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<td>Author(s)</td>
<td>Mizuta, Atsushi</td>
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<td>Citation</td>
<td>Kyushu University (京都大学)</td>
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<td>Issue Date</td>
<td>2014-05-23</td>
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<tr>
<td>URL</td>
<td><a href="https://doi.org/10.14989/doctor.k18446">https://doi.org/10.14989/doctor.k18446</a></td>
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Universality of Kolmogorov’s Cascade Picture in Inverse Energy Cascade Range of Two-dimensional Turbulence

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Submitted to the Division of Physics and Astronomy, Graduate School of Science, Kyoto University on February 7, 2014 in partial fulfillment of the requirements for the degree of Doctor of Philosophy in physics.
Universality of Kolmogorov’s Cascade Picture in Inverse Energy Cascade Range of Two-dimensional Turbulence

Abstract: We numerically investigate the inverse energy cascade range of two-dimensional Navier-Stokes turbulence. Our focus is on the universality of Kolmogorov’s phenomenology. In our direct numerical simulations, two types of forcing processes, the random forcing and the deterministic forcing, are employed besides the systematically varied numerical parameters. We first calculate the two-dimensional Navier-Stokes equations and confirm that the results obtained in the quasi steady state are consistent with Kolmogorov’s phenomenology for both types of forcing processes. It is also found that the difference in the evolution for the forcing process appears after the inverse energy cascade range reaches the system size; the dipole coherent vortices emerge and grow only when the random forcing is adopted. Then we add a large-scale drag term to the Navier-Stokes equations to obtain the statistically stationary state. When the random forcing is used, the scaling exponent of the energy spectrum in the stationary state starts to differ from the predicted $-5/3$ in the inverse energy cascade range as the infrared Reynolds number $Re_d$ increases, where $Re_d$ is defined as $k_f/k_d$ with the forcing wave number $k_f$ and the large-scale drag wave number $k_d$. This can be interpreted as a transition phenomenon in which the local maximum vorticity grows like an order parameter caused by excitation of strong coherent vortices. Strong coherent vortices emerge, grow after the quasi steady state, and then destroy the scaling law when $Re_d$ is over a critical value. These coherent vortices are not due to the finite-size effect, unlike the dipole coherent vortices. On the other hand, when the deterministic forcing is adopted, strong coherent vortices are hardly seen and the $k^{-5/3}$ scaling law holds independently of $Re_d$. We also examine the cases of the combination of both types of forcing processes and find that formation of such coherent vortices is sensitive to the mechanism of the external forcing process as well as the numerical parameters. Several types of large-scale drag terms are also tested and their insignificant influence on these qualitative properties is revealed. Our results apparently imply the non-universality of the Kolmogorov’s phenomenology in the two-dimensional Navier-Stokes turbulence. However, even when the $k^{-5/3}$ scaling law is not observed at a glance, the background vorticity field, where the strong coherent vortices are filtered out, always holds the $k^{-5/3}$ scaling law. The inverse energy cascade process may be universal and exist even when the $k^{-5/3}$ scaling law is destroyed.

Keywords: K41, KLB theory, two-dimensional turbulence, direct numerical simulation, inverse energy cascade, statistically stationary state, coherent vortices
Acknowledgments

First of all, I would like to acknowledge Professor Sadayoshi Toh who is my supervisor of PhD course of science in Kyoto University. He has supported my research for over six years even after I left the university. He gave me a lot of insightful comments and suggestions.

I am grateful to Professor Takeshi Matsumoto. The main code for the direct numerical simulation of two dimensional turbulence is one of his great work. Moreover, his meticulous comments were an enormous help to me.

I would also like to thank my colleagues of Fluid Physics Laboratory of Kyoto University, Kentaro Kanatani, Hiroki Yatou, Shunsuke Kato, Yusuke Okahashi, Shunsuke Kouno, Kentaro Takagi, and Toshiki Teramura for their discussion, comments, and suggestions. I also thank all staffs of Yukawa institute for theoretical physics (YITP) and Division of Physics and Astronomy, Graduate School of Science, Kyoto University for their kindness and hospitality.

Most of numerical computation in this thesis was carried out on SX-8 and SR16000 at YITP in Kyoto University. My research was supported by the Grand-in-Aid for the Global COE program "The Next Generation of Physics, Spun from University and Emergence" from the Ministry of Education, Culture, Sports, Science, and Technology(MEXT) of Japan. The \LaTeX template made by Olivier Comnowick (http://olivier.commowick.org/thesis_template.php) is used for this thesis.

Finally I would like to express my thanks to my family for their constant encouragement and tireless support.
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Chapter 1

General introduction

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1.1 K41 theory

Turbulence is ubiquitous in science, technology, and daily life. Examples of turbulence are everywhere: wind motion in the atmosphere, ocean currents, flames, boundary layers on aircraft wings, jet engine exhaust, water flow from a faucet, and mixing milk into a cup of coffee. In fully developed turbulence, velocity fluctuations of a wide range of spatial and time scales interact strongly forming some coherence and self-similarity and then make flows quite complicated. Moreover these turbulent fluctuations have a substantial influence on technologically significant matters such as hydrodynamic forces, heat transfer, mixing, and acoustics. Thus, turbulence is fundamentally interesting and of great practical importance.

The modern theory of fully developed Navier-Stokes turbulence originates from the Kolmogorov’s phenomenological theory [Kolmogorov 1941c, Kolmogorov 1941a, Kolmogorov 1941b], now referred to as K41 theory. Kolmogorov then proposed the notion of an “inertial range” of scales, in which the inertia, or advection, term dominates the dynamics. It is based on Richardson’s picture of the energy cascade, in which nonlinearity transfers energy from larger to smaller eddies. In Richardson’s words, “Big whorls have little whorls that feed on their velocity, and little whorls have lesser whorls and so on to viscosity”. If kinetic energy is injected at the large scales ($L_f$), it cascades to smaller scales via nonlinear inertial (energy conserving) processes until it reaches a scale of order $l_\nu$, where viscous dissipation becomes dominant and the kinetic energy is converted into heat. In such case, the intermediate spatial scales $r$, $l_\nu \ll r \ll L_f$, is in an inertial range, where large-scale forcing and viscosity have negligible effects. In an inertial range, physical quantities exhibit universal behaviors, which are independent of the mechanism by which the turbulence is driven.

Kolmogorov further assumed the self-similarity of the cascade process. That is, eddies of a given size behave statistically the same as eddies of a different size. From
this similarity hypothesis, it is predicted that the energy spectrum takes the form,

\[ E(k) = C_K \varepsilon^{2/3} k^{-5/3} \]

in an inertial range. Here, \( k \) is the wave number, \( C_K \) the dimensionless universal constant called the Kolmogorov constant, and \( \varepsilon \) the constant energy transfer rate. This is called Kolmogorov’s \(-5/3\) law and well supported by experiments and direct numerical simulations. A schematic view of it is shown in Fig. 1.1.

### 1.2 KLB theory

In the two-dimensional system, the vorticity of each fluid parcel is conserved in the inviscid case. It means that, in the inviscid limit, there are an infinite number of conserved quantities including kinetic energy and enstrophy defined as a half of mean square vorticity. Thus, the K41 theory in the three-dimensional system cannot be applied to the two-dimensional system in its original form. However, the conceptual framework for turbulence in the K41 theory is applied to homogeneous, isotropic and statistically (quasi-)stationary two-dimensional forced turbulence in the Kraichnan-Leith-Batchelor (KLB) theory developed in Refs. [Kraichnan 1967, Leith 1968, Batchelor 1969]. Kraichnan [Kraichnan 1967], based on Fjørtoft’s work [Fjørtoft 1953], first predicted the double cascade process and explained that turbulence may be compatible with the presence of several conservation laws.
In the double cascade process, there are two different scaling ranges; the inverse energy cascade range, where energy, injected by external forcing at intermediate scales, transfers to ever larger scales, and the direct enstrophy cascade range, where injected enstrophy transfers from forcing scales to smaller scales. From the dimensional analysis, in the inertial subranges of inverse and direct cascade ranges,

\[ E(k) = C_I \varepsilon^{2/3} k^{-5/3} \]  \hspace{1cm} (1.2)

and

\[ E(k) = C_D \eta^{2/3} k^{-3} \]  \hspace{1cm} (1.3)

(with a possible logarithmic correction [Kraichnan 1971]), respectively. Here \( C_I \) and \( C_D \) are the dimensionless universal constants and \( \eta \) is the constant enstrophy transfer rate. A schematic view of it is shown in Fig. 1.2.

picture. For typical classes of the deterministic forcings, by assuming that a statistically steady state is attained, such constraints are obtained, outside of which the energy spectrum differs from $k^{-5/3}$ and $k^{-3}$ separated by the forcing wave number [Constantin 1994, Tran 2002]. However one must bear in mind that the assumption of the statistical steadiness is not made in the original KLB argument. In the random forcing cases [Eyink 1996, Kuksin 2012, Constantin 2013], moments of the vorticity and the invariant measure have been studied as well as the constraints. To our knowledge, theoretical comparison between the two types of forcing is not very common, perhaps due to difference of the theoretical and mathematical tools. In this thesis, we do this comparison numerically.

