again that $W$ is a submanifold of $\mathbb{C}^3$. Since $\mu$ is constant on $W$ we see that $W$ is then invariant under the $T^2$-action, and so contained in a fibre of $F$. Actually $W$ need to be a submanifold of $\mathbb{C}^3$, but we can again modify the argument and so $Spt\|W\|$ is contained in a fibre of $F$.

For each fibre of $F$ we have an explicit description of the topology and asymptotic behaviour at infinity, and so we shall be able to complete the proof by an elementary argument.

We begin now with a review of some basic properties of Laplacians over cones.

We suppose that $\Sigma$ is a compact manifold of dimension $m - 1$. We put $C = (0, \infty) \times \Sigma$. We denote by $r$ the projection of $C$ onto $(0, \infty)$. We suppose that $g_\Sigma$ is a Riemannian metric on $\Sigma$. We put $g_C = dr^2 + r^2 g_\Sigma$. With respect to $g_C$ we can define the Laplacian $\Delta_C : C^\infty(C; \mathbb{R}) \to C^\infty(C; \mathbb{R})$.

We also have the Laplacian $\Delta_\Sigma : C^\infty(\Sigma; \mathbb{R}) \to C^\infty(\Sigma; \mathbb{R})$ with respect to $g_\Sigma$. We put $t = \log r : C \to \mathbb{R}$ and $\partial_t = \frac{\partial}{\partial t}$. It is easy then to see $-e^{2t}\Delta_C = \Delta_\Sigma = \partial_t^2 + (m - 2)\partial_t - \Delta_\Sigma$. We denote by $0 = \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots$ the eigenvalues of $\Delta_\Sigma$. For each integer $i \geq 0$ we consider the equation $x^2 + (m - 2)x - \gamma_i = 0$ in $x$. We denote by $\alpha_i$ and $\beta_i$ the two solutions with $\alpha_i \geq \beta_i$. We suppose $m \geq 2$. Since $\gamma_0 = 0$ we get $\beta_0 = 2 - m \leq 0 = \alpha_0$. Since $\gamma_0 \leq \gamma_1 \leq \gamma_2 \cdots$ we get $\alpha_0 \leq \alpha_1 \leq \alpha_2 \cdots$ and $\beta_0 \geq \beta_1 \geq \beta_2 \cdots$. Since $\gamma_i$ tends to $\infty$ as $i \to \infty$ we see that $\alpha_i$ tends to $\infty$ and $\beta_i$ tends to $-\infty$ as $i \to \infty$. We also put $\Lambda = \{\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \cdots\}$. For each $\lambda \in \mathbb{R}$ we put

$$E_\lambda = \{f \in C^\infty(\Sigma; \mathbb{R}) : \Delta_\Sigma f = \lambda(\lambda + m - 2)f\}.$$  

By the definition of the eigenvalues we have $E_\lambda \neq \{0\}$ if and only if $\lambda \in \Lambda$.

We can also take a complete orthonormal basis $\{v_0, v_1, v_2, \cdots\}$ for $L^2(\Sigma; \mathbb{R})$ such that $\Delta_\Sigma v_i = \gamma_i v_i$ for each integer $i \geq 0$. We have then:

**Proposition 5.** Let $I$ be an open interval in $\mathbb{R}$, let $u \in C^\infty(I \times \Sigma; \mathbb{R})$ and suppose $\Delta_C u = 0$. Then there exist $a_0, b_0, a_1, b_1, a_2, b_2, \cdots \in \mathbb{R}$ such that $u = \sum_{i=0}^{\infty}(a_i e^{\alpha_i t} + b_i e^{\beta_i t})v_i$ where the series converge in the local $C^\infty$-topology.

We give a proof for the sake of clarity:

**Proof.** We put $\log I = \{t \in \mathbb{R} : e^t \in I\}$. For each $t \in \log I$ and $i \in \{0, 1, 2, \cdots\}$ we define $u_i(t)$ as the inner product of $u|_{\{e^t\} \times \Sigma}$ and $v_i$ in $L^2(\Sigma; \mathbb{R})$. Then $t \mapsto u_i(t)$ is a smooth function on $\log I$. Since $\Delta_C u = 0$ we get $(\partial_t^2 + (m - 2)\partial_t - \lambda_i)u_i = 0$. Hence we find $a_i, b_i \in \mathbb{R}$ such that $u_i = a_i e^{\alpha_i t} + b_i e^{\beta_i t}$. It suffices therefore to prove that $\sum_{i=0}^{\infty} u_i v_i$ converges to $u$ in the local $C^\infty$-topology. We denote by $d\mu_\Sigma$ the Riemannian measure on $\Sigma$ with respect to $g_\Sigma$, and by $\|\cdot\|$ the norm of $L^2(\Sigma; \mathbb{R})$. We have then

$$\int_{\log I} \|u_i\|^2 d\mu_\Sigma = \int_{\log I} \sum_{i=0}^{\infty} \|u_i(t)\|^2 dt = \sum_{i=0}^{\infty} \int_{\log I} \|u_i(t)\|^2 dt < \infty.$$ 

Putting $w_n = \sum_{i=0}^{n} u_i v_i$ we get $\int_{\log I} \|u - w_n\|^2 d\mu_\Sigma = \sum_{i=n+1}^{\infty} \int_{\log I} \|u_i(t)\|^2 dt$ which tends to $0$ as $n \to \infty$. Thus $w_n$ tends to $u$ in the $L^2$-topology. Applying elliptic regularity to $u - w_n$ we see that $w_n$ tends to $u$ in the local $C^\infty$-topology. \[\square\]
Let \( u \in C^\infty(C; \mathbb{R}) \) and \( \alpha \in \mathbb{R} \). Then we shall write \( u = O(r^\alpha) \) as \( r \to \infty \) if there exists \( R > 0 \) such that
\[
\sup_{(R, \infty) \times \Sigma} |r^{-\alpha+k}\nabla^k u| < \infty \text{ for every } k = 0, 1, 2, 3, \ldots
\]
where \( \nabla \) denotes the Levi-Civita connexion with respect to \( g_C \), and \( |\cdot| \) denotes the norm with respect to \( g_C \).

By a result of Simon [23, Part I, Lemma 5.9] we have:

**Proposition.** Suppose \( q \in (2 - m, \infty) \setminus \Lambda \) and \( f \in C^\infty((R, \infty) \times \Sigma; \mathbb{R}) \) with \( f = O(r^{q-2}) \). Then there exists \( u \in C^\infty((R, \infty) \times \Sigma; \mathbb{R}) \) with \( u = O(r^q) \) such that \( \Delta_C u = f \).

We give a corollary to this:

**Corollary 2.** Suppose \( R > 0 \), \( u \in C^\infty((R, \infty) \times \Sigma; \mathbb{R}) \), \( p, q \in \mathbb{R} \), \( 2 - m < q < p \), \( q \notin \Lambda \), \( u = O(r^p) \) and \( \Delta_C u = O(r^{q-2}) \). Then there exists \( \{f_\lambda\}_{\lambda \in \Lambda} \in \bigoplus_{\lambda \in \Lambda} E_\lambda \) such that \( u = \sum_{\lambda \in \Lambda \setminus \{q,p\}} f_\lambda r^\lambda + O(r^q) \) where \( \Lambda \setminus \{q,p\} \) is possibly empty and in that case we set \( \bigoplus_{\lambda \in \emptyset} E_\lambda = \{0\} \).

**Proof.** Applying the proposition above to \( \Delta_C u \) in place of \( f \) we find \( u' \in C^\infty((R, \infty) \times \Sigma; \mathbb{R}) \) with \( u' = O(r^q) \) such that \( \Delta_C u' = \Delta_C u \). By Proposition 5 we can find some \( a_0, 0_0, a_1, b_1, a_2, b_2, \ldots \in \mathbb{R} \) such that \( u - u' = \sum_{i=0}^\infty (a_i r^{\alpha_i} + b_i r^{\beta_i}) v_i \).

Since \( u' = O(r^q) \), we get \( u = \sum_{i=0}^\infty (a_i r^{\alpha_i} + b_i r^{\beta_i}) v_i + O(r^q) \). Since \( u = O(r^p) \) and \( p > q \) we get \( a_i = 0 \) if \( \alpha_i > p \). Since \( a_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \) tend to \( \infty \) we can find a unique integer \( i(p) \) such that \( \alpha_{i(p)+1} > p \) and \( \alpha_{i(p)} \leq p \). Since \( a_i = 0 \) for every \( i > i(p) \) we see that \( \sum_{i=0}^\infty b_i r^{\beta_i} v_i \) converges in the local \( C^\infty \)-topology. We put \( w = \sum_{i=0}^\infty b_i r^{\beta_i} v_i \).

Since \( \sum_{i=0}^\infty b_i r^{\beta_i} v_i \) converges in the local \( C^\infty \)-topology, we have
\[
\int_R^{2R} \int_{|\cdot|} \left| w \right| \frac{2 \, dr \, d\mu_\Sigma}{r} \leq \left( \frac{R}{R} \right)^{2-m} \int_R^{2R} \int_{\Sigma} \left| w \right| \frac{2 \, dr \, d\mu_\Sigma}{r}.
\]
Applying elliptic regularity to \( w \) we get \( w = O(r^{2-m}) \). Since \( u = \sum_{i=0}^\infty a_i r^{\alpha_i} v_i + w + O(r^q) \) and \( q > 2 - m \) we get \( u = \sum_{i=0}^\infty a_i r^{\alpha_i} v_i + O(r^q) \), completing the proof. \( \square \)

We suppose now that \( C \) is a smooth special Lagrangian cone in \( (\mathbb{R}^{2m} \setminus \{0\}, \omega_0, J_0, \Omega_0) \) and \( \Sigma = C \cap S^{2m-1} \). For each \( \rho > 0 \) we put \( \beta_\rho = \{ y \in \mathbb{R}^n : |y| < \rho \} \). We denote by \( i_C : C \to \mathbb{R}^{2m} \) the inclusion map of \( C \) into \( \mathbb{R}^{2m} \).

