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Canonical Operator Formalism for Quantum Stochastic Differential Equations

— An Introduction to Non-Equilibrium Thermo Field Dynamics —*

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Abstract

量子力学では，状態ベクトルの時間発展を記述する Schrödinger 方程式の時間推進演算子（エネルギー演算子 Hamiltonian）を使って，演算子の時間発展を記述する Heisenberg 方程式が書き下され（表示の変更を伴う），正準交換関係が保存する正準演算子形式の理論体系が構成されている。量子解放系（散逸系）を記述する正準演算子形式の理論体系 Non-Equilibrium Thermo Field Dynamics（NETFD）では，統計演算子（密度演算子）を真空状態のケット・ベクトルと見立て直すことにより，確率 Liouville 方程式の量子版が，確率 Schrödinger 方程式の形式で与えられる。その時間推進演算子（確率 hat-Hamiltonian と呼ばれ，量子 Brown 運動を乱雑力演算子として含む）を使って Heisenberg 方程式を書き下すと量子系の Langevin 方程式（確率 Heisenberg 方程式とも呼ばれる）が得られる。乱雑力演算子に関して平均をとると，確率 Schrödinger 方程式は散逸 Schrödinger 方程式（量子マスター方程式とも呼ばれる）に，確率 Heisenberg 方程式は散逸 Heisenberg 方程式になる。得られた散逸 Schrödinger 方程式の時間推進演算子（hat-Hamiltonian と呼ばれる）は，散逸 Heisenberg 方程式の hat-Hamiltonian にもなっている（表示の変更を伴う）。確率的生成消滅演算子と散逸的生成消滅演算子の正準交換関係は何れも保存し，確率微分方程式系および散逸微分方程式系に対して，正準演算子形式の理論体系が構成されている（図 1 参照）。集中セミナーでは，NETFD の上記体系の構成からくりを詳しく解説し，NETFD ならではの新しい自然認識や技巧，応用について紹介する。

1 Introduction

1.1 Flow Chart of Contents

In this paper, we will introduce a *canonical operator formalism* for quantum systems in far-from-equilibrium state, named Non-Equilibrium Thermo Field Dynamics (NETFD) [1, 2, 3, 4, 5, 6, 7, 8, 9], which provides us with a unified framework composed of the dissipative Schrödinger equation, the dissipative Heisenberg equation (quantum master equation), the quantum stochastic Liouville equation and the stochastic Heisenberg equation (quantum Langevin equation) (see Fig. 1). NETFD treats dissipative quantum open systems by the method similar to the usual quantum mechanics and quantum field theory which accommodate the concept of the dual structure in the interpretation of nature, i.e. in terms of the *operator algebra* and the *representation space*. The representation space of NETFD (named *thermal space*) is composed of the direct product of two Hilbert spaces, the one for *non-tilde* fields and the other for *tilde* fields.[1] It was revealed that *dissipation* is taken into account by a *rotation* in whole the two Hilbert spaces. The terms with the product of tilde and non-tilde fields in the infinitesimal time-evolution generator (we will call it *hat-Hamiltonian*) take care of dissipative (i.e. irreversible) time-evolution. This notion was discovered first when NETFD was constructed [1, 2].[2] Throughout this paper, we confine ourselves to the case of boson fields, for simplicity. The extension to the case of fermion fields are rather straightforward [8].

In sub-section 1.2, we will explain the functioning of random force operator within quantum mechanics if one requires a canonical operator formalism to be satisfied even for quantum dissipative systems

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[1] Within NETFD, any operator A is associated with its tilde field $\tilde{A}$ (see **Tool 1** in section 2.1).

[2] This notion had not appeared in the formulation of the equilibrium thermo field dynamics (TFD) [10] which is an operator formalism of the Gibbs ensembles. This is one of the essential difference between NETFD and TFD.

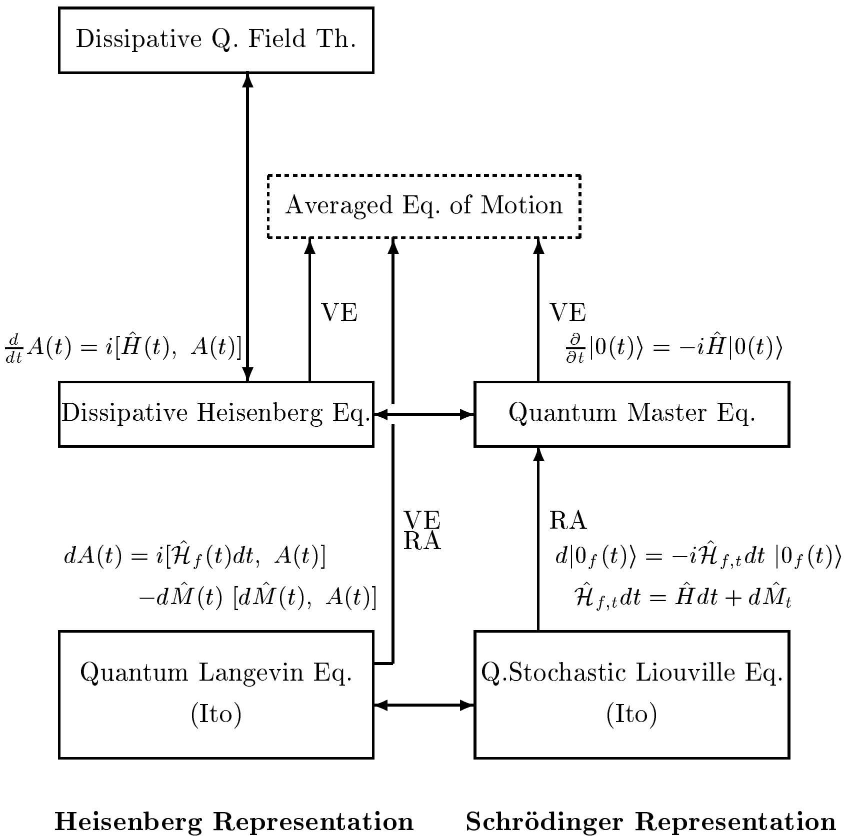

Figure 1: System of the Stochastic Differential Equations within Non-Equilibrium Thermo Field Dynamics. RA stands for the random average. VE stands for the vacuum expectation.

described by a quantum Langevin equation. This raises a question about the origin of dissipation in the universe, which is one of the motivations of the present paper. In sub-section 1.3, we go over two treatments of Schrödinger equation, i.e., one by the partial differential equation for wave function, and the other by the algebra for annihilation and creation operators with the definition of vacuum. In sub-section 1.4, we will show how to treat the quantum master equation by mapping it to a partial differential equation for a c-number function, and how to construct the system of stochastic differential equations in the c-number function space. This approach is quite similar to handle with the Schrödinger equation for wave function in quantum mechanics. In section 2, we will construct the canonical operator formalism for dissipative quantum open systems, i.e., NETFD, which enables us to treat dissipative systems with the equal-time canonical commutation relation between annihilation and creation operators with a definition of unstable time-dependent vacuum. With this new formalism, we obtain a novel viewpoint that the time-evolution of dissipative quantum systems is controlled by a condensation of particle pairs into the thermal vacuum. In section 3, we will introduce two kind of interaction hat-Hamiltonians, i.e., one is hermitian and the other non-hermitian. This process is necessary to construct the hermitian and non-hermitian martingale operators in the following section. In section 4, we will construct a unified canonical operator formalism for stochastic quantum systems within NETFD. Applications to the system of damped harmonic oscillator (section 5) and of quantum Kramers equation (section 6) are performed. The essential difference of these two systems stems from the difference in the structures of martingale. An answer to the question raised in sub-section 1.2 is given in section 7 within the unified framework of NETFD. Appendices are provided in order to make this paper self-contained at least for the parts necessary to construct NETFD, which may be convenient for those who try to follow the derivations of formulae by their own hands.

1.2 Motivation

The studies of the Langevin equation for quantum systems were started in connection with the development of laser [11, 12, 13], and are still continued in order to develop a satisfactory formulation

[14, 15, 16, 17, 18, 19] (see comments in [20]). Most of the mathematical approaches for quantum Langevin equation, in which the canonical commutation relation between annihilation and creation operetors preserves in time, are created based on the non-commutativity of the random force operators.

For dissipative systems, for example, we have equations for the operators $\langle a(t)\rangle$ and $\langle a^\dagger(t)\rangle$ averaged with respect to random force operators of the forms

$$d/dt\ \langle a(t)\rangle = -i\omega\langle a(t)\rangle - \kappa\langle a(t)\rangle, \qquad d/dt\ \langle a^\dagger(t)\rangle = i\omega\langle a^\dagger(t)\rangle - \kappa\langle a^\dagger(t)\rangle \tag{1.1}$$

with the initial condition $\langle a(0)\rangle = a$ and $\langle a^\dagger(0)\rangle = a^\dagger$ where a and $a^\dagger$ satisfy the canonical commutation relation

$$[a,\ a^\dagger] = 1. \tag{1.2}$$

The equal-time commutation relation for these operators decays in time: $[\langle a(t)\rangle,\ \langle a^\dagger(t)\rangle] = \mathrm{e}^{-2\kappa t}$.

Random force operators $df(t)$ and $df^\dagger(t)$ are introduced in order to rescue this situation. If the random force operators in the Langevin equations

$$da(t) = -i\omega a(t)dt - \kappa a(t)dt + \sqrt{2\kappa}\ df(t), \quad da^\dagger(t) = i\omega a^\dagger(t)dt - \kappa a^\dagger(t)dt + \sqrt{2\kappa}\ df^\dagger(t) \tag{1.3}$$

satisfy

$$[df(t),\ df^\dagger(t)] = dt, \tag{1.4}$$

the equal-time commutation relation for the stochastic operators $a(t)$ and $a^\dagger(t)$ preserves in time, i.e., $d\left([a(t),\ a^\dagger(t)]\right) = 0$. Its left-hand side can be calculated with (1.4) as

$$\begin{aligned} d\left([a(t),\ a^\dagger(t)]\right) &= [da(t) \,\overset{\circ}{,}\, a^\dagger(t)] + [a(t) \,\overset{\circ}{,}\, da^\dagger(t)] \\ &= -2\kappa dt\ [a(t),\ a^\dagger(t)] + \sqrt{2\kappa}\left([df(t) \,\overset{\circ}{,}\, a^\dagger(t)] + [a(t) \,\overset{\circ}{,}\, df^\dagger(t)]\right) \\ &= -2\kappa dt\ [a(t),\ a^\dagger(t)] + 2\kappa\ [df(t),\ df^\dagger(t)] \\ &= -2\kappa dt\left\{[a(t),\ a^\dagger(t)] - 1\right\} \end{aligned} \tag{1.5}$$

where $[X \,\overset{\circ}{,}\, Y] = X \circ Y - Y \circ X$ is the commutator with the *stochastic multiplication of the Stratonovich type* (Stratonovich product) [21] (see Appendix D). At the third equality, we used

$$[df(t) \,\overset{\circ}{,}\ a^\dagger(t)] = [df(t);\ a^\dagger(t)] + [df(t),\ da^\dagger(t)]/2 = \sqrt{\kappa/2}\ [df(t),\ df^\dagger(t)] \tag{1.6}$$

and its hermitian conjugate. We also used the commutativity $[df(t);\ a^\dagger(t)] = 0$ for the *stochastic multiplication of the Ito type* (Ito product) [22] and the connection formula between the Ito and Stratonovich products (see Appendix D). Then, with the initial condition (1.2), we obtain the consistent solution $[a(t),\ a^\dagger(t)] = 1$.

The above argument is of zero temperature related only to the zero-point quantum fluctuation. However, it has been extended to include the situations for finite temperature. Then, we have a crucial question. Should we interpret that the origin of thermal dissipation is quantum mechanical? In this paper, we will investigate this question [23, 24] in the course of the introduction of the system of quantum stochastic differential equations within NETFD.

1.3 Quantum Mechanics

Let us go over briefly the Schrödinger equation ($\hbar = 1$)

$$i\partial/\partial t\ |\psi(t)\rangle = \check{H}|\psi(t)\rangle \tag{1.7}$$

with the Hamiltonian of a harmonic oscillator:

$$\check{H} = \check{p}^2/2m + m\omega^2\check{x}^2/2. \tag{1.8}$$

The operators $\check{x}$ and $\check{p}$ satisfy the canonical commutation relation $[\check{x},\ \check{p}] = i$.

Coordinate Representation

In the x-representation, the Hamiltonian reduces to

$$\langle x|\check{H}|x'\rangle = H(x', \partial/\partial x')\delta(x - x'), \qquad H(x, \partial/\partial x) = -(1/2m)\partial^2/\partial x^2 + m\omega^2 x^2/2 \tag{1.9}$$

with the properties $\langle x|\check{x}|x'\rangle = x\delta(x - x')$ and $\langle x|\check{p}|x'\rangle = (1/i)\,\partial/\partial x'\ \delta(x - x')$ where the states $|x\rangle$ are defined by $\check{x}|x\rangle = x|x\rangle$ for $x \in \mathbf{R}$, and form the orthogonal complete set satisfying $\langle x'|x\rangle = \delta(x' - x)$ and $\int dx|x\rangle\langle x| = \mathbf{1}$. Then, the Schrödinger equation (1.7) reads

$$i\partial/\partial t\ \psi(x,t) = H(x, \partial/\partial x)\psi(x,t) \tag{1.10}$$

with the wave function defined by $\langle x|\psi(t)\rangle = \psi(x,t)$.

Substituting $\psi(x,t) = u(x)\mathrm{e}^{-iEt}$ into (1.10), we have an eigen-value equation

$$\left[-(1/2m)\,d^2/dx^2 + m\omega^2x^2/2\right]u(x) = Eu(x). \tag{1.11}$$

This is solved with the energy eigen-value

$$E_n = (n+1/2)\,\omega \qquad (n = 0,1,2,3,\cdots) \tag{1.12}$$

to give the eigen-function belonging to the energy state (see Appendix A.1)

$$u_n(x) = N_nH_n(\sqrt{m\omega}x)\mathrm{e}^{-m\omega x^2/2} \tag{1.13}$$

with the normalization $\int_{-\infty}^{\infty}dx|u_n(x)|^2 = 1$. Here, $H_n(\xi)$ is the Hermite polynomials which satisfies the differential equation

$$H_n''(\xi) - 2\xi H_n'(\xi) + 2nH_n(\xi) = 0. \tag{1.14}$$

The variable x does not represent the coordinate of real space but of "phase-space" in the sense that $|\psi(x,t)|^2dx$ gives the probability of finding a particle within the range $x \sim x+dx$ at time t.

Number Representation

As is well-known, by introducing the annihilation and creation operators $a,\ a^\dagger$ through the relations [3] $\check{x} = \sqrt{1/2m\omega}\left(a^\dagger + a\right)$ and $\check{p} = i\sqrt{m\omega/2}\left(a^\dagger - a\right)$, the Hamiltonian (1.8) becomes $\check{H} = \left(a^\dagger a + 1/2\right)\omega$. The annihilation and creation operators satisfy the equal-time canonical commutation relation $[a,\ a^\dagger] = 1$.

The eigen-state $|n\rangle$ of the number operator $a^\dagger a$, i.e., $a^\dagger a|n\rangle = n|n\rangle$ $(n = 0,1,2,3,\cdots)$, is simultaneously the energy eigen-state satisfying $\check{H}|n\rangle = E_n|n\rangle$ with (1.12). The states $|n\rangle$'s are generated cyclically on a vacuum $|0\rangle$ as $|n\rangle = \left(a^\dagger\right)^n/\sqrt{n!}\ |0\rangle$. The vacuum state $|0\rangle$ is defined by $a|0\rangle = 0$.

The algebraic reconstruction of quantum mechanics in terms of the annihilation and creation operators had put forward our deeper understanding of nature considerably in addition to provide us with its technical transparency. It led us to the construction of quantum field theory. Within the formulation of quantum statistical mechanics, a similar reconstruction was performed for dissipative non-equilibrium quantum systems. We will show in this paper the development towards the construction of canonical operator formalism.

1.4 Quantum Statistical Mechanics

1.4.1 Quantum Master Equation

Let us investigate what is the most fundamental characteristics of the quantum master equation for the statistical operator (density operator) $\rho(t)$, i.e., $\partial/\partial t\ \rho(t) = -iL\rho(t)$. The characteristics of the Liouville operator L are given as follows (see also Appendix B).