In the above-mentioned previous numerical simulations and experiments, the $k^{-5/3}$ scaling [Frisch 1984, Sommeria 1986, Smith 1993, Smith 1994, Paret 1997, Sukoriansky 1999, Boffetta 2000, Tran 2004a, Chen 2006, Chertkov 2007, Xiao 2009] and the $k^{-3}$ scaling [Borue 1993, Pasquero 2002, Chen 2003, Bracco 2010] have been confirmed independently. Recently, both scaling laws have also simultaneously observed in laboratory experiments [Rutgers 1998, Bruneau 2005] and direct numerical simulations [Boffetta 2007, Boffetta 2010, Farazmand 2011], even though the scaling ranges cannot be so wide in these studies and the $k^{-3}$ scaling is only asymptotically achieved as the Reynolds number increases in Refs. [Boffetta 2007, Boffetta 2010].

However, the departure from the $k^{-5/3}$ scaling in the inverse energy cascade range is also recognized in some numerical studies [Smith 1994, Borue 1994, Sukoriansky 1999, Danilov 2001a, Danilov 2001b, Tran 2004b, Scott 2007, Vallgren 2011, Fontane 2013]. In such cases, strong coherent vortices are usually observed in vorticity field and they are supposed to be the cause of this departure [Smith 1994, Borue 1994, Sukoriansky 1999, Danilov 2001a, Danilov 2001b, Scott 2007, Vallgren 2011, Fontane 2013]. The mechanism of the formation of such coherent vortices has not been well understood and there seems to be no general agreement on the conditions for emergence of these strong coherent vortices. It is stated in Refs. [Scott 2007, Vallgren 2011, Fontane 2013] that when the direct enstrophy cascade range is well resolved, strong coherent vortices emerge and the $k^{-5/3}$ scaling is destroyed. On the other hand, the $k^{-5/3}$ scaling is obtained with a relatively wide direct enstrophy cascade range in Refs. [Boffetta 2007, Boffetta 2010]. It is also demonstrated in Ref. [Vallgren 2011] that high resolution in the inverse energy cascade range causes the emergence of strong coherent vortices. In Ref. [Sukoriansky 1999], the importance of employing the appropriate large-scale drag formulation to obtain the statistically stationary structureless turbulent flow field that holds the $k^{-5/3}$ energy spectrum is illustrated by using a specially devised large-scale drag. Nevertheless, relatively wide $k^{-5/3}$ scaling range is achieved with a hypodrag term in Ref. [Xiao 2009].

The KLB theory outlines the universal features which should be independent of the details of energy input and output mechanism. Under circumstances where a lot of data support the KLB theory, it has been usual to adopt only one type of forcing process in each previous numerical study. However, previous numerical
1.2. KLB theory

simulation results imply that the formation of strong coherent vortices, which causes the departure from the $k^{-5/3}$ scaling law, is not a universal phenomenon. Thus, it should be tested if the forcing mechanism can influence on the formation of coherent vortices and the $k^{-5/3}$ scaling law, and, if so, how it does. The fact that the type of forcing process modifies the slope of scaling in the enstrophy inertial subrange was shown in Ref. [Farazmand 2011].

In our numerical study, we use two typical types of forcing process. In each case, we investigate the conditions for formation of strong coherent vortices and the departure of the $k^{-5/3}$ scaling law by varying numerical parameters to mainly control the resolution in the inverse energy cascade range. We will show that formation of strong coherent vortices can be described as a transition phenomenon for the change in the numerical resolution in the inverse energy cascade range and that highly depends on the type of forcing process.

As is usual with numerical studies of two-dimensional Navier-Stokes turbulence, a large-scale drag term is added to the Navier-Stokes equations in this study to dissipate the energy transferred from the forcing scale and attain a statistically stationary state. However, this additional term of course can influence statistical features not only in large scales but also in the inverse energy cascade range. To see the influence of the drag term, the Navier-Stokes equations (without any large-scale drag term) are also calculated in this study. It is reported in Refs. [Smith 1993, Smith 1994, Chertkov 2007] that strong dipole coherent vortices emerge and grow after the inverse energy cascade range reaches the system size when no drag term is added. We reproduce this dipole structure and compare it with the coherent structures observed when a large-scale drag term is added. We also employ several types of large-scale drag term to check the influence of them.
Chapter 2

Numerical simulation

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2.1 Introduction

The complex behavior of turbulence is thought to be described by a simple set of equations; the Navier-Stokes equations. However the exact solutions of them even under simple boundary conditions cannot be represented by closed forms. Thus, direct numerical simulation is one of the most powerful tools for studying turbulence, which provides an approximate solutions. It has the advantage that a complete description of a turbulent flow, where all the flow variables are known as a function of space and time, can be obtained, which is very difficult in real experiments.

The computational simulation of two-dimensional turbulence is first carried out by Lilly [Lilly 1969] to test the validity of Kraichnan’s predictions on the structure of two-dimensional turbulence. A finite difference method on $64^2$ grids is used in this early attempt. The first direct numerical simulation of Navier-Stokes equations in three-dimensional system, on the other hand, was performed by Orszag and Patterson [Orszag 1972]. In this pioneering work, a spectral method using fast Fourier transforms is employed on $32^3$ grid points.

In spectral methods, the solution of the differential equation is approximated with a sum of certain basis functions which are nonzero over the whole calculation domain. The Fourier basis functions are usually used when periodic boundary conditions are specified as in this thesis. The coefficients in the sum are determined in order to satisfy the differential equation as well as possible. Consequently, unlike in finite difference methods, the solution or its derivative at a given point is evaluated by using information from the entire calculation domain. This feature is very important when the solution varies drastically in time or space, when very high spatial resolution is needed, and when the function must be integrated for long time, which are very common situation in turbulence research.

Owing to the advent of supercomputers and the development of spectral numerical technique, the achievable spatial resolutions for homogeneous turbulence under periodic boundary conditions has reached $4096^3$ [Kaneda 2003]. The Reynolds number achieved in such a simulation is comparable with the largest Reynolds number
in laboratory experiments. Recently, two-dimensional turbulence is calculated by using the fully dealiased pseudospectral method on a double periodic square domain at spatial resolution up to $32768^2$ to resolve two inertial ranges for both inverse and direct cascades simultaneously [Boffetta 2010]. The width of each inertial range is narrow even in such a spatial resolution, though.