Joyce [13, Definition 7.1] defines special Lagrangian submanifolds of \( (\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0) \) asymptotic to \( C \) with multiplicity 1 at infinity with some rate \( \lambda < 2 \). We extend it to varifolds as follows. We denote by \( W \) the set of all special Lagrangain integral varifolds \( W \) with \( \partial W = \emptyset \) in \( (\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0) \) asymptotic at infinity to \( C \) with multiplicity 1. For each \( \lambda < 2 \) we denote by \( W_\lambda \) the set of all \( W \in W \)
such that we can find a compact subset $K$ of $W$, an $R > 0$ and a diffeomorphism $f : C \setminus \overbar{B_R} \to \text{Spt}\|W\| \setminus K$ such that $f - i_C = O(r^{\lambda - 1})$.

We suppose that $C$ is Jacobi-integrable in the sense of Joyce [13, Definition 6.7]. In a way similar to Simon [23, Part II, §§5 and 6], then, we can prove that if $W \in \mathcal{W}_\lambda$, then there exists $\lambda < 2$ such that $W \in \mathcal{W}_\lambda$.

Joyce [13, Theorem 4.3] proves a version of Weinstein’s theorem [27, Corollary 6.2], which we shall recall next. We denote by $T^*C$ the cotangent bundle over $C$, by $0_C$ the zero-section of $T^*C$, and by $\omega_C$ the canonical symplectic form on $T^*C$. We regard $0 \in (0, \infty)$ as a multiplicative group acting upon $C$ and $\mathbb{R}^{2m}$ as re-scaling. We can lift the $(0, \infty)$-action uniquely to $T^*C$ so that for each $t \in (0, \infty)$ we shall have $t^*\omega_C = t^2\omega_C$ (on the left-hand side we regard $t$ as a map of $C$ into itself). We have then:

**Lemma 1.** There exist a neighbourhood $UC$ of $\text{Im} 0_C$ in $T^*C$ invariant under $(0, \infty)$, and a diffeomorphism $\Phi_C$ of $UC$ into $\mathbb{R}^{2m}$ equivariant under $(0, \infty)$ with $\Phi_C \circ 0_C = i_C$ and $\Phi_C^*\omega_0 = \omega_C$.

Let $W \in \mathcal{W}_\lambda$. Then we can take a compact subset $K_\lambda$ of $W$ and a closed 1-form $\omega$ on $C \setminus \overbar{B_R}$ such that $\text{Spt}\|W\| \setminus K_{\lambda} = \Phi_C(\text{Graph} \omega)$. We denote by $\pi_C$ denotes the projection of $C$ onto $\Sigma$. We may write $w = \pi_C^*\eta_W + dh_W$ for some 1-form $\eta_W$ on $\Sigma$ and some $h_W : C \setminus \overbar{B_R} \to \mathbb{R}$. By results of Joyce [13, Equations (7.7) and (7.8)] we have:

**Lemma 2.** If $\alpha < 2$ and $h_W = O(r^\alpha)$ then we have $\Delta_{\lambda} h_W = O(r^{2(\alpha - 2)})$.

We can extend a result of Joyce [13, Theorem 7.11] as follows:

**Lemma 3.** Let $W \in \mathcal{W}_\lambda$, $\lambda' < \lambda < 2$ and $[\lambda', \lambda] \cap \lambda = \emptyset$. Then we have $W \in \mathcal{W}_{\lambda'}$.

**Proof.** For each integer $n \geq 0$ we put $\lambda(n) = 2^n(\lambda - 2) + 2$. We can take a unique integer $\nu$ such that $\lambda(\nu + 1) < \lambda' \leq \lambda(\nu)$. By an induction on $n = 0, 1, \cdots, \nu$, we shall prove $h_W = O(r^{\lambda(n)})$.

By the property of $h_W$ we have $h_W = O(r^{\lambda}) = O(r^{\lambda(0)})$. If $\nu = 0$ we can then complete the induction automatically. We suppose therefore $\nu > 0$. Suppose also that we have $h_W = O(r^{\lambda(n)})$ for some $n = 0, 1, \cdots, \nu - 1$. By Lemma 2 we have $\Delta h_W = O(r^{\lambda(n) - 2}) = O(r^{\lambda(n + 1) - 2})$. Since $n < \nu$ we get $\lambda(n + 1) \geq \lambda'$ and so $[\lambda(n + 1), \lambda] \cap \lambda = \emptyset$. Applying Corollary 2 to $h_W, \lambda, \lambda(n + 1)$ in place of $u, p, q$ respectively, we get $h_W = O(r^{\lambda(n + 1)})$, completing the induction.

We have thus proved $h_W = O(r^{\lambda(n)})$ for every $n = 0, 1, \cdots, \nu$. Putting $n = \nu$ we get $h_W = O(\nu^{\lambda(\nu)})$. By Lemma 2 we have $\Delta_{\lambda} h_W = O(r^{\nu^{\lambda(\nu) - 2}})$. By the definition of $\nu$ we have $\lambda(\nu + 1) \leq \lambda'$ and so $\Delta_{\lambda} h_W = O(r^{\lambda' - 2})$. Applying Corollary 2 to $h_W, \lambda, \lambda'$ in place of $u, p, q$ respectively, we get $h_W = O(r^{\lambda'})$, completing the proof.

Joyce [14, Definition 3.6] defines the stability of $C$. We have:

**Theorem 6.** Let $C$ be stable in the sense of Joyce. Then there exists $b \in \mathbb{R}^{2m}$ such that $W - b \in \mathcal{W}_0$. 

16
Proof. By the stability of $C$ we have $\Lambda \cap (1, 2) = \emptyset$ and $E_1 = \{b \cdot x : b \in \mathbb{R}^{2m}\}$. We take $\lambda \in (1, 1 + \varepsilon)$. Let $h_W$ be as above. Then by Lemma 3 we have $h_W = O(r^\lambda)$. By Lemma 2, therefore, we have $\Delta_C h_W = O(r^{2(\lambda - 2)})$. Applying Corollary 2 to $h_W, \lambda, 2(\lambda - 2)$ in place of $u, p, q$ we get $h_W = b \cdot x |_C + O(r^{2(\lambda - 2) + 2})$.

We may suppose that for each $t \in [0, 1]$ there exist a compact subset $K_t$ of $\mathbb{R}^{2m}$ and a 1-form $w_t$ on $C \setminus \overline{B_R}$ such that $\Phi \in \Phi_C(\text{Graph} w_t)$. We put $\beta = (\tilde{b} \cdot x) \circ \Phi_C$. We have then a function $\beta : UC \to \mathbb{R}$. Notice that $\Phi_C^{-1}(W - tb)$ is the image of the time-one map of the flow generated by $d\beta \omega_C$. Then we have $\partial w_t / \partial t = -w_t^\beta \beta$ and so $w_t = w_0 - d\int_0^1 w_t^\beta \beta dt = \pi_2^h \eta_{w_0} + d(h_W - \int_0^1 w_t^\beta \beta dt)$. Hence we get $h_W - b = h_W - \int_0^1 w_t^\beta \beta dt = h_W - \beta |_C - \int_0^1 (w_t^\beta \beta - \beta |_C) dt = O(r^{2\lambda - 2}) + O(r^{\lambda - 1}) = O(r^{2\lambda - 2})$. By results of Joyce [13, Equations (7.7) and (7.8)] we have $\Delta_C h_W - b = O(r^{4\lambda - 8})$. Applying Corollary 2 to $h_W - b, 2\lambda - 2, 4\lambda - 8$ in place of $u, p, q$ we get $c \in \mathbb{R}$ such that $h_W - b = c + O(r^{4\lambda - 8}) = O(r^0)$ as we may suppose $4\lambda - 6 < 0$. This completes the proof.

Let $\lambda < 2$ and $W \in W_\lambda$. Take a compact subset $K_W$ of $\mathbb{R}^{2m}$, an $R > 0$ and a diffeomorphism $f_W : C \setminus \overline{B_R} \to \text{Spt} ||W|| \setminus K_W$ with $f_W^* i_C = O(\lambda - 1)$. Then we have a Riemannian metric $f_W^* g_0$ over $C \setminus \overline{B_R}$. With respect to $f_W^* g_0$ we can define the Laplacian $\Delta_W : C^\infty(C \setminus \overline{B_R}; \mathbb{R}) \to C^\infty(C \setminus \overline{B_R}; \mathbb{R})$. We have then:

**Proposition.** If $u \in C^\infty(C \setminus \overline{B_R}; \mathbb{R})$ and $u = O(r^a)$ then we have $\Delta_W u = \Delta_C u + O(r^{\alpha + \lambda - 4})$.

**Proof.** Since $f_W - i_C = O(r^{\lambda - 1})$ we get $f_W^* g_0 = i_C^* g_0 + O(r^{\lambda - 2})$. We denote by $\nabla_W$ and $\nabla_C$ the Levi-Civita connexions over $C \setminus \overline{B_R}$ with respect to $f_W^* g_0$ and $i_C^* g_0$ respectively. We have then $\nabla_W = \nabla_C + O(r^{\lambda - 3})$ and so $\Delta_W u = \Delta_C u + O(r^{\alpha + \lambda - 4})$ as we want.