D1. The hermiticy of the Liouville operator iL in the sense that $(iL\ \bullet)^\dagger = iL\bullet$, e.g.,

$$(i[H,\ \bullet\ ])^\dagger = -i[\ \bullet,\ H^\dagger] = i[H,\ \bullet\ ], \tag{1.15}$$

$$(\mathit{\Pi}\ \bullet)^\dagger = \kappa\left\{[a\ \bullet\ ,\ a^\dagger] + [a,\ \bullet\ a^\dagger]\right\}^\dagger + 2\kappa\bar{n}\left([a,\ [\ \bullet\ ,\ a^\dagger]]\right)^\dagger = \mathit{\Pi}\ \ \bullet\ . \tag{1.16}$$

Here, $\bullet$ indicates an operand operator.

D2. The conservation of probability (tr $\rho = 1$): tr $L\ \bullet = 0$.

D3. The hermiticy of the density operator: $\rho^\dagger(t) = \rho(t)$. The eigenvalues of $\rho(t)$ are non-negative.

In the trace formalism, the expectation value of an observable operator A is given by

$$\langle A\rangle_t = \mathrm{tr}\ A\rho(t) = \mathrm{tr}\ A\mathrm{e}^{-iLt}\rho(0) = \mathrm{tr}\ \mathrm{e}^{iLt}A\mathrm{e}^{-iLt}\rho(0) \tag{1.17}$$

where we used the formal solution $\rho(t) = \mathrm{e}^{-iLt}\rho(0)$ and the property **D2**.

Then, the "Heisenberg operator" $A(t)$ is defined by

$$A(t) = \mathrm{e}^{iLt}A\mathrm{e}^{-iLt} \tag{1.18}$$

which satisfies the "Heisenberg equation" with the Liouville operator $L(t) = \mathrm{e}^{iLt}L\mathrm{e}^{-iLt} = L$:

$$d/dt\ A(t) = i[L(t),\ A(t)]. \tag{1.19}$$

[3] In the following in this paper, we will not put "ˇ" for the symbol of operator, unless it is confusing.

Damped Harmonic Oscillator

Here, we show how we had been dealing with quantum dissipative systems within the density operator formalism before NETFD was constructed. The quantum master equation for a damped harmonic oscillator is given by [25]

$$\partial/\partial t\ \rho_S(t) = -iL\rho_S(t), \qquad L = H_S^{\times} + i\mathit{\Pi} \tag{1.20}$$

with the symbol $H_S^{\times}X = [H_S,\ X]$ where H_S is the Hamiltonian of the system we are interested in, i.e.,

$$H_S = \omega a^{\dagger}a, \quad \omega = \epsilon - \mu, \quad [a,\ a^{\dagger}] = 1 \tag{1.21}$$

with ϵ and μ being the one-particle energy and the chemical potential, respectively. $\mathit{\Pi}$ is the damping operator defined by (1.16), i.e.,

$$\mathit{\Pi}X = \kappa\left([aX,\ a^{\dagger}] + [a,\ Xa^{\dagger}]\right) + 2\kappa\bar{n}[a,\ [X,\ a^{\dagger}]] \tag{1.22}$$

with the boson distribution function $\bar{n} = 1/(\mathrm{e}^{\omega/T}-1)$ and $\kappa = \Re\mathrm{e}\, g^2\int_0^{\infty}dt\sum_{\mathbf{k}}\langle[R_{\mathbf{k}}(t),\ R_{\mathbf{k}}^{\dagger}(0)]\rangle_R e^{i\omega t}$. Here, we have introduced the average, $\langle\cdots\rangle_R = \mathrm{tr}_R\cdots\rho_R$, with the density operator for a thermal reservoir which is given by $\rho_R = e^{-H_R/T}/Z_R$ with $Z_R = \mathrm{tr}_R\ e^{-H_R/T}$. Throughout this paper, we use the unit in which the Boltzmann constant is equal to unity. The coupling constant g represents the strength of the interaction between the harmonic oscillator and the thermal reservoir whose temperature is T. We see that the one-particle distribution function, defined by

$$n(t) = \mathrm{tr}\ a^{\dagger}a\ \rho_S(t), \tag{1.23}$$

satisfies the Boltzmann equation

$$d/dt\ n(t) = -2\kappa\left[n(t) - \bar{n}\right]. \tag{1.24}$$

The above quantum master equation (1.20) can be obtained by projecting out the reservoir by means of the damping theory [25, 26, 27], starting with the quantum Liouville equation (Liouville-von Neumann equation)

$$\partial/\partial t\ \rho(t) = -iH^{\times}\rho(t), \qquad H = H_S + H_R + H_I \tag{1.25}$$

where $H_I = g\sum_{\mathbf{k}}(aR_{\mathbf{k}}^{\dagger} + \mathrm{h.c.})$ is the Hamiltonian describing the interaction between the system and the thermal reservoir (called a linear dissipative coupling). $R_{\mathbf{k}}^{\dagger}$ and $R_{\mathbf{k}}$ are the operators of the thermal reservoir. H_R is the Hamiltonian of the reservoir the explicit form of which needs not be specified to get the master equation (1.20). The coarse-grained density operator $\rho_S(t)$ is defined by $\rho_S(t) = \mathrm{tr}_R\ \rho(t)$.

1.4.2 Mapping to C-number Function Space

Fokker-Planck Equation within Coherent Stare Representation

Introducing the boson coherent state representation of the anti-normal ordering [28, 29, 30] through

$$\rho_S(t) = \int\frac{d^2z}{\pi}P(z,t)\ |z\rangle\langle z| \tag{1.26}$$

with the boson coherent state $|z\rangle$, defined by $a|z\rangle = z|z\rangle$, we can map the master equation (1.20) into a partial differential equation for the c-number function $P(z,t)$ as [25]

$$\partial/\partial t\ P(z,t) = \left[-i\omega\left(\partial/\partial z^*\ z^* - \mathrm{c.c.}\right) + \kappa\left(\partial/\partial z^*\ z^* + \mathrm{c.c.}\right) + 2\kappa\bar{n}\ \partial^2/\partial z^*\partial z\right]P(z,t) \tag{1.27}$$

where the symbol c.c. indicates to take complex conjugate. The normalization of $P(z,t)$ is given from (1.26) with $\mathrm{tr}\ \rho_S(t) = 1$ as

$$\int\frac{d^2z}{\pi}P(z,t) = 1. \tag{1.28}$$

With the help of (1.27), we have the averaged equations of motion

$$d/dt\ \langle z\rangle_t = -i\omega\langle z\rangle_t - \kappa\langle z\rangle_t, \quad d/dt\ \langle z^*\rangle_t = i\omega\langle z^*\rangle_t - \kappa\langle z^*\rangle_t, \quad d/dt\ \langle |z|^2\rangle_t = -2\kappa(\langle |z|^2\rangle_t - \bar{n}) \tag{1.29}$$

where $\langle\cdots\rangle_t = \int(d^2z/\pi)\cdots P(z,t)$.

Solving the Fokker-Planck Equation

Let us solve (1.27) here for the case where $P(z,t)$ depends only on $|z|$. With the transformation

$$F(z,t) = \mathrm{e}^{it\omega(\frac{\partial}{\partial z^*}z^* - \frac{\partial}{\partial z}z)}P(z,t), \tag{1.30}$$

the Fokker-Planck equation (1.27) is transformed into

$$\partial/\partial t\ F(\xi,t) = 2\kappa\,(\partial/\partial\xi\ \xi + \bar{n}\partial/\partial\xi\ \xi\partial/\partial\xi)\,F(\xi,t) \tag{1.31}$$

where $\xi = |z|^2$. The normalization (1.28) reduces to $\int_0^\infty d\xi F(\xi,t) = 1$. The Fokker-Planck equation (1.31) is solved by expanding the desired function $F(\xi,t)$ as (see Appendix A.2)

$$F(\xi,t) = \sum_{\ell=0}^{\infty} a_\ell R_\ell(\zeta)\mathrm{e}^{-2\kappa\ell t} \tag{1.32}$$

with the right-hand side eigen-function $R_\ell(\zeta)$ of (A.2) belonging to the eigen-value $\lambda = \ell$ ($\ell = 0, 1, 2, \cdots$). $R_\ell(\zeta)$ is related to the Laguerre polynomials $L_\ell(\zeta)$ satisfying the differential equation (A.5) by the relation

$$R_\ell(\zeta) = L_\ell(\zeta)\mathrm{e}^{-\zeta}. \tag{1.33}$$

Note that $L_\ell(\zeta)$ can be seen as the left-hand side eigen-function of (A.3) belonging to the eigen-value $\lambda = \ell$. The left and right eigen-functions form an ortho-normal complete set, i.e.,

$$\int_0^\infty d\zeta L_\ell(\zeta)R_{\ell'}(\zeta) = \delta_{\ell,\ell'}, \qquad \sum_{\ell=0}^{\infty} R_\ell(\zeta)L_\ell(\zeta') = \delta(\zeta-\zeta'). \tag{1.34}$$

It may be worthwhile to note here that the right-hand side eigen-functions $R_\ell(\zeta)$ are of $L_2(\mathbf{R}_+)$, whereas the left-hand side eigen-functions $L_\ell(\zeta)$ are not. We may say that $R_\ell(\zeta)$ and $L_\ell(\zeta)$ belong, respectively, to the *nuclear space* and its *conjugate space* in the Gel'fand triplet (or the rigged Hilbert space) [31].

For the case of the initial condition (see Appendix C for its physical meaning)

$$F(\xi,0) = f_S(\xi,0) = \mathrm{e}^{-\xi/n}/n \tag{1.35}$$

the coefficient a_ℓ in (1.32) is obtained in the form $a_\ell = \int_0^\infty d\zeta' L_\ell(\zeta')\mathrm{e}^{-\zeta'\bar{n}/n}/n$. Substituting this into (1.32), we can derive the solution of (1.31) as (see Appendix A.3)

$$F(\xi,t) = \mathrm{e}^{-\xi/n(t)}/n(t), \qquad n(t) = \bar{n} + (n-\bar{n})\,\mathrm{e}^{-2\kappa t}. \tag{1.36}$$

Note that $n(t)$ satisfies the Boltzmann equation (1.24) with the initial condition $n(0) = n$.

1.4.3 Stochastic Differential Equations within the C-number Function Space

Langevin Equation

The dynamics given by the Fokker-Planck equation (1.27) can be described by the Langevin equations within the c-number function space

$$dz(t) = -i\omega z(t) - \kappa z(t)dt + dW(t), \qquad dz(t)^* = i\omega z(t)^* - \kappa z(t)^* dt + dW(t)^* \tag{1.37}$$

where the random process $dW(t)$ is supposed to satisfy

$$\langle dW(t)\rangle = 0, \qquad \langle dW(t)dW(t)^*\rangle = 2\kappa\bar{n}dt. \tag{1.38}$$

The average $\langle\cdots\rangle$, here, indicates to take a random average with respect to the random process $dW(t)$. The latter equation in (1.38) is the fluctuation dissipation theorem of the second kind.

Stochastic Liouville Equation

The stochastic Liouville equation of the Stratonovich type for the same system is given by [32, 33, 34]

$$df(z,t) = \Omega(z,t)dt \circ f(z,t), \qquad \Omega(z,t)dt = -(\partial/\partial z^*\ dz^* + \partial/\partial z\ dz) \tag{1.39}$$

with $f(z,t) = f(z,t;\Omega(z,t)dt, P(z,0))$, the symbol of the Stratonovich product "$\circ$" and

$$dz = -i\omega z dt - \kappa z dt + dW(t), \qquad dz^* = i\omega z^* dt - \kappa z^* dt + dW(t)^*. \tag{1.40}$$

The stochastic Liouville equation of the Ito type for (1.39) has the expression (see Appendix A.4)

$$df(z,t) = \Omega(z,t)dt\cdot f(z,t) + 2\kappa\bar{n}\ \partial^2/\partial z^*\partial z\ f(z,t) \tag{1.41}$$

with the symbol of the Ito product "$\cdot$".

The easiest way to obtain the Fokker-Planck equation (1.27) for $P(z,t) = \langle f(z,t;\Omega(z,t)dt, P(z,0))\rangle$ is realized by averaging the stochastic Liouville equation of the Ito type (1.41) over all possibilities of $\Omega(z,t)dt$ with respect to the stochastic process $dW(t)$.

Comments

As can be seen from (1.39), $f(z,t)$ satisfies the conservation of probability within the relevant system: $\int d^2z/\pi\ f(z,t) = 1$. Note that the Langevin equation (1.37) of the Stratonovich type does *not* contain the diffusion term. The stochastic differential equation of the Stratonovich type [21] allows us to proceed calculation as if the stochastic function $z(t)$ were an analytic one. The fluctuation-dissipation theorem of the second kind given in (1.38) is introduced in order that the Langevin equation (1.37) is consistent with the Fokker-Planck equation (1.27).

2 Canonical Operator Formalism for Dissipative Systems

2.1 Basics of NETFD

Let us list up first the technical basics of NETFD.

Tool 1. Any operator A in NETFD is accompanied by its partner (tilde) operator $\tilde{A}$. Here, the tilde conjugation $\sim$ is defined by

$$(A_1A_2)^{\sim} = \tilde{A}_1\tilde{A}_2, \quad (c_1A_1 + c_2A_2)^{\sim} = c_1^*\tilde{A}_1 + c_2^*\tilde{A}_2, \quad (\tilde{A})^{\sim} = A, \quad (A^\dagger)^{\sim} = \tilde{A}^\dagger. \tag{2.1}$$

Tool 2. Equal-time commutativity between the tilde and non-tilde operators: $[A,\ \tilde{B}] = 0$.

Tool 3. Thermal state condition for the thermal bra-vacuum $\langle 1|$: $\langle 1|A^\dagger = \langle 1|\tilde{A}$.

The general characteristics of the Liouville equation in sub-section 1.4 are rephrased in NETFD as follows. Dynamics of the system is described in NETFD by the Schrödinger equation (quantum master equation) ($\hbar = 1$)

$$\partial/\partial t\ |0(t)\rangle = -i\hat{H}\ |0(t)\rangle. \tag{2.2}$$

The property of the time-evolution generator (hat-Hamiltonian) is specified as follows.

B1. The hat-Hamiltonian $\hat{H}$ satisfies the characteristics named *tildian*: $(i\hat{H})^{\sim} = i\hat{H}$.
$\hat{H}$ is not necessarily hermitian operator.

B2. The hat-Hamiltonian has zero eigenvalue for the thermal bra-vacuum: $\langle 1|\hat{H} = 0$.
This is the manifestation of the conservation of probability, i.e. $\langle 1|0(t)\rangle = 1$.

B3. The thermal vacuums $\langle 1|$ and $|0\rangle$ are *tilde invariant*: $\langle 1|^{\sim} = \langle 1|$, $|0\rangle^{\sim} = |0\rangle$.
They are normalized as $\langle 1|0\rangle = 1$.

Within NETFD, the expectation value of an observable operator A is given by

$$\langle A\rangle_t = \langle 1|A|0(t)\rangle = \langle 1|A\hat{V}(t)|0\rangle = \langle 1|\hat{V}(t)^{-1}A\hat{V}(t)|0\rangle \tag{2.3}$$

where we used the property $\langle 1|\hat{H} = 0$ and the formal solution $|0(t)\rangle = \hat{V}(t)|0\rangle$ with the time-evolution operator $\hat{V}(t) = \mathrm{e}^{-i\hat{H}t}$ satisfying

$$d/dt\ \hat{V}(t) = -i\hat{H}\hat{V}(t) \tag{2.4}$$

with the initial condition $\hat{V}(0) = 1$. We see that the Heisenberg operator

$$A(t) = \hat{V}^{-1}(t)A\hat{V}(t) \tag{2.5}$$

satisfies the Heisenberg equation for dissipative systems

$$d/dt\ A(t) = i[\hat{H}(t),\ A(t)] \tag{2.6}$$

with $\hat{H}(t) = \hat{V}(t)^{-1}\hat{H}\hat{V}(t)$. Note that $\hat{V}(t)^{\sim} = \hat{V}(t)$.