2.2 Numerical method

In our direct numerical simulation study, the two-dimensional Navier-Stokes equations are solved by using the pseudospectral method with the $2/3$ dealiasing rule in a doubly periodic square domain of each side length $2\pi$. The fourth-order Runge-Kutta method is employed for time integration. In practice, the vorticity equation in Fourier space,

$$\frac{\partial \hat{\omega}(\mathbf{k}, t)}{\partial t} + [(u \cdot \nabla)\hat{\omega}](\mathbf{k}, t) = -(\nu k^{2h} + dk^{-2q})\hat{\omega}(\mathbf{k}, t) + \hat{f}(\mathbf{k}, t),$$

is integrated with the incompressible condition ($\nabla \cdot u(x, t) = 0$), where $\hat{\omega}$ denotes the Fourier transform, $u$ is the fluid velocity, $\omega = -\nabla \times u$ the vorticity, $\nu$ the hyperviscosity coefficient, $h$ the hyperviscosity exponent, $d$ the hypodrag coefficient, $q$ the hypodrag exponent, and $f$ the forcing term. The large-scale drag term, the second term on the right-hand side of Eq.(2.1), is added to the Navier-Stokes equations in this study to dissipate the energy transferred from small forcing scales and obtain a statistically stationary turbulent flow field. Note that this term is dropped by reducing $d$ to zero in Sec. 3.1 to see the influence of it.

The forcing term is band-limited in Fourier space; the forcing wave range is $k_f - \Delta k \leq k \leq k_f + \Delta k$ for a small constant $\Delta k (< 0.01k_f)$. Two types of forcing processes are employed in this study. One is the white-in-time random forcing process (here denoted as $f_R$ for brevity), which is

$$\hat{f}(\mathbf{k}, t) = \begin{cases} \varepsilon_{in} \frac{k^2_f}{(\Delta t n_f)} (\xi^R_k + i \xi^I_k) & \text{if } |\mathbf{k}| \in [k_f - \Delta k, k_f + \Delta k], \\ 0 & \text{if } |\mathbf{k}| \notin [k_f - \Delta k, k_f + \Delta k]. \end{cases}$$

in Eq.(2.1). Here $\xi^R_k$ and $\xi^I_k$ are both independent, zero-mean, unit-variance, Gaussian random variables which are independent for every $\mathbf{k}$ at every time step, $\Delta t$ is a time step, and $n_f$ is the number of the spectral modes in the forcing range. A control parameter $\varepsilon_{in}$ is supposed to be an energy input rate. We have confirmed that the short-time averaged energy input rate is equal to $\varepsilon_{in}$ from the results of the energy budget in each simulation. This type of forcing is widely used in the previous studies [Frisch 1984, Smith 1993, Borue 1993, Smith 1994, Borue 1994, Sukoriansky 1999, Pasquero 2002, Scott 2007, Vallgren 2011, Fontane 2013] and strong coherent vortices are formed in some cases. The other one is the deterministic forcing process (here denoted as $f_D$), which is

$$\hat{f}(\mathbf{k}, t) = \begin{cases} \varepsilon_{in} |\mathbf{k}|^2 / (n_f \hat{\omega}^*(\mathbf{k}, t)) & \text{if } |\mathbf{k}| \in [k_f - \Delta k, k_f + \Delta k], \\ 0 & \text{if } |\mathbf{k}| \notin [k_f - \Delta k, k_f + \Delta k]. \end{cases}$$

2.2. Numerical method

in Eq. (2.1). Here $\hat{\omega}^*$ denotes the complex conjugate of $\hat{\omega}$. With this forcing, a relatively wide $k^{-5/3}$ scaling range is achieved without coherent vortices in [Xiao 2009]. The same [Chen 2003, Chen 2006, Xiao 2009] and a similar type of forcing [Shepherd 1987, Tran 2004a, Tran 2004b] are used in previous studies. The attractive aspect of this formulation is that it provides a constant energy input rate $\varepsilon_{in}$ at every time step.

The initial condition is a homogeneous zero vorticity field when $f_R$ is used. On the other hand, a random vorticity field with a small variance is prepared as an initial state when $f_D$ is used, since the forcing term cannot be calculated when the vorticity is zero. This random vorticity field consists of Fourier modes generated with independent Gaussian distributions and scaled to give the energy spectrum,

$$E(k, 0) = \begin{cases} e_0 & \text{if } k \leq k_f + \Delta k, \\ 0 & \text{if } k > k_f + \Delta k, \end{cases} \quad (2.4)$$

for a small constant $e_0$ ($= 1.0 \times 10^{-8}$) as in [Xiao 2009]. Here the energy spectrum $E(k, t)$ is defined as

$$E(k, t) \equiv \frac{1}{\delta k} \sum_{k \leq |k'| < k + \delta k} \frac{1}{2} |\hat{u}(k')|^2, \quad (2.5)$$

for small $\delta k$ ($= 1$).

Numerical parameters in our main simulations are listed in Table 2.1. We employ $512^2$, $1024^2$, $2048^2$, and $4096^2$ for the number $N^2$ of spatial grid points in the calculation domain. While the energy input rate $\varepsilon_{in}$ is fixed at 0.1, the forcing wave number $k_f$ is set in proportion to $N$. To concentrate the computational resources on the inverse energy cascade range, the eighth-order hyperviscosity ($h = 8$) is used and the localized forcing scale is set in the vicinity of the viscous range. Consequently, the direct enstrophy cascade range is not resolved well.