We give a corollary to this:

**Corollary 3.** If $u \in C^\infty(C \setminus \overline{B_R}; \mathbb{R})$, $\Delta_W u = 0$ and $u = O(r^0)$ then we have $u = c + O(r^{2 - m})$ for some $c \in \mathbb{R}$.

**Proof.** By the proposition above we have $\Delta_C u = O(r^{0 + \lambda - 4}) = O(r^{\lambda - 4})$. If $\lambda - 2 < 2 - m$ then we can complete the proof by applying Corollary 2 to $0, \lambda - 2$ in place of $p, q$ respectively.

We suppose therefore $\lambda - 2 > 2 - m$. We take $\lambda' \in (\lambda, 2)$ such that $\lambda' - 2 > 2 - m$. Applying Corollary 2 to $0, \lambda' - 2$ in place of $p, q$ respectively, we get $u = c + O(r^{\lambda' - 2})$ for some $c > 0$. We have thus improved the decay order estimate for $u$, and so we can complete the proof in a way similar to the proof of Lemma 3.

We suppose now

$$C = \{(z_1, \cdots, z_m) \in \mathbb{C}^m \setminus \{0\} : |z_1| = \cdots = |z_m|, z_1 \cdots z_m \in (0, \infty)\}.$$
This is an extension of (1) in §1 to dimension $m$. Harvey and Lawson [9, Chapter III.3.A, Theorem 3.1] prove that $C$ is a special Lagrangian submanifold of $(\mathbb{R}^{2m} \setminus \{0\}, \omega_0, J_0, \partial_0)$.

We define a $T^{m-1}$-action on $\mathbb{C}^m$ as follows. We write $T^{m-1} = S^1 \times \cdots \times S^1$ and $S^1 = \{ t \in \mathbb{C} : |t| = 1 \}$. For each $j \in \{ 2, \ldots, m \}$ we define the $j$-th $S^1$-action on $\mathbb{C}^m$ by setting $t \cdot (z_1, \ldots, z_j, \ldots, z_m) = (tz_1, \ldots, t^{-1}z_j, \ldots, z_m)$ for each $(z_1, \ldots, z_m) \in \mathbb{C}^m$.

We also define a map $\mu_j : \mathbb{C}^m \to \mathbb{R}$ by $2\mu_j(z_1, \ldots, z_m) = |z_1|^2 - |z_j|^2$. We shall identify $\mathbb{R}$ with the Lie algebra of $S^1$ so that $\mu_j$ will be the moment map on $(\mathbb{C}^m, \omega_0)$ with respect to the $S^1$-action. We have then:

**Theorem.** If $W \in \mathcal{W}_0$ and $j \in \{ 2, \ldots, m \}$ then we have $\text{grad}_{TW} \mu_j = 0$ almost everywhere on $\mathbb{R}^{2m}$ with respect to $\|W\|$. 

**Proof.** We can take a compact subset $K_W$ of $\mathbb{R}^{2m}$, an $R > 0$ and a diffeomorphism $f_W : C \setminus \overline{B_R} \to \text{Spt} \|W\| \setminus K_W$ with $f_W - iC = O(r^{-1})$. Since $\mu_j = O(r^2)$ and $\mu_j \circ iC = 0$ we get $f_W(\mu_j) = \mu_j \circ f_W = \mu_j \circ f_W - \mu_j \circ iC = \int_0^1 df_W(\mu_j)(1-iC)dt = O(r^{-1})$ and so $f_W(\mu_j) = O(r^0)$. By a result of Joyce [14, Lemma 3.4] we have $\Delta_W f_W(\mu_j) = 0$ for each $j = 2, \ldots, m$.

By Corollary 3 we can find $c_j \in \mathbb{R}$ such that $f_W(\mu_j) - c_j = O(r^{2-m})$. Putting $\mu_j' = \mu_j - c_j$ we get $f_W(\mu_j') = O(r^{2-m})$. We have clearly $\text{grad} \mu_j' = \text{grad} \mu_j$.

Take a smooth function $\chi : \mathbb{R} \to [0, 1]$ with $\chi = 1$ on $B_1$ and $\chi = 0$ on $B_2$.

Let $R > 0$, and define a function $\chi_R : \mathbb{R}^{2m} \to [0, 1]$ by setting $\chi_R(x) = \chi(|x|/R)$. Since $W$ has first variation 0 in $B_{2R}$ we get then

$$\int_{B_{2R}} \text{div}_{TW}(\chi_R \mu_j' \text{grad} \mu_j)d\|W\| = 0.$$ 

Also by a result of Joyce [14, Lemma 3.4] we have $\text{div}_{TW} \mu_j = 0$ and so

$$\int_{B_{2R}} \mu_j'(\text{grad} \chi_R, \text{grad}_{TW} \mu_j) + \chi_R|\text{grad}_{TW} \mu_j|^2d\|W\| = 0. \quad (12)$$

Notice that $|\text{grad} \chi_R, \text{grad}_{TW} \mu_j| = (d\chi_R, d\mu_j'|_{\text{Spt} \|W\| \setminus K_W})$ on $\text{Spt} \|W\| \setminus K_W$. Take $R$ sufficiently large so that $K_W \subset B_R$. Then we have $\chi_R = 0$ on $K_W$ and so $|\text{grad} \chi_R, \text{grad}_{TW} \mu_j| = (d\chi_R, d\mu_j'|_{\text{Spt} \|W\| \setminus K_W})$ on $\text{Spt} \|W\|$. We have therefore

$$\int_{B_{2R}} -\mu_j'(\text{grad} \chi_R, \text{grad}_{TW} \mu_j)d\|W\| \leq \sup_{\text{Spt} \|W\| \setminus K_W} |\mu_j'(d\chi_R, d\mu_j)| \int_{B_{2R}} d\|W\|$$

and so by (12) we have

$$\int_{B_{2R}} |\text{grad}_{TW} \mu_j|^2d\|W\| \leq \sup_{\text{Spt} \|W\| \setminus K_W} |\mu_j'(d\chi_R, d\mu_j)| \int_{B_{2R}} d\|W\|. \quad (13)$$

Since $f_W : C \setminus \overline{B_R} \to \text{Spt} \|W\| \setminus K_W$ is a diffeomorphism we get

$$\sup_{\text{Spt} \|W\| \setminus K_W} |\mu_j'(d\chi_R, d\mu_j)| = \sup_{C \setminus \overline{B_R}} |f_W^\ast \mu_j'(d\chi_R, d\mu_j)|. \quad (14)$$
Since \( \chi_R = \chi(r/R) \) we get \( |d\chi_R| \leq kR^{-1} \) for some \( k > 0 \) independent of \( R \). Since \( f^i_W \mu_j^i = O(r^{2-m}) \) we get
\[
\sup_{C \setminus B_R} |f^i_W \mu_j^i(d\chi_R, df^i_W \mu_j)| \leq kR^{2-m} R^{-1} R^{1-m} = kR^{2-2m}
\]
for some \( k > 0 \) independent of \( R \). By (13)–(15) we have
\[
\int_{B_{2R}} |\chi_R| |\text{grad}_{TW} \mu_j|^2 d\|W\| \leq kR^{2-2m} \int_{B_{2R}} d\|W\|.
\]
On the other hand, by the monotonicity formula, we have \( \int_{B_{2R}} d\|W\| \leq kR^m \) for some \( k > 0 \) depending only on \( m \) and \( T_\infty W \). We have therefore
\[
\int_{B_{2R}} |\chi_R| |\text{grad}_{TW} \mu_j|^2 d\|W\| \leq kR^{2-m}, k > 0 \text{ independent of } R.
\]
Letting \( R \to \infty \) we get \( \int_{R^{2m}} |\text{grad}_{TW} \mu_j|^2 d\|W\| = 0 \), completing the proof. \( \square \)

We give a corollary to the theorem above. We define a map \( f : \mathbb{C}^m \to \mathbb{C} \) by setting \( f(z_1, \cdots, z_m) = i^{m+1}z_1 \cdots z_m \) for each \( (z_1, \cdots, z_m) \in \mathbb{C}^m \). We put \( \text{Im } f = (f - \bar{f})/2i : \mathbb{C}^m \to \mathbb{R} \) and \( F = (\mu_2, \cdots, \mu_m, \text{Im } f) : \mathbb{C}^m \to \mathbb{R}^m \). We have then:

**Corollary 4.** If \( W \in W_0 \) then we have \( T_y W = \text{Ker } dF|_y \) for \( \|W\| \)-almost every \( y \in \mathbb{R}^{2m} \).

This follows readily from the proof of Harvey and Lawson [9, Chapter III.3.A, Theorem 3.1].

We have moreover:

**Corollary 5.** For every \( W \in W_0 \) there exists \( c \in \mathbb{R}^m \) such that \( F = c \) on \( \text{Spt } \|W\| \).

**Proof.** By Corollary 4 we have \( dF|_{\text{Spt } \|W\| \setminus K_W} = 0 \), and \( F|_{\text{Spt } \|W\| \setminus K_W} \) is therefore locally constant. Since \( \text{Spt } \|W\| \setminus K_W \cong C \setminus B_R \cong (R, \infty) \times T_{\infty}^{-1} \) we see that \( \text{Spt } \|W\| \setminus K_W \) is connected, and \( F|_{\text{Spt } \|W\| \setminus K_W} \) is therefore constant; i.e. we have \( F|_{\text{Spt } \|W\| \setminus K_W} = c \) for some \( c \in \mathbb{R}^m \).