The equation of motion for the averaged quantity $\langle 1|A(t)|0\rangle$ is derived by means of the Heisenberg equation (2.6) by taking its vacuum expectation:

$$d/dt\ \langle 1|A(t)|0\rangle = i\langle 1|[\hat{H}(t),\ A(t)]|0\rangle. \tag{2.7}$$

The same equation can be also derived with the help of the master equation (2.2) as

$$d/dt\ \langle 1|A|0(t)\rangle = -i\langle 1|A\hat{H}|0(t)\rangle. \tag{2.8}$$

We emphasize here that the existence of the Heisenberg equation of motion (2.6) for coarse grained operators is one of the notable features of NETFD. This enabled us to construct a *canonical formalism of the dissipative quantum field theory*, where the coarse grained operators $a(t)$ etc. in the Heisenberg representation preserve the equal-time canonical commutation relation

$$[a(t),\ a^\dagger(t)] = 1, \qquad [\tilde{a}(t),\ \tilde{a}^\dagger(t)] = 1. \tag{2.9}$$

2.2 Dissipative Unstable Particles (Semi-free Fields)

Quantum Master Equation

Let us consider the quantum master equation (2.2) with the hat-Hamiltonian

$$\hat{H} = \hat{H}_S + i\hat{\Pi}, \qquad \hat{H}_S = \omega \left(a^\dagger a - \tilde{a}^\dagger \tilde{a} \right), \tag{2.10}$$

$$\hat{\Pi} = -\kappa \left[(1+2\bar{n}) \left(a^\dagger a + \tilde{a}^\dagger \tilde{a} \right) - 2(1+\bar{n})\, a\tilde{a} - 2\bar{n} a^\dagger \tilde{a}^\dagger \right] - 2\kappa\bar{n} \tag{2.11}$$

which can be derived axiomatically [3] or by means of the principle of correspondence [1, 2] (see Appendix A.5.1).[4]

Note that, from the Heisenberg equation (2.6) with (2.10), we have an equation of motion for the vector $\langle 1|A(t)$:

$$\begin{aligned} d/dt\ \langle 1|A(t) &= i\langle 1|[\hat{H}(t),\ A(t)] = i\langle 1|[H_S(t),\ A(t)] \\ &-\kappa \left\{ \langle 1|[A(t),\ a^\dagger(t)]a(t) + \langle 1|a^\dagger(t)[a(t),\ A(t)] \right\} + 2\kappa\bar{n}\langle 1|[a(t),\ [A(t),\ a^\dagger(t)]] \end{aligned} \tag{2.13}$$

written in terms of non-tilde operators only with the help of **Tool 3**, i.e., $\langle 1|\tilde{a}(t) = \langle 1|a^\dagger(t)$. Applying the ket-vacuum $|0\rangle$ to (2.13), we obtain the equation of motion for the averaged quantity (2.7).

Thermal Doublet

Let us introduce the thermal doublet notation by $a(t)^{\mu=1} = a(t)$, $a(t)^{\mu=2} = \tilde{a}^\dagger(t)$, $\bar{a}(t)^{\mu=1} = a^\dagger(t)$ and $\bar{a}(t)^{\mu=2} = -\tilde{a}(t)$. Then, the canonical commutation relation (2.9) can be written as $[a(t)^\mu,\ \bar{a}(t)^\nu] = \delta^{\mu\nu}$ with $a(t)^\mu = \hat{V}^{-1}(t) a^\mu \hat{V}(t)$ and $\bar{a}(t)^\mu = \hat{V}^{-1}(t)\bar{a}^\mu \hat{V}(t)$. Making use of the thermal doublet notation, the hat-Hamiltonian (2.10) reduces to

$$\hat{H} = \omega \bar{a}^\mu a^\mu + i\hat{\Pi} + \omega, \quad \hat{\Pi} = -\kappa \bar{a}^\mu A^{\mu\nu} a^\nu + \kappa, \quad A^{\mu\nu} = \begin{pmatrix} 1+2\bar{n} & -2\bar{n} \\ 2(1+\bar{n}) & -(1+2\bar{n}) \end{pmatrix}, \tag{2.14}$$

and the Heisenberg equations for the semi-free particle become

$$d/dt\ a(t)^\mu = i[\hat{H}(t),\ a(t)^\mu] = -i(\omega\delta^{\mu\nu} - i\kappa A^{\mu\nu}) a(t)^\nu, \tag{2.15}$$

$$d/dt\ \bar{a}(t)^\mu = i[\hat{H}(t),\ \bar{a}(t)^\mu] = \bar{a}(t)^\nu i(\omega\delta^{\nu\mu} - i\kappa A^{\nu\mu}). \tag{2.16}$$

Annihilation and Creation Operators

Let us introduce the annihilation and creation operators $\gamma(t)^{\mu=1} = \gamma(t)$, $\gamma(t)^{\mu=2} = \tilde{\gamma}^{\natural}(t)$, $\bar{\gamma}(t)^{\mu=1} = \gamma^{\natural}(t)$ and $\bar{\gamma}(t)^{\mu=2} = -\tilde{\gamma}(t)$ by

$$\gamma(t)^\mu = B(t)^{\mu\nu} a(t)^\nu, \quad \bar{\gamma}(t)^\mu = \bar{a}(t)^\nu B^{-1}(t)^{\nu\mu}, \quad B(t)^{\mu\nu} = \begin{pmatrix} 1+n(t) & -n(t) \\ -1 & 1 \end{pmatrix} \tag{2.17}$$

with the *time-dependent Bogoliubov transformation* $B(t)^{\mu\nu}$ where $n(t)$ is the one-particle distribution function satisfying the Boltzmann equation (1.24).

The annihilation and creation operators satisfy the canonical commutation relation $[\gamma(t)^\mu,\ \bar{\gamma}(t)^\nu] = \delta^{\mu\nu}$, and annihilate the bra- and ket-vacuums at the initial time:

$$\gamma(t)|0\rangle\ = 0, \qquad \langle 1|\tilde{\gamma}^{\natural}(t) = 0. \tag{2.18}$$

The equation of motion for the thermal doublet $\gamma(t)^\mu$ has the form (see appendix A.7)

$$d/dt\ \gamma(t)^\mu = -i \left[\omega\delta^{\mu\nu} - i\kappa\tau_3^{\mu\nu} \right] \gamma(t)^\nu \tag{2.19}$$

where the matrix $\tau_3^{\mu\nu}$ is defined by $\tau_3^{11} = -\tau_3^{22} = 1,\ \tau_3^{12} = \tau_3^{21} = 0$. The solution of (2.19) is given by

$$\gamma(t)^\mu = \exp\left\{ -i \left(\omega\delta^{\mu\nu} - i\kappa\tau_3^{\mu\nu} \right) (t-t') \right\} \gamma(t')^\nu. \tag{2.20}$$

Two-Point Function (Propagator)

The time-ordered two-point function $G(t,t')^{\mu\nu}$ has the form

$$G(t,t')^{\mu\nu} = -i\langle 1|T \left[a(t)^\mu \bar{a}(t')^\nu \right] |0\rangle = \left[B^{-1}(t)\mathcal{G}(t,t')B(t') \right]^{\mu\nu} \tag{2.21}$$

[4] We are dealing with the case in which the initial ket-vacuum $|0\rangle = |0(t=0)\rangle$ is specified by $a|0\rangle = f\tilde{a}^\dagger|0\rangle$ with a real quantity f. Here, we are neglecting the *initial correlation* [35]. The initial condition of the one-particle distribution function $n = n(t=0)$ is related to f by the relation (see Appendix A.6)

$$n = f/(1-f). \tag{2.12}$$

where

$$\mathcal{G}(t,t')^{\mu\nu} = -i\langle 1|T\left[\gamma(t)^{\mu}\bar{\gamma}(t')^{\nu}\right]|0\rangle = \begin{pmatrix} G^R(t,t') & 0 \\ 0 & G^A(t,t') \end{pmatrix}, \tag{2.22}$$

$$G^R(t,t') = -i\theta(t-t')\mathrm{e}^{-i(\omega-i\kappa)(t-t')}, \qquad G^A(t,t') = i\theta(t'-t)\mathrm{e}^{-i(\omega+i\kappa)(t-t')}. \tag{2.23}$$

In deriving the above expression, we used the elements of the solution (2.20) with some algebraic manipulations. For example,

$$\begin{aligned}\mathcal{G}(t,t')^{11} &= -i\langle 1|T[\gamma(t)\gamma(t')^{\dagger}]|0\rangle = -i\left[\theta(t-t')\langle 1|\gamma(t)\gamma^{\dagger}(t')|0\rangle + \theta(t'-t)\langle 1|\gamma^{\dagger}(t')\gamma(t)|0\rangle\right] \\ &= -i\theta(t-t')\mathrm{e}^{-i(\omega-i\kappa)(t-t')} = G^R(t,t').\end{aligned} \tag{2.24}$$

In the third equality, we used

$$\langle 1|\gamma(t)\gamma^{\dagger}(t')|0\rangle = \mathrm{e}^{-i(\omega-i\kappa)(t-t')}\langle 1|\gamma(t')\gamma^{\dagger}(t')|0\rangle = \mathrm{e}^{-i(\omega-i\kappa)(t-t')}, \tag{2.25}$$

$$\langle 1|\tilde{\gamma}(t')\tilde{\gamma}^{\dagger}(t)|0\rangle = \langle 1|\gamma(t')\gamma^{\dagger}(t)|0\rangle^{\sim} = \mathrm{e}^{-i(\omega+i\kappa)(t-t')}. \tag{2.26}$$

Miscellaneous

The representation space (thermal space) of NETFD is the vector space spanned by the set of bra and ket state vectors which are generated, respectively, by cyclic operations of the annihilation operators $\gamma(t)$ and $\tilde{\gamma}(t)$ on $\langle 1|$, and of the creation operators $\gamma^{\dagger}(t)$ and $\tilde{\gamma}^{\dagger}(t)$ on $|0\rangle$.

The normal product is defined by means of the annihilation and the creation operators, i.e. $\gamma^{\dagger}(t)$, $\tilde{\gamma}^{\dagger}(t)$ stand to the left of $\gamma(t)$, $\tilde{\gamma}(t)$. The process, rewriting physical operators by means of the normal product with respect to the annihilation and creation operators, leads to a Wick-type formula, which in turn leads to Feynman-type diagrams for multi-point functions in the renormalized interaction representation. The internal line in the Feynman-type diagrams is the unperturbed two-point function (2.21).

Condensation of Particle Pairs

Introducing the annihilation and creation operators in the Schrödinger representation $\gamma_t^{\mu=1} = \gamma_t$, $\gamma_t^{\mu=2} = \tilde{\gamma}^{\dagger}$, $\bar{\gamma}_t^{\mu=1} = \gamma^{\dagger}$ and $\bar{\gamma}_t^{\mu=2} = -\tilde{\gamma}_t$ through the relation

$$\gamma(t)^{\mu} = \hat{V}^{-1}(t)\gamma_t^{\mu}\hat{V}(t), \qquad \bar{\gamma}(t)^{\mu} = \hat{V}^{-1}(t)\bar{\gamma}_t^{\mu}\hat{V}(t) \tag{2.27}$$

with $\hat{V}(t)$ being specified by (2.4), we can rewrite the hat-Hamiltonian (2.10) as

$$\hat{H} = \omega\left(\gamma^{\dagger}\gamma_t - \tilde{\gamma}^{\dagger}\tilde{\gamma}_t\right) + i\hat{\Pi}, \qquad \hat{\Pi} = -\kappa\left(\gamma^{\dagger}\gamma_t + \tilde{\gamma}^{\dagger}\tilde{\gamma}_t + 2\left[n(t)-\bar{n}\right]\gamma^{\dagger}\tilde{\gamma}^{\dagger}\right). \tag{2.28}$$

It is easy to see from this normal product form of $\hat{H}$ that it satisfies **B2** in sub-section 2.1, since the annihilation and creation operators satisfy

$$\gamma_t|0(t)\rangle = 0, \qquad \langle 1|\tilde{\gamma}^{\dagger} = 0. \tag{2.29}$$

Substituting (2.28) into the quantum master equation (2.2), we have

$$\partial/\partial t\ |0(t)\rangle = -2\kappa\left[n(t)-\bar{n}\right]\gamma^{\dagger}\tilde{\gamma}^{\dagger}|0(t)\rangle = dn(t)/dt\ \gamma^{\dagger}\tilde{\gamma}^{\dagger}|0(t)\rangle. \tag{2.30}$$

It is solved to give

$$|0(t)\rangle = \exp\left[\int_0^t dt'\frac{dn(t')}{dt'}\gamma^{\dagger}\tilde{\gamma}^{\dagger}\right]|0\rangle = \exp\left[\left[n(t)-n(0)\right]\gamma^{\dagger}\tilde{\gamma}^{\dagger}\right]|0\rangle. \tag{2.31}$$

This expression tells us that the time-evolution of the unstable vacuum is realized by the condensation of $\gamma_k^{\dagger}\tilde{\gamma}_k^{\dagger}$-pairs into the vacuum. The attractive expression (2.31), which was obtained first in [36], led us to the notion of a mechanism named the *spontaneous creation of dissipation* [37]. We can obtain the result (2.31) only by algebraic manipulations. This technical convenience of the operator algebra in NETFD, which is very much similar to that of the usual quantum mechanics and quantum field theory, enables us to treat dissipative quantum systems simpler and more transparent [8, 9].

The expression (2.30) also shows that the vacuum is a functional of the one-particle distribution function $n_k(t)$. The dependence of the thermal vacuum on $n_k(t)$ is given by

$$\delta/\delta n_k(t)\ |0(t)\rangle = \gamma_k^{\dagger}\tilde{\gamma}_k^{\dagger}|0(t)\rangle. \tag{2.32}$$

We see that the vacuum $|0(t)\rangle$ represents the state containing the macroscopic object described by the one-particle distribution function $n_k(t)$. The quantum master equation (2.2) can be rewritten as

$$\left[\frac{\partial}{\partial t} - \int d^3k\ \frac{dn_k(t)}{dt}\frac{\delta}{\delta n_k(t)}\right]|0(t)\rangle = 0. \tag{2.33}$$

This shows that the unstable vacuum, in this case, is migrating in the super-representation space spanned by the one-particle distribution function $\{n_k(t)\}$ with the *velocity* $\{dn_k(t)/dt\}$ as a conserved quantity.

Diagonal Representation

The hat-Hamiltonian (2.10) can be also written in the form

$$\hat{H} = \omega(d^\dagger d - \tilde{d}^\dagger \tilde{d}) - i\kappa(d^\dagger d + \tilde{d}^\dagger \tilde{d}) \tag{2.34}$$

where $d^{\mu=1} = d$, $d^{\mu=2} = \tilde{d}^\dagger$, $\bar{d}^{\mu=1} = d^\dagger$ and $\bar{d}^{\mu=2} = -\tilde{d}$ are defined through the Bogoliubov transformation

$$d^\mu = \bar{B}^{\mu\nu} a^\nu, \quad \bar{d}^\mu = \bar{a}^\nu \bar{B}^{-1\nu\mu}, \quad \bar{B}^{\mu\nu} = \begin{pmatrix} 1+\bar{n} & -\bar{n} \\ -1 & 1 \end{pmatrix}. \tag{2.35}$$

The initial ket-thermal vacuum specified by $a|0\rangle = f\tilde{a}^\dagger|0\rangle$ can be expressed in terms of d and $\tilde{d}^\dagger$ as $d|0\rangle = (n-\bar{n})\ \tilde{d}^\dagger|0\rangle$. It is easy to see from the diagonalized form (2.34) of $\hat{H}$ that

$$d(t) = \hat{V}^{-1}(t)\ d\ \hat{V}(t) = d\ e^{-(i\omega+\kappa)t}, \qquad \tilde{d}^{\dagger\!\dagger}(t) = \hat{V}^{-1}(t)\ \tilde{d}^\dagger\ \hat{V}(t) = \tilde{d}^\dagger\ e^{-(i\omega-\kappa)t}. \tag{2.36}$$

The difference between the operators which diagonalize $\hat{H}$ and the ones which make $\hat{H}$ in the form of normal product is one of the features of NETFD, and shows the point that the formalism is quite different from usual quantum mechanics and quantum field theory. This is a manifestation of the fact that the hat-Hamiltonian is a time-evolution generator for irreversible processes. In thermal equilibrium state $n(t) = \bar{n}$, they coincide.

Irreversibility

Let us check here the irreversibility of the system. The entropy of the system is given by

$$\mathcal{S}(t) = -\left\{n(t)\ln n(t) - [1+n(t)]\ln[1+n(t)]\right\}, \tag{2.37}$$

whereas the heat change of the system by

$$d'Q = \omega dn. \tag{2.38}$$

Thermodynamics tells us that

$$d\mathcal{S} = d\mathcal{S}_e + d\mathcal{S}_i, \qquad d\mathcal{S}_e = d'Q/T_R, \qquad d\mathcal{S}_i \geq 0. \tag{2.39}$$

The latter inequality in (2.39) is the second law of thermodynamics. Putting (2.37) and (2.38) into (2.39), for $d\mathcal{S}$ and $d\mathcal{S}_e$, respectively, we have a relation for the entropy production rate [38]

$$d/dt\ \mathcal{S}_i(t) = d/dt\ \mathcal{S}(t) - d/dt\ \mathcal{S}_e(t) = 2\kappa\,[n(t)-\bar{n}]\ln\{n(t)[1+\bar{n}]/\bar{n}[1+n(t)]\} \geq 0. \tag{2.40}$$

It is easy to check that the expression on the right-hand side of the second equality satisfies the last inequality which is consistent with the last equation in (2.39). The equality realizes either for the thermal equilibrium state $n(t) = \bar{n}$, or for the quasi-stationary process $\kappa \to 0$.