The hyperviscosity coefficient $\nu$ is varied to obtain an approximately constant energy transfer rate $\varepsilon_{tr}$ in these simulations. It is empirically found that to have almost the same $\varepsilon_{tr}$ with fixed $\varepsilon_{in}$, the viscous wave number $k_v$ defined as $(\varepsilon_{in}/\nu^3)^{1/(6h-2)}$ should be proportional to $k_f$ and $N$. We also use various numerical parameters for a large-scale drag term, $d$ and $q$, to see the effect of the drag term, however, it is revealed that these parameters influence little on $\varepsilon_{tr}$. Note that the ratio of energy transfer rate $\varepsilon_{tr}$ to energy input rate $\varepsilon_{in}$ is denoted as the inverse-cascade strength in [Tran 2004a] and treated as the key quantity to attain the $k^{-3}$ scaling of energy spectrum in the enstrophy cascading range. It is also stated in [Tran 2004a, Tran 2004b] that the $k^{-5/3}$ inverse cascading range is realizable even for small $\varepsilon_{tr}/\varepsilon_{in}$. In all simulations shown in Table 2.1, $\varepsilon_{tr}/\varepsilon_{in} = 0.18 \pm 0.01$. 

```
Table 2.1: Numerical settings in each simulation are listed below: $N^2$ is the number of spatial grid points, $f$ the type of forcing process, $\varepsilon_m$ energy input rate, $k_f$ the forcing wave number, $h$ the hyperviscosity exponent, $\nu$ the hyperviscosity coefficient, $q$ the hypodrag exponent, and $d$ the hypodrag coefficient.

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3.1 Two-dimensional Navier-Stokes turbulence

3.1.1 Quasi steady state

We begin with calculating the Navier-Stokes equations (without any large-scale drag term) to set a baseline for simulations with a large-scale drag term shown in the following sections. On the basis of the KLB theory, there would be a quasi steady state [Kraichnan 1967], where the inertial subrange is observed, before the inverse energy cascade range reaches the system size. In the inertial subrange of inverse energy cascade range, energy should transfer scale-locally toward large scales at a constant rate $\varepsilon_{tr}$ and the energy spectrum $E(k)$ should take the form

$$E(k) = C_I \varepsilon_{tr}^{2/3} k^{-5/3},$$

(3.1)

as described in Sec. 1.2. The dimensionless universal constant $C_I$ is called the Kolmogorov-Kraichnan constant.

In this section, the results of AR1 and AD1 in Table 2.1 are presented. Two types of forcing processes, $f_R$ and $f_D$, are used independently in each simulation. Time evolution of the total kinetic energy, defined as

$$K(t) \equiv \sum_k \frac{1}{2} \left| \hat{u}(k, t) \right|^2,$$

(3.2)

is shown in Fig. 3.1. In both cases, the total kinetic energy grows linearly with time after the initial stage ($t \gtrsim T_L$), where the time scale $T_L$ defined as $\varepsilon_{in}^{-1/3}(2\pi)^{2/3}$ is 7.3. The quasi steady state is observed in both cases in an intermediate time region, $T_L \lesssim t \lesssim 3T_L$. In this time region, the energy peak wave number $k_p$ goes
down toward lower wave numbers and the $k^{-5/3}$ energy spectrum is observed for $k_p \lesssim k \lesssim k_f$ as shown in Fig. 3.2.

The energy flux function, defined as

$$\Pi_E(k, t) \equiv \sum_{|k'|<k} \frac{1}{k'^2} Re[\overline{\omega^*}(k', t)(u \cdot \nabla)\omega(k', t)],$$

(3.3)
in this time region is also shown in Fig. 3.3. Energy conservation following the Navier-Stokes equations (with using hyperviscosity) is expressed with this function as follows:

$$\frac{\partial}{\partial t} \int_k^\infty E(k', t)dk' = \Pi_E(k, t) - 2\nu \int_k^\infty k^{2h} E(k', t)dk' + \int_k^\infty F(k', t)dk',$$

(3.4)

where $F(k, t)$ is the energy input rate by the external forcing. Since $F(k, t)$ is nonzero only for $k_f - \Delta k \leq k \leq k_f + \Delta k$ and the value of $2\nu k^{2h} E(k, t)$ is very small for $k \ll k_f$, the growth rate of the kinetic energy at intermediate or large scales is estimated as follows:

$$\frac{\partial}{\partial t} \int_k^{k'} E(k'', t)dk'' \sim \Pi_E(k, t) - \Pi_E(k', t),$$

(3.5)

for $k < k' < k_f$. As shown in Fig. 3.3, $\Pi_E(k, t)$ gives a negative constant $-\varepsilon_{tr}$ with some fluctuation in the $k^{-5/3}$ energy spectrum range. The constant value $\varepsilon_{tr}$ is approximately 0.018 in both cases and obviously coincides with the growth rate of total kinetic energy in this time region.
Figure 3.2: Snapshots of energy spectrum at $t = 20, 15, 10$ and $5$ (from top to bottom) in AR1(a) and AD1(b). Initial state $(t = 0)$ is also shown in (b).
Figure 3.3: Snapshots of energy flux function at $t = 20, 15, 10,$ and 5 (from left to right) in AR1(a) and AD1(b).
3.1. Two-dimensional Navier-Stokes turbulence

These facts imply that only a small part of kinetic energy input by the external forcing is transferred at a constant rate $\varepsilon_{tr}$ to large scales around the energy peak wave number $k_p$, which goes down toward lower wave numbers in time. Because of the closeness between the forcing range and the viscous range, the most part $[\sim 0.82 (= 1 - \varepsilon_{tr}/\varepsilon_{in})]$ of injected energy is dissipated in the viscous range. In the original inverse energy cascade theory [Kraichnan 1967], most of the input energy supposed to be carried down toward the lower wave number; however, this cascading process observed in our simulations is qualitatively consistent with the theory. Almost the same results have been obtained in the previous direct numerical simulations [Smith 1993, Smith 1994]. There is no distinct difference in both cases, $f_R$ and $f_D$, in the quasi steady state.

3.1.2 After quasi steady state

The linear growth of total kinetic energy, $K(t) \propto 0.018t$, is observed for a long time period as shown in Fig. 3.4. Since the calculation domain size is finite, the energy cascading to large scales eventually piles up at the smallest wave number. This condensation process was first studied numerically in detail by Smith and Yakhot [Smith 1993, Smith 1994]. They employed a small-scale, white-in-time Gaussian forcing and the eighth-order hyperviscosity, which is quite similar to the numerical setting in AR1. It is stated in it that “In physical space, the observed distribution of vorticity is structureless before the formation of the condensate. In the condensate state, as energy piles up in wave number $k = 1$, the vorticity localizes in space until it is eventually concentrated in two vortices of opposite sign.” Similar results are also observed in Ref. [Chertkov 2007], in which energy condensation in two-dimensional turbulence is investigated with a band-limited stochastic forcing with fixed amplitude and random phase.

These results are observed in AR1. The vorticity field is structureless in the quasi steady state and two coherent vortices of opposite sign are observed in the condensate state as illustrated in Fig. 3.5. These coherent vortices get to be distinguishable at $t \sim 8T_L$, when the energy peak wave number reaches the fundamental mode ($k = 1$). The intensity of their vorticity grows in time after that, while the sizes of them remain comparable with the forcing scale ($\sim 2\pi/k_f$). As stated in Ref. [Chertkov 2007], time growth of the maximum value of vorticity in AR1 is proportional to $\sqrt{t}$ in late simulation time ($t \gtrsim 50T_L$).

However, the vorticity field remains structureless after the quasi steady state when $f_D$ is used. Strong coherent vortices are hardly seen as demonstrated in Fig. 3.6 and Fig. 3.7. This difference in physical space reveals that the mechanism of the external forcing process has a strong influence on the vorticity field after the quasi steady state. Even when the capital numerical parameters are fixed, a completely different vorticity field can be obtained if the forcing process is changed.

Here we filter out the strong coherent vortices in physical space and calculate the background vorticity field in AR1. The filtering approach is as follows: (i) We prepare the function $g(\mathbf{x}) = \sum_{i=1}^{N_{\Omega}} \exp\left(-\frac{(\mathbf{x} - \mathbf{c}_i)^2}{2\sigma^2}\right)$, where $N_{\Omega} (= 2)$ is...
The number of the strong coherent vortices, $c$, the center position of $i$-th coherent vortex, and $\sigma$ is fixed at the forcing scale $2\pi/k_f$. (ii) The background vorticity field is calculated as $\omega(x) - g(x)\omega(x)$, where $\omega(x)$ is the original vorticity field. After this filtering, we return to Fourier space and calculate the energy spectrum of it. As shown in Fig. 3.8, the scaling exponent of the energy spectrum of the background vorticity field gets smaller than $-5/3$ in the intermediate scale range. This value approaches asymptotically to $-1$ with time. This result is consistent with that in Ref. [Chertkov 2007], where the $k^{-1}$ spectrum is observed for the background vorticity field at a much later calculation time. It is also found that the energy spectrum of the background vorticity field in AR1 is almost the same with the energy spectrum in AD1. Through the entire calculation time, formation of the strong coherent vortices in AR1 seems to be the only difference between the results in AR1 and AD1.