Put \( \phi = |F - c|^2 \). Then we have \( \phi = 0 \) on \( \text{Spt } \|W\| \setminus K_W \) and so \( \text{Spt } \phi \cap \text{Spt } \|W\| \subseteq K_W \). By Corollary 4 we have \( \text{grad}_{TW} \phi = 0 \) almost everywhere on \( \mathbb{R}^{2m} \) with respect to \( \|W\| \).

We shall now use a result of Michael and Simon [20, Theorem 2.1], who prove a Poincaré–Sobolev inequality for varifolds; we refer also to Simon [24, Theorem 18.6], who uses varifolds more explicitly. We are going to use the following version:

**Lemma.** Let \( W \) be a stationary integral varifold of dimension \( m \) in \( (\mathbb{R}^n, g_0) \). Suppose that we have a smooth function \( \phi : \mathbb{R}^n \to [0, \infty) \) with \( \text{Spt } \phi \cap \text{Spt } \|W\| \) compact and \( \text{grad}_{TW} \phi = 0 \) almost everywhere on \( \mathbb{R}^n \) with respect to \( \|W\| \). Then we have \( \phi = 0 \) on \( \text{Spt } \|W\| \).
We give a proof for the sake of clarity:

**Proof.** We define $r : \mathbb{R}^n \to [0, \infty)$ by setting $r(y) = |y|$. We can then define $\partial_r = \partial/\partial r$ as a smooth vector field on $\mathbb{R}^n \setminus \{0\}$. It is easy to see that $r\partial_r$ extends smoothly to $\mathbb{R}^n$. Let $\phi$ be as above, and let $\chi$ be a compactly-supported smooth function on $\mathbb{R}^n$ with $\chi = 1$ on $\text{Spt} \phi \cap \text{Spt} \|W\|$. Then we can define $\phi r\partial_r$ as a smooth vector field on $\mathbb{R}^n$. Since $W$ has first variation 0 we get

$$\int_{\mathbb{R}^n} \text{div}_W \chi \phi r\partial_r d\|V\| = 0.$$ 

Since $\chi = 1$ on $\text{Spt} \phi \cap \text{Spt} \|W\|$ we get $\int_{\text{Spt} \phi \cap \text{Spt} \|W\|} \text{div}_W \chi \phi r\partial_r d\|W\| = 0.$

Since $\text{grad}_W \phi = 0$ on $\text{Spt} \|W\|$ we get $\int_{\text{Spt} \phi \cap \text{Spt} \|W\|} \phi \text{div}_W r\partial_r d\|W\| = 0.$

Since $\text{div}_W r\partial_r = m$ we get $\int_{\text{Spt} \phi \cap \text{Spt} \|W\|} m \phi d\|W\| = 0.$ Since $\phi \geq 0$ we get $\phi = 0$ on $\text{Spt} \phi \cap \text{Spt} \|V\|$, completing the proof. \hfill \Box

Hence we get $\phi = |F - c|^2 = 0$ on $\text{Spt} \|W\|$, completing the proof of Corollary 5. \hfill \Box

We suppose now $m = 3$. For the fibres of $F : \mathbb{C}^3 \to \mathbb{R}^3$ we have an explicit description of the topology and asymptotic behaviour at infinity, which we shall use next. The behaviour of $F : \mathbb{C}^m \to \mathbb{R}^m$ is rather complicated if $m > 3$, which we shall not discuss.

We put $Y = \{(a,0,0) \in \mathbb{R}^3 : a \geq 0\} \cup \{(0,a,0) \in \mathbb{R}^3 : a \geq 0\} \cup \{(-a,-a,0) \in \mathbb{R}^3 : a \geq 0\}$. We note that if $c \in \mathbb{R}^3 \setminus Y$ then $F^{-1}(c)$ has no fixed point with respect to the $T^2$-action

$$(e^{i\theta},e^{i\phi}) \cdot (z_1,z_2,z_3) = (e^{i\theta}z_1,e^{i\phi}z_2,e^{-i\theta-i\phi}z_3).$$

We have:

**Proposition 6.** Let $W \in \mathcal{W}$ and suppose $\text{Spt} \|W\| \subset F^{-1}(c)$ for some $c \in \mathbb{R}^3$. Then we have $c \in Y$.

For the proof we shall use:

**Lemma 4.** If $c \in \mathbb{R}^3 \setminus Y$ then $F^{-1}(c)$ is a submanifold of $\mathbb{R}^3$ diffeomorphic to $\mathbb{R} \times S^1 \times S^1$ and asymptotic to $C \cup -C$ with multiplicity 1 at infinity.

**Proof.** We put $c = (c_1,c_2,c_3)$. We may suppose $c_1 \geq c_2 \geq 0$ without loss of generality. Since $(c_1,c_2,c_3) \in \mathbb{R}^3 \setminus Y$ we get $c_3 \neq 0$ or $c_2 > 0$. For each $t \in \mathbb{R}$ we can find a unique $\phi_t(t) \in [0,\infty)$ such that

$$(\phi_t(t) + c_1)(\phi_t(t) + c_2)\phi_t(t) = |t + ic_3|^2.$$ 

It is easy to see that $\phi_t(t)$ depends smoothly on $t$. We put $\psi_c(t) = \sqrt{\phi_t(t)} + c_1\sqrt{\phi_t(t)} + c_2$. Since $c_3 \neq 0$ or $c_2 > 0$ we get $\psi_c(t) > 0$ for every $t \in \mathbb{R}$, and so we can define
$\psi^{-1}(t) > 0$ for every $t \in \mathbb{R}$. Define a smooth map $\Phi_e : \mathbb{R} \times S^1 \times S^1 \to f^{-1}(c)$ by setting

$$\Phi_e(t, u, v) = (\sqrt{c_1}u + c_2v, \frac{t + ic_3}{\psi_e(t)uv}),$$

and a smooth map $\Psi_e(t) : F^{-1}(c) \to \mathbb{R} \times S^1 \times S^1$ by setting

$$\Psi_e(z_1, z_2, z_3) = (\text{Re } z_1 z_2 z_3, \frac{z_1}{\phi_e(z_1 z_2 z_3)}, \frac{z_2}{\phi_e(z_1 z_2 z_3)}).$$

Then $\Psi_e \circ \Phi_e$ is clearly the identity map of $\mathbb{R} \times S^1 \times S^1$. It is also easy to see that $\Phi_e \circ \Psi_e$ is the identity map of $F^{-1}(c)$, and so $F^{-1}(c)$ is a submanifold of $\mathbb{C}^3$ diffeomorphic to $\mathbb{R} \times S^1 \times S^1$. It is also easy to see that $\Phi_e \circ \Psi_e$ is asymptotic to $C \cup -C$ with multiplicity 1 at infinity. \hfill $\square$

**Proof of Proposition 6.** We give a proof by contradiction, and so suppose $c \notin Y$. By Lemma 4, then, $F^{-1}(c)$ will be a connected submanifold of $\mathbb{R}^6$. By a constancy theorem of Allard [1, Theorem 4.6(3)] or Simon [24, Theorem 41.1], therefore, $\Theta_W$ will be constant on $F^{-1}(c)$. On the other hand we have $\Theta_W = 1$ near infinity on $\text{Spt } |W|$ and so we shall have $\Theta_W = 1$ on $F^{-1}(c)$, which implies $|W| = |F^{-1}(c)|$. By Lemma 4, however, $F^{-1}(c)$ is asymptotic at infinity to $C \cup -C$ with multiplicity 1, which contradicts that $W$ is asymptotic at infinity to $C$ with multiplicity 1. This completes the proof. \hfill $\square$

Let $c \in Y \subset \mathbb{R}^2 \times \{0\}$. Then we have $z_1 z_2 z_3 \in \mathbb{R}$ on $F^{-1}(c)$. For each $\epsilon > 0$ we put $L'_\epsilon = F^{-1}(c) \cap \{z_1 z_2 z_3 > \epsilon\}$. We also put $L_\epsilon = F^{-1}(c) \cap \{z_1 z_2 z_3 \geq 0\}$. By the definition of $C$ we have $L_0 = C$. By (11) we have $L_1 = L_2(0,2,0)$ and $L_3 = L_2(0, -2, -2)$.

We have:

**Proposition 7.** Let $W \in W$ and suppose $\text{Spt } |W| \subset F^{-1}(c)$ for some $c \in Y$. Then we have $|W| = |sL|$ for some $s > 0$ and $L \in \{C, L_1, L_2, L_3\}$.

For the proof we shall use:

**Lemma 5.** $L_1, L_2, L_3$ are submanifolds of $\mathbb{C}^3$ diffeomorphic to $S^1 \times \mathbb{R}^2$ and asymptotic to $C$ with multiplicity 1 at infinity.

**Proof of Proposition 7.** We give the proof only for $L_1$; we can do it likewise for $L_2$ and $L_3$. By definition we have $L_1 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - 1 = |z_2|^2 = |z_3|^2, z_1 z_2 z_3 \in [0, \infty)\}$.