3 Interaction Hat-Hamiltonian

3.1 Hermitian Interaction Hat-Hamiltonian

The simplest hermitian interaction hat-Hamiltonian may be given by

$$\hat{H}_t' = H_t' - \tilde{H}_t', \qquad H_t' = i\left[a^\dagger b(t) - b^\dagger(t) a\right] \tag{3.1}$$

where $b(t)$ and $b^\dagger(t)$ are operators in the external system and are assumed to commute with the operators a, $a^\dagger$ etc. of the relevant system. The tilde and non-tilde operators of the external system are related with each other by $\langle|b^\dagger(t) = \langle|\tilde{b}(t)$.

Applying the bra-vacuum $\langle 1|$ for the relevant system on (3.1), we have

$$\langle 1|\hat{H}_t' = -i\langle 1|[a\beta^{\natural}(t) + a^\dagger\tilde{\beta}^{\natural}(t)]. \tag{3.2}$$

Here, we introduced a new operator

$$\beta^{\natural}(t) = b^\dagger(t) - \tilde{b}(t) \tag{3.3}$$

which annihilates the bra-vacuum $\langle|$ for the external system, i.e., $\langle|\beta^{\natural} = 0$. If we apply the bra-vacuum $\langle|$ on (3.1) in addition to $\langle 1|$, we observe that $\langle\!\langle 1|\hat{H}_t' = 0$ where we introduced $\langle\!\langle 1| = \langle|\cdot\langle 1|$.

The above investigation shows that a simple introduction of an interaction hat-Hamiltonian of the form (3.1) violates the conservation of probability within the relevant system. It can be understood by considering the quantum master equation

$$\partial/\partial t\ |0(t)\rangle = -i(\hat{H} + \hat{H}_t')|0(t)\rangle, \tag{3.4}$$

and apply $\langle 1|$. Note that the conservation of probability is satisfied for the total system, i.e., the relevant system and the external system, as can be seen by applying $\langle\!\langle 1|$ to (3.4).

3.2 Non-Hermitian Interaction Hat-Hamiltonian

Let us consider the case where the quantum master equation

$$\partial/\partial t\ |0(t)\rangle = -i(\hat{H} + \hat{H}_t'')|0(t)\rangle \tag{3.5}$$

satisfies the conservation of probability *within* the relevant system, i.e.,

$$\langle 1|\hat{H}_t'' = 0. \tag{3.6}$$

It is satisfied with the interaction hat-Hamiltonian (see appendix A.8)

$$\hat{H}_t'' = i[\gamma^{\dagger}\beta(t) + \text{t.c.}], \tag{3.7}$$

$$\gamma^{\dagger} = a^{\dagger} - \tilde{a}, \qquad \beta(t) = \mu b(t) + \nu \tilde{b}^{\dagger}(t), \qquad \mu + \nu = 1. \tag{3.8}$$

The creation operator $\gamma^{\dagger}$ annihilates the ket-vacuum $\langle 1|$, i.e., $\langle 1|\gamma^{\dagger} = 0$. The annihilation operator $\beta(t)$ satisfies the equal-time commutation relation

$$[\beta(t),\ \beta^{\dagger}(t)] = 1. \tag{3.9}$$

3.3 Relation between the Two Interaction Hat-Hamiltonian

Note that the hermitian hat-Hamiltonian $\hat{H}_t'$ of (3.1) and the non-hermitian one $\hat{H}_t''$ of (3.7) are related with each other by

$$\hat{H}_t' = \hat{H}_t'' - i[\gamma_\nu \beta^{\dagger}(t) + \text{t.c.}] \tag{3.10}$$

where we introduced

$$\gamma_\nu = \mu a + \nu \tilde{a}^{\dagger} \tag{3.11}$$

which forms a canonical set with $\gamma^{\dagger}$ in (3.8), i.e., $[\gamma_\nu,\ \gamma^{\dagger}] = 1$. Here t.c. stands for tilde conjugation.

4 Unified System of Quantum Stochastic Differential Equations

4.1 Quantum Brownian Motion (QBM)

QBM in Hilbert Space

Let us introduce b_t and $b_t^{\dagger}$ denoting, respectively, boson annihilation and creation operators at time $t \in [0, \infty)$ satisfying the canonical commutation relations $[b_t, b_s^{\dagger}] = \delta(t-s)$ and $[b_t, b_s] = [b_t^{\dagger}, b_s^{\dagger}] = 0$. The bra- and ket-vacuums $(|$ and $|)$ are defined, respectively, by $(|b_t^{\dagger} = 0$ and $b_t|) = 0$. Note that $(| = |)^{\dagger}$ since here we are considering the unitary representation of b_t and $b_t^{\dagger}$.

We see that the quantum Brownian motion defined by $B_t = \int_0^t dt'\ b_{t'}$ and $B_t^{\dagger} = \int_0^t dt'\ b_{t'}^{\dagger}$ with $B_0 = 0$ and $B_0^{\dagger} = 0$ has the characteristics of the Brownian motion [17], i.e., $[B_t,\ B_s^{\dagger}] = \min(t, s)$. The increments $dB_t = B_{t+dt} - B_t = b_t dt$ and $dB_t^{\dagger} = B_{t+dt}^{\dagger} - B_t^{\dagger} = b_t^{\dagger} dt$ annihilate the vacuum, i.e., $(|dB_t^{\dagger} = 0$ and $dB_t|) = 0$, which are consistent with the definition of random force, i.e., $(|dB_t|) = (|dB_t^{\dagger}|) = 0$. The boson Brownian motion is specified by the multiplication formulae within the *weak relation*, expecting that the vacuum expectation is taken later on, for the increments dB_t and $dB_t^{\dagger}$ in the form $dB_t\ dB_t^{\dagger} = dt$, while other multiplications are zero.

QBM in Thermal Space

Let us introduce operators b_t, $b_t^{\dagger}$ and their tilde conjugates which satisfy the commutation relations among them, i.e., $[b_t, b_s^{\dagger}] = [\tilde{b}_t, \tilde{b}_s^{\dagger}] = \delta(t-s)$ and $[b_t, \tilde{b}_s] = [b_t, \tilde{b}_s^{\dagger}] = 0$.

Thermal degrees of freedom can be introduced by the Bogoliubov transformation under the demand that the expectation value of $b_t^{\dagger} b_s$ should be $\langle b_t^{\dagger} b_s \rangle = \bar{n}\delta(t-s)$ with $\bar{n} \geq 0$ which is consistent with the thermal state conditions $\langle|\tilde{b}_t^{\dagger} = \langle|b_t$ and $\tilde{b}_t|\rangle = [\bar{n}/(1+\bar{n})]\, b_t^{\dagger}|\rangle$. Here, $\langle\cdots\rangle = \langle|\cdots|\rangle$ indicates an expectation with respect to the tilde invariant thermal vacuums satisfying $\langle|^{\sim} = \langle|$ and $|\rangle^{\sim} = |\rangle$.

Now, we introduce new operators c_t, $\tilde{c}_t^{\dagger}$ and their tilde conjugates, which annihilate the thermal vacuums, i.e., $\langle|c_t^{\dagger} = \langle|\tilde{c}_t^{\dagger} = 0$ and $c_t|\rangle = \tilde{c}_t|\rangle = 0$, through the Bogoliubov transformation

$$c_t^{\mu} = \bar{B}^{\mu\nu} b_t^{\nu}, \qquad \bar{c}_t^{\mu} = \bar{b}_t^{\nu} \left(\bar{B}^{-1}\right)^{\nu\mu} \tag{4.1}$$

with (2.35). Here, we used the thermal doublet notations $c_t^{\mu=1} = c_t$, $c_t^{\mu=2} = \tilde{c}_t^{\dagger}$, $\bar{c}_t^{\mu=1} = c_t^{\dagger}$, $\bar{c}_t^{\mu=2} = \tilde{c}_t$ and $b_t^{\mu=1} = b_t$, $b_t^{\mu=2} = \tilde{b}_t^{\dagger}$, $\bar{b}_t^{\mu=1} = b_t^{\dagger}$, $\bar{b}_t^{\mu=2} = \tilde{b}_t$. This new operators satisfy the canonical commutation relations $[c_t, c_s^{\natural}] = \delta(t-s)$. In the following, we will use the representation space constructed on thermal vacuums $\langle|$ and $|\rangle$.

With the increments $dC_t^\sharp = c_t^\sharp dt$, $d\tilde{C}_t^\sharp = \tilde{c}_t^\sharp dt$ and $dB_t^\sharp$, $d\tilde{B}_t^\sharp$ for the Quantum Brownian motions with $\sharp$ standing for null or $\natural$ or $\dagger$, the Bogoliubov transformation (4.1) reduces to $dB_t = dC_t + \bar{n} d\tilde{C}_t^{\natural}$, $d\tilde{B}_t^\dagger = (1+\bar{n}) d\tilde{C}_t^{\natural} + dC_t$ and their tilde conjugates. Since the moments of $dC_t^\sharp$ and $d\tilde{C}_t^\sharp$ in thermal space satisfy $\langle dC_t\rangle = \langle d\tilde{C}_t\rangle = \langle dC_t^{\natural}\rangle = \langle d\tilde{C}_t^{\natural}\rangle = 0$, $\langle dC_t^{\natural} dC_t\rangle = \langle d\tilde{C}_t^{\natural} d\tilde{C}_t\rangle = 0$ and $\langle dC_t dC_t^{\natural}\rangle = \langle d\tilde{C}_t d\tilde{C}_t^{\natural}\rangle = dt$, the calculation of moments of quantum Brownian motion in the thermal space can be performed, for instance, as $\langle dB_t dB_t^\dagger\rangle = \langle (dC_t + \bar{n} d\tilde{C}_t^{\natural})[(1+\bar{n}) dC_t^{\natural} + d\tilde{C}_t]\rangle = (1+\bar{n})\langle dC_t dC_t^{\natural}\rangle = (1+\bar{n})dt$. We finally arrive at the *weak relations*

$$dB_t^\dagger\, dB_t = \bar{n} dt, \quad dB_t\, dB_t^\dagger = (\bar{n}+1)\, dt, \quad d\tilde{B}_t\, dB_t = \bar{n} dt, \quad d\tilde{B}_t^\dagger\, dB_t^\dagger = (\bar{n}+1)\, dt \tag{4.2}$$

and their tilde conjugates under the agreement that the expectation should be taken with the vacuum states $\langle|$ and $|\rangle$ representing the thermal quantum Brownian motion.

4.2 System of Quantum Stochastic Differential Equations

Quantum Stochastic Liouville Equation

Let us start the consideration with the stochastic Liouville equation of the Ito type:

$$d|0_f(t)\rangle = -i\hat{\mathcal{H}}_{f,t} dt\ |0_f(t)\rangle. \tag{4.3}$$

For the type of the stochastic multiplications, see Appendix D. The generator $\hat{V}_f(t)$, defined by $|0_f(t)\rangle = \hat{V}_f(t)|0\rangle$, satisfies

$$d\hat{V}_f(t) = -i\hat{\mathcal{H}}_{f,t} dt\ \hat{V}_f(t) \tag{4.4}$$

with $\hat{V}_f(0) = 1$. The stochastic hat-Hamiltonian $\hat{\mathcal{H}}_{f,t}dt$ is a tildian operator satisfying $(i\hat{\mathcal{H}}_{f,t}dt)^\sim = i\hat{\mathcal{H}}_{f,t}dt$. The thermal ket-vacuum is tilde invariant, i.e., $|0_f(t)\rangle^\sim = |0_f(t)\rangle$.

From the knowledge of the stochastic integral, we know that the required form of the hat-Hamiltonian should be

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}dt + :d\hat{M}_t:, \quad \hat{H} = \hat{H}_S + i\hat{\Pi}, \quad \hat{H}_S = H_S - \tilde{H}_S, \quad \hat{\Pi} = \hat{\Pi}_R + \hat{\Pi}_D \tag{4.5}$$

where $\hat{\Pi}_R$ and $\hat{\Pi}_D$ are, respectively, the *relaxational* and the *diffusive* parts of the damping operator $\hat{\Pi}$. The martingale $d\hat{M}_t$ is the term containing *linearly* the operators representing the quantum Brownian motion dB_t, $d\tilde{B}_t^\dagger$ and their tilde conjugates, and satisfies $\langle| :d\hat{M}_t: |\rangle = 0$. The symbol $:d\hat{M}_t:$ indicates to take the normal ordering with respect to the annihilation and the creation operators both in the relevant and the irrelevant systems (see (5.2) below). It is assumed that, at $t = 0$, a relevant system starts to contact with the irrelevant system representing the stochastic process included in the martingale $d\hat{M}_t$.[5]

Taking the random average by applying the bra-vacuum $\langle|$ of the irrelevant sub-system to (4.3), we can obtain the quantum master equation (2.2) with $\hat{H}dt = \langle|\hat{\mathcal{H}}_{f,t}dt|\rangle$ and $|0(t)\rangle = \langle|0_f(t)\rangle$.

Quantum Langevin Equation

The dynamical quantity $A(t)$ of the relevant system is defined by the operator in the Heisenberg representation, $A(t) = \hat{V}_f^{-1}(t)\ A\ \hat{V}_f(t)$, where $\hat{V}_f^{-1}(t)$ satisfies (see Appendix A.9)

$$d\hat{V}_f^{-1}(t) = \hat{V}_f^{-1}(t)\ i\hat{\mathcal{H}}_{f,t}^{-}dt, \qquad \hat{\mathcal{H}}_{f,t}^{-}dt = \hat{\mathcal{H}}_{f,t}dt + i :d\hat{M}_t::d\hat{M}_t: . \tag{4.6}$$

Within NETFD, the Heisenberg equation of the Ito type for $A(t)$ is the quantum Langevin equation of the form (see Appendix A.10)

$$dA(t) = i[\hat{\mathcal{H}}_f(t)dt,\ A(t)] - :d'\hat{M}(t):\ [:d'\hat{M}(t):,\ A(t)], \tag{4.7}$$

$$\hat{\mathcal{H}}_f(t)dt = \hat{V}_f^{-1}(t)\ \hat{\mathcal{H}}_{f,t}dt\ \hat{V}_f(t), \qquad d'\hat{M}(t) = \hat{V}_f^{-1}(t)\ d\hat{M}_t\ \hat{V}_f(t). \tag{4.8}$$

The martingale $d'\hat{M}(t)$ satisfies $\langle| :d'\hat{M}(t): |\rangle = 0$, and includes the operators of the quantum Brownian motion of the form $d'B(t) = \hat{V}_f^{-1}(t)\ dB_t\ \hat{V}_f(t)$, $d'\tilde{B}^\dagger(t) = \hat{V}_f^{-1}(t)\ d\hat{B}_t\ \hat{V}_f(t)$ and their tilde conjugates. Since $A(t)$ is an arbitrary observable operator in the relevant system, (4.7) can be the Ito formula generalized to quantum systems.

[5] Within the formalism, the random force operators dB_t and $dB_t^\dagger$ are assumed to commute with any relevant system operator A in the Schrödinger representation: $[A,\ dB_t] = [A,\ dB_t^\dagger] = 0$ for $t \geq 0$.

Applying the bra-vacuum $\langle\!\langle 1| = \langle|\langle 1|$ to (4.7) from the left, we obtain the Langevin equation for the bra-vector $\langle\!\langle 1|A(t)$ in the form

$$d\langle\!\langle 1|A(t) = i\langle\!\langle 1|[H_S(t),\ A(t)]dt + \langle\!\langle 1|A(t)\hat{\Pi}(t)dt - i\langle\!\langle 1|A(t)\ \ {:}d'\hat{M}(t){:}\ . \tag{4.9}$$

In the derivation, use had been made of the properties $\langle 1|\tilde{A}^\dagger(t) = \langle 1|A(t)$, $\langle|d'\tilde{B}^\dagger(t) = \langle|d'B(t)$ and $\langle\!\langle 1|d'\hat{M}(t) = 0$.