3.1. Two-dimensional Navier-Stokes turbulence

Figure 3.5: Vorticity field in physical space at $t = 20(a)$ and 300(b) in AR1.

Figure 3.6: Vorticity field in physical space at $t = 300$ ($\sim 40T_L$) in AD1.
Figure 3.7: Side view of Fig. 3.5(b)(fR) and Fig. 3.6(fD).

Figure 3.8: Snapshots of energy spectrum at $t = 300$ ($\sim 40T_L$) in AR1 ($f_R$) and AD1 ($f_D$). The energy spectrum of the background vorticity field in AR1 is also shown. Note the overlap between AD1 and that.
3.2 Addition of the first-order hypodrag term

In this section, the first-order hypodrag term \( q = 1 \) in Eq. (2.1) is added as an energy sink at large scales in order to dissipate the energy transferred from small forcing scales and obtain a statistically stationary state. It is claimed in Ref. [Sukoriansky 1999] that to employ the appropriate large scale drag formulation is important to obtain a statistically stationary turbulent flow field that holds phenomenological statistical laws, such as the \(-5/3\) energy spectrum and constant energy flux function in the intermediate scale range. In fact, such a flow field is obtained in Ref. [Sukoriansky 1999] by using a specially devised large scale drag. However, in this section, the first-order hypodrag term is simply employed as suggested in Ref. [Goto 2004], in which the phenomenological statistical laws are also observed in statistically stationary turbulent flow. The influence of the type of drag (various order \( q \) of hypodrag term) on statistical features is tested in Sec. 3.4.

Time evolution of the total kinetic energy is shown in Fig. 3.9 for \( d = 0.0, 0.2, 2.0, \) and \( 10.0 \) in both cases, \( f_R \) and \( f_D \). These simulations correspond to AR1, BR1, BR3, BR7, AD1, BD1, BD3, and BD4 in Table 2.1. Since the influence of the first-order hypodrag term diminishes at high wave numbers, the effect of it is negligible at the beginning of calculation and the quasi steady state is also observed even in these cases. As the energy peak wave number \( k_p \) goes down to a lower wave number, the drag effect is intensified mainly around \( k_p \) and the total drag effect grows in time. Eventually, \( k_p \) is stabilized at a certain wave number \(( \sim 2.7k_d \)) where \( k_d \) is the drag wave number defined as \((d^3/\varepsilon_{tr})^{1/(2+6q)}\). Up to this simulation time \(( \lesssim 3T_L \)) the difference in the results of \( f_R \) and \( f_D \) is hardly seen when the same numerical parameters are used. Moreover, when a sufficiently large drag coefficient is set, no distinct difference is observed between the results of \( f_R \) and \( f_D \) throughout the entire simulation.

The statistically stationary state is shortly subsequent to the quasi steady state when \( f_D \) is used or when a sufficiently large drag coefficient is set as illustrated in Fig. 3.9. In these stationary states (in BR7, BD1, BD3, and BD4), statistical features such as the energy spectrum and the energy flux function in adequately small scales \([k > k_p (\sim 2.7k_d)]\) are almost the same with those in quasi steady state in the simulations AR1 and AD1 (shown in Sec. 3.1.1). The long-time averaged energy flux function gives a constant value \( \varepsilon_{tr} (\sim 0.018) \) in the intermediate scale range (see Fig. 3.10) and the \( k^{-5/3} \) scaling law is clearly observed in the long-time averaged energy spectrum for this scale range (see Fig. 3.11). The Kolmogorov-Kraichnan constant \( C_I \) evaluated from these results are comparable with the estimated value, 6.5 \pm 1, in the laboratory experiment [Paret 1997] which has attained a relatively wide inertial subrange. This value is also consistent with the previous numerical studies. These results apparently imply that the K41 phenomenology can also prevail in statistically stationary two-dimensional turbulence with a large-scale drag term.

However, when \( f_R \) is used and a small drag coefficient is set, the total kinetic energy gradually grows after the quasi steady state as demonstrated with simulations.
Figure 3.9: Time evolution of total kinetic energy for four drag coefficients, $d = 0, 0.2, 2.0$ and 10 (from top to bottom) with $f_R$ (AR1, BR1, BR3, and BR7) and $f_D$ (AD1, BD1, BD3, and BD4). Note overlaps between AR1 and AD1 and between BR6 and BD3. Time scale $T_L(\equiv \varepsilon_{in}^{-1/3}(2\pi)^{2/3})$ is about 7.3 in all cases.

BR1 and BR3 in Fig. 3.9. In this second growth process, several strong coherent vortices as illustrated in Fig. 3.12 emerge and the intensity of them increases. (See also Appx. A.1.1 for details about the formation process for strong coherent vortices.) A bulge in the energy spectrum also appears and grows in the intermediate scale range in this time region, which destroys the $k^{-5/3}$ scaling. After this long-term second growth process, a statistically stationary state is eventually attained even in these cases.

The long-time averaged energy flux functions and the energy spectra in such statistically stationary states (in BR1 and BR3) are shown in Fig. 3.10(a) and Fig. 3.11(a), respectively. From the comparison between Fig. 3.10(a) and Fig. 3.10(b), it is obvious that the long-time averaged energy flux function depends little on the type of forcing process. On the other hand, when only $f_R$ is used, a spectral bulge is formed in the energy spectrum for a small drag coefficient. The spectral bulge is more intense for lower drag coefficient and the $k^{-5/3}$ scaling is apparently destroyed. Even when the spectral bulge is formed, the energy peak wave number $k_p$ stays at about $2.7k_d$, where $k_p$ is stabilized at the end of the quasi steady state. The linear relationship between $k_p$ and $k_d$ is also pointed out in the previous studies [Smith 2002, Danilov 2001a].

In such statistically stationary states where the spectral bulge is formed in the energy spectrum, several strong coherent vortices with almost the same intensity are observed in physical space, as illustrated in Fig. 3.12(a) and Fig. 3.13. The intensity of these coherent vortices fluctuates little in time. (See Fig. A.1 in Appx. A.1.1.)
3.2. Addition of the first-order hypodrag term

Figure 3.10: Long-time averaged energy flux functions in the statistically stationary state in BR1, BR3, and BR7 (a); $d = 0.2, 2.0, 10$ (from left to right) with $f_R$, and in BD1, BD2, and BD3 (b); $d = 0.2, 2.0, 10$ (from left to right) with $f_D$. Here $\langle \cdot \rangle$ denotes long-time average.
Figure 3.11: Long-time averaged energy spectra in the statistically stationary state in BR1, BR3, and BR7 (a); $d = 0.2, 2.0,$ and $10$ (from top left to bottom right) with $f_R,$ and in BD1, BD2, and BD3 (b); $d = 0.2, 2.0,$ and $10$ (from top left to bottom right) with $f_D.$ Insets are compensated plots; $\langle E(k,t) \rangle \propto k^{-5/3}$ vs $k.$ Here the energy transfer rate $\varepsilon_{tr}$ is 0.018.
3.2. Addition of the first-order hypodrag term

Figure 3.12: Snapshots of vorticity field in the statistically stationary state of calculation BR3(a) and BD3(b).

Figure 3.13: Side view of Fig. 3.12.
Figure 3.14: The energy spectra of an instantaneous vorticity field in the statistically stationary state in BR1 and its background vorticity field. Long-time averaged energy spectrum in the statistically stationary state in BD1 is also shown. Inset is a compensated plot; $\langle E(k,t) \rangle \sim k^{-5/3}$ vs $k$.

This value depends on the drag coefficient; it is more intense for small drag coefficient. The size of these coherent vortices are comparable with the forcing scale $(2\pi/k_f)$, which is similar to the dipole coherent vortices seen in Sec. 3.1.2 (see Fig. 3.7). These coherent vortices are not observed in the $f_D$ cases and in the $f_R$ cases with sufficiently large drag coefficients. Even when the calculation is started with the vorticity field where strong coherent vortices exist, they are dissipated and vorticity fields get to be structureless, as shown in Fig. 3.12(b) and Fig. 3.13, in these cases. (See also Appx. A.1.2 for details.)