Define a smooth map $\phi : L_1 \to S^1 \times \mathbb{C}$ by setting $\phi(z_1, z_2, z_3) = (z_1/\sqrt{|z_2|^2 + 1}, z_2)$, and a smooth map $\psi : S^1 \times \mathbb{C} \to L_1$ by setting $\psi(e^{\theta}, z) = (\sqrt{|z|^2 + e^{i\theta}}, z, e^{-i\theta} \overline{z})$. Then $\phi \circ \psi$ is clearly the identity map of $S^1 \times \mathbb{C}$. It is also easy to see that $\psi \circ \phi$ is the identity map of $L_1$. This implies that $L_1$ is a submanifold of $\mathbb{C}^3$ diffeomorphic to $S^1 \times \mathbb{C}$. It is also easy to see that $L_1$ is asymptotic to $C$ with multiplicity 1 at infinity. \hfill $\square$
Proof of Proposition 7. We have only to prove \( \|W\| = |L_c| \). For each \( \epsilon > 0 \) we put \( U_\epsilon^+ = \{ \text{Re } z_1 z_2 z_3 > \epsilon \} \) and \( U_\epsilon^- = \{ \text{Re } z_1 z_2 z_3 < -\epsilon \} \). In a way similar to the proof of Lemma 5 we can take \( \epsilon_0 > 0 \) such that if \( \epsilon \in (0, \epsilon_0) \) then \( F^{-1}(c) \cap U_\epsilon^+ \) and \( F^{-1}(c) \cap U_\epsilon^- \) will be connected submanifolds of \( \mathbb{R}^6 \). By the constancy theorem, therefore, \( \Theta_W \) is constant on \( F^{-1}(c) \cap U_\epsilon^+ \) and \( F^{-1}(c) \cap U_\epsilon^- \), respectively.

In a way similar to the proof of Lemma 5 we can prove that \( F^{-1}(c) \cap U_\epsilon^+ \) is asymptotic at infinity to \( C \) with multiplicity 1 and \( F^{-1}(c) \cap U_\epsilon^- \) is asymptotic at infinity to \( -C \) with multiplicity 1. Notice also that \( W \) is asymptotic at infinity to \( C \) with multiplicity 1. Then we see that on \( F^{-1}(c) \cap U_\epsilon^+ \) we have \( \Theta_W = 1 \) and on \( F^{-1}(c) \cap U_\epsilon^- \) we have \( \Theta_W = 0 \). This holds for every \( \epsilon > 0 \), and so \( \|W\| = |F^{-1}(c) \cap \{ \text{Re } z_1 z_2 z_3 \geq 0 \}| = |L_c| \) as we want.

We shall now complete the proof of Theorem 5. We suppose that \( W \) is as in Theorem 5. By Theorem 6, then, we can find \( b \in \mathbb{R}^6 \) such that \( W - b \in W_0 \). By Corollary 5, therefore, we can find \( c \in \mathbb{R}^3 \) such that \( \text{Spt } \|W - b\| \subset F^{-1}(c) \). By Proposition 6, therefore, we have \( c \in Y \). By Proposition 7, therefore, we have \( \|W - b\| = |sL| \) for some \( a > 0 \) and \( L \in \{ C, L_1, L_2, L_3 \} \) as we want.

5 Proof of Theorem 1

As in §1 the main steps in the proof of Theorem 1 are the bubbling-off analysis in §3 and the classification result in §4. In order to complete the proof, however, we have to do some technical things in addition. We begin with a review of the theory of Joyce [13, 14, 15, 16, 17].

We suppose that \((M, \omega, J, \Omega)\) is an almost Calabi–Yau manifold of dimension 3 over \( C \). We suppose that \( M \) is compact. We take \( a \in H_3(M; \mathbb{Z}) \). As in §2 we denote by \( V \) the space of all special Lagrangian varifolds \( V \) in \((M, \omega, J, \Omega)\) with \( \text{Spt } \|V\| \) compact, \( \partial \bar{V} = 0 \) and \( [\bar{V}] = a \).

We denote by \( X \) the set of all elements of \( V \) with support connected, and singularity only at one point and modelled on \( C \) with multiplicity 1. We take \( X \in X \). We denote by \( x \) the singular point of \( X \). We take a linear isomorphism \( \gamma : \mathbb{R}^6 \rightarrow T_xM \) with \( \gamma^*g|_x = g_0 \), \( \gamma^*J|_x = J|_0 \), \( \gamma^*\Omega|_x = \Omega_0 \) and \( \gamma(C) \) a multiplicity 1 smooth tangent cone to \( X \) at \( x \). We may regard \( X \) as a submanifold of \( M \setminus \{ x \} \), which we shall denote by \( X' \).

As in §2 we define \( \psi : M \rightarrow (0, \infty) \) by (2). We have then \( \omega_0 = \psi^2(x)\gamma^*\omega|_x \). By a theorem of Darboux we can find a real number \( \delta > 0 \) and a diffeomorphism \( \Gamma \) of \( B_\delta \) into \( M \) with \( \Gamma(0) = x \), \( d\Gamma|_0 = \psi(x)\gamma \) and \( \Gamma^*\omega|_x = \omega_0 \).

Joyce [14, §3.2] proves that \( C \) is rigid and so special Lagrangian Jacobi-integrable, which implies that \( x \) is an isolated conical singularity in the sense of Joyce [13, Definition 3.6]; for the details we refer to Joyce [13, Theorem 6.8], Allard–Almgren [2], Simon [23, Part II, §6] and Adams–Simon [4].

We define \( T^*C, \omega_C, U_C \) and \( \Phi_C \) as in Lemma 1. From results of Joyce [13, Theorem 4.4 and Lemma 4.5] we see that making \( \delta > 0 \) smaller if necessary we can take an embedding \( f_X \) of \( C \cap B_\delta \) into \( X' \), a function \( h_X : C \cap B_\delta \rightarrow \mathbb{R} \)
and an $\alpha > 2$ with $h_X = O(r^\alpha)$ such that $f_X^{-1}(X') = \Phi_C(\Gamma(\text{Graph } dh_X))$. We put $Z = X' \setminus \text{Im } f_X$ where $\text{Im } f_X = f_X(C \cap B_3)$. It is clear that $Z$ is a compact submanifold of $X'$ of co-dimension 0 with boundary diffeomorphic to $\Sigma$.

We denote by $T^*X'$ the cotangent bundle over $X'$, by $0_C$ the zero-section of $T^*X'$, and by $\omega_X$ the canonical symplectic form on $T^*X'$. Since $f_X$ maps $C \cap B_3$ diffeomorphically onto $X' \setminus Z$ we get a vector-bundle isomorphism $T^*f_X : T^*(X' \setminus Z) \to T^*(C \cap B_3)$ covering $f_X^* : X' \setminus Z \to C \cap B_3$. By a result of Joyce [13, Theorem 4.6] we can find a neighbourhood $UX$ of $\text{Im } 0_X$ in $T^*X'$ and a diffeomorphism $\Phi_X$ of $UX$ into $M$ with $\Phi_X \circ 0_X = \iota_X$, $\Phi_X^*\omega = \omega_X$ and

$$\Phi_X|_{UX\setminus T^*Z} = \Gamma \circ \Phi_C \circ (+dh_X) \circ T^*f_X$$

where $+dh_X$ denotes the fibrewise translation of $T^*C$ by $dh_X$.

For each $y \in M$ we denote by $P|_y$ the set of all linear isomorphisms $\phi : \mathbb{R}^6 \to T_yM$ with $\phi^*g|_y = g_0$, $\phi^*J|_y = J_0$ and $\phi^*\Omega|_y = \Omega_0$. We put $P = \bigcup_{y \in M} P|_y$. It is clear that $P$ is a principal bundle over $M$ with structure group $SU_3$. We have $(x, \gamma) \in P$. By a result of Joyce [14, Theorem 5.2] we can take a neighbourhood $U_{x, \gamma}$ of $(x, \gamma)$ in $P$ such that for all $p = (\gamma, \phi) \in U_{x, \gamma}$ we can construct embeddings $\Gamma_p$ of $B_3$ into $M$ depending smoothly on $p$ with $\Gamma_p(0) = y$, $d\Gamma_p|_0 = \phi$ and $\psi^\gamma(y)\Gamma^*_p\omega = \omega_0$, and embeddings $\Phi_p$ of $UX$ into $M$ depending smoothly on $p$ with $\Phi_p \circ 0_X = \iota_X$, $\Phi_p^*\omega = \omega_X$ and

$$\Phi_p|_{UX\setminus T^*Z} = \Gamma_p \circ \Phi_C \circ (+dh_X) \circ T^*f_X$$

Joyce [14, Definition 5.6] defines a topology on $\mathcal{X}$. On the other hand $\mathcal{X}$ has also the topology induced from $\mathcal{V}$. By a result of Joyce [14, Theorem 5.3], Allard’s regularity theorem and Theorem 3, the two topologies on $\mathcal{X}$ are the same.

By the stability of $C$ we can apply a result of Joyce [14, Corollary 6.11], who proves that we can take a neighbourhood $\mathcal{Y}$ of $X$ in $\mathcal{X}$ which has a structure of manifold, and $T_X\mathcal{Y}$ is isomorphic to the image of $H^1_\mathcal{X}(X'; \mathbb{R}) \to H^1(X'; \mathbb{R})$. We can parametrize $\mathcal{Y}$ as follows. Since $\text{Spt } ||X||$ and $\Sigma$ are connected we see that $X'$ is also connected, and so $H^1_\mathcal{X}(X'; \mathbb{R}) \to H^1(X'; \mathbb{R})$ is injective. We denote by $\Omega^1_\mathcal{X}$ the space of all 1-forms on $X'$ supported in $Z$; we can then define the natural projection $\pi_Z : \Omega^1_\mathcal{X} \to H^1(X'; \mathbb{R})$; we can take a linear subspace $H^1_Z$ of $\Omega^1_\mathcal{X}$ such that $\pi_Z$ maps $H^1_Z$ isomorphically onto $H^1(X'; \mathbb{R})$. We take a real number $\mu \in (2, 3)$ as in Joyce [14, Definition 3.7] and define a function space $C^\infty_\mu(X'; \mathbb{R})$ as in [14, Definition 4.2]. From the proof of Joyce [14, Corollary 6.11] we get:

**Lemma 6.** There exist a neighbourhood $Z$ of 0 in $H^1_Z$ and an embedding $\mathcal{H}$ of $Z$ into $\mathcal{X}$ with the following property: for all $\zeta \in Z$ there exist $p(\zeta) \in P$ and $h_\zeta \in C^\infty(\mathcal{X}; \mathbb{R})$ depending smoothly on $\zeta$ with $\mathcal{H}(\zeta) = \Phi_{p(\zeta)}(\text{Graph}(\zeta + dh_\zeta))$.