By making use of the relation between the Ito and Stratonovich stochastic calculus, we can rewrite the Ito stochastic Liouville equation (4.3) and the Ito Langevin equation (4.7) into the Stratonovich ones, respectively, i.e., (see Appendix A.11)

$$d|0_f(t)\rangle = -i\hat{H}_{f,t}dt \circ |0_f(t)\rangle, \qquad \hat{H}_{f,t}dt = \hat{H}_S dt + i(\hat{\Pi}dt + {:}d\hat{M}_t{:}\ {:}d\hat{M}_t{:}\ /2) + d\hat{M}_t, \tag{4.10}$$

$$dA(t) = i[\hat{H}_f(t)dt \overset{\circ}{;} A(t)], \quad \hat{H}_f(t)dt = \hat{H}_S(t)dt + i[\hat{\Pi}(t)dt + {:}d'\hat{M}(t){:}\ {:}d'\hat{M}(t){:}\ /2] + {:}d'\hat{M}(t){:}\ . \tag{4.11}$$

Fluctuation-Dissipation Relation

The fluctuation-dissipation theorem of the second kind for the multiple of martingales, ${:}d\hat{M}_t{:}\ {:}d\hat{M}_t{:}$, is determined by the criterion that the term $\hat{\Pi}dt + {:}d\hat{M}_t{:}\ {:}d\hat{M}_t{:}\ /2$ in $\hat{H}_{f,t}dt$ of (4.10) should not have a diffusive term, i.e.,

$${:}d\hat{M}_t{:}\ {:}d\hat{M}_t{:} = -2\hat{\Pi}_D dt. \tag{4.12}$$

The origin of this criterion is attributed to the way how the Langevin equation was introduced in physics, i.e., relaxation term and random force term were introduced in mechanical equation within the Stratonovich calculus. Therefore, there is no dissipative terms in stochastic equations of the Stratonovich type. We adopted this criterion in quantum cases. The weak relation (4.12) may be called a generalized fluctuation-dissipation theorem of the second kind.

Heisenberg Operator for QBM

The Heisenberg operators of the quantum Brownian motion are defined by $B(t) = \hat{V}_f^{-1}(t)\ B_t\ \hat{V}_f(t)$, $B^\dagger(t) = \hat{V}_f^{-1}(t)\ B_t^\dagger\ \hat{V}_f(t)$ and their tilde conjugates. Their increments $dB^\#(t) = d(\hat{V}_f^{-1}(t)\ B_t^\#\ \hat{V}_f(t))$ with $\#$ being nul, $\dagger$ and/or tilde are given by (see Appendix A.12)

$$dB^\#(t) = d'B^\#(t) + i[{:}d'\hat{M}(t){:},\ d'B^\#(t)] = dB_t^\# + i[{:}d'\hat{M}(t){:},\ dB_t^\#]. \tag{4.13}$$

In the second equality, we used the property $d'B^\#(t) = dB_t^\#$ which is due to the commutativity $[dB_t^\#;\ \hat{V}_f(t)] = 0$. For the increment of the martingale operator $d\hat{M}(t) = d(\hat{V}_f^{-1}(t)\hat{M}_t\hat{V}_f(t))$ in the Heisenberg representation, we obtain, by the similar process as (4.13), the important relation

$${:}d\hat{M}(t){:} = {:}d'\hat{M}(t){:} + i[{:}d'\hat{M}(t){:},\ {:}d'\hat{M}(t){:}] = {:}d'\hat{M}(t){:}, \tag{4.14}$$

which shows that the martingale operator in the Heisenberg representation satisfies the condition for the martingale operator, i.e., $\langle|{:}d\hat{M}(t){:}|\rangle = 0$.

It may be worthy to note that

$$d'B^\#(t) = dB_t^\# = \hat{V}_f^{-1}(t) \circ dB_t^\# \circ \hat{V}_f(t) - (i/2)\ [{:}d'\hat{M}(t){:},\ d'B^\#(t)], \tag{4.15}$$

$${:}d'\hat{M}(t){:} = \hat{V}_f^{-1}(t) \circ {:}d\hat{M}_t{:} \circ \hat{V}_f(t). \tag{4.16}$$

5 Application to Stochastic Semi-Free Particle

Model

We will apply the formalism in section 4 to the model of a harmonic oscillator embedded in an environment with temperature T. The Hamiltonian H_S of the relevant system is given by $H_S = \omega a^\dagger a$ where a, $a^\dagger$ and their tilde conjugates are stochastic operators of the relevant system satisfying the canonical commutation relation $[a,\ a^\dagger] = 1$ and $[\tilde{a},\ \tilde{a}^\dagger] = 1$. The tilde and non-tilde operators are related with each other by the relation $\langle 1|a^\dagger = \langle 1|\tilde{a}$ where $\langle 1|$ is the thermal bra-vacuum of the relevant system.

Since we are interested in the system of stochastic semi-free particles, we will confine ourselves to the case where the stochastic hat-Hamiltonian $\hat{\mathcal{H}}_t dt$ is bi-linear in a, $a^\dagger$, dB_t, $dB_t^\dagger$ and their tilde conjugates, and is invariant under the phase transformation $a \to a\mathrm{e}^{i\theta}$, and $dB_t \to dB_t\ \mathrm{e}^{i\theta}$. This gives us the system of linear-dissipative coupling. Then, we find that $\hat{\Pi}_R$ and $\hat{\Pi}_D$ consisting of $\hat{\Pi}$ introduced in (2.28) have, respectively, the expressions

$$\hat{\Pi}_R = -\kappa(\gamma^{\overset{\circ}{\dagger}}\gamma_\nu + \tilde{\gamma}^{\overset{\circ}{\dagger}}\tilde{\gamma}_\nu), \qquad \hat{\Pi}_D = 2\kappa(\bar{n} + \nu)\gamma^{\overset{\circ}{\dagger}}\tilde{\gamma}^{\overset{\circ}{\dagger}} \tag{5.1}$$

where we introduced a set of canonical stochastic operators $\gamma_\nu = \mu a + \nu \tilde{a}^\dagger$ and $\gamma^{\ddagger} = a^\dagger - \tilde{a}$ with $\mu + \nu = 1$ which satisfy the commutation relation $[\gamma_\nu,\ \gamma^{\ddagger}] = 1$. The parameter ν (or μ) is closely related to the ordering of operators when they are mapped to c-number function space with the help of the coherent state representation [6], i.e., $\nu = 1$ for the normal ordering, $\nu = 0$ for the anti-normal ordering and $\nu = 1/2$ for the Weyl ordering. The new operators $\gamma^{\ddagger}$ and $\tilde{\gamma}^{\ddagger}$ annihilate the relevant bra-vacuum, i.e., $\langle 1|\gamma^{\ddagger} = 0$ and $\langle 1|\tilde{\gamma}^{\ddagger} = 0$.

Martingale Operator

Let us adopt the martingale operator:

$$:d\hat{M}_t: = :d\hat{M}_t^{(-)}: \ + \lambda :d\hat{M}_t^{(+)}: \tag{5.2}$$

$$:d\hat{M}_t^{(-)}: = i(\gamma^{\ddagger} dW_t + \tilde{\gamma}^{\ddagger} d\tilde{W}_t), \qquad :d\hat{M}_t^{(+)}: = -i(dW_t^{\ddagger}\gamma_\nu + d\tilde{W}_t^{\ddagger}\tilde{\gamma}_\nu). \tag{5.3}$$

Here, the annihilation and creation random force operators dW_t and $dW_t^{\ddagger}$ are defined, respectively, by

$$dW_t = \sqrt{2\kappa}(\mu dB_t + \nu d\tilde{B}_t^\dagger), \qquad dW_t^{\ddagger} = \sqrt{2\kappa}(dB_t^\dagger - d\tilde{B}_t). \tag{5.4}$$

The latter annihilates the bra-vacuum $\langle|$ of the irrelevant system, i.e, $\langle|dW_t^{\ddagger} = 0$ and $\langle|d\tilde{W}_t^{\ddagger} = 0$. Note that the normal ordering $:\cdots:$ is defined with respect to γ's and dW's.

The real parameter λ measures the degree of non-commutativity among the martingale operators, i.e.,

$$[\ :d\hat{M}_t^{(-)}:,\ :d\hat{M}_t^{(+)}:\ ] = -2\hat{\Pi}_R dt. \tag{5.5}$$

In deriving this, we used the weak relations

$$dW_t\ d\tilde{W}_t = d\tilde{W}_t\ dW_t = 2\kappa\,(\bar{n} + \nu)\,dt, \qquad dW_t\ dW_t^{\ddagger} = d\tilde{W}_t\ d\tilde{W}_t^{\ddagger} = 2\kappa dt \tag{5.6}$$

and

$$[dW_t,\ dW_t^{\ddagger}] = 2\kappa dt \tag{5.7}$$

which should be compared with (1.4).

There exist at least two physically attractive cases [6, 23, 24], i.e., one is the case for $\lambda = 0$ giving non-hermitian martingale $d\hat{M}_t = i\sqrt{2\kappa}[(a^\dagger - \tilde{a})d\tilde{B}_t^\dagger + \text{t.c.}]$, and the other for $\lambda = 1$ giving hermitian martingale $d\hat{M}_t = i\sqrt{2\kappa}[(a^\dagger dB_t - dB_t^\dagger a) + \text{t.c.}]$. The former case follows the characteristics of the classical stochastic Liouville equation where the stochastic probability density function satisfies the conservation of probability within the phase-space of a relevant system (see section 1.4), i.e.,

$$\langle 1|d\hat{M}_t = 0 \tag{5.8}$$

which means that the stochastic Liouville equation preserves its probability just within the relevant system. Whereas the latter employs the characteristics of the Schrödinger equation in quantum mechanics where the norm of the stochastic wave function preserves itself. In this case, the consistency with the structure of classical system is destroyed [6, 23, 24].

The fluctuation-dissipation theorem of the system is given by

$$:d\hat{M}_t::d\hat{M}_t: = -2(\lambda\hat{\Pi}_R + \hat{\Pi}_D)dt \tag{5.9}$$

where we used the weak relations $:d\hat{M}_t^{(-)}::d\hat{M}_t^{(-)}: = -2\hat{\Pi}_D dt$, $:d\hat{M}_t^{(-)}::d\hat{M}_t^{(+)}: = -2\hat{\Pi}_R dt$ and $:d\hat{M}_t^{(+)}::d\hat{M}_t^{(+)}: = :d\hat{M}_t^{(+)}::d\hat{M}_t^{(-)}: = 0$ which can be derived by making use of (5.6).

For the present model, the hat-Hamiltonian for $\hat{V}_f^{-1}$, (4.6), is given by

$$\hat{\mathcal{H}}_{f,t}^{-}dt = \hat{H}_S dt + i((1-2\lambda)\hat{\Pi}_R - \hat{\Pi}_D)dt + d\hat{M}_t, \tag{5.10}$$

the one for the stochastic Liouville equation, (4.10), of the Stratonovich type by

$$\hat{H}_{f,t}dt = \hat{H}_S dt + i(1-\lambda)\hat{\Pi}_R dt + d\hat{M}_t \tag{5.11}$$

and the one for the Langevin equation, (4.11), by

$$\hat{H}_f(t)dt = \hat{H}_S(t)dt + i(1-\lambda)\hat{\Pi}_R(t)dt + :d'\hat{M}(t):. \tag{5.12}$$

Heisenberg Operators for QBM

The increments of the Heisenberg operators of the Quantum Brownian motion are given by

$$dB(t) = d'B(t) + \sqrt{2\kappa}\left[(1-\lambda)\,\nu\left(\tilde{a}^\dagger(t) - a(t)\right) - \lambda a(t)\right]dt, \tag{5.13}$$

$$dB^\dagger(t) = d'B^\dagger(t) - \sqrt{2\kappa}\left[(1-\lambda)\,\mu\left(a^\dagger(t) - \tilde{a}(t)\right) + \lambda a^\dagger(t)\right]dt \tag{5.14}$$

and their tilde conjugates. Then, we have

$$dW(t) = d'W(t) - \lambda 2\kappa\gamma_\nu(t)dt, \qquad dW^{\ddagger}(t) = d'W(t)^{\ddagger} - 2\kappa\gamma^{\ddagger}(t)dt. \tag{5.15}$$

Note that $d'B(t) = dB_t$, $d'B^\dagger(t) = dB_t^\dagger$, $d'W(t) = dW_t$ and $d'W(t)^{\ddagger} = dW_t^{\ddagger}$. By making use of (5.15), we see from (5.2) that

$$\begin{aligned} :d\hat{M}(t): &= i[\gamma^{\ddagger}(t)dW(t) + \tilde{\gamma}^{\ddagger}(t)d\tilde{W}(t)] - i\lambda[dW^{\ddagger}(t)\gamma_\nu(t) + d\tilde{W}^{\ddagger}(t)\tilde{\gamma}_\nu(t)] \\ &= i[\gamma^{\ddagger}(t)d'W(t) + \tilde{\gamma}^{\ddagger}(t)d'\tilde{W}(t)] - i\lambda[d'W^{\ddagger}(t)\gamma_\nu(t) + d'\tilde{W}^{\ddagger}(t)\tilde{\gamma}_\nu(t)] \\ &= :d'\hat{M}(t):. \end{aligned} \tag{5.16}$$

Quantum Langevin Equations

The quantum Langevin equation of the Ito type is given by

$$\begin{aligned} dA(t) = &\ i[\hat{H}_S(t),\ A(t)]dt \\ &+\kappa\{(1-2\lambda)(\gamma^{\ddagger}(t)[\gamma_\nu(t),\ A(t)] + \tilde{\gamma}^{\ddagger}(t)[\tilde{\gamma}_\nu(t),\ A(t)]) \\ &\quad +[\gamma^{\ddagger}(t),\ A(t)]\gamma_\nu(t) + [\tilde{\gamma}^{\ddagger}(t),\ A(t)]\tilde{\gamma}_\nu(t)\}dt \\ &+2\kappa(\bar{n}+\nu)[\tilde{\gamma}^{\ddagger}(t),\ [\gamma^{\ddagger}(t),\ A(t)]]dt \\ &-\{[\gamma^{\ddagger}(t),\ A(t)]dW_t + [\tilde{\gamma}^{\ddagger}(t),\ A(t)]d\tilde{W}_t\} \\ &\quad +\lambda\{dW_t^{\ddagger}[\gamma_\nu(t),\ A(t)] + d\tilde{W}_t^{\ddagger}[\tilde{\gamma}_\nu(t),\ A(t)]\} \end{aligned} \tag{5.17}$$

$$\begin{aligned} = &\ i[\hat{H}_S(t),\ A(t)]dt \\ &+\kappa\{\gamma^{\ddagger}(t)[\gamma_\nu(t),\ A(t)] + \tilde{\gamma}^{\ddagger}(t)[\tilde{\gamma}_\nu(t),\ A(t)] \\ &\quad +(1-2\lambda)([\gamma^{\ddagger}(t),\ A(t)]\gamma_\nu(t) + [\tilde{\gamma}^{\ddagger}(t),\ A(t)]\tilde{\gamma}_\nu(t))\}dt \\ &+2\kappa(\bar{n}+\nu)[\tilde{\gamma}^{\ddagger}(t),\ [\gamma^{\ddagger}(t),\ A(t)]]dt \\ &-\{[\gamma^{\ddagger}(t),\ A(t)]dW(t) + [\tilde{\gamma}^{\ddagger}(t),\ A(t)]d\tilde{W}(t)\} \\ &\quad +\lambda\{dW^{\ddagger}(t)[\gamma_\nu(t),\ A(t)] + d\tilde{W}^{\ddagger}(t)[\tilde{\gamma}_\nu(t),\ A(t)]\} \end{aligned} \tag{5.18}$$

with $\hat{H}_S(t) = \hat{V}_f^{-1}(t)\hat{H}_S\hat{V}_f(t) = H_S(t) - \tilde{H}_S(t)$. Note that the quantum Langevin equation is written by means of the quantum Brownian motion in the Schrödinger (the interaction) representation (the input field [39]) in (5.17), and by means of that in the Heisenberg representation (the output field [39]) in (5.18).

The quantum Langevin equation for the bra-vector state, $\langle\!\langle 1|A(t)$, reduces to

$$\begin{aligned} d\langle\!\langle 1|A(t) = &\ i\langle\!\langle 1|[H_S(t),\ A(t)]dt \\ &-\kappa\left\{\langle\!\langle 1|[A(t),\ a^\dagger(t)]a(t) + \langle\!\langle 1|a^\dagger(t)[a(t),\ A(t)]\right\}dt \\ &+2\kappa\bar{n}\langle\!\langle 1|[a(t),\ [A(t),\ a^\dagger(t)]]dt \\ &+\langle\!\langle 1|[A(t),\ a^\dagger(t)]\sqrt{2\kappa}\ dB_t + \langle\!\langle 1|\sqrt{2\kappa}\ dB_t^\dagger[a(t),\ A(t)] \end{aligned} \tag{5.19}$$

$$\begin{aligned} = &\ i\langle\!\langle 1|[H_S(t),\ A(t)]dt \\ &-\kappa(1-2\lambda)\left\{\langle\!\langle 1|[A(t),\ a^\dagger(t)]a(t) + \langle\!\langle 1|a^\dagger(t)[a(t),\ A(t)]\right\}dt \\ &+2\kappa\bar{n}\langle\!\langle 1|[a(t),\ [A(t),\ a^\dagger(t)]]dt \\ &+\langle\!\langle 1|[A(t),\ a^\dagger(t)]\sqrt{2\kappa}\ dB(t) + \langle\!\langle 1|\sqrt{2\kappa}\ dB^\dagger(t)[a(t),\ A(t)]. \end{aligned} \tag{5.20}$$

The relation between the expression (5.19) and (5.20) can be interpreted as follows. Substituting the *solution* of the Heisenberg random force operators (5.13) and (5.14) for $dB(t)$ and $dB^\dagger(t)$, respectively, into (5.20), we obtain the quantum Langevin equation (5.19) which does not depend on the non-commutativity parameter λ.