To see the relation between the strong coherent vortices and the spectral bulge, we filter out the strong coherent vortices in an instantaneous vorticity field in statistically stationary state. The filtering approach is the same as used in Sec. 3.1.2. In the energy spectrum of these remaining fluctuations, the $k^{-5/3}$ scaling is observed as repeatedly reported in the previous study [Smith 1994, Scott 2007, Vallgren 2011]. This energy spectrum is almost the same as that of the simulation in which $f_D$ and the same numerical parameters are used, as demonstrated in Fig. 3.14. It is natural to think that the departure from the $k^{-5/3}$ scaling comes from the formation of strong coherent vortices after the quasi steady state. Note that the $k^{-1}$ scaling is obtained in the intermediate scale range when the dipole coherent vortices are filtered out in Sec. 3.1.2. Thus the difference between the coherent vortices in this subsection and the dipole vortices is in the energy spectrum of the background field. The similarities are that both are formed after the quasi steady state and the sizes
of them are comparable with the forcing scale \((2\pi/k_f)\).

Strong coherent vortices are also observed in simulations CR1, CR2, CR3, CR4, DR1, DR2, DR3, DR4, DR5, and DR6 at higher spatial resolutions. They are formed only when \(f_R\) and an insufficiently small drag coefficient are used. Radii of coherent vortices are comparable with the forcing scale in all simulations if they are formed. More coherent vortices are observed for higher forcing wave numbers at higher spatial resolutions. Each intensity of strong coherent vortices in statistically stationary state is at a comparable level around the maximum vorticity in each simulation as illustrated in Fig. 3.13. The maximum vorticity highly depends on the numerical parameters such as \(k_f\) and \(d\).

To investigate the relation between the strong coherent vortices and the numerical parameters, we plot the maximum absolute value of vorticity normalized with the root-mean-square vorticity \(|\omega|_{\text{max}}/\omega_{\text{rms}}\) in a statistically stationary state against the infrared Reynolds number \(Re_d\), defined as \(k_f/d\) according to Ref. [Vallgren 2011], in Fig. 3.15. All the data for \(f_R\) with \(q = 1\) at the resolution 512\(^2\), 1024\(^2\), and 2048\(^2\) are shown in Fig. 3.15. Obviously there is a critical point around \(Re_d \approx 40\) and these data give a single curve \(a_1(Re_d - b_1)^{1/2} + 5\) for \(Re_d \gtrsim 40\), with \((a_1, b_1) = (3.1, 41)\), as if the supercritical (pitchfork) bifurcation might occur. This result suggests that not \(k_d\) but \(Re_d\) is a key factor for formation of strong coherent vortices. Note that the infrared region \([1, k_d]\) is enlarged twice and four times for the same \(Re_d\) when the resolution gets higher from 512\(^2\) to 1024\(^2\) and 2048\(^2\), respectively. Thus, the shortage of spectral modes in the infrared region is not the cause of the formation of strong coherent vortices. That is, strong coherent vortices are not due to the finite-size effect. When \(f_R\) is used with \(q = 1\), the spectral bulge is also observed for \(Re_d \gtrsim 40\). It gets to be more distinguishable as \(Re_d\) increases.

These results imply that a wide \(k^{-5/3}\) scaling range cannot be observed when \(f_R\) is adopted, while a wide inertial subrange can be easily attained when \(f_D\) is used. To check this, we employ the maximum spatial resolution 4096\(^2\) in this study in ER1 and ED1. In these simulations, the forcing wave number is shifted to a higher wave number in proportion to the maximum wave number. Both the energy input rate \(\varepsilon_in\) and the energy transfer rate \(\varepsilon_tr\) are also fixed as in the previous cases. The results are what we expected; strong coherent vortices and a spectral bulge, which destroys the \(k^{-5/3}\) scaling, are observed for \(f_R\), while the structureless vorticity field with a relatively wide \(k^{-5/3}\) scaling range is attained for \(f_D\). The comparison of the instantaneous energy spectrum when the statistically stationary state is nearly attained in ER1 with the long-time averaged energy spectrum in stationary state in ED1 is illustrated in Fig. 3.16. Because of a high computational cost to trace all the energy growing process after the quasi steady state, we stop the calculation of ER1 when the stationary state is nearly obtained.

At the end of the simulation ER1, we evaluate the maximum absolute value of the vorticity normalized with the root-mean-square vorticity \(|\omega|_{\text{max}}/\omega_{\text{rms}}\) and add it to the previous data in Fig. 3.17. This value is much smaller than the estimated value from the fitting function \(a_1(Re_d - b_1)^{1/2} + 5\), where \((a_1, b_1) = (3.1, 41)\), from the previous data. Judging from the little growth of this value in final stage, this
Figure 3.15: Maximum absolute value of vorticity normalized by root-mean-square vorticity in a statistically stationary state vs the infrared Reynolds number $Re_d$ when $f_R$ is employed, $k_f = 124, 249, 498$, and $q = 1$. Error bars show each of maximum and minimum values in several realizations. These data can be fitted onto a single curve $a_1(Re_d - b_1)^{1/2} + 5$ with $(a_1, b_1) = (3.1, 41)$ for $Re_d \gtrsim 40$.

Figure 3.16: Instantaneous energy spectrum when the statistically stationary state is nearly attained in ER1 (upper) and long-time averaged energy spectrum in stationary state in ED1 (lower).
3.2. Addition of the first-order hypodrag term

Figure 3.17: Maximum absolute value of vorticity normalized by root-mean-square vorticity vs the infrared Reynolds number $Re_d$ when $f_R$ is employed, $k_f = 124, 249, 498$, and $997 (ER1)$, and $q = 1$. Fitting lines for $k_f/k_d \gtrsim 40$, $a_1(Re_d - b_1)^{1/2} + 5$ (dotted black upper line) and $a_2(Re_d - b_2)^{1/3} + 5$ (solid gray lower line), are estimated with the data for $k_f = 124, 249, 498$, where $(a_1, b_1) = (3.1, 41)$ and $(a_2, b_2) = (4.7, 41)$. However, the estimated value from this fitting curve is much smaller than the calculated value at the end of the simulation ER1, as shown in Fig. 3.17. While the growth of $|\omega|_{\text{max}}/\omega_{\text{rms}}$ with $Re_d$ near the critical point $Re_d^c$ seems to be expressed with $a_1(Re_d - Re_d^c)^{1/2} + 5$, we are not sure how $|\omega|_{\text{max}}/\omega_{\text{rms}}$ depends on $Re_d$ at much higher $Re_d$. Even so, we expect that the coherent vortices would emerge and destroy the $k^{-5/3}$ scaling law for high $Re_d$ when $f_R$ is adopted. On the other hand, much wider inertial subrange would be attained when $f_D$ is employed.

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gap would not be made up for even if the calculation of ER1 continued.

If the coherent vortices in these parameters have something to do with the dipole coherent structures mentioned in Sec. 3.1.2, the $\sqrt{t}$ growth of the intensity of the coherent vortices may be prevented by the drag term. Since the time scale $\tau_d$ at the drag scale $1/k_d$ is estimated as $\tau_d \sim \varepsilon_{tr}^{-1/3}k_d^{-2/3}$, the intensity of the coherent vortices $|\omega|_{\text{max}}/\omega_{\text{rms}}$ may depend on $\sqrt{\tau_d} \sim \varepsilon_{tr}^{-1/6}k_d^{-1/3} \sim Re_d^{1/3}$. Here $Re_d = k_f/k_d$. Thus, we also fit the previous data with the function $a_2(Re_d - b_2)^{1/3} + 5$ and evaluate $(a_2, b_2)$ as $(4.7, 41)$. However, the estimated value from this fitting curve is much smaller than the calculated value at the end of the simulation ER1, as shown in Fig. 3.17.
3.3 Combination of two types of forcing

To bridge a gap between the results of \( f_R \) and \( f_D \), we examine some cases of the combination of both two types of forcing process. Here the random forcing ratio \( r_{RF} \) to the total forcing is defined by \( \frac{\varepsilon_R^m}{\varepsilon_m} \) where \( \varepsilon_R^m \) and \( \varepsilon_D^m \) are energy input rates for \( f_R \) and \( f_D \), respectively. In the following, the total energy input rate \( \varepsilon_m = \varepsilon_R^m + \varepsilon_D^m \) is fixed. We carried out simulations changing \( r_{RF} \) with using the parameters of BR1 and BR3 in Table 2.1. The infrared Reynolds number \( R_{ed} \) in BR1 and BR3 are approximately 137 and 58, respectively. In Fig. 3.18, \( |\omega|_{max}/\omega_{rms} \) in each statistically stationary state is plotted against \( r_{RF} \). The results of the simulations BR7 and BD4, which correspond to \( r_{RF} = 1 \) and \( 0 \) when \( R_{ed} = 32 \), are also shown in this figure. At a large and fixed value of \( R_{ed} \) such that the coherent vortices are formed in the case of \( r_{RF} = 1 \), the coherent vortices also emerge and destroy the \( k^{-5/3} \) scaling law if \( r_{RF} \) is relatively close to 1. Decreasing \( r_{RF} \) to a critical value \( r_{RF}^c \), which depends on the fixed value of \( R_{ed} \), the intensity \( |\omega|_{max}/\omega_{rms} \) also decreases. This behavior can be expressed with the function \( \alpha(r_{RF} - r_{RF}^c)^{1/2} + 5 \), where \( (\alpha, r_{RF}^c) \) is \((35, 0.2)\) when \( R_{ed} = 137 \) and \((17, 0.6)\) when \( R_{ed} = 58 \) as shown in Fig. 3.18.