We define $L_1$, $L_2$ and $L_3$ by (11) in §4. We take $L \in \{L_1, L_2, L_3\}$. We can then take a compact submanifold $K$ of $L$ of co-dimension 0 with boundary diffeomorphic to $\Sigma$, a real number $R > 0$ and a diffeomorphism $f_L$ of $C \setminus \overline{B}_R$
onto \(L \setminus K\) with \(f_L - i_C = O(r^{-1})\). Making \(R > 0\) larger if necessary we can find a 1-form \(\eta_L\) on \(C \setminus \overline{B_R}\) with \(\Phi_C(\text{Graph} \eta_L) = L \setminus K\).

We denote by \(T^*L\) the cotangent bundle over \(L\), by \(0_L\) the zero-section of \(T^*L\), and by \(\omega_L\) the canonical symplectic form on \(T^*L\). Since \(f_L\) maps \(C \setminus \overline{B_R}\) diffeomorphically onto \(L \setminus K\) we get a vector-bundle isomorphism \(T^*f_L : T^*(L \setminus K) \to T^*(C \setminus \overline{B_R})\) covering \(f_L^{-1} : L \setminus K \to C \setminus \overline{B_R}\). By a result of Joyce [13, Theorem 7.5] we can find a neighbourhood \(UL\) of \(\text{Im}0_L\) in \(T^*L\) and a diffeomorphism \(\Phi_L\) of \(UL\) into \(\mathbb{R}^6\) with \(\Phi_L^*\omega_0 = \omega_L\), \(\Phi_L \circ 0_L = i_L\) and

\[
\Phi_{L|UL \setminus T\cdot K} = \Phi_C \circ (+\eta_L) \circ T^*f_L
\]

where \(+\eta_L\) denotes the fibrewise translation of \(T^*C\) by \(\eta_L\).

We have a homomorphism \(f_X^* : H^1(X'; \mathbb{R}) \to H^1(C \cap B; \mathbb{R}) \cong H^1(\Sigma; \mathbb{R})\). We denote by \([\eta_L]\) the de Rham cohomology class of \(\eta_L\). We have then \([\eta_L] \in H^1(C \setminus \overline{B_R}; \mathbb{R}) \cong H^1(\Sigma; \mathbb{R})\). By Hodge theory we can take a harmonic 1-form \(\eta_L''\) on \(\Sigma\) with \([\eta_L'']\) = \([\eta_L]\). We denote by \(\pi_\Sigma\) the projection of \(C\) onto \(\Sigma\). We put \(\tilde{\eta}_L = \pi_\Sigma\eta_L''\).

We denote by \(\text{Im} f_X^*\) the image of the linear map \(f_X^* : H^1(X'; \mathbb{R}) \to H^1(C \cap B; \mathbb{R}) \cong H^1(\Sigma; \mathbb{R})\). Joyce [17, Lemma 10.1] proves also that if \(L \neq L' \in \{L_1, L_2, L_3\}\) then \([\eta_L]\) and \([\eta_L']\) are linearly independent. Hence we get:

**Lemma 7.** There exists at most one \(L \in \{L_1, L_2, L_3\}\) with \([\eta_L]\) \in \(\text{Im} f_X^*\).

As in §1 we suppose \(X \in \partial N\). By Corollary 6 below, then, we shall have \(L \in \{L_1, L_2, L_3\}\) with \([\eta_L]\) \in \(\text{Im} f_X^*\). Hence we can find a closed 1-form \(\xi_L\) on \(X'\) with \(f_X^*[\xi_L] = [\eta_L]\). By a result of Joyce [17, Theorem 10.4] there exist a real number \(\tau > 0\), an open submanifold \(\tilde{K}\) of \(L\) with \(K \subset \tilde{K}\) and an open submanifold \(\tilde{Z}\) of \(X\) with \(Z \subset \tilde{Z}\) such that for all \(t \in (0, \tau)\) we can find \(u_C : C \cap A_{1, R, \delta} \to \mathbb{R}\) with \(|du_C|_{1, R, \delta} = o(t)\), \(u_L : \tilde{K} \to \mathbb{R}\) with \(|du_L|_{C^1(\tilde{K})} = o(t)\), and \(u_X : \tilde{Z} \to \mathbb{R}\) with \(|du_X|_{C^1(\tilde{Z})} = o(t)\) such that on \(f_X^{-1}(\tilde{K}) \cap A_{1, R, \delta}\) we have \(t^2(f^*_X du_L + \eta_L)\), on \(f_X^{-1}(\tilde{Z})\) we have \(f_X^*(du_X + t^2\xi_L) + dh_X = du_C + t^2\tilde{\eta}_L\), and

\[
\Gamma(\Phi_L(\text{Graph} du_L)) \cup \Gamma \circ \Phi_C(\text{Graph} (du_C + t^2\tilde{\eta}_L)) \cup \Phi_X(\text{Graph} (du_X + t^2\xi_L))
\]

will be a compact special Lagrangian submanifold of \((M, \omega, J, \Omega)\). From the proof Joyce [17, Theorem 10.4] it follows readily that we can extend the result as follows:

**Lemma 8.** There exist a real number \(\tau > 0\), a neighbourhood \(U_{x, \gamma}\) of \((x, \gamma)\) in \(P\), a neighbourhood \(Z\) of 0 in \(H^2_x\), an open submanifold \(\tilde{K}\) of \(L\) with \(K \subset \tilde{K}\) and an open submanifold \(\tilde{Z}\) of \(X\) with \(Z \subset \tilde{Z}\) such that for all \((t, p, \zeta) \in (0, \tau) \times U_{x, \gamma} \times Z\) we can find \(u_C \in C^\infty(C \cap A_{1, R, \delta}; \mathbb{R})\) with \(|u_C|_{1, R, \delta} = o(t)\), \(u_L \in C^\infty(\tilde{K}; \mathbb{R})\) with \(|u_L|_{C^1(\tilde{K})} = o(t)\), and \(u_X \in C^\infty(\tilde{Z}; \mathbb{R})\) with \(|u_X|_{C^1(\tilde{Z})} = o(t)\) such that on
We take \((t, p, \zeta) \in (0, \tau) \times U_{x, \gamma} \times Z\). We denote by \(i_{t, p, \zeta}\) the inclusion map of \(G(t, p, \zeta)\) into \(M\). We denote by \(T^* G(t, p, \zeta)\) the cotangent bundle over \(G(t, p, \zeta)\), and by \(0_{t, p, \zeta}\) the zero-section of \(T^* G(t, p, \zeta)\). We shall define a neighbourhood \(U_{t, p, \zeta}\) of \(\text{Im} \ 0_{t, p, \zeta}\) and an embedding \(\Phi_{t, p, \zeta}\) of \(U_{t, p, \zeta}\) into \(M\) with \(\Phi_{t, p, \zeta} \circ 0_{t, p, \zeta} = i_{t, p, \zeta}\).

We put \(\phi_{t, p, \zeta}^{L, p, \zeta} = \Gamma_p \circ t \circ \Phi_L \circ du_L\) where \(du_L\) is regarded as a map of \(\tilde{K}\) into \(T^* \tilde{K}\). Thus we get an embedding \(\phi_{t, p, \zeta}^{L, p, \zeta} : \tilde{K} \rightarrow G(t, p, \zeta)\). We put \(G_{t, p, \zeta}^{t, p, \zeta} = \text{Im} \phi_{t, p, \zeta}^{L, p, \zeta}\). Since \(\phi_{t, p, \zeta}^{L, p, \zeta}\) maps \(\tilde{K}\) diffeomorphically onto \(G_{t, p, \zeta}^{t, p, \zeta}\) we get a vector bundle isomorphism \(T^* \phi_{t, p, \zeta}^{L, p, \zeta} : T^* G_{t, p, \zeta}^{t, p, \zeta} \rightarrow T^* \tilde{K}\). On a neighbourhood of \(\text{Im} (0_{t, p, \zeta} \circ \phi_{t, p, \zeta}^{L, p, \zeta})\) in \(T^* G(t, p, \zeta)\) we set \(\Phi_{t, p, \zeta} = \Gamma_p \circ t \circ \Phi_L \circ (+du_L) \circ T^* \phi_{t, p, \zeta}^{L, p, \zeta}\).