The quantum Langevin equations for $a(t)$ and $a^\dagger(t)$ of the system reduce to

$$da(t) = (-i\omega - \kappa)\,a(t)dt + dW_t - 2(1-\lambda)\nu\kappa\left[\tilde{a}^\dagger(t) - a(t)\right]dt - \lambda\nu d\tilde{W}_t^{\ddagger}, \tag{5.21}$$

$$da^\dagger(t) = (i\omega - \kappa)\,a^\dagger(t)dt + d\tilde{W}_t + 2(1-\lambda)\mu\kappa\left[a^\dagger(t) - \tilde{a}(t)\right]dt + \lambda\mu dW_t^{\ddagger}. \tag{5.22}$$

Note that the last two terms in the above equations disappear when one applies $\langle\!\langle 1|$ to them, i.e., applying $\langle\!\langle 1| = \langle 1|\langle|$ to (5.21) and (5.22), we obtain, for any values of λ, μ and ν, the quantum Langevin equations of the vectors $\langle\!\langle 1|a(t)$ and $\langle\!\langle 1|a^\dagger(t)$ in the forms

$$d\langle\!\langle 1|a(t) = -i\omega\langle\!\langle 1|a(t)dt - \kappa\langle\!\langle 1|a(t)dt + \sqrt{2\kappa}\langle\!\langle 1|dB_t, \tag{5.23}$$

$$d\langle\!\langle 1|a^\dagger(t) = i\omega\langle\!\langle 1|a^\dagger(t)dt - \kappa\langle\!\langle 1|a^\dagger(t)dt + \sqrt{2\kappa}\langle\!\langle 1|dB_t^\dagger. \tag{5.24}$$

Note that these equations have the same structure as those in (1.3) and in (1.37).

For $\lambda = 0$, (5.21) and (5.22) become, respectively, to

$$da(t) = -i\omega a(t)dt - \kappa\tilde{a}^\dagger(t)dt + dW_t, \quad da^\dagger(t) = i\omega a^\dagger(t)dt - \kappa\tilde{a}(t)dt + d\tilde{W}_t \tag{5.25}$$

where we put $\mu = \nu = 1/2$, for simplicity. For $\lambda = 1$, (5.21) and (5.22) read, for any value of μ,

$$da(t) = -i\omega a(t)dt - \kappa a(t)dt + \sqrt{2\kappa}dB_t, \quad da^\dagger(t) = i\omega a^\dagger(t)dt - \kappa a^\dagger(t)dt + \sqrt{2\kappa}dB_t^\dagger, \tag{5.26}$$

respectively, which should be compared with (1.37).

6 Application to Quantum Kramers Equation

Quantum Master Equation

Let us find out the general structure of hat-Hamiltonian which is bilinear in $(x,\ p,\ \tilde{x},\ \tilde{p})$. The operators x, p and their tilde conjugates satisfy the canonical commutation relations $[x,\ p] = i$ and $[\tilde{x},\ \tilde{p}] = -i$.

The conditions $(i\hat{H})^\sim = i\hat{H}$ and $\langle 1|\hat{H} = 0$ give us the general expression

$$\hat{H} = \hat{H}_S + i\hat{\Pi}, \quad \hat{H}_S = H_S - \tilde{H}_S, \quad H_S = p^2/2m + m\omega^2x^2/2, \tag{6.1}$$

$$\hat{\Pi} = \hat{\Pi}_R + \hat{\Pi}_D, \quad \hat{\Pi}_R = -i\kappa\,(x-\tilde{x})\,(p+\tilde{p})\,/2, \quad \hat{\Pi}_D = -\kappa m\omega(1+2\bar{n})\,(x-\tilde{x})^2\,/2. \tag{6.2}$$

Here, we neglected the diffusion in x-space. The Schrödinger equation (2.2) with (6.1) gives the *quantum Kramers equation* [40].

The Heisenberg equation for the dissipative system is given by

$$d/dt\ x(t) = i[\hat{H}(t),\ x(t)] = p(t)/m + \kappa\,[x(t)-\tilde{x}(t)]\,/2, \tag{6.3}$$

$$d/dt\ p(t) = -m\omega^2x(t) - \kappa\,[p(t)+\tilde{p}(t)]\,/2 + i\kappa m\omega(1+2\bar{n})\,[x(t)-\tilde{x}(t)]\,. \tag{6.4}$$

Applying the bra-vacuum $\langle 1|$ of the relevant system, we have the equations for the vectors:

$$d/dt\ \langle 1|x(t) = \langle 1|p(t)/m, \quad d/dt\ \langle 1|p(t) = -m\omega^2\langle 1|x(t) - \kappa\langle 1|p(t). \tag{6.5}$$

Non-unitary Stochastic Time-Evolution

The stochastic Liouville equation of the Ito type (4.3) is given with

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}dt + d\hat{M}_t, \quad d\hat{M}_t = (x-\tilde{x})(dX_t + d\tilde{X}_t), \quad dX_t = \sqrt{\kappa m\omega}\ (dB_t + dB_t^\dagger)/2. \tag{6.6}$$

Here, dB_t, $dB_t^\dagger$ and their tilde conjugates are the operators representing quantum Brownian motion. The martingale operator $d\hat{M}_t$ satisfies (5.8), and the generalized fluctuation-dissipation theorem of the second kind (4.12). Taking the random average with respect to the stochastic process dB_t, the stochastic Liouville equation (4.3) reduces to the quantum master equation (2.2) for $|0(t)\rangle = \langle|0_f(t)\rangle\rangle$ with (6.1).

The the Langevin equation for this hat-Hamiltonian is given by

$$dx(t) = i[\hat{\mathcal{H}}_f(t)dt,\ x(t)] - d\hat{M}(t)\ [d\hat{M}(t),\ x(t)] = p(t)dt/m + \kappa\,[x(t)-\tilde{x}(t)]\,dt/2, \tag{6.7}$$

$$dp(t) = -m\omega^2x(t)dt - \kappa\,[p(t)+\tilde{p}(t)]\,dt/2 - (dX_t + d\tilde{X}_t) \tag{6.8}$$

where we used the properties $dX(t) = dX_t$ and $d\tilde{X}(t) = d\tilde{X}_t$. Applying the bra vacuum $\langle\langle 1|$ to (6.7) and (6.8), we have the Langevin equations for vectors

$$d\langle\langle 1|x(t) = \langle\langle 1|p(t)dt/m, \qquad d\langle\langle 1|p(t) = -m\omega^2\langle\langle 1|x(t)dt - \kappa\langle\langle 1|p(t)dt - 2\langle\langle 1|dX_t. \tag{6.9}$$

The averaged equation of motion is given by applying $|0\rangle$ to (6.9) in the forms

$$d/dt\ \langle\langle x(t)\rangle\rangle = \langle\langle p(t)\rangle\rangle/m, \qquad d/dt\ \langle\langle p(t)\rangle\rangle = -m\omega^2\langle\langle x(t)\rangle\rangle - \kappa\langle\langle p(t)\rangle\rangle \tag{6.10}$$

where $\langle\langle\cdots\rangle\rangle = \langle 1|\langle|\cdots|\rangle|1\rangle$. These averaged equations can be also derived from (6.3) and (6.4) by taking the average $\langle\langle\cdots\rangle\rangle$.

Unitary Stochastic Time-Evolution

The martingale operator representing position-position interaction may be given by $dM_t^U = xdX_t - \tilde{x}d\tilde{X}_t$. We did not include the crossing terms between tilde and non-tilde operators to be consistent with the microscopic interaction Hamiltonian. The generalized fluctuation-dissipation theorem for this martingale operator is given by $dM_t^U dM_t^U = -2\hat{\Pi}^U dt$ with $\hat{\Pi}^U = -\kappa m\omega(1+2\bar{n})\,(x-\tilde{x})^2\,/8$. Then, the Ito stochastic hat-Hamiltonian becomes $\hat{\mathcal{H}}_{f,t}^U dt = \hat{H}^U dt + dM_t^U$ with $\hat{H}^U = \hat{H}_S + i\hat{\Pi}^U$. Here, $\hat{H}^U$ is the hat-Hamiltonian for the quantum master equation. Note that the present quantum master equation is different from (2.2) with (6.1) and (6.2).

The Langevin equations for $x(t)$ and $p(t)$ become

$$dx(t) = p(t)dt/m, \qquad dp(t) = -m\omega^2 x(t)dt - dX_t \tag{6.11}$$

where we used the fact $dX(t) = dX_t$. Applying $\langle\!\langle 1|$ to (6.11), we have the Langevin equations for the vectors $\langle\!\langle 1|x(t)$ and $\langle\!\langle 1|p(t)$ in the forms

$$d\langle\!\langle 1|x(t) = \langle\!\langle 1|p(t)dt/m, \qquad d\langle\!\langle 1|p(t) = -m\omega^2\langle\!\langle 1|x(t)dt - \langle\!\langle 1|dX_t \tag{6.12}$$

which are different from (6.9).

7 What is the Origin of Dissipation?

Within the system of *non-unitary* time-evolution generator $V_f(t)$ $(\lambda = 0)$ which is constituted of the commutative random force operators dW_t and $d\tilde{W}_t$, the random force operators $dW(t)$ and $dX(t)$ in the Heisenberg representation is, respectively, equal to dW_t and dX_t in the Schrödinger representation, i.e.,

$$dW(t) = dW_t, \qquad dX(t) = dX_t. \tag{7.1}$$

In the application of the system of *unitary* time-evolution generator $(\lambda = 1)$ to the damped harmonic oscillator, the random force operators in the Heisenberg representation are related to those in the Schrödinger representation by

$$dW(t) = dW_t - \kappa\gamma_\nu(t)dt, \qquad dW^{\text{♀}}(t) = dW_t^{\text{♀}} - \kappa\gamma^{\text{♀}}(t)dt. \tag{7.2}$$

The second terms show up because of the non-commutativity, whose appearance is essential in order to make the unitary system consistent with corresponding quantum master equation. The martingale operator is constituted by non-commutative random force operators due to the linear dissipative coupling between the relevant and irrelevant sub-systems.

On the contrary, in the application of the *unitary* time-evolution generator $(\lambda = 1)$ to the quantum Kramers equation, where the martingale operator is constituted only by commutative random force operators because of the position-position coupling between the relevant and irrelevant sub-systems, the random force operators in the Heisenberg representation is equal to those in the Schrödinger representation, i.e.,

$$dX(t) = dX_t. \tag{7.3}$$

Therefore, the unitary system cannot be consistent with corresponding quantum master equation.

The above investigation tells us that the origin of dissipation cannot be quantum mechanical. In spite of this unsatisfactory nature of the unitary system, it is attractive since hat-Hamiltonian for microscopic system is hermitian and there is no mixing terms between tilde and non-tilde operators, i.e., the hat-Hamiltonian should have the structure $\hat{H} = H - \tilde{H}$ with $H^\dagger = H$ for microscopic systems. In fact, we succeeded to extract the correct stochastic hat-Hamiltonian for the stochastic Kramers equation by an appropriate coarse graining of operators (the *stochastic mapping*) in time and corresponding renormalization of physical quantities [41]. The simple limit [42] does not give us the correct Kramers equation. This something touchy situation should be investigated based on the unified system of stochastic differential equations shown in this paper. It will be reported in the future publications.

A Derivations

A.1 Solution of (1.11)

With the non-dimensional parameters $\xi = \sqrt{m\omega}x$ and $\lambda = 2E/\omega$, (1.11) reduces to

$$d^2/d\xi^2\ u + (\lambda - \xi^2)u = 0. \tag{A.1}$$

With the transformation $u(\xi) = H(\xi)\mathrm{e}^{-\xi^2/2}$, (A.1) further reduces to $H'' - 2\xi H' - (\lambda - 1)H = 0$ which reminds us with the Hermite polynomials defined by $\mathrm{e}^{-s^2+2s\xi} = \sum_{n=0}^{\infty}[H_n(\xi)/n!]s^n$ or $H_n(\xi) = \mathrm{e}^{\xi^2}(-d/d\xi)^n\mathrm{e}^{-\xi^2}$. Comparing with the differential equation (1.14) for the Hermite polynomials, we see that the energy is quantized as (1.12) and that the eigen-function belonging to the energy state is given by (1.13).

A.2 Solution of (1.31)

Putting $F(\xi,t) = R(\zeta)\mathrm{e}^{-2\kappa\lambda t}$ in (1.31), we have an eigen-value equation for the right-hand side eigen-functions in the form

$$\zeta R''(\zeta) + (1+\zeta)R'(\zeta) + R(\zeta) = -\lambda R(\zeta). \tag{A.2}$$

The differential equation (an eigen-value equation for the left-hand side eigen-functions) adjoint of (A.2) turns out to be

$$\zeta L''(\zeta) + (1-\zeta)L'(\zeta) = -\lambda L(\zeta). \tag{A.3}$$

Note that a further transformation $R(\zeta) = \mathrm{e}^{-\zeta} f(\zeta)$ in (A.2) gives us $\zeta f'' + (1-\zeta)f' = -\lambda f$.

Now, we remember that the Laguerre polynomials defined by

$$\sum_{\ell=0}^{\infty} L_\ell(\zeta)x^\ell = \frac{1}{1-x}\mathrm{e}^{-\zeta\frac{x}{1-x}}, \qquad L_\ell(\zeta) = \frac{1}{\ell!}\,\mathrm{e}^{\zeta}\,\frac{d^\ell}{d\zeta^\ell}(\mathrm{e}^{-\zeta}\zeta^\ell) = \sum_{k=0}^{\ell}(-)^k \begin{pmatrix} \ell \\ k \end{pmatrix} \frac{\zeta^k}{k!} \tag{A.4}$$

satisfy the differential equation

$$\zeta L_\ell''(\zeta) + (1-\zeta)L_\ell'(\zeta) + \ell L_\ell(\zeta) = 0 \qquad (\ell = 0,1,2,\cdots). \tag{A.5}$$

For example, $L_\ell(\zeta)$'s are given by $L_0(\zeta) = 1$, $L_1(\zeta) = 1-\zeta$, $L_2(\zeta) = 1 - 2\zeta + \zeta^2/2$, $L_3(\zeta) = 1 - 3\zeta + 3\zeta^2/2 - \zeta^3/6$, and satisfy $L_\ell(0) = 1$, $L_\ell'(0) = -\ell$.

We notice in the comparison of (A.5) with (A.3) that the eigen-value λ should be $\lambda = \ell$ ($\ell = 0,1,2,\cdots$).

A.3 Derivation of (1.36)

The substitution of the expression a_ℓ into (1.32) gives

$$\begin{aligned}
F(\xi,t) &= \frac{1}{n}\sum_{\ell=0}^{\infty}\int_0^\infty d\zeta' L_\ell(\zeta')R_\ell(\zeta)\ \mathrm{e}^{-\zeta'\bar{n}/n}\ \mathrm{e}^{-2\kappa\ell t} \\
&= \frac{1}{n}\sum_{\ell=0}^{\infty}\int_0^\infty d\zeta' L_\ell(\zeta')R_\ell(\zeta)\ \mathrm{e}^{-\zeta'}\ \mathrm{e}^{-\zeta'(\bar{n}-n)/n}\ \mathrm{e}^{-2\kappa\ell t} \\
&= \frac{1}{\bar{n}}\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} R_\ell(\zeta)\left(\frac{\bar{n}-n}{\bar{n}}\right)^k \mathrm{e}^{-2\kappa\ell t}\int_0^\infty d\zeta' L_\ell(\zeta')R_k(\zeta') \\
&= \frac{1}{\bar{n}}\sum_{\ell=0}^{\infty} R_\ell(\zeta)\left(\frac{\bar{n}-n}{\bar{n}}\right)^\ell \mathrm{e}^{-2\kappa\ell t} = \frac{1}{\bar{n}}\mathrm{e}^{-\zeta}\sum_{\ell=0}^{\infty} L_\ell(\zeta)\left(\frac{\bar{n}-n}{\bar{n}}\mathrm{e}^{-2\kappa t}\right)^\ell = \frac{1}{n(t)}\mathrm{e}^{-\xi/n(t)}
\end{aligned} \tag{A.6}$$

with $n(t)$ in (1.36). For the second equality, $\mathrm{e}^{-\zeta'\bar{n}/n}$ was divided into two exponentials. For the third equality, we used the generating function (A.4) of the Laguerre polynomials for $\mathrm{e}^{-\zeta'(\bar{n}-n)/n}$ and (1.33). For the fourth equality, the orthogonality (1.34) was used. For the final equality, we used the formulae (1.33) and (A.4) again.