From these results, the critical infrared Reynolds numbers \( R_{ed}^c \) when \( r_{RF} = 0.2 \) and 0.6 are roughly estimated to be 137 and 58, respectively. As shown in Fig. 3.15, \( R_{ed}^c \sim 40 \) when \( r_{RF} = 1 \). This dependence of \( R_{ed}^c \) on \( r_{RF} \) is schematically shown in Fig. 3.19. From the result of ED1, \( R_{ed}^c \) is at least over 462 when \( r_{RF} = 0 \), although we expect it to be infinite. It is obvious that including even a small fraction of the random component to the forcing process can drastically change the vorticity field and destroy the \( k^{-5/3} \) scaling law when \( R_{ed} \) is high.
3.3. Combination of two types of forcing

Figure 3.18: Maximum absolute value of vorticity normalized by root-mean-square vorticity in statistically stationary state as a function of the random forcing ratio $r_{RF}$ when $Re_d = 137$, 58, and 32. Error bars show each of maximum and minimum values in several realizations. Dotted black upper line is $35(r_{RF} - 0.2)^{1/2} + 5$ and solid gray lower line is $17(r_{RF} - 0.6)^{1/2} + 5$.

Figure 3.19: Roughly estimated critical infrared Reynolds number $Re_d^c$ against the random forcing ratio $r_{RF}$.
3.4 Effect of the order of hypodrag term

In our direct numerical simulations, a statistically stationary turbulent flow field that holds phenomenological statistical laws, such as the $-5/3$ energy spectrum and constant energy flux function in the intermediate scale range, is obtained with the first-order hypodrag term. However it was also found that the statistical features depend on the type of forcing process. In this section, we test the influence on the statistical features of the type of drag term by varying its inverse-Laplacian order $q$.

We carried out simulations FR1, FR2, FD1, FD2, GR1, GR2, GD1, HR1, HR2, HR3, HD1, ID1, JR1, JR2, JD1, and KD1 in Table 2.1 employing $q = 0, 2,$ and $8$. Strong coherent vortices are observed only in the simulations GR1, HR1, HR2, and JR1. The results are quite similar to those with $q = 1$: (i) Both the energy spectrum and the energy flux function are consistent with the K41 phenomenology in the inverse energy cascade range and the vorticity field is structureless when $f_D$ is adopted. (ii) Those also hold for the case of $f_R$ only when the infrared Reynolds number $Re_d$ is low. (iii) Strong coherent vortices and a spectral bulge are formed when $f_R^F$ is used at high $Re_d$. Note that the constant energy flux range is narrow when $q = 0$ as usual [Boffetta 2000, Danilov 2001a], since the linear drag ($q = 0$) affects all scales uniformly as pointed out in Ref. [Danilov 2001b]. (See the results for FR2 in Fig. 3.20(a).) Strong coherent vortices cannot be observed at the 512$^2$ resolution when $q = 0$, while they are seen in GR1 at the resolution of 2048$^2$.

The analytical form in the drag range of the energy spectrum depends on $q$ and the large-scale bottleneck effect gets to be prominent for large $q$. The spectral bump in the energy spectrum is formed in the vicinity of $k_p$ for large $q$ as predicted with a closure model in Ref. [Bos 2009]. (See Fig. 3.20(b).) It is observed for both types of forcing process. This effect, however, does not change the $k^{-5/3}$ scaling in the inertial subrange, but just reduces the width of the $k^{-5/3}$ scaling range. The critical infrared Reynolds number $Re_d^c$ for $f_R^F$ also depends on $q$; small $Re_d^c$ is evaluated for large $q$.

To see the results in another numerical setting, we also reproduce RUN3 in Ref. [Xiao 2009], where a relatively wide $k^{-5/3}$ scaling range is attained with $f_D$ at the resolution of 2048$^2$. The numerical parameters of this simulation LD1 are shown in Table 3.1. Clear $k^{-5/3}$ scaling is observed over one decade as in Ref. [Xiao 2009], with the inverse-cascade strength $\varepsilon_{tr}/\varepsilon_{in}$ is approximately 0.30. We then replace the forcing process by $f_R$ in the simulation LR1 in Table 3.1. The spectral bulge and coherent vortices, which are never seen in LD1, are formed and the $k^{-5/3}$ scaling is destroyed, as expected. From these results, we infer that it may be impossible to have a wide $k^{-5/3}$ scaling range when $f_R$ is adopted.
Table 3.1: Numerical settings in simulations LR1 and LD1.

<table>
<thead>
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<th>Name</th>
<th>( N )</th>
<th>( f )</th>
<th>( k_f )</th>
<th>( \varepsilon_{in} )</th>
<th>( h )</th>
<th>( \nu )</th>
<th>( q )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
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<td>2048</td>
<td>( f_R ) 485</td>
<td>0.00016</td>
<td>8</td>
<td>2.651 \times 10^{-44}</td>
<td>2</td>
<td>277.103</td>
<td></td>
</tr>
<tr>
<td>LD1</td>
<td>2048</td>
<td>( f_D ) 485</td>
<td>0.00016</td>
<td>8</td>
<td>2.651 \times 10^{-44}</td>
<td>2</td>
<td>277.103</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3.20: Long-time averaged energy flux functions (a) and the energy spectra (b) in statistically stationary states for \( q = 0(FR2), 1(BR6), 2(HR3), \) and 8(JR2). Inset in (b) is a compensated plot; \( \langle E(k, t) \rangle \varepsilon_{tr}^{-2/3} k^{5/3} \) vs \( k \), where the energy transfer rate \( \varepsilon_{tr} \) is assumed to be 0.018.
4.1 Summary

We numerically investigate the inverse energy cascade range of two-dimensional Navier-Stokes turbulence focusing on the universality of the K41 phenomenology in it. Our results show that the $k^{-5/3}$ scaling law of the inertial subrange in the inverse energy cascade range is sensitive to not only the numerical parameters but also the mechanism (deterministic or random) of the forcing process, which apparently imply the non-universality of the K41 picture. This sensitivity is traced to the formation of strong coherent vortices. These coherent vortices have been observed in many preceding numerical works [Smith 1994, Borue 1994, Sukoriansky 1999, Danilov 2001a, Danilov 2001b, Scott 2007, Vallgren 2011, Fontane 2013]. However, it has also been repeatedly pointed out [Smith 1994, Borue 1994, Scott 2007, Vallgren 2011] that, even in the cases where the strong coherent vortices destroy the scaling law, once they are filtered out, the $k^{-5/3}$ scaling can be observed in the energy spectrum of the background vorticity field.

In this study, we employed two types of forcing, the random forcing $f_R$ and the deterministic forcing $f_D$, and found that strong coherent vortices are formed only when $f_R$ is adopted. The energy spectrum of the background field in these cases is almost the same as that of the entire flow field when $f_D$ is used with the same numerical parameters, in which the $k^{-5/3}$ scaling law is clearly observed. When $f_D$ is substituted for $f_R$ after the strong coherent vortices are formed, they are dissipated and the $k^{-5/3}$ scaling is recovered. The time scale of the formation and dissipation process of these coherent vortices are much longer than that of inverse energy cascade such as the large eddy turnover time. These observations lead a following picture: The inverse energy cascade turbulent flow field with deviation from the $k^{-5/3}$ energy spectrum can be decomposed to the coherent vortices and the background field; the former causes the deviation and the latter has the $k^{-5/3}$ energy spectrum. In spite of the discrepancy of the energy spectrum, the universality of the K41 phenomenology holds for the latter in fact.
To characterize the coherent vortices, a parameter $|\omega|_{\text{max}}/\omega_{\text{rms}}$ is introduced in this study, because the strong coherent vortices, once they emerge, are long-lived and of the same size in general. It is found that this parameter grows like an order parameter in the case of a transition phenomenon with increase in the infrared Reynolds number $Re_d(= k_f/k_d)$ when $f_R$ is adopted. With this parameter, we can detect more precisely when the coherent vortices are formed in a series of simulations. Taking advantage of this methodology, we revealed that the finite-size effect is not the cause of the formation of strong coherent vortices with similar results for resolution improvement. Strong coherent vortices are formed when $f_R$ is used and $Re_d$ exceeds a critical value. This is the case at least when the order $q$ of hypodrag term is 0, 1, 2, or 8. These results imply that a wide $k^{-5/3}$ scaling range would never be observed with $f_R$.

When the deterministic forcing $f_D$ is adopted, strong coherent vortices are hardly seen and the results are consistent with the K41 phenomenology even when $Re_d = 462$ at the highest spatial resolution $4096^2$ in this study. This suggests that the inertial subrange can be extended with no limit like in the case of three-dimensional homogeneous isotropic turbulence. We have verified that the $k^{-5/3}$ energy spectrum does not depend on the order $q$ of hypodrag term, even though the scaling range is shortened for large $q$ by the large-scale bottleneck effect where the spectral bump is formed in the vicinity of the energy peak wave number.