We put \(\phi_{t, p, \zeta}^{C, p, \zeta} = \Gamma_p \circ \phi_C \circ (du_C + t^2 \tilde{\eta}_L)\) where \(du_C + t^2 \tilde{\eta}_L\) is regarded as a map of a compact special Lagrangian submanifold of \((M, \omega, J, \Omega)\). Thus we get an embedding \(\phi_{t, p, \zeta}^{C, p, \zeta} : \tilde{Z} \rightarrow G(t, p, \zeta)\). We put \(G_{t, p, \zeta}^{t, p, \zeta} = \text{Im} \phi_{t, p, \zeta}^{C, p, \zeta}\). Since \(\phi_{t, p, \zeta}^{C, p, \zeta}\) maps \(\tilde{Z}\) diffeomorphically onto \(G_{t, p, \zeta}^{t, p, \zeta}\) we get a vector bundle isomorphism \(T^* \phi_{t, p, \zeta}^{C, p, \zeta} : T^* G_{t, p, \zeta}^{t, p, \zeta} \rightarrow T^* (C \cap A_{t, t, \delta})\). On a neighbourhood of \(\text{Im} \ 0_{t, p, \zeta} \circ \phi_{t, p, \zeta}^{C, p, \zeta}\) in \(T^* G(t, p, \zeta)\) we set \(\Phi_{t, p, \zeta} = \Gamma_p \circ \phi_C \circ (+du_C) \circ T^* \phi_{t, p, \zeta}^{C, p, \zeta}\). Thus we get:

**Lemma 9.** There exists a neighbourhood \(U_{t, p, \zeta}\) of \(\text{Im} \ 0_{t, p, \zeta}\) and an embedding \(\Phi_{t, p, \zeta}\) of \(U_{t, p, \zeta}\) into \(M\) with \(\Phi_{t, p, \zeta} \circ 0_{t, p, \zeta} = i_{t, p, \zeta}\).

We shall now take a neighbourhood \(U\) of \(X\) in \(V\) and define a map \(F : U \cap N \rightarrow (0, \tau) \times Z\). By Allard's regularity theorem we may suppose that for every \(N \in U \cap N\) we can find a unique 1-form \(\beta_X\) on \(Z\) such that \(\Phi_X(T^* (C \cap B_\delta)) = \Phi_X(\text{Graph} \beta_X)\). Since \(0 \neq [\eta_L] \in \text{Im} f_X^1\) and \(f_X^1\) is of dimension 1 we can find a unique \(s \in \mathbb{R}\) with \(s[\eta_L] = f_X^1[\beta_X]\). We have then a compactly-supported 1-form \(\beta_X - s \xi_X\) on \(Z\). As \([\beta_X - s \xi_X]\) has zero mass we may write \([\beta_X - s \xi_X] = H_X^1(X; \mathbb{R})\). As \(\pi_Z\) maps \(H_X^1\) isometrically onto \(H_X^2\) we may write \([\beta_X - s \xi_X] \in H_X^2\). We set \(F(N) = (s, [\beta_X - s \xi_X])\).

It is easy to see that for each \((t, \zeta) \in (0, \tau) \times Z\) we have \(F \circ G(t, \zeta) = (t^2, \zeta)\). Hence we see that \(G : (0, \tau) \times \mathcal{Y} \rightarrow N\) is injective. By an argument of Joyce [17, 25].
§8.2], moreover, $F: \mathcal{U} \cap \mathcal{N} \to (0, \tau) \times \mathcal{Z}$ is a local diffeomorphism. Hence we see that $G(t, p, \zeta)$ is independent of $p \in U_{x, \gamma}$ as we may suppose that $U_{x, \gamma}$ is connected. We may therefore write $G: (0, \tau) \times \mathcal{Z} \to \mathcal{N}$.

We can extend $G$ continuously to $[0, \tau) \times \mathcal{Z}$ by setting $G(0, 0) = 0$ for each $0 \in \mathcal{Z}$. We can re-state Theorem 1 as follows:

**Theorem.** $G$ maps $[0, \tau) \times \mathcal{Z}$ onto a neighbourhood of $X$ in $\mathcal{V}$.

We are going now to prove the theorem above.

Using the results of §3–4 we shall determine a neighbourhood of $X$ in $\mathcal{V}$:

**Theorem 7.** Let $\epsilon > 0$, let $\hat{K}$ be an open submanifold of $L$ with $K \subset \hat{K}$, and let $\hat{Z}$ be an open submanifold of $X$ with $\mathcal{Z} \subset \hat{Z}$. Then there exists a neighbourhood $\mathcal{U}$ of $X$ in $\mathcal{V}$ such that for every $V \in \mathcal{U} \setminus \mathcal{X}$ we can find an $L \in \{L_1, L_2, L_3\}$, a real number $s > 0$, a point $p \in P$, a 1-form $\beta$ on $C \cap A_{sR, \delta}$ with $|\beta_C|_{[\mathcal{C}, \mathcal{V}]} < \epsilon$, a 1-form $\beta_L$ on $\hat{K}$ with $|\beta_L|_{C^1(K)} < \epsilon$, and a 1-form $\beta_X$ on $\hat{Z}$ with $|\beta_X|_{C^1(\hat{Z})} < \epsilon$ such that on $f^{-1}_L(\hat{K}) \cap A_{sR, \delta}$ we have $s^2(f^{-1}_L \beta_L + \eta_L) = \beta_C$, on $f^{-1}_X(\hat{Z}) \cap A_{sR, \delta}$ we have $f^{-1}_X \beta_X + dh_X = \beta_C$, and if we put

$$N = \left\{ \Gamma_p(s \Phi_L(\text{Graph } \beta_L)) \cup \Gamma_p \circ \Phi_C(\text{Graph } \beta_C) \cup \Phi_X(\text{Graph } \beta_X) \right\}$$

then we have $|V| = |N|$ where $|N| = \mathcal{H}^3 \cap \mathcal{N}$ where $\mathcal{H}^3$ denotes the Hausdorff 3-dimensional measure on $(M, g)$.

**Remark.** Since $s^2(f^{-1}_L \beta_L + \eta_L) = \beta_C$ on $f^{-1}_L(\hat{K}) \cap A_{sR, \delta}$ and $f^{-1}_X(\hat{Z}) \cap A_{sR, \delta}$ on $f^{-1}_X \beta_X + dh_X = \beta_C$ we see that $N$ will be a compact submanifold of $M$.

**Proof.** Let $(V_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{V} \setminus \mathcal{X}$ converging to $X$. Then we have only to prove that for each $\nu = 1, 2, 3, \cdots$ we can find an integer $n > \nu$ such that with $V_n$ in place of $V$ the property above will hold.

We consider $\gamma^\ast \exp_x^\ast V_n$. To $\gamma^\ast \exp_x^\ast V_n$ in place of $V_n$ we can apply Theorem 4, and so we can find a sequence $(\delta_n)_{n=1}^{\infty}$ in $(0, \infty)$ converging to 0, a sequence $(y_n)_{n=1}^{\infty}$ in $B_P$ converging to 0, and a subsequence of $(\delta_n^{\ast -1}(V_n - y_n))_{n=1}^{\infty}$ converging to some special Lagrangian varifold $W$ in $(\mathbb{R}^6, \omega_0, J_0, \Omega_0)$ asymptotic at infinity to some $C' \subset C$, with multiplicity 1 and satisfying $E(W - \delta) > 0$ for each $\delta \in \mathbb{R}$. We may suppose that $(\delta_n^{\ast -1}(V_n - y_n))_{n=1}^{\infty}$ itself converges to $W$.

Since $C$ is rigid in the sense of Joyce we can find $a \in SU_3$ with $aC = C'$. To $a^{\ast -1}W$ we can apply Theorem 5 and so we can find $\lambda > 0$, $L \in \{C, L_1, L_2, L_3\}$, and $b \in \mathbb{R}$ such that $|W| = |\lambda L + b|$. Since $E(W - \delta) > 0$ we get $E(L) > 0$ and so $L \neq C$, which implies $L \in \{L_1, L_2, L_3\}$.

We define an embedding $E_n$ of $B_P$ into $M$ by setting $E_n(y) = \exp_x \gamma a(y + \delta_n b)$. We put $y(n) = E_n(0) = \exp_x \gamma a(\delta_n b)$. It is easy to see that $y(n)$ tends as $n \to \infty$ to $\exp_x \gamma a(0) = x$. We put also $\phi(n) = dE_n|_0 = \gamma \circ a$. It is easy to see that $\phi(n)$ is close to $\gamma$.

We put $s_n = \delta_n \lambda$. From Allard’s regularity theorem and Theorem 2 we see that for $n$ sufficiently large we can find an embedding $e_{L,n}$ of $\hat{K}$ into $\mathbb{R}^6$ with $|e_{L,n} - i_L|_{C^1(\hat{K})} < \epsilon$, an embedding $e_{C,n}$ of $C \cap A_{s,R,\delta}$ into $\mathbb{R}^6$, with
This implies that on such that and an embedding of $\tilde{Z}$ into $X$ with $|e_{X,n} - iX|_{C^1(\tilde{Z})} < \epsilon$ such that

$$||V'|| = ||E_n(s_n \operatorname{Im} e_{L,n}) \cup E_n(\operatorname{Im} e_{C,n}) \cup \operatorname{Im} e_{X,n}||.$$

We put $p(n) = (y(n), \phi(n))$. We have then $\Gamma_p(n)(0) = E_n(0)$ and $d\Gamma_p(n)|_0 = dE_n|_0$. Hence we can find an embedding $f_{L,n}$ of $\tilde{K}$ into $\mathbb{R}^6$ with $|f_{L,n} - iL|_{C^1(\tilde{K})} < \epsilon$, an embedding $f_{C,n}$ of $C \cap A_{s,n,R,\sigma}$ into $\mathbb{R}^6$, with $|f_{C,n} - iC|_{C^1_{(s_n,R,\sigma)}} < \epsilon$ and an embedding of $\tilde{Z}$ into $X$ with $|f_{X,n} - iX|_{C^1(\tilde{Z})} < \epsilon$ such that

$$||V|| = ||\Gamma_p(n)(s_n \operatorname{Im} f_{L,n}) \cup \Gamma_p(n)(\operatorname{Im} f_{C,n}) \cup \operatorname{Im} f_{X,n}||.$$

Hence we can find a 1-form $\beta_{L,n}$ on $\tilde{K}$ with $|\beta_{L,n}|_{C^1(\tilde{K})} < \epsilon$, a 1-form $\beta_{C,n}$ on $C \cap A_{s,n,R,\delta}$ with $|\beta_{C,n}|_{C^1_{(s_n,R,\delta)}} < \epsilon$, and a 1-form $\beta_{X,n}$ on $\tilde{Z}$ with $|\beta_{X,n}|_{C^1(\tilde{Z})} < \epsilon$ such that

$$||V|| = ||\Gamma_p(s_n \Phi_L(\operatorname{Graph} \beta_L)) \cup \Phi_C(\operatorname{Graph} \beta_C) \cup \Phi_X(\operatorname{Graph} \beta_X)||.$$ 

This implies that on $f_{L,n}^{-1}(\tilde{K}) \cap A_{s,n,R,\delta}$ we have $s^2(f_{L,n}^{-1} \beta_{L,n} + \eta_L) = \beta_{C,n}$, on $f_{X}^{-1}(\tilde{Z}) \cap A_{s,n,R,\delta}$ we have $f_X^{-1} \beta_{X,n} + dh_X = \beta_{C,n}$. This contradicts our hypothesis, and so completes the proof.