A.4 Derivation of (1.41)

By making use of the connection formula (D.4) between the Ito and Stratonovich products, one has

$$\begin{aligned}
dW(t)\circ f(z,t) &= dW(t)\cdot f(z,t) + dW(t)df(z,t)/2 \\
&= dW(t)\cdot f(z,t) - \partial/\partial z^*\ dW(t)dW(t)^* f(z,t)/2 - \partial/\partial z\ dW(t)dW(t)f(z,t)/2 \\
&= dW(t)\cdot f(z,t) - \kappa\bar{n}\ \partial/\partial z^*\ f(z,t),
\end{aligned} \tag{A.7}$$

$$dW(t)^*\circ f(z,t) = dW(t)^*\cdot f(z,t) - \kappa\bar{n}\ \partial/\partial z\ f(z,t) \tag{A.8}$$

where we used the fluctuation-dissipation relations

$$dW(t)dW(t)^* = 2\kappa\bar{n}dt, \quad dW(t)dW(t) = 0, \quad dW(t)^*dW(t)^* = 0 \tag{A.9}$$

within the stochastic convergence. Substituting (A.7) and (A.8) into the Stratonovich type (1.39), i.e.,

$$\begin{aligned}
df(z,t) = {}& [-i\omega(\partial/\partial z^*\ z^* - \text{c.c.}) + \kappa(\partial/\partial z^*\ z^* + \text{c.c.})]dtf(z,t) \\
&-\partial/\partial z^*\ dW(t)^*\circ f(z,t) - \partial/\partial z\ dW(t)\circ f(z,t),
\end{aligned} \tag{A.10}$$

we obtain the stochastic Liouville equation of the Ito type (1.41) with the properties $\langle dW(t)\cdot f(z,t)\rangle = 0$ and $\langle dW(t)^*\cdot f(z,t)\rangle = 0$ which provide us with

$$\langle dW(t)\circ f(z,t)\rangle = -\kappa\bar{n}\ \partial/\partial z^*\ P(z,t), \quad \langle dW(t)^*\circ f(z,t)\rangle = -\kappa\bar{n}\ \partial/\partial z\ P(z,t). \tag{A.11}$$

A.5 Derivations of the Semi-Free Hat-Hamiltonian

A.5.1 The Principle of Correspondence

The principle of correspondence is defined by [43, 1, 2] $\rho_S(t) \longleftrightarrow |0(t)\rangle$, $A_1\rho_S(t)A_2 \longleftrightarrow A_1\tilde{A_2}^\dagger|0(t)\rangle$. It is easy to see that the master equation (1.20) with the damping operator (1.22) reduces to the Schrödinger equation (2.2) accompanied by the hat-Hamiltonian (2.10) and (2.11).

It is Crawford [44] who noticed first that the introduction of *two* kinds of operators enables us to handle the quantum Liouville equation as the Schrödinger equation.

A.5.2 Axiomatic Derivation

The hat-Hamiltonian of the semi-free field is bi-linear in $(a, \tilde{a}, a^\dagger, \tilde{a}^\dagger)$, and is invariant under the phase transformation $a \to a\mathrm{e}^{i\theta}$:

$$\hat{H} = g_1 a^\dagger a + g_2 \tilde{a}^\dagger \tilde{a} + g_3 a\tilde{a} + g_4 a^\dagger \tilde{a}^\dagger + g_0, \tag{A.12}$$

where g's are time-dependent c-number complex functions. The operators a, $\tilde{a}^\dagger$, etc. satisfy the canonical commutation relation $[a_{\mathbf{k}},\ a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$ and $[\tilde{a}_{\mathbf{k}},\ \tilde{a}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$. The tilde and non-tilde operators are mutually commutative. Throughout this paper, we do not label explicitly the operators a, $\tilde{a}^\dagger$, etc. with a subscript $\mathbf{k}$ for specifying a momentum and/or other degrees of freedom. However, remember that we are dealing with a *dissipative quantum field*.

B1 in section 2.1 makes (A.12) tildian:

$$\hat{H} = \omega(a^\dagger a - \tilde{a}^\dagger \tilde{a}) + i\hat{\Pi}, \qquad \hat{\Pi} = c_1(a^\dagger a + \tilde{a}^\dagger \tilde{a}) + c_2 a\tilde{a} + c_3 a^\dagger \tilde{a}^\dagger + c_4 \tag{A.13}$$

where we introduced new quantities $\omega = \Re\mathrm{e}\ g_1 = -\Re\mathrm{e}\ g_2$, $c_1 = \Im\mathrm{m}\ g_1 = \Im\mathrm{m}\ g_2$, $c_2 = \Im\mathrm{m}\ g_3$, $c_3 = \Im\mathrm{m}\ g_4$ and $c_4 = \Im\mathrm{m}\ g_0$. With the help of **Tool 3** in sub-section 2.1, i.e., $\langle 1|a^\dagger = \langle 1|\tilde{a}$, **B2** gives us relations $2c_1 + c_2 + c_3 = 0$ and $c_3 + c_4 = 0$. Then, (A.13) reduces to $\hat{\Pi} = c_1(a^\dagger a + \tilde{a}^\dagger \tilde{a}) + c_2 a\tilde{a} - (2c_1 + c_2)\, a^\dagger \tilde{a}^\dagger + (2c_1 + c_2)$.

Let us write down here the Heisenberg equations for a and $a^\dagger$ (see (2.6)):

$$d/dt\ a(t) = -i\omega a(t) + c_1 a(t) - (2c_1 + c_2)\, \tilde{a}^{\dagger\!\!\dagger}(t), \qquad d/dt\ a^{\dagger\!\!\dagger}(t) = i\omega a^{\dagger\!\!\dagger}(t) - c_1 a^{\dagger\!\!\dagger}(t) - c_2 \tilde{a}(t). \tag{A.14}$$

Since the semi-free hat-Hamiltonian $\hat{H}$ is not necessarily hermitian, we introduced the symbol $\dagger\!\!\dagger$ in order to distinguish it from the hermite conjugation $\dagger$. However in the following, we will use $\dagger$ instead of $\dagger\!\!\dagger$, for simplicity, unless it is confusing. By making use of the Heisenberg equations in (A.14), we obtain the equation of motion for a vector $\langle 1|a^\dagger(t)a(t)$ in the form

$$d/dt\ \langle 1|a^\dagger(t)a(t) = -2\kappa\langle 1|a^\dagger(t)a(t) + i\Sigma^<\langle 1| \tag{A.15}$$

where we introduced κ and $\Sigma^<$, respectively, by

$$\kappa = c_1 + c_2, \qquad \Sigma^< = i(2c_1 + c_2). \tag{A.16}$$

In deriving (A.15), we used **Tool 3** in order to eliminate tilde operators.

Applying the thermal ket-vacuum to (A.15), we obtain the equation of motion for the *one-particle distribution function* $n(t) = \langle 1|a^{\dagger\!\!\dagger}(t)a(t)|0\rangle = \langle 1|a^\dagger a|0(t)\rangle$ as

$$d/dt\ n(t) = -2\kappa n(t) + i\Sigma^<. \tag{A.17}$$

The equation (A.17) is the Boltzmann equation of the system. The function $\Sigma^<$ is given when the interaction hat-Hamiltonian is specified.

If it is assumed that there is only one stationary state, we can refer the stationary state as a thermal equilibrium state. We will assign the thermal equilibrium state to be specified by the Planck distribution function with temperature T: $n(t \to \infty) = \bar{n} = 1/(\mathrm{e}^{\omega/T} - 1)$. Then, we have from (A.17)

$$i\Sigma^< = 2\kappa\bar{n}. \tag{A.18}$$

In this case, the Boltzmann equation (A.17) reduces to (1.24).

Solving (A.16) with respect to c_1 and c_2, and substituting them into (A.13) with (A.18) for $\Sigma^<$, we finally arrive at the most general form of the semi-free hat-Hamiltonian (2.10) and (2.11).

A.6 Derivation of (2.12)

Let us treat $\langle 1|a\tilde{a}|0\rangle$ in two ways. In the first place,

$$\langle 1|a\tilde{a}|0\rangle = \langle 1|afa^\dagger|0\rangle = f\left(\langle 1|a^\dagger a|0\rangle + \langle 1|0\rangle\right) = f\,(n+1) \tag{A.19}$$

where we used the tilde conjugate of $a|0\rangle = f\tilde{a}^\dagger|0\rangle$ for the first equality, and the canonical commutation relation $[a, a^\dagger] = 1$ for the second. On the other hand,

$$\langle 1|a\tilde{a}|0\rangle = \langle 1|\tilde{a}a|0\rangle = \langle 1|a^\dagger a|0\rangle = n. \tag{A.20}$$

Here, for the first equality, we used **Tool 2** in sub-section 2.1 and, for the second equality, **Tool 3** with $A = a$. Equating (A.19) and (A.20) leads us to (2.12).

A.7 Derivation of (2.19)

The equation of motion (2.19) for the thermal doublet $\gamma(t)^\mu$ is derived as

$$\begin{aligned} d/dt\ \gamma(t)^\mu &= (dB(t)^{\mu\nu}/dt)a(t)^\nu + B(t)^{\mu\nu}da(t)^\nu/dt \\ &= \left[(dB(t)/dt)B^{-1}(t)\right]^{\mu\nu}\gamma(t)^\nu - i\left[B(t)\left(\omega\ 1 - i\kappa A\right)B^{-1}(t)\right]^{\mu\nu}\gamma(t)^\nu \\ &= -i\left[\omega\delta^{\mu\nu} - i\kappa\tau_3^{\mu\nu}\right]\gamma(t)^\nu \end{aligned} \tag{A.21}$$

where the matrix $\tau_3^{\mu\nu}$ is defined by $\tau_3^{11} = -\tau_3^{22} = 1$, $\tau_3^{12} = \tau_3^{21} = 0$. For the third equality, we used the relations

$$\frac{dB(t)}{dt} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\frac{dn(t)}{dt}, \qquad \frac{dB(t)}{dt}B^{-1}(t) = -\frac{dn(t)}{dt}\tau_+, \tag{A.22}$$

$$B(t)AB^{-1}(t) = \tau_3 + 2\left[n(t) - \bar{n}\right]\tau_+, \qquad \tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{A.23}$$

The Boltzmann equation (1.24) has been used also.

A.8 Derivation of (3.7)

We assume that the interaction hat-Hamiltonian is globally gauge invariant and bilinear:

$$\begin{aligned} \hat{H}_t'' = i\{&h_1 a^\dagger b(t) + h_2 a^\dagger\tilde{b}^\dagger(t) + h_3\tilde{a}\ b(t) + h_4\tilde{a}\ \tilde{b}^\dagger(t) \\ &+ h_5\tilde{a}^\dagger\tilde{b}(t) + h_6\tilde{a}^\dagger b^\dagger(t) + h_7 a\ \tilde{b}(t) + h_8 a\ b^\dagger(t)\} \end{aligned} \tag{A.24}$$

where the quantities h's are time-independent complex c-numbers.

The tildian (**B1** in section 2.1) $(i\hat{H}_t'')^\sim = i\hat{H}_t''$ gives us

$$h_1^* = h_5, \quad h_2^* = h_6, \quad h_3^* = h_7, \quad h_4^* = h_8. \tag{A.25}$$

The requirement (3.6) that the Schrödinger equation (3.5) has the characteristics of the conservation of probability within the relevant system leads us to the relations

$$h_1 + h_3 = 0, \quad h_2 + h_4 = 0. \tag{A.26}$$

With (A.25) and (A.26), (A.24) reduces to

$$\hat{H}_t'' = i[\gamma^{\natural}\beta(t) + \text{t.c.}] \tag{A.27}$$

where we introduced $\gamma^{\natural} = a^\dagger - \tilde{a}$ and $\beta(t) = h_1 b(t) + h_2\tilde{b}^\dagger(t)$.

Let us consider the moments

$$\langle\beta(t)\tilde{\beta}(t)\rangle = (h_1 + h_2)\{h_1^*\langle b^\dagger(t)b(t)\rangle + h_2^*\langle b(t)b^\dagger(t)\rangle\}, \tag{A.28}$$

$$\langle\tilde{\beta}(t)\beta(t)\rangle = (h_1^* + h_2^*)\{h_1\langle b^\dagger(t)b(t)\rangle + h_2\langle b(t)b^\dagger(t)\rangle\} \tag{A.29}$$

where we used **Tool 2** and **Tool 3** in sub-section 2.1 for $b(t)$, $b^\dagger(t)$ etc.. We are using the symbol $\langle\cdots\rangle = \langle|\cdots|t\rangle$ without specifying the dynamics which determines the ket-vacuum $|t\rangle$ of the external system. For the present purpose, the details of its dynamics are not required. With the further use of the property **Tool 2** of the commutativity $\langle\beta(t)\tilde{\beta}(t)\rangle = \langle\tilde{\beta}(t)\beta(t)\rangle$ provides us with the relations $(h_1 + h_2)h_1^* = (h_1^* + h_2^*)h_1$ and $(h_1 + h_2)h_2^* = (h_1^* + h_2^*)h_2$ which reduce to $h_1^*h_2 = h_1h_2^* = (h_1^*h_2)^*$, and allow us to put $h_1 = \mu e^{i\theta}$ and $h_2 = \nu e^{i\theta}$ where $\mu = |h_1|$ and $\nu = |h_2|$.

The vector $\langle|\beta(t)$ is calculated as $\langle|\beta(t) = (\mu + \nu)e^{i\theta}\langle|b(t)$. The further requirement that the *norm* of $\langle|\beta(t)$ should be equal to that of $\langle|b(t)$, i.e., $\|\langle|\beta(t)\| = \|\langle|b(t)\|$, leads us to the relation (3.8). This requirement of norms indicates that the *intensities* of the external operators $\beta(t)$ and $b(t)$ should be equal, and makes the operators $\beta^{\natural}(t)$, defined by (3.3) and (3.8) with the real numbers μ and ν satisfying the last equation in (3.8), a set of canonical operators, i.e., (3.9). Putting the phase factor $e^{i\theta}$ on $b(t)$ and $\tilde{b}^\dagger(t)$, we have the non-hermitian interaction hat-Hamiltonian (3.7) with $\gamma^{\natural}$ and $\beta(t)$ defined in (3.8).

A.9 Derivation of (4.6)

Let us begin with the differentiation of $\hat{V}_f(t)\hat{V}_f^{-1}(t) = 1$ with respect to time t within Stratonovich product, i.e., $d[\hat{V}_f(t)\hat{V}_f^{-1}(t)] = d\hat{V}_f(t) \circ \hat{V}_f^{-1}(t) + \hat{V}_f(t) \circ d\hat{V}_f^{-1}(t) = 0$ whose structure is the same as the one for the analytic function or the "analytic operator" within quantum mechanics and quantum field theory. Rewriting this into the differential formula within Ito product with the help of the connection formula (D.4) between the stochastic products, we have $d[\hat{V}_f(t)\hat{V}_f^{-1}(t)] = d\hat{V}_f(t)\hat{V}_f^{-1}(t) + \hat{V}_f(t)d\hat{V}_f^{-1}(t) + d\hat{V}_f(t)d\hat{V}_f^{-1}(t) = 0$ which can be arranged up to the order of $\mathcal{O}(dt)$ as

$$d\hat{V}_f^{-1}(t) = -[\hat{V}_f(t) + d\hat{V}_f(t)]^{-1} d\hat{V}_f(t)\hat{V}_f^{-1}(t) \approx [1 - \hat{V}_f^{-1}(t)d\hat{V}_f(t)]\hat{V}_f^{-1}(t)\ d\hat{V}_f(t)\hat{V}_f^{-1}(t). \tag{A.30}$$

Substituting (4.4) into (A.30), we obtain $d\hat{V}_f^{-1}(t) = \hat{V}_f^{-1}(t)\ i[\hat{\mathcal{H}}_{f,t}dt + i\hat{\mathcal{H}}_{f,t}dt\ \hat{\mathcal{H}}_{f,t}dt]$ which reduces to (4.6) by making use of $\hat{\mathcal{H}}_{f,t}dt\ \hat{\mathcal{H}}_{f,t}dt \approx\ :d\hat{M}_t:\ :d\hat{M}_t:$ that is correct up to the order of $\mathcal{O}(dt)$.[6]

A.10 Derivation of (4.7)

The Substitution of (4.4) and (4.6) into

$$dA(t) = d\hat{V}_f^{-1}(t)A\hat{V}_f(t) + \hat{V}_f^{-1}(t)Ad\hat{V}_f(t) + d\hat{V}_f^{-1}(t)Ad\hat{V}_f(t) \tag{A.31}$$

gives the Ito formula (4.7) up to the order of $\mathcal{O}(dt)$. Note that (A.31) is derived by applying the connection formula (D.4) to the differential formula $dA(t) = d\hat{V}_f^{-1}(t) \circ A\hat{V}_f(t) + \hat{V}_f^{-1}(t)A \circ d\hat{V}_f(t)$ within the Stratonovich calculus which has the same structure as the one for the analytic function or for the "analytic operator" within quantum mechanics and quantum field theory.