From the results of simulations with the combination of both types of forcing process with $q = 1$, it is also found that even a small fraction of the random forcing process can cause the formation of strong coherent vortices and destroy the $k^{-5/3}$ scaling law when $Re_d$ is high. We currently do not have an explanation why only the simulation with $f_D$ can be free from the coherent vortices and completely consistent with the K41 phenomenology.

### 4.2 Some remarks and future work

In this study, we restrict both the inverse-cascade strength $\varepsilon_{tr}/\varepsilon_{in}$ and the ratio $k_{\text{max}}/k_f$ to low values, where $k_{\text{max}}$ is the maximum wave number. Judging from the results in Refs. [Smith 1994, Tran 2004b, Scott 2007, Vallgren 2011, Fontane 2013], this confinement on the parameter space may have a positive effect on avoiding the formation of coherent vortices. Tran [Tran 2004b] has shown that, even with the deterministic forcing process similar to $f_D$, the $k^{-5/3}$ scaling is destroyed when $\varepsilon_{tr}/\varepsilon_{in}$ is raised. It is also shown in Ref. [Scott 2007] that strong coherent vortices are formed and destroy the $k^{-5/3}$ scaling even in quasi steady state when $k_{\text{max}}/k_f > 16$. Since raising $\varepsilon_{tr}/\varepsilon_{in}$ or $k_{\text{max}}/k_f$ means the elongation of the dissipation-free spectral range smaller than the forcing scale. Thus we infer that a wide dissipation-free spectral range might make it easier for the strong coherent vortices to emerge.

The coherence distance is always limited to the forcing scale in our numerical simulations. The size distribution of strong coherent vortices seems to be different from that in the previous results such as shown in Ref. [Borue 1994]. If the excita-
tion process of the strong coherent vortices is altered by the change of the forcing and dissipation terms, the size distribution of strong coherent vortices may differ and much weird energy spectrum may be obtained. Even when $\varepsilon_{tr}/\varepsilon_{in}$ and $k_{max}/k_f$ are restricted to low values, our simulation results show that the formation of coherent vortices highly depends on the external forcing mechanism. We, therefore, conclude that a more comprehensive investigation of the formation mechanism of strong coherent vortices is required to elucidate the universality and robustness of the K41 phenomenology in two-dimensional Navier-Stokes turbulence. We will leave it for future works.
Appendix A

Appendix

A.1 Dynamics of strong coherent vortices

A.1.1 Formation process

Here we demonstrate the time development of the strong coherent vortices. The results of the simulation BR2 in Table 2.1 are shown as an example. We also show the results of the simulation BD2 for comparison. In both cases, the infrared Reynolds number $Re_d$ is approximately 75. As described in Sec. 3.2, with such a high $Re_d$, the total kinetic energy gradually grows after the quasi steady state when $f_R$ is used, while the statistically stationary state is soon subsequent to the quasi steady state when $f_D$ is used. In this long-term growth process in the case of $f_R$, the strong coherent vortices, as illustrated in Fig. 3.12(a), are formed.

We plot the local extrema of the vorticity field at every $T_L (\equiv \varepsilon^{-1/3}(2\pi)^{2/3}) \approx 7.3$ in Fig. A.1(b). The data whose value are less than $5\omega_{\text{rms}}$ are omitted. Time evolution of the total kinetic energy is also shown in Fig. A.1(a). The number of the strong coherent vortices gradually increases while the total kinetic energy grows. They are newly formed even after the total kinetic energy gets saturated. As the number of them increases, the merger of two coherent vortices of the same sign occurs, which decreases the number of coherent vortices. Eventually gain and loss in the number of them balance in the long term. Note that both the formation and the merger are rare events on the basis of the time scale $T_L$. The intensity of each strong coherent vortex grows monotonically in time. The mature coherent vortices have almost the same intensity as if there is an upper limit for the intensity. Even after the merger events, the intensity of a new coherent vortex does not exceed that value.

We briefly describe the dynamics of the strong coherent vortices in physical space. The trajectories of strong coherent vortices during a time interval $T_L$ are plotted in Fig. A.2. In the early stage, while the number of strong coherent vortices is small and each intensity is immature, each coherent vortex independently totters along in the calculation domain. As the number of coherent vortices increases, two coherent vortices of the opposite sign occasionally pair up and the pair rapidly travels. Once such a pair is formed, the pair travel and the pair reconnection are alternately observed. Two coherent vortices of the same sign sometimes swirl around each other, however, the merger of the two is rare.
Figure A.1: (a) Time evolution of total kinetic energy for BR2 (upper) and BD2 (lower). Inset is a zoom of the initial stage. (b) Local extrema in vorticity field at every $T_L$ for BR2 (red circle) and BD2 (green square).
Figure A.2: Trajectories of strong coherent vortices in BR2. Local extremal points in vorticity field are plotted at every 0.3 in time range $[t_s, t_s + T_L]$, where $t_s = 250(a)$, 500(b), and 1500(c). Local extremal points at $t_s$ are shown with relatively large black dots.
A.1.2 Dissipation process

The statistically stationary state in each simulation depends little on the initial condition. The strong coherent vortices and the spectral bulge in energy spectrum developed in the case of $f_R$ disappear when $f_D$ is substituted for $f_R$ or when the drag coefficient $d$ is adequately increased.

In order to demonstrate the above, we show the results for the simulations BR7 and BD2 in Table 2.1 whose initial conditions are the statistically stationary state in BR2. Time evolution of the total kinetic energy and the intensities of strong coherent vortices in these dissipation process are shown in Fig. A.3.

In both dissipation processes, the strong coherent vortices are axisymmetric and their radii remain comparable with the forcing scale $2\pi/k_f$. However, there is an distinct difference in dissipation process between raising $d$ (BR7) and employing $f_D$ (BD2). The intensities of coherent vortices monotonically decrease when $d$ is raised. On the other hand, when $f_D$ is used, the strong coherent vortices are rather intensified for a while soon after the forcing term is switched to $f_D$. Even after that, the intensities of them occasionally grow, while they decrease on average in long term.

To see the dynamics of the strong coherent vortices while they are dissipating, the trajectories of them are plotted for each simulation in Fig. A.4 and Fig. A.5. When $d$ is intensified, the coherent vortices apparently lose their mobility. Even the pair of coherent vortices of the opposite sign ends up in swirling in a small part of the calculation domain as shown in Fig. A.4(b). Finally pairs of coherent vortices disappear and each coherent vortex also dissipates. When $f_D$ is used, on the other hand, the pair travel of coherent vortices are observed even after $10T_L$ as illustrated in Fig. A.5(b). The mobility of each coherent vortex seems to be maintained at the same level with that in the statistically stationary state for $f_R$. (See also Fig. A.2.)
Figure A.3:  (a) Time evolution of total kinetic energy for BR7 (lower) and BD2 (upper). Both simulations are started with the statistically stationary state in BR2. (b) Local extrema in vorticity field at every $T_L$ for BR7 (blue triangle) and BD2 (green square).
Figure A.4: Trajectories of coherent vortices in BR7. This simulation is started with the statistically stationary state in BR2. Local extremal points in vorticity field are plotted at every 0.3 in time range \([t_s, t_s + T_L]\), where \(t_s = 0\) (a), 20 (b), and 100 (c). Local extremal points at \(t_s\) are shown with relatively large black dots.
A.1. Dynamics of strong coherent vortices

Figure A.5: Trajectories of coherent vortices in BD2. This simulation is started with the statistically stationary state in BR2. Local extremal points in vorticity field are plotted at every 0.3 in time range $[t_s, t_s + T_L]$, where $t_s = 0$(a), 100(b), and 200(c). Local extremal points at $t_s$ are shown with relatively large black dots.
Bibliography