We give a corollary to Theorem 7:

**Corollary 6.** Suppose that for every neighbourhood $\mathcal{U}$ of $X$ in $\mathcal{V}$ we have $\mathcal{U} \setminus X \neq \emptyset$. Then there exists an $L \in \{L_1, L_2, L_3\}$ with $|\eta_L| \in \operatorname{Im} f_X$.

**Proof.** Let $\epsilon > 0$, and let $\mathcal{U}$ be as in Theorem 7. Then we have by assumption $\mathcal{U} \setminus X \neq \emptyset$, and so we can find some $V \in \mathcal{U} \setminus X$. Let $s$, $\beta_L$, $\beta_C$ and $\beta_X$ be as in Theorem 7. Then we have $s^2(f_L^{-1} \beta_{L,n} + \eta_L) = [f_X^{-1} \beta_X + dh_X] \in H^4(\Sigma; \mathbb{R})$ and so $f_X f_L \beta_{L,n} + \eta_L \in \operatorname{Im} f_X$. Notice also that $|\beta_L|$ tends to $0$ as $\epsilon \to 0$. Then we get $|\eta_L| \in \operatorname{Im} f_X$, completing the proof.

We shall give another corollary to Theorem 7. By assumption we have $X \in \partial N$. By Corollary 6, therefore, we can find $L \in \{L_1, L_2, L_3\}$ with $|\eta_L| \in \operatorname{Im} f_X$. We take $\epsilon > 0$ and suppose that $\tilde{K}$ and $\tilde{Z}$ are as in Theorem 7. Making $K$ and $\tilde{Z}$ smaller if necessary we can construct $G : (0, \tau) \times U_{z,\gamma} \times Z \to N$ as in Lemma 8. We define $\Phi_{t,p,\zeta}$ as in Lemma 9. We have then:

**Corollary 7.** There exists a neighbourhood $\mathcal{U}$ of $X$ in $\mathcal{V}$ such that for every $V \in \mathcal{U} \setminus X$ there exist $t > 0$, $p \in P$, $\zeta \in H^2_P$ and $w \in C^\infty(G(t,p,\zeta); \mathbb{R})$ with $|dw|_{C^1(G(t,p,\zeta))} < \epsilon$ such that $||V|| = ||\Phi_{t,p,\zeta}(\operatorname{Graph} dw)||$.

**Proof.** Let $\mathcal{U}$ be as in Theorem 7, and let $V \in \mathcal{U} \setminus X$. Let $\beta_C$ and $\beta_X$ be as in Theorem 7. Then we have $[\beta_C] = [dh_X + f_X^{-1} \beta_X] = [f_X^{-1} \beta_X]$ and so $[\beta_C] \in \operatorname{Im} f_X$. Let $s$ and $\beta_L$ be as in Theorem 7. Then we have $s^2(f_L^{-1} \beta_{L,n} + \eta_L) = [\beta_C]$. We have also $|\beta_L|_{C^1(\tilde{K})} < \epsilon$ and $|\eta_L| \neq 0$. We may therefore suppose $|\beta_C| \neq 0$. Notice that $\operatorname{Im} f_X$ is generated by $|\eta_L|$ and $|\beta_C|$ is sufficiently close to $s^2|\eta_L|$. Then we can find a unique $t > 0$ with $t^2|\eta_L| = |\beta_C|$ and $t$ sufficiently close to $s$. 

27
Notice that \( f^*_L [\beta_X - t^2 \xi_L] = [\beta_C] - t^2 [\eta_L] = 0 \). Notice also that we have an exact sequence \( H^1_2(X'; \mathbb{R}) \to H^1(X'; \mathbb{R}) \to H^1(\Sigma; \mathbb{R}) \). Then we get \( \beta_X - t^2 \xi_L \in \text{Im}(H^1_2(X'; \mathbb{R}) \to H^1(X'; \mathbb{R})) \). Hence we can find \( \alpha : X' \to \mathbb{R} \) with \( \beta_X - t^2 \xi_L - da \) compactly-supported. We denote its compactly-supported de Rham cohomology class by \( [\pi_X - t^2 \xi_L - da]_c \). Let \( \pi_Z \) and \( H^1_Z \) be as in Lemma 6. Then \( \pi_Z \) maps \( H^1_Z \) isomorphically onto \( H^1(X'; \mathbb{R}) \). Hence we can find a unique \( \zeta \in H^1_Z \) with \( [\zeta]_c = [\beta_X - t^2 \xi_L - da]_c \) where \( [\zeta]_c \) denotes compactly-supported de Rham cohomology class of \( \zeta \). Let \( \phi^{t,p,\zeta}_L \) be as in Lemma 9. Then we can find an exact 1-form \( \alpha_X \) on \( \hat{Z} \) such that \( \text{Spt} \|V\| \cap \Phi_p(T^*Z) = \Phi_{t,p,\zeta}(\text{Graph} \, \alpha_X) \).

Let \( p \) be as in Theorem 7. Then we can find a 1-form \( \alpha \) on \( G(t,p,\zeta) \) with \( |\alpha|_{C^1(G(t,p,\zeta))} < \epsilon \) such that \( \|V\| = |\Phi_{t,p,\zeta}(\text{Graph} \, w)| \).

We have to prove that \( \alpha \) is an exact 1-form.

Since \( L \) and \( X' \) are connected we get an exact sequence \( 0 \to H^1_1(G(t,p,\zeta)) \to H^1_1(\hat{K}) \oplus H^1_1(\hat{Z}) \). It suffices therefore to prove \( [\alpha] \) maps to 0 under \( H^1_1(G(t,p,\zeta)) \to H^1_1(\hat{K}) \oplus H^1_1(\hat{Z}) \).

The map \( H^1_1(G(t,p,\zeta)) \to H^1_1(\hat{Z}) \) is given by \( \phi^{t,p,\zeta}_L \) and we have \( \phi^{t,p,\zeta}_L \circ \alpha = \alpha_X \), which is an exact 1-form.

Let \( \phi^{t,p,\zeta}_L \) be as in Lemma 9, and let \( \alpha = \phi^{t,p,\zeta}_L \circ \alpha \). Then we have only to prove \( [\alpha|_L] = 0 \in H^1(\hat{K}) \).

Let \( u_L \) be as in Lemma 8. Then we have

\[
\int (f^*_L (\alpha|_L + du_L) + \eta_L) = \beta_C.
\]

on \( f^*_L (\hat{K}) \cap A_{1,R,\delta} \) We have also \( t^2 [\eta_L] = [\beta_C] \) and so \( f^*_L [\alpha|_L] = 0 \in H^1(\Sigma; \mathbb{R}) \). As in the proof of Lemma 5 we can construct a diffeomorphism of \( L \) onto \( S^1 \times \mathbb{R}^2 \) and we may suppose that \( \tilde{K} \) is diffeomorphic to \( S^1 \times \mathbb{R}^2 \). Hence we get \( H^1_1(\hat{K}; \mathbb{R}) = 0 \).

Notice also that we have an exact sequence \( H^1_1(\hat{K}; \mathbb{R}) \to H^1_1(\hat{K}; \mathbb{R}) \to H^1_1(\Sigma; \mathbb{R}) \).

Then we see that \( f^*_L : H^1_1(\hat{K}; \mathbb{R}) \to H^1_1(\Sigma; \mathbb{R}) \) is injective. On the other hand we have \( f^*_L [\alpha|_L] = 0 \) and so \( [\alpha|_L] = 0 \in H^1(\hat{K}; \mathbb{R}) \) as we want. \( \square \)

We shall now complete the proof of Theorem 1. Let \( U \) be as in Corollary 7, and let \( V \in U \setminus X \). Then we have only to prove \( \text{Spt} \|V\| = G(t,\zeta) \) for some \( (t,\zeta) \in (0,\tau) \times Z \). Let \( (t,p,\zeta) \) be as in Corollary 7. Then we can find a function \( w : G(t,\zeta) \to \mathbb{R} \) with \( \Phi_{t,p,\zeta}(dw) = \text{Spt} \|V\| \). As \( \|V\| \) is special Lagrangian \( w \) satisfies an elliptic equation to which we can apply a maximum principle of Hopf [11]. Hence we see that \( w \) is constant, and so \( \|V\| = |G(t,\zeta)| \).

References