A.11 Derivation of (4.10) and (4.11)

Rewriting the stochastic Liouville equation of Ito type (4.3) into the one of Stratonovich type as $d|0_f(t)\rangle = -i\hat{\mathcal{H}}_{f,t}dt \circ |0_f(t)\rangle + i\hat{\mathcal{H}}_{f,t}dt\ d|0_f(t)\rangle$ with the help of the connection formula in the Schrödinger representation which has the same structure as (D.4), we obtain up to the order of $\mathcal{O}(dt)$

$$d|0_f(t)\rangle = -i(1 - i\hat{\mathcal{H}}_{f,t}dt/2)^{-1}\hat{\mathcal{H}}_{f,t}dt \circ d|0_f(t)\rangle \approx -i[\hat{\mathcal{H}}_{f,t}dt + i\hat{\mathcal{H}}_{f,t}dt\ \hat{\mathcal{H}}_{f,t}dt/2] \circ |0_f(t)\rangle. \tag{A.32}$$

The substitution of $\hat{\mathcal{H}}_{f,t}dt\ \hat{\mathcal{H}}_{f,t}dt \approx\ :d\hat{M}_t:\ :d\hat{M}_t:$ into (A.32) gives (4.10) within the accuracy up to the order of $\mathcal{O}(dt)$.

Substituting (4.5) into (4.7), we have

$$dA(t) = i[\hat{H}(t)dt,\ A(t)] + i[:d'\hat{M}(t):,\ A(t)] -\ :d'\hat{M}(t):\ \ [:d'\hat{M}(t):,\ A(t)] \tag{A.33}$$

where we omitted the symbol "·" for simplicity. Rewriting the term $[:d'\hat{M}(t):,\ A(t)]$ as $[:d'\hat{M}(t):,\ A(t)] = [:d'\hat{M}(t) :\ \circ\!\!,\ A(t)] - [:d'\hat{M}(t):,\ dA(t)]/2$, and substituting the quantum Langevin equation of Ito type (4.7) for $dA(t)$, we have

$$[:d'\hat{M}(t):,\ A(t)] = [:d'\hat{M}(t):\ \circ\!\!,\ A(t)] - [:d'\hat{M}(t):,\ [:d'\hat{M}(t):,\ A(t)]]/2 \tag{A.34}$$

within the accuracy up to the order of $\mathcal{O}(dt)$. Here, we introduced the notation $[X \circ\!\!, Y] = X \circ Y - Y \circ X$. The substitution of (A.34) into (A.33) gives (4.11) as

$$\begin{aligned} dA(t) &= i[\hat{H}(t)dt,\ A(t)] + i[:d'\hat{M}(t):\ \circ\!\!,\ A(t)] - \{:d'\hat{M}(t):,\ [:d'\hat{M}(t):,\ A(t)]\}/2 \\ &= i[\hat{H}(t)dt,\ A(t)] + i[:d'\hat{M}(t):\ \circ\!\!,\ A(t)] - [:d'\hat{M}(t):\ :d'\hat{M}(t):,\ A(t)]/2. \end{aligned} \tag{A.35}$$

Note that the quantum Langevin equation of the Stratonovich type has the same structure as the Heisenberg equation for the "analytic operator" within quantum mechanics and quantum field theory.

A.12 Derivation of (4.13) and (4.15)

Let us begin with the relation

$$d(\Gamma_t\ \hat{V}_f(t)) = d\Gamma_t\ \hat{V}_f(t) + \Gamma_t\ d\hat{V}_f(t) + d\Gamma_t\ d\hat{V}_f(t) = d\Gamma_t\ \hat{V}_f(t) + \Gamma_{t+dt}\ d\hat{V}_f(t) \tag{A.36}$$

[6] One can proceed the derivation similarly starting with $d[\hat{V}_f^{-1}(t)\hat{V}_f(t)] = 0$.

an operator Γ_t which is linearly dependent on $B_t^{\#}$ (# = null, † and/or tilde) where we introduced the notation $\Gamma_{t+dt} = \Gamma_t + d\Gamma_t$. The increment $d\Gamma(t)$ of the Heisenberg operator $\Gamma(t) = \hat{V}_f^{-1}(t)\ \Gamma_t\ \hat{V}_f(t)$ is estimated as follows:

$$\begin{aligned} d\Gamma(t) &= (d\hat{V}_f^{-1}(t))\Gamma_t\ \hat{V}_f(t) + \hat{V}_f^{-1}(t)\ d(\Gamma_t\ \hat{V}_f(t)) + (d\hat{V}_f^{-1}(t))d(\Gamma_t\ \hat{V}_f(t)) \\ &= d'\Gamma(t) + \hat{V}_f^{-1}(t)\{\hat{V}_f(t)\ d\hat{V}_f^{-1}(t)\ \Gamma_{t+dt} + \Gamma_{t+dt}\ d\hat{V}_f(t)\ \hat{V}_f^{-1}(t) \\ &\qquad + \hat{V}_f(t)\ d\hat{V}_f^{-1}(t)\ \Gamma_{t+dt}\ d\hat{V}_f(t)\ \hat{V}_f^{-1}(t)\}\hat{V}_f(t) \\ &= d'\Gamma(t) + \hat{V}_f^{-1}(t)\{i\hat{\mathcal{H}}_{f,t}^{-}dt\ \Gamma_{t+dt} - \Gamma_{t+dt}\ i\hat{\mathcal{H}}_{f,t}dt + \hat{\mathcal{H}}_{f,t}^{-}dt\ \Gamma_{t+dt}\ \hat{\mathcal{H}}_{f,t}dt\}\hat{V}_f(t) \\ &= d'\Gamma(t) + i[:d'\hat{M}(t):,\ d'\Gamma(t)] \end{aligned} \tag{A.37}$$

with $d'\Gamma(t) = \hat{V}_f^{-1}(t)\ d\Gamma_t\ \hat{V}_f(t)$. At the second equality in (A.37) we used (A.36), and at the third equality we have substituted (4.4) and (4.6). At the last equality, we substituted (4.5) with (4.6), and kept the terms up to the order of $\mathcal{O}(dt)$ by considering $dB_t \sim \mathcal{O}(\sqrt{dt})$, $:d\hat{M}_t:\sim \mathcal{O}(\sqrt{dt})$ and $\hat{H}dt \sim \mathcal{O}(dt)$. In the derivation, we used the properties

$$[\hat{H}dt,\ B_{t+dt}^{\#}] = 0, \quad [:d\hat{M}_t:,\ B_t^{\#}] = 0, \quad [:d\hat{M}_t:,\ B_{t+dt}^{\#}] = [:d\hat{M}_t:,\ dB_t^{\#}] \sim \mathcal{O}(dt). \tag{A.38}$$

We are omitting the Ito product symbol "·" for simplicity. For $\Gamma_t = B_t$ and $\Gamma_t = \hat{M}_t$, we have (4.13) and (4.14), respectively.

Similar estimation gives

$$\hat{V}_f^{-1}(t)\cdot d\Gamma_t \cdot \hat{V}_f(t) = \hat{V}_f^{-1}(t)\circ d\Gamma_t \circ \hat{V}_f(t) - (i/2)\ [:d'\hat{M}(t):,\ d'\Gamma(t)]. \tag{A.39}$$

For $\Gamma_t = B_t$ and $\Gamma_t = \hat{M}_t$, we have (4.15) and (4.16), respectively.

B Comment on Liouville Operator

Let us investigate the characteristics of the Liouville operator $iL \bullet = i[H,\ \bullet\ ]$ by considering the super-operator $A(t_2, t_1)$ defined by $A(t_2, t_1)X = \mathrm{e}^{iLt_2}A\mathrm{e}^{-iLt_1}X$. The part $\mathrm{e}^{-iLt_1}X$ can be analyzed as follows. Differentiate it with respect to t_1, we have $d/dt_1\ \mathrm{e}^{-iLt_1}X = -iL\mathrm{e}^{-iLt_1}X = -i[H,\ \mathrm{e}^{-iLt_1}X]$ and obtain $\mathrm{e}^{-iLt_1}X = \mathrm{e}^{-iHt_1}X\mathrm{e}^{iHt_1}$. Similar analysis with respect to t_2 gives $A(t_2, t_1)X = \mathrm{e}^{iHt_2}A\mathrm{e}^{-iHt_1}X\mathrm{e}^{iH(t_1-t_2)}$.

Then, we see that the Heisenberg superoperator

$$A(t)X = \mathrm{e}^{iLt}A\mathrm{e}^{-iLt}X = \mathrm{e}^{iHt}A\mathrm{e}^{-iHt}X \tag{B.1}$$

reduces to the usual Heisenberg operator $A(t) = \mathrm{e}^{iHt}A\mathrm{e}^{-iHt}$. We can interpret that the operation e^{-iLt} on X in (B.1) is canceled by the operation e^{iLt}.

C Boson Coherent State

The boson coherent state $|\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}|0)$ is the eigen-state of the annihilation operator a satisfying $a|\alpha) = \alpha|\alpha)$, and therefore its adjoint $(\alpha| = |\alpha)^\dagger$ satisfies $(\alpha|a^\dagger = (\alpha|\alpha^*$. The vacuums $|0)$ and $(0|$ are defined by $a|0) = 0$ and $(0|a^\dagger = 0$. The coherent state has the overlapping and consists of an over-completeness set, i.e.,

$$(\alpha_1|\alpha_2) = \mathrm{e}^{-\frac{1}{2}|\alpha_1|^2 - \frac{1}{2}|\alpha_2|^2 + \alpha_1^*\alpha_2}, \qquad \int \frac{d^2\alpha}{\pi}|\alpha)(\alpha| = 1. \tag{C.1}$$

Using the property $\mathrm{e}^{\alpha a^\dagger - \alpha^* a} = \mathrm{e}^{\alpha a^\dagger}\mathrm{e}^{-\alpha^* a}\mathrm{e}^{-|\alpha|^2/2}$, we have $|\alpha) = \mathrm{e}^{-|\alpha|^2/2}\sum_{\ell=0}^{\infty}(\alpha^\ell/\sqrt{\ell!})|\ell)$ where

$$|\ell) = [(a^\dagger)^\ell/\sqrt{\ell!}]|0) \tag{C.2}$$

is the eigen-function of the number operator $a^\dagger a$ satisfying $a^\dagger a|\ell) = \ell|\ell)$.

Now, let us derive the c-number function $P(z, t=0)$ introduced in (1.26) for the statistical operator of the canonical ensemble

$$\rho = \mathrm{e}^{-\omega a^\dagger a/T_0}/Z. \tag{C.3}$$

Representing it by the number states (C.2), we have

$$(\ell|\rho|m) = \delta_{\ell,m}\mathrm{e}^{-\omega m/T_0}/(1+n), \qquad n = 1/(\mathrm{e}^{\omega/T_0} - 1). \tag{C.4}$$

Here, n is the boson distribution function at temperature T_0.

Within the coherent state representation, we can calculate the matrix element $(\alpha|\rho|\alpha)$ as follows:

$$
\begin{aligned}
(\alpha|\rho|\alpha) &= \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}(\alpha|\ell)(\ell|\rho|m)(m|\alpha) = \frac{1}{1+n}\sum_{\ell=0}^{\infty}(\alpha|\ell)\mathrm{e}^{-\omega\ell/T_0}(\ell|\alpha) \\
&= \frac{1}{1+n}\mathrm{e}^{-|\alpha|^2/2}\sum_{\ell=0}^{\infty}\mathrm{e}^{-\omega\ell/T_0}\frac{|\alpha|^{2\ell}}{\ell!} = \frac{1}{1+n}\mathrm{e}^{-|\alpha|^2/(1+n)}.
\end{aligned}
\tag{C.5}
$$

Here, we used the matrix element (C.4) for the second equality, and the overlapping (C.1) for the third equality. Substituting (C.5) into

$$
(\alpha|\rho|\alpha) = \int\frac{d^2\alpha_1}{\pi}\ P(\alpha_1)\ (\alpha|\alpha_1)(\alpha_1|\alpha) = \int\frac{d^2\alpha_1}{\pi}\ P(\alpha_1)\ \mathrm{e}^{-(\alpha_1-\alpha)(\alpha_1^*-\alpha^*)},
\tag{C.6}
$$

we have the integral equation

$$
\frac{1}{1+n}\mathrm{e}^{-x^2/(1+n)}\mathrm{e}^{-y^2/(1+n)} = \int\frac{dx_1dy_1}{\pi}\ P(x_1,y_1)\ \mathrm{e}^{-(x_1-x)^2}\mathrm{e}^{-(y_1-y)^2},
\tag{C.7}
$$

where we have changed the integration variables by $\alpha = x+iy$. Inspecting (C.7), we can put the Gaussian form $P(x_1,y_1) = A\mathrm{e}^{-Kx_1^2}\mathrm{e}^{-Ky_1^2}$ without loss of generality. Then, the integral equation (C.7) reduces to $\mathrm{e}^{-x^2/(1+n)}\mathrm{e}^{-y^2/(1+n)}/(1+n) = A\mathrm{e}^{-|\alpha|^2K/(K+1)}/(K+1)$ which determines the coefficients A and K as $A = 1/n$ and $K = 1/n$. We finally have the c-number function corresponding to the statistical operator (C.3) as $P(\alpha) = \mathrm{e}^{-|\alpha|^2/n}/n$.

D Stochastic Multiplications

The definitions of the Ito [22] and the Stratonovich [21] multiplications are given, respectively, by

$$
X(t)\cdot dY(t) = X(t)\left[Y(t+dt)-Y(t)\right], \quad dX(t)\cdot Y(t) = \left[X(t+dt)-X(t)\right]Y(t),
\tag{D.1}
$$

$$
X(t)\circ dY(t) = \{[X(t+dt)+X(t)]/2\}\left[Y(t+dt)-Y(t)\right],
\tag{D.2}
$$

$$
dX(t)\circ Y(t) = \left[X(t+dt)-X(t)\right]\left[Y(t+dt)+Y(t)\right]/2
\tag{D.3}
$$

for arbitrary stochastic operators $X(t)$ and $Y(t)$ in the Heisenberg representation. From (D.1), (D.2) and (D.3), we have the formulae which connect the Ito and the Stratonovich products in the differential form as

$$
X(t)\circ dY(t) = X(t)\cdot dY(t) + dX(t)\cdot dY(t)/2, \quad dX(t)\circ Y(t) = dX(t)\cdot Y(t) + dX(t)\cdot dY(t)/2.
\tag{D.4}
$$

This can be proven by the identity

$$
\begin{aligned}
&\left[X(t+dt)-X(t)\right]\left[Y(t+dt)+Y(t)\right]/2 \\
&\quad = \left[X(t+dt)-X(t)\right]Y(t) + \left[X(t+dt)-X(t)\right]\left[Y(t+dt)-Y(t)\right]/2.
\end{aligned}
\tag{D.5}
$$

The connection formulae for the stochastic operators in the Schrödinger representation are given, in the same form as (D.4) for X_t and Y_t.

With the help of (D.1), (D.2) and (D.3), we obtain the differentiation formula for the product of two stochastic operators $X(t)Y(t)$ in the form

$$
\begin{aligned}
d[X(t)Y(t)] &= X(t+dt)Y(t+dt) - X(t)Y(t) \\
&= dX(t)\circ Y(t) + X(t)\circ dY(t) \qquad \text{(D.6)} \\
&= dX(t)\cdot Y(t) + X(t)\cdot dY(t) + dX(t)\ dY(t). \qquad \text{(D.7)}
\end{aligned}
$$

Therefore, if one uses the Stochastic multiplication of the Stratonovich type, the differentiation formula for the product of two stochastic operators turns out to be the same as those for “analytic” operators.

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